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A Numerical Analysis of the Nash Strategy for Weakly Coupled Large-Scale Systems

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***Abstract*—This note discusses the feedback Nash equilibrium of linear quadratic N-player Nash games for infinite-horizon large-scale interconnected systems. The asymptotic structure along with the uniqueness and positive semidefiniteness of the solutions of the cross-coupled algebraic Riccati equations (CAREs) is newly established via the Newton-Kantorovich theorem. The main contribution of this study is the proposal of a new algorithm for solving the CAREs. In order to improve the convergence rate of the algorithm, Newton's method is combined with a new decoupling algorithm; it is shown that the proposed algorithm attains quadratic convergence. Moreover, it is shown for the first time that solutions to the CAREs can be obtained by solving the independent algebraic Lyapunov equation (ALE) by using the reduced-order calculation.**

***Index Terms*—Cross-coupled algebraic Riccati equations (CARE), fixed-point algorithm, Nash games, Newton's method, weakly coupled large-scale systems.**

I. Introduction

The stability analysis and control of large-scale systems has been extensively investigated (see, e.g., [1]). For example, these control problems can be illustrated by multiarea power systems [2], [3]. The control problems of large-scale interconnected systems is parameterized by a small weak coupling parameter ε. This has been extensively studied in [2], [3], [5].

The linear quadratic Nash games and their applications have been widely investigated in many literatures (see, e.g., [9], [24] and the references therein). In particular, the definition and standard results given in [24] will be used for reference. There exist two different types of Nash equilibria: a) open-loop equilibria and b) closed-loop no-memory and feedback equilibria. The existence of open-loop Nash equilibria has been studied in [16], [19], and [20] for both continuous and discrete time systems. Further, in the case of memoryless perfect-state information structure, the asymptotic analysis of the linear feedback Nash equilibria has also been investigated [21]–[23]. These researches have focused on the analysis of the existence of Nash equilibria. Although a numerical algorithm for solving the feedback Nash algebraic Riccati equations has been presented, only a scalar feedback has been considered [23].

It is well known that in order to obtain Nash strategies, the cross-coupled algebraic Riccati equations (CARE) must be solved. In [10], a Newton-type algorithm for solving the CARE has been applied. A variant of the classical eigenvector approach for solving the CARE of the open-loop Nash games has also been studied in [16]. However, the computing workspace for these techniques requires very large matrix dimensions. Therefore, the reduction in the matrix dimensions poses a crucial problem because the weakly coupled systems include numerous subsystems. In contrast, in [4], the recursive algorithms have been developed for solving the CARE of the weakly coupled systems. Recently, an algorithm that is based on the Lyapunov iterations for solving the CARE has been introduced [6], [7]. Although such algorithms can be computed by using the dimension of each subsystem, the convergence rate is given by the linear convergence. Moreover, there is no proof of whether the abovementioned algorithms fail to converge in the case of strongly coupled systems.

This note investigates the feedback Nash equilibrium of linear quadratic N-player Nash games for infinite-horizon large-scale interconnected systems by using Newton's method. It should be noted that this study considers the linear feedback strategy [18] with memoryless perfect-state information structure [21]–[24]. This note is an extension of [8] in the sense that the convergence criteria for ε is derived for the first time. Such a condition is derived by applying the Newton–Kantorovich theorem. The Newton–Kantorovich theorem plays an important role in showing that the uniqueness and positive semidefiniteness of the convergence solutions are guaranteed in the neighborhood of the initial conditions. Moreover, the asymptotic structure of the solutions of the CARE is established without the implicit function theorem. This note also proposes a new decoupling algorithm for computing Newton's iterations as another important feature. As a result, a reduction in the computation is attained by using the existing fixed-point algorithm [6], [7].

Notation: The notations used in this note are fairly standard. **block diag** denotes the block diagonal matrix. The superscript T denotes the matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes the Euclidean norm for a matrix. $\mathrm{vec}M$ denotes the column vector of the matrix M [15]. $\det M$ denotes the determinant of the matrix M. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta. $\mathrm{Re}\lambda M$ denotes the real part of the eigenvalue of the matrix M.

II. Problem Formulation

Consider weakly coupled large-scale linear systems with N players

$$\dot{x}_i(t) = A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1, j\neq i}^{N} A_{ij}x_j(t) + \varepsilon \sum_{j=1, j\neq i}^{N} B_{ij}u_j(t) \quad x_i(0) = x_i^0, \qquad i = 1, 2, \ldots, N \quad (1)$$

where $x_i \in \mathbf{R}^{n_i}$, $i = 1, 2, \ldots, N$ represent the ith state vectors. $u_i \in \mathbf{R}^{m_i}$, $i = 1, 2, \ldots, N$ represent the ith control inputs. ε denotes a small weak coupling parameter that connects the other subsystems. Each player attempts to minimize its cost performance subject to (1) by exploiting the available information in order to take the correct decision

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in accordance with the sought strategy. The cost performance for each strategy subset is defined by

$$J_i(u_1,\ldots,u_N)=\int_0^\infty\Big[x^T(t)Q_{i\varepsilon}x(t)+u_i^T(t)R_{ii}u_i(t)+\varepsilon\sum_{j=1,j\neq i}^{N}u_j^T(t)R_{ij}u_j(t)\Big]\,dt \quad (2)$$

where $R_{ii}=R_{ii}^T>0\in\mathbf{R}^{m_i\times m_i}$, $R_{ij}=R_{ij}^T\geq 0\in\mathbf{R}^{m_j\times m_j}$, $x(t):=[x_1(t)^T\ \ldots\ x_N(t)^T]^T\in\mathbf{R}^{\bar{n}}$,

$$Q_{i\varepsilon}=\begin{bmatrix}\varepsilon^{1-\delta_{i1}}Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N}\\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}}Q_{i2} & \cdots & \varepsilon Q_{i2N}\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}Q_{iN}\end{bmatrix}\geq 0\in\mathbf{R}^{\bar{n}\times\bar{n}}\quad \bar{n}:=\sum_{i=1}^{N}n_i,\qquad i,j=1,2,\ldots,N.$$

