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Phase Structure of Four-Fermion Theories at Finite Temperature and Chemical Potential in Arbitrary Dimensions

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Abstract

The phase structure of four-fermion theories is thoroughly investigated with varying temperature and chemical potential for arbitrary space-time dimensions ($2 \leq D < 4$) by using the $1/N$ expansion method. It is shown that the chiral symmetry is restored in the theory under consideration for sufficiently high temperature and/or chemical potential. The critical line dividing the symmetric and broken phase is given explicitly. It is found that for space-time dimension $2 \leq D < 3$ both the first-order and second-order phase transition occur depending on the value of temperature and chemical potential while for $3 \leq D < 4$ only the second-order phase transition exists.

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1 INTRODUCTION

The idea of spontaneous symmetry breaking has played a decisive role in recent developments of particle physics. In fact the spontaneous breaking of the gauge symmetry $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ is the basic ingredient of the standard electroweak theory and grand unified theories (GUT) are constructed on the basis of the Higgs mechanism. In these approaches, however, the spontaneous breaking of symmetries is introduced phenomenologically in terms of the Higgs field. In other words the search for the dynamical origin of the spontaneous symmetry breaking is passed over by introducing elementary Higgs fields. Under this circumstance it is very interesting to look for a possible dynamical mechanism of the spontaneous symmetry breaking.

More than 30 years ago Y. Nambu and G. Jona-Lasinio first introduced the idea of dynamical symmetry breaking in particle physics[1] and the idea has attracted a vast number of researchers in particle physics. One of the major applications of this idea has been made on the construction of composite Higgs models such as the technicolor model.[2] It is important to test phenomenologically whether any of these composite Higgs models is a candidate of the underlying theory of the standard electroweak theory.

One of the possible environments where the composite Higgs models may be tested is found in the early universe where the symmetry of the primary unified theory is broken down to yield lower-level theories which describe phenomena at lower energy scales. In the early universe it is not adequate to neglect the effect of the curvature, temperature and density. Hence it is important to examine a theory in the finite curvature, temperature and density. In our study we are interested in the spontaneous breaking of the electroweak symmetry and hence we consider the scale much smaller than the Planck scale. In such a situation the essential feature of the early universe may be described only with the effects of finite temperature and density, the curvature effect being negligible. In the present communication we would like to report our investigation in a simple model field theory for composite Higgs fields at finite temperature and chemical potential.

We work in a four-fermion theory in the flat space-time with arbitrary dimensions. The theory is composed of N-component fermions with four-fermion interactions. We employ the $1/N$ expansion method to estimate the effective potential for the fermion-

antifermion composite field. In two space-time dimensions the theory reduces to the Gross-Neveu model[3] which possesses the discrete chiral symmetry that is broken dynamically. In three space-time dimensions the theory defines a model which is solvable and renormalizable in the sense of the $1/N$ expansion.[4] Beyond four space-time dimensions the theory is nonrenormalizable. We confine ourselves to the space-time dimensions greater or equal to 2 and less than 4.

We introduce the temperature and chemical potential in the theory through the standard procedure. We investigate the behavior of the effective potential in the leading order of the $1/N$ expansion by varying the temperature and chemical potential for arbitrary space-time dimensions. Through the study of the shape of the effective potential and the detailed analysis of the gap equation we observe the phase transition from the phase with the broken chiral symmetry to the phase with the chiral symmetry when the temperature and chemical potential vary. With this analysis we shall be able to derive explicitly the critical curve on the temperature-chemical potential plane. We find that for space-time dimensions less than 3 both the first-order and second-order phase transition exist while for space-time dimensions greater or equal to 3 only the second-order phase transition is realized. At some specific points on the critical curve we obtain an explicit expression for the critical temperature and/or chemical potential. We also calculate the dynamical fermion mass as a function of the temperature and chemical potential.

In section 2 we briefly review the general properties of four-fermion theories in space-time dimensions $2 \leq D < 4$ with vanishing temperature and chemical potential. In section 3 we introduce the temperature and chemical potential in the theory. We calculate the effective potential in the leading order of the $1/N$ expansion and observe the change of the phase. We derive the critical curve which divide the chiral symmetric phase and the chiral asymmetric phase. The section 4 is devoted for the concluding remarks.

2 FOUR-FERMION