Let us define the following matrices:

$$A_\varepsilon:=\begin{bmatrix}A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N}\\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN}\end{bmatrix}$$

$$B_{i\varepsilon}:=\begin{bmatrix}\varepsilon^{1-\delta_{1i}}B_{1i}\\ \varepsilon^{1-\delta_{2i}}B_{2i}\\ \vdots\\ \varepsilon^{1-\delta_{Ni}}B_{Ni}\end{bmatrix}.$$

It should be noted that the delta scaling represents the dominant portion of the matrix $B_{i\varepsilon}$ for each subsystem. This assumption is imposed on the weak coupling structure. In other words, it is assumed that the influence of the control input of each player on the subsystem is almost negligible.

Using the abovementioned notations, the weakly coupled large-scale linear systems (1) can be rewritten as

$$\dot{x}(t)=A_\varepsilon x(t)+\sum_{j=1}^{N}B_{j\varepsilon}u_j(t)\quad x(0)=x^0. \quad (3)$$

The problem considered here requires the strategy space and information structure available to the players to be specified. In this note, a full state information structure is assumed and the number of players that stabilize the constant linear feedback strategies is restricted. Thus, only the controls $u_i(t)$ of type

$$u_i(t)=K_{i\varepsilon}x(t),\qquad i=1,2,\ldots,N \quad (4)$$

are considered, where $K_\varepsilon=(K_{1\varepsilon},\ldots,K_{N\varepsilon})$ with $K_{i\varepsilon}\in\mathbf{R}^{m_i\times\bar{n}}$ belongs to the set

$$\mathcal{K}_\varepsilon:=\left\{(K_{1\varepsilon},\ldots,K_{N\varepsilon})\,|\,A_\varepsilon+\sum_{j=1}^{N}B_{j\varepsilon}K_{i\varepsilon}\ \text{is stable}\right\}. \quad (5)$$

According to the feedback information structure, a set of equilibrium strategies should be independent of the initial state. Furthermore, the strategies should satisfy the typical equilibrium inequalities. A formal definition is given here (see also, e.g., [23]).

Definition 1: The strategy set $(u_1^*,\ldots,u_N^*)$ is a Nash equilibrium strategy set if, for each $i=1,\ldots,N$, the following inequality:

$$J_i\left(u_1^*,\ldots,u_N^*\right)\leq J_i\left(u_1^*,\ldots,u_{i-1}^*,u_i,u_{i+1}^*,\ldots,u_N^*\right),\qquad i=1,2,\ldots,N \quad (6)$$

holds, where $u_i^*(t)=K_{i\varepsilon}^*x(t)$ for all K_ε that satisfy

$$(K_{1\varepsilon}^*,\ldots,K_{N\varepsilon}^*),\ \ (K_{1\varepsilon}^*,\ldots,K_{i-1\varepsilon}^*,K_{i\varepsilon},K_{i+1\varepsilon}^*,\ldots,K_{N\varepsilon}^*)\in\mathcal{K}_\varepsilon.$$

The optimal linear feedback strategies for the Nash games are given by

$$u_i^*(t)=-R_{ii}^{-1}B_{i\varepsilon}^TP_{i\varepsilon}x(t),\qquad i=1,2,\ldots,N \quad (7)$$

where $P_{i\varepsilon}$ are the positive–semidefinite solutions of the following N-CAREs:

$$\begin{aligned}&\mathcal{F}_i(P_{1\varepsilon},\ldots,P_{N\varepsilon})\\ &:=P_{i\varepsilon}\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}\right)+\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}\right)^TP_{i\varepsilon}\\ &\quad+P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon}+\varepsilon\sum_{j=1,j\neq i}^{N}P_{j\varepsilon}G_{ij\varepsilon}P_{j\varepsilon}+Q_{i\varepsilon}=0\end{aligned} \quad (8)$$

with $S_{i\varepsilon}:=B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T$, $G_{ij\varepsilon}:=B_{j\varepsilon}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{j\varepsilon}^T$.

It is well known that even if the problem is linear and quadratic, the linearity of the closed-loop Nash solutions is not guaranteed [18]. It should be noted that in this study, the strategies $u_i^*(t)$ are restricted as the linear feedback strategies.

It should be noted that for the special problem that is considered here, the specific strategy that relaxes the Nash strategy, which has been studied in the previous few decades, is introduced by scaling the diagonal elements of the matrix $Q_{i\varepsilon}$.

Since A_ε and $S_{i\varepsilon}$ include ε, the solution $P_{i\varepsilon}$ of the CARE (8), if it exists, must contain terms of the order ε. By taking this fact into account, the solution $P_{i\varepsilon}$ of the CARE (8) with the following structure is considered [7], [11]:

$$P_{i\varepsilon}:=\begin{bmatrix}\varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N}\\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{i2N}\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN}\end{bmatrix}\in\mathbf{R}^{\bar{n}\times\bar{n}}.$$

Although the structure of $P_{i\varepsilon}$ appears to be an assumption, it can be easily proved by directly applying the implicit function theorem to (8). The detailed proof is omitted here due to the limitation on the number of pages. The following analysis requires a basic assumption.

Assumption 1: The triples $(A_{ii},B_{ii},\sqrt{Q_{ii}})$, $i=1,2,\ldots,N$ are stabilizable and detectable.

III. Existence of a Unique Solution

First, in order to obtain Nash strategies, the asymptotic structure of CARE (8) is established. By substituting matrices $A_\varepsilon, S_{i\varepsilon}, G_{ij\varepsilon}, Q_{i\varepsilon}$, and $P_{i\varepsilon}$ into CARE (8), setting $\varepsilon = 0$, and partitioning CARE (8), the following reduced-order algebraic Riccati equations (AREs) are obtained; here, $\bar{P}_{ii}, i = 1, \ldots, N$ are the limiting solutions of the CARE (8) as $\varepsilon \to +0$

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}S_{ii}\bar{P}_{ii} + Q_{ii} = 0 \tag{9}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$.

The limiting behavior of $P_{i\varepsilon}$ when $\varepsilon \to +0$ is described by the following lemma.

Lemma 1: [6], [7] Under Assumption 1, there exists a small σ^* such that for all $\varepsilon \in (0, \sigma^*)$, CARE (8) allows for a positive–semidefinite solution $P_{i\varepsilon}^*$, which can be written as

$$\begin{aligned} P_{i\varepsilon} &:= P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon) \\ &= \mathbf{block\ diag}(0\ \cdots\ \bar{P}_{ii}\ \cdots\ 0) + O(\varepsilon). \end{aligned} \tag{10}$$

Moreover, there exists a unique solution $P_{i\varepsilon}^*$ of the CARE (8) in the neighborhood of the solution $\bar{P}_i$.

Using the Newton–Kantorovich theorem [13], [14], which will be presented later in this note, it is clear that there exists a small σ^* such that for all $\varepsilon \in (0, \sigma^*)$, the CARE (8) has positive semidefinite solutions within the limits of the sufficiency condition. Moreover, it should be noted that the asymptotic structure of (10) can also be obtained by applying the Newton–Kantorovich theorem.

Although it has been generally shown that there exist several solutions to the CARE [16], it should be noted that for the weakly coupled systems, both positive semidefiniteness and uniqueness of the solutions are guaranteed as long as ε is small.

In order to obtain the optimal strategies, a useful algorithm that is based on Newton's method is given as follows:

$$\begin{aligned} &P_{i\varepsilon}^{(k+1)}\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right) \\ &+ \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right)^T P_{i\varepsilon}^{(k+1)} - \sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(k+1)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} \\ &- \sum_{j=1, j\neq i}^N P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k+1)} + \varepsilon\sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(k+1)}G_{ij\varepsilon}P_{j\varepsilon}^{(k)} \\ &+ \varepsilon\sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(k)}G_{ij\varepsilon}P_{j\varepsilon}^{(k+1)} + \sum_{j=1, j\neq i}^N P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k)} \\ &+ \sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(k)}S_{j\varepsilon}P_{i\varepsilon}^{(k)} + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} \\ &- \varepsilon\sum_{j=1, j\neq i}^N P_{j\varepsilon}^{(k)}G_{ij\varepsilon}P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} = 0 \end{aligned} \tag{11a}$$

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}}P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN}^{(k)} \end{bmatrix} \tag{11b}$$

$$\bar{A}_\varepsilon := A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}, \qquad k = 0, 1, \ldots \tag{11c}$$

with the initial conditions $P_{i\varepsilon}^{(0)} = \bar{P}_i = \mathbf{block\ diag}\ (0\ \cdots\ \bar{P}_{ii}\ \cdots\ 0)$.

The new algorithm (11) can be constructed by setting $P_{i\varepsilon}^{(k+1)} = P_{i\varepsilon}^{(k)} + \Delta P_{i\varepsilon}^{(k)}$ and ignoring the quadratic $O(\Delta P_{i\varepsilon}^{(k)T}\Delta P_{i\varepsilon}^{(k)})$ term (see, e.g., [17] in detail). Newton's method is well known and it is widely used to obtain solutions to algebraic equations; moreover, its local convergence properties are well understood.

In order to establish the main result, the following fact must be considered.

Newton–Kantorovich Theorem [13], [14]: Assume that $F : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable on a convex set D. Suppose that the inverse of map F exists and moreover it is differentiable on set D and that $\|F'(\mathbf{x}) - F'(\mathbf{y})\| \le \gamma\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in D$. Suppose that there is an $\mathbf{x}^0 \in D$ such that $\|F'(\mathbf{x}^0)^{-1}\| \le \beta$, $\|F'(\mathbf{x}^0)^{-1}F(\mathbf{x}^0)\| \le \eta$ and $\theta := \beta\gamma\eta < 1/2$. Assume that $S := \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}^0\| \le t^*\} \subset D, t^* = (1 - \sqrt{1 - 2\theta})/\beta\gamma$. Then, Newton iterations $\mathbf{x}^{k+1} = \mathbf{x}^k - F'(\mathbf{x}^k)^{-1}F(\mathbf{x}^k)$, $k = 0, 1, \ldots$, are well defined and converge to a solution $\mathbf{x}^*$ of $F(\mathbf{x}) = 0$ in S. Moreover, the solution $\mathbf{x}^*$ is unique in $\tilde{S} \cap D$, where $\tilde{S} := \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}^0\| \le \tilde{t}\} \subset D, \tilde{t} = (1 + \sqrt{1 - 2\theta})/\beta\gamma$ and error estimate is given by $\|\mathbf{x}^* - \mathbf{x}^k\| \le ((2\theta)^{2^k}/2^k\beta\gamma) = 2^{1-k}(2\theta)^{2^k - 1}\eta$, $k = 0, 1, \ldots$.

The main result for algorithm (11) is stated as follows.

Theorem 1: Under Assumption 1, there exists a small σ^* such that for all $\varepsilon \in (0, \sigma^*)$, Newton's method (11) converges to the exact solution of $P_{i\varepsilon}$ at the same rate as that of the quadratic convergence; here, $P_{i\varepsilon}^{(k)}$ is positive semidefinite and $A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}$ is stable. Moreover, the convergence solutions attain a unique solution $P_{i\varepsilon}^*$ of the CARE (8) in the neighborhood of the initial condition $P_{i\varepsilon}^{(0)} = \bar{P}_i$. In other words, the following conditions are satisfied:

$$\left\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\right\| \le \frac{O\left(\varepsilon^{2^k}\right)}{2^k\bar{\beta}\bar{\gamma}}, \quad k = 0, 1, \ldots \tag{12a}$$

$$\mathrm{Re}\lambda\left[A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right] = \mathrm{Re}\lambda\bar{A}_\varepsilon < 0 \tag{12b}$$

where

$$\begin{aligned} \bar{\gamma} &:= 2(2N-1)\sum_{i=1}^N\|S_{i\varepsilon}\| + 2\varepsilon\sum_{i=1}^N\sum_{j=1, j\neq i}^N\|G_{ij\varepsilon}\| \\ \bar{\beta} &:= \left\|\left[\nabla\mathcal{F}\left(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)}\right)\right]^{-1}\right\| \\ \nabla\mathcal{F}(P_{1\varepsilon}, \ldots, P_{N\varepsilon}) &:= \begin{bmatrix} \frac{\partial\mathrm{vec}\mathcal{F}_1}{\partial(\mathrm{vec}P_{1\varepsilon})^T} & \cdots & \frac{\partial\mathrm{vec}\mathcal{F}_1}{\partial(\mathrm{vec}P_{N\varepsilon})^T} \\ \vdots & \ddots & \vdots \\ \frac{\partial\mathrm{vec}\mathcal{F}_N}{\partial(\mathrm{vec}P_{1\varepsilon})^T} & \cdots & \frac{\partial\mathrm{vec}\mathcal{F}_N}{\partial(\mathrm{vec}P_{N\varepsilon})^T} \end{bmatrix}. \end{aligned}$$

Proof: Now, let us define a matrix function as follows:

$$\begin{aligned} \mathcal{Z}(\mathcal{P}) &:= \mathcal{P}\left[\mathcal{A} - \sum_{j=2}^N\mathcal{T}^{(j-1)T}\mathcal{S}\mathcal{P}\mathcal{T}^{(j-1)}\right] \\ &+ \left[\mathcal{A} - \sum_{j=2}^N\mathcal{T}^{(j-1)T}\mathcal{S}\mathcal{P}\mathcal{T}^{(j-1)}\right]^T\mathcal{P} - \mathcal{P}\mathcal{S}\mathcal{P} \\ &+ \varepsilon\sum_{j=2}^N\mathcal{T}^{(j-1)T}\mathcal{P}\mathcal{G}_j\mathcal{P}\mathcal{T}^{(j-1)} + \mathcal{Q} \\ &= 0 \end{aligned} \tag{13}$$

where

$$\mathcal{T} := \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{n_N} \\ I_{n_1} & 0 & \cdots & 0 & 0 \\ 0 & I_{n_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_{N-1}} & 0 \end{bmatrix}$$

$$\mathcal{P} := \mathbf{block\ diag}(P_{1\varepsilon} \ \cdots \ P_{N\varepsilon})$$

$$\mathcal{A} := \mathbf{block\ diag}(A_\varepsilon \ \cdots \ A_\varepsilon)$$

$$\mathcal{S} := \mathbf{block\ diag}(S_{1\varepsilon} \ \cdots \ S_{N\varepsilon})$$

$$\mathcal{Q} := \mathbf{block\ diag}(Q_{1\varepsilon} \ \cdots \ Q_{N\varepsilon})$$

$$\mathcal{G}_2 := \mathbf{block\ diag}\left(G_{N1\varepsilon}\, G_{12\varepsilon}\, G_{23\varepsilon}\, G_{34\varepsilon}\, G_{45\varepsilon} \ \cdots \ G_{(N-1)N\varepsilon}\right)$$

$$\mathcal{G}_3 := \mathbf{block\ diag}\left(G_{(N-1)1\varepsilon}\, G_{N2\varepsilon}\, G_{13\varepsilon}\, G_{24\varepsilon} \ \cdots \ G_{(N-3)(N-1)\varepsilon}\, G_{(N-2)N\varepsilon}\right)$$

$$\vdots$$

$$\mathcal{G}_N := \mathbf{block\ diag}\left(G_{21\varepsilon}\, G_{32\varepsilon}\, G_{43\varepsilon}\, G_{54\varepsilon} \ \cdots \ G_{N(N-1)\varepsilon}\, G_{1N\varepsilon}\right).$$

Since the function $\mathcal{Z}(\mathcal{P})$ is continuous at any $\mathcal{P}$, taking the partial derivative of the function $\mathcal{Z}(\mathcal{P})$ with respect to $\mathcal{P}$ yields

$$\begin{aligned} \nabla \mathcal{Z}(\mathcal{P}) &:= \frac{\partial \mathrm{vec}\mathcal{Z}(\mathcal{P})}{\partial (\mathrm{vec}\mathcal{P})^T} \\ &= \left(\mathcal{A} - \mathcal{S}\mathcal{P} - \sum_{j=2}^{N} \mathcal{T}^{(j-1)T}\mathcal{S}\mathcal{P}\mathcal{T}^{(j-1)}\right)^T \otimes I_{\bar{n}} \\ &\quad + I_{\bar{n}} \otimes \left(\mathcal{A} - \mathcal{S}\mathcal{P} - \sum_{j=2}^{N} \mathcal{T}^{(j-1)T}\mathcal{S}\mathcal{P}\mathcal{T}^{(j-1)}\right)^T \\ &\quad - \sum_{j=2}^{N} \mathcal{T}^{(j-1)T} \otimes \left(\mathcal{P}\mathcal{T}^{(j-1)T}\mathcal{S}\right) \\ &\quad - \sum_{j=2}^{N} \left(\mathcal{P}\mathcal{T}^{(j-1)T}\mathcal{S}\right) \otimes \mathcal{T}^{(j-1)T} \\ &\quad + \varepsilon \sum_{j=2}^{N} \left(\mathcal{G}_j \mathcal{P}\mathcal{T}^{(j-1)}\right)^T \otimes \mathcal{T}^{(j-1)T} \\ &\quad + \varepsilon \sum_{j=2}^{N} \mathcal{T}^{(j-1)T} \otimes \left(\mathcal{G}_j \mathcal{P}\mathcal{T}^{(j-1)}\right)^T. \end{aligned} \tag{14}$$

Thus, for any matrices $\mathcal{X}$ and $\mathcal{Y}$ that belong to $\mathcal{P}$, it is immediately obtained from (14) that

$$\|\nabla \mathcal{Z}(\mathcal{X}) - \nabla \mathcal{Z}(\mathcal{Y})\| \le \bar{\gamma}\|\mathcal{X} - \mathcal{Y}\|. \tag{15}$$

Moreover, using (14), it is easy to derive that

$$\nabla \mathcal{F}\left(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)}\right) = \begin{bmatrix} \mathbf{J}_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{J}_0 \end{bmatrix} + O(\varepsilon) \tag{16}$$

where

$$\begin{aligned} \mathbf{J}_0 &= \mathbf{block\ diag}(\mathbf{D}_{11} \ \cdots \ \mathbf{D}_{NN}) \\ \mathbf{D}_{ii} &:= (A_{ii} - S_{ii}\bar{P}_{ii})^T \otimes I_{n_i} + I_{n_i} \otimes (A_{ii} - S_{ii}\bar{P}_{ii})^T. \end{aligned} \tag{17}$$

Evidently, $D_{ii} := A_{ii} - S_{ii}\bar{P}_{ii}$ is nonsingular because the ARE (9) has a positive semidefinite stabilizing solution under Assumption 1. Therefore, the matrix in (16) is invertible for sufficiently small ε. Consequently, $\bar{\beta} = \|[\nabla \mathcal{F}(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)})]^{-1}\|$ exists as a finite number that is independent of ε. On the other hand, since $\mathcal{F}_i(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)}) = O(\varepsilon)$, there exists $\bar{\eta}$ such that $\bar{\eta} = \|[\nabla \mathcal{F}(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)})]^{-1}\| \cdot \|\mathcal{F}(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)})\| = O(\varepsilon)$, where

$$\mathcal{F}(P_{1\varepsilon}, \ldots, P_{N\varepsilon}) := [\mathcal{F}_1(P_{1\varepsilon}, \ldots, P_{N\varepsilon}), \ldots, \mathcal{F}_N(P_{1\varepsilon}, \ldots, P_{N\varepsilon})].$$

Thus, for a sufficiently small ε, there exists $\bar{\theta}$ such that $\bar{\theta} = \bar{\beta}\bar{\eta}\bar{\gamma} < 2^{-1}$ because of $\bar{\eta} = O(\varepsilon)$, $\lim_{\varepsilon \to +0} \|G_{ij\varepsilon}\| < \infty$, and

$$\lim_{\varepsilon \to +0} \bar{\gamma} = \lim_{\varepsilon \to +0} \left[2(2N-1)\sum_{i=1}^{N} \|S_{i\varepsilon}\| + 2\varepsilon \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \|G_{ij\varepsilon}\|\right] < \infty.$$

Using the Newton–Kantorovich theorem, the error estimate is given by

$$\left\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\right\| \le \frac{(2\bar{\theta})^{2^k}}{2^k \bar{\beta}\bar{\gamma}}, \qquad k = 0, 1, \ldots. \tag{18}$$

Substituting $2\bar{\theta} = O(\varepsilon)$ into (18), (12a) holds.

Second, the uniqueness of the solution is discussed. Now, let us define $\bar{t}^* \equiv (1/\bar{\gamma}\bar{\beta})[1 - \sqrt{1 - 2\bar{\theta}}]$. Clearly, $S \equiv \{P_{i\varepsilon} : \|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| \le \bar{t}^*\}$ is in the convex set D. In the sequel, since $\|P_{i\varepsilon} - P_{i\varepsilon}^{(0)}\| = O(\varepsilon)$ holds for a small ε, the uniqueness of $P_{i\varepsilon}^*$ is guaranteed for a subset S by applying the Newton–Kantorovich theorem.

Since the remainder of the proofs for positive semidefiniteness and stability are similar to the proof given in [6] and [7], they have been omitted. ■

The subject of this note is closely related to the result of [25]. From this result, it is clear that for a small ε, the considered game will have a unique feedback Nash equilibrium. However, it will not be possible to obtain a supremum σ^* such that the game has a unique feedback Nash equilibrium for all $\varepsilon \in (0, \sigma^*)$.

It should be noted that no proof exists of whether the proposed algorithm fails to converge for strongly coupled systems. In this note, the convergence criteria for ε is established for the first time. Such a condition is derived from the Newton–Kantorovich theorem.

Corrollary 1: If the following inequality holds for any small parameter: $\varepsilon = \varepsilon_0$

$$\begin{aligned} \bar{\theta}(\varepsilon) &= \bar{\beta}\bar{\eta}\bar{\gamma} \\ &:= \left\|\left[\nabla \mathcal{F}\left(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)}\right)\right]^{-1}\right\|^2 \\ &\quad \cdot \left\|\mathcal{F}\left(P_{1\varepsilon}^{(0)}, \ldots, P_{N\varepsilon}^{(0)}\right)\right\| \\ &\quad \cdot \left[2(2N-1)\sum_{i=1}^{N}\|S_{i\varepsilon}\| + 2\varepsilon\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\|G_{ij\varepsilon}\|\right] \\ &< 2^{-1} \end{aligned} \tag{19}$$

algorithm (11) guarantees quadratic convergence.

Proof: Since it is clear that this proof can be derived by applying the Newton–Kantorovich theorem, it has been omitted. ■

IV. A Numerical Algorithm for Solving the CARE

When the cross-coupled algebraic Lyapunov equation (CALE) (11a) is solved, the existence of the cross-coupled term

$$-\sum_{j=1,j\neq i}^{N} P_{j\varepsilon}^{(k+1)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} - \sum_{j=1,j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k+1)} + \varepsilon \sum_{j=1,j\neq i}^{N} P_{j\varepsilon}^{(k+1)} G_{ij\varepsilon} P_{j\varepsilon}^{(k)} + \varepsilon \sum_{j=1,j\neq i}^{N} P_{j\varepsilon}^{(k)} G_{ij\varepsilon} P_{j\varepsilon}^{(k+1)}$$

in CALE (11a) makes it difficult to solve this equation directly. Thus, in order to avoid the cross-coupled term, a new decoupling algorithm that is based on the fixed-point algorithm is established. Taking into account the fact that $S_{j\varepsilon}P_{i\varepsilon}^{(k)} = O(\varepsilon)$, $i \neq j$, let us consider CALE (20) in its general form

$$X_{i\varepsilon}\Lambda_\varepsilon + \Lambda_\varepsilon^T X_{i\varepsilon} + \varepsilon \sum_{j=1,j\neq i}^{N} \left(X_{j\varepsilon}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon} \right) + U_{i\varepsilon} = 0, \quad i = 1, \ldots, N \tag{20}$$

where

$$X_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} X_{i1} & \varepsilon X_{i12} & \cdots & \varepsilon X_{i1N} \\ \varepsilon X_{i12}^T & \varepsilon^{1-\delta_{i2}} X_{i2} & \cdots & \varepsilon X_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{i1N}^T & \varepsilon X_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} X_{iN} \end{bmatrix}$$

$$\Lambda_\varepsilon := \begin{bmatrix} \Lambda_{11} & \varepsilon\Lambda_{12} & \cdots & \varepsilon\Lambda_{1N} \\ \varepsilon\Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon\Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Lambda_{N1} & \varepsilon\Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix}$$

$$\Phi_{i\varepsilon} := \begin{bmatrix} \Phi_{i1} & \varepsilon\Phi_{i12} & \cdots & \varepsilon\Phi_{i1N} \\ \varepsilon\Phi_{i21} & \Phi_{i2} & \cdots & \varepsilon\Phi_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Phi_{iN1} & \varepsilon\Phi_{iN2} & \cdots & \Phi_{iN} \end{bmatrix}$$

$$U_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} U_{i1} & \varepsilon U_{i12} & \cdots & \varepsilon U_{i1N} \\ \varepsilon U_{i12}^T & \varepsilon^{1-\delta_{i2}} U_{i2} & \cdots & \varepsilon U_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{i1N}^T & \varepsilon U_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} U_{iN} \end{bmatrix}.$$

It should be noted that

$$P_{i\varepsilon}^{(k+1)} \Rightarrow X_{i\varepsilon} \quad P_{j\varepsilon}^{(k+1)} \Rightarrow X_{j\varepsilon}$$

$$A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon} P_{j\varepsilon}^{(k)} \Rightarrow \Lambda_\varepsilon \quad -S_{j\varepsilon}P_{i\varepsilon}^{(k)} + \varepsilon G_{ij\varepsilon} P_{j\varepsilon}^{(k)} \Rightarrow \varepsilon\Phi_{j\varepsilon}$$

$$\sum_{j=1,j\neq i}^{N} P_{i\varepsilon}^{(k)} S_{j\varepsilon} P_{j\varepsilon}^{(k)} + \sum_{j=1,j\neq i}^{N} P_{j\varepsilon}^{(k)} S_{j\varepsilon} P_{i\varepsilon}^{(k)} + P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} - \varepsilon \sum_{j=1,j\neq i}^{N} P_{j\varepsilon}^{(k)} G_{ij\varepsilon} P_{j\varepsilon}^{(k)} + Q_{i\varepsilon} \Rightarrow U_{i\varepsilon}$$

where $\Rightarrow$ represents the replacement.

Without loss of generality, the following condition is assumed for CALE (20).

Asumption 2: $\Lambda_{11}, \ldots, \Lambda_{NN}$ are stable.

Algorithm (21) for solving CALE (20) is given as follows:

$$X_{i\varepsilon}^{(n+1)}\Lambda_\varepsilon + \Lambda_\varepsilon^T X_{i\varepsilon}^{(n+1)} + \varepsilon \sum_{j=1,j\neq i}^{N} \left(X_{j\varepsilon}^{(n)}\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T X_{j\varepsilon}^{(n)} \right) + U_{i\varepsilon} = 0, \quad i = 1, \ldots, N,\ n = 0, 1, \ldots \tag{21}$$

where $X_{i\varepsilon}^{(0)} = 0$, $i = 1, \ldots, N$.

It should be noted that since (20) is modified as (21), the numerical computation can be carried out independently for each solution. The following theorem indicates the convergence of algorithm (21).

Theorem 2: Under Assumption 2, the fixed-point algorithm (21) converges to an exact solution $X_{i\varepsilon}$ with a rate of

$$\left\| X_{i\varepsilon}^{(n)} - X_{i\varepsilon} \right\| = O(\varepsilon^n), \qquad n = 1, 2, \ldots. \tag{22}$$

Proof: The proof of Theorem 2 can be derived by using mathematical induction. When $n = 0$ for algorithm (21), it is easy to verify that the first-order approximations $X_{i\varepsilon}$ corresponding to ε are $X_{i\varepsilon}^{(1)}$. When $n = h$, $h \geq 2$, it is assumed that

$$\left\| X_{i\varepsilon}^{(h)} - X_{i\varepsilon} \right\| = O(\varepsilon^h). \tag{23}$$

By subtracting (20) from (21) and substituting h into n, the following equations hold:

$$\left(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon} \right)\Lambda_\varepsilon + \Lambda_\varepsilon^T \left(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon} \right) = -\varepsilon \sum_{j=1,j\neq i}^{N} \left[\left(X_{j\varepsilon}^{(h)} - X_{j\varepsilon} \right)\Phi_{j\varepsilon} + \Phi_{j\varepsilon}^T \left(X_{j\varepsilon}^{(h)} - X_{j\varepsilon} \right) \right]. \tag{24}$$

Using assumption (23), the following equations are satisfied:

$$\left(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon} \right)\Lambda_\varepsilon + \Lambda_\varepsilon^T \left(X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon} \right) = O(\varepsilon^{h+1}). \tag{25}$$

Since from Assumption 2, Λ_{ii}, $i = 1, 2, \ldots, N$ are stable, Λ_ε is stable. Using the standard properties of the algebraic Lyapunov equation (ALE) [12], it is easy to verify that

$$\left\| X_{i\varepsilon}^{(h+1)} - X_{i\varepsilon} \right\| = O(\varepsilon^{h+1}). \tag{26}$$

Consequently, error (22) hold for all $n \in \mathbf{N}$. This completes the proof of Theorem 2. ∎

When ALE (21) is solved, a large computational dimension $\bar{n} := \sum_{i=1}^{N} n_i$ is required to be compared with the small computational dimensions n_i, $i = 1, \ldots, N$. Thus, in order to reduce the computational dimension, a fixed-point algorithm can be applied (see, e.g., [6], [7]).

It is well known that it is very difficult to solve the CARE. For example, in the case of Newton's method, it is necessary to solve the large linear equations that depend on the other iterative solutions. In this study, the decoupling algorithm that combines Newton's method with the fixed-point algorithm has been provided for the first time in order to avoid such a dependence. This novel idea is based on the property of $P_{j\varepsilon}^{(k+1)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} = O(\varepsilon)$, where ε is a weakly coupled perturbation parameter. As a result, since the iterative solutions $P_{1\varepsilon}^{(k+1)}, P_{2\varepsilon}^{(k+1)}, \ldots$ do not depend on other equations, each solution can be solved independently.

TABLE I
CONVERGENCE CRITERIA FOR VARIOUS PARAMETERS

	$\varepsilon = 1.0e-01$	$\varepsilon = 1.0e-02$	$\varepsilon = 1.0e-03$	$\varepsilon = 1.0e-04$	$\varepsilon = 1.0e-05$	$\varepsilon = 1.0e-06$	$\varepsilon = 1.0e-07$
$\bar{\theta}(\varepsilon)$	$2.5314e+05$	$1.7548e+04$	$1.8337e+03$	$1.8430e+02$	$1.8439e+01$	1.8440	$1.8440e-01$

TABLE II
ERROR PER ITERATIONS

k	$\mathcal{E}(1.0e-01)$	$\mathcal{E}(1.0e-02)$	$\mathcal{E}(1.0e-03)$	$\mathcal{E}(1.0e-04)$
0	3.3361	$3.3361e-01$	$3.3361e-02$	$3.3361e-03$
1	$6.0610e-01$	$4.6495e-03$	$4.6553e-05$	$4.6580e-07$
2	$3.1527e-02$	$2.2313e-06$	$2.2363e-10$	$2.4667e-12$
3	$2.0453e-05$	$3.4355e-11$	$1.1435e-12$	
4	$2.3771e-11$			

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is provided. The system matrices are given as follows:

$$A_{11} = \begin{bmatrix} 0 & 1 & -0.2 & 0 \\ -0.7 & -1.7 & -1.3 & -0.3 \\ 0 & 0 & 0 & 1 \\ -1.9 & 0 & -0.5 & -0.3 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0.2 & 0 & -0.8 & 0.2 \\ -0.1 & 0 & 1.1 & -0.1 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0 & 1.1 & 0.4 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 0.7 & 0 & -0.2 & 0.3 \\ -0.4 & 0 & 2.8 & -0.2 \\ 0 & 0 & 0 & 0 \\ 0.9 & 0 & 1.7 & 0.2 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0.2 & 0 & 0.1 & 0.3 \\ -1.1 & 0 & -1.6 & -0.1 \\ 0 & 0 & 0 & 0 \\ -1.4 & 0 & 1.3 & -0.3 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -0.2 & 1 & -1.6 & -0.5 \\ -1.9 & -1.8 & 1.3 & -0.1 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -0.5 & 0.3 \end{bmatrix}$$

$$A_{23} = \begin{bmatrix} 0.6 & 0 & 0.4 & 0.2 \\ -1 & 0 & 1.4 & -0.4 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0.2 & -0.2 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} -0.2 & 0 & 0.8 & 0 \\ -0.6 & 0 & -1.0 & 0.9 \\ 0 & 0 & 0 & 0 \\ -1.2 & 0 & 0.4 & -0.1 \end{bmatrix}$$

$$A_{32} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0 \\ -1.1 & 0 & 1.7 & -0.1 \\ 0 & 0 & 0 & 0 \\ -0.7 & 0 & 0.6 & -0.1 \end{bmatrix}$$

$$A_{33} = \begin{bmatrix} -0.1 & 1 & -1.2 & -0.3 \\ -0.5 & -0.2 & 0.7 & -1.3 \\ 0 & 0 & 0 & 1 \\ -0.3 & 0 & -0.2 & -0.7 \end{bmatrix}$$

TABLE III
NUMBER OF ITERATIONS

ε	Newton's Method	Lyapunov iterations
$1.0e-01$	4	9
$1.0e-02$	3	5
$1.0e-03$	3	3
$1.0e-04$	2	2

$$B_{11} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 0 \\ 0.7 \\ 0 \\ 0 \end{bmatrix} \quad B_{33} = \begin{bmatrix} 0 \\ 1.0 \\ 0 \\ 0 \end{bmatrix}$$

$$Q_{11} = Q_{22} = Q_{33} = 0.1 \times I_4$$

$$R_{11} = R_{22} = R_{33} = 0.1 \quad R_{12} = R_{13} = 0.2$$

$$R_{23} = R_{21} = 0.3 \quad R_{31} = R_{32} = 0.1.$$

Table I shows the values of $\bar{\theta}$ for various values of ε. Since the convergence criteria (19) of Newton's method is satisfied for $\varepsilon = 1.0e-07$, the asymptotic structure of the solutions with uniqueness, positive semidefiniteness and quadratic convergence is attained for any ε value that is smaller than $\varepsilon = 1.0e-07$. It should be noted that convergence criteria (19) is a conservative condition. Hence, even if such a condition is not satisfied, a required solution that attains quadratic convergence might exist. In fact, the existence of a unique convergence solution for some parameter will be verified later.

In order to verify the exactitude of the solution, the remainder per iteration is computed for several values of ε by substituting $P_{i\varepsilon}^{(k)}$ into CARE (8). Table II shows the errors $\mathcal{E}(\varepsilon)$ per iteration for various values of ε, where $\mathcal{E}(\varepsilon) := \sum_{i=1}^{3} \|\mathcal{F}_i(P_{1\varepsilon}^{(k)}, P_{2\varepsilon}^{(k)}, P_{3\varepsilon}^{(k)})\|$. It should be noted that when $\varepsilon = 1.0e-01$, algorithm (11a) converges to the exact solution with an accuracy of $\mathcal{E}(\varepsilon) < 1.0e-10$ after four iterations. Hence, it can be observed from Table II that algorithm (11a) attains quadratic convergence.

The required iterations of the proposed algorithm (11a) versus the Lyapunov iterations [6], [7] are presented in Table III. It can be observed from Table III that as compared with the Lyapunov iterations, the proposed algorithm (11a) succeeds in reducing the number of iterations for different values of ε. In particular, for a large ε value, the required iterations are small. Hence, the resulting algorithm in this note is very attractive for a sufficiently small ε.

Table IV presents the results of the CPU time with regard to the comparison between the new method and Lyapunov iterations [6], [7]. The

TABLE IV
CPU TIME [SEC]

ε	Newton's Method	Lyapunov iterations
$1.0e-01$	$3.0450e-01$	$8.0100e-02$
$1.0e-02$	$1.2320e-01$	$4.8100e-02$
$1.0e-03$	$1.0320e-01$	$3.1000e-02$
$1.0e-04$	$5.3100e-02$	$2.4000e-02$

CPU time represents the average based on the computations of ten runs. The CPU time is the total time that has been used for all the iterations. In other words, it should be noted that the CPU time used for the decoupling algorithm is included in the total CPU time used in Newton's method. It can be observed from Table IV that as compared to the Lyapunov iterations [6], [7], the iterative algorithm (11) requires considerably more CPU time. This is because the computation of algorithm (21) involves many procedures. However, it should be noted that the proposed algorithm is useful because the positive semidefiniteness and uniqueness of the proposed algorithm can be guaranteed, whereas those of the Lyapunov iterations cannot be guaranteed.

From the viewpoint of this example, it should be noted that when the fixed-point algorithm is applied, even if the number of subsystems is greater than four, the computing workspace required for the strategies is the same as the dimension of the subsystems. In other words, even if the large-scale systems (1) are composed of N four-dimensional subsystems, the required workspace is four.

VI. CONCLUSION

In this note, Nash games for large-scale systems that are connected by ε have been studied. A new algorithm that combines Newton's method and fixed point iterations for solving the large-scale CAREs has been proposed. It should be noted that the proposed design method is rather different from the existing methods such as the recursive approach [11] and the Lyapunov iterations [6], [7]. As a result, the convergence rate has been dramatically improved because the proposed algorithm attains quadratic convergence. Another important feature is that the asymptotic structure with uniqueness and positive semidefiniteness has been proved by using the Newton–Kantorovich theorem. Moreover, the convergence criteria of Newton's method has been derived for the first time.

Finally, by using the new decoupling algorithm, a solution can be obtained by solving the independent ALE. Moreover, when the fixed-point algorithm [6], [7] is applied, the required workspace is the same as the dimension of each subsystem. Thus, the proposed algorithm is expected to be very useful and reliable for a sufficiently small ε.

It should be noted that there are no results for a large ε. In particular, for a large ε, a unique equilibrium will not exist. Therefore, it is impossible to verify that even if the proposed algorithm converges to a solution, more than one solution exists. The principal result of this note is that for a given a priori number σ^*, the game will have a unique equilibrium for all ε values smaller than this number. These problems and their extensions will be addressed in future investigations.

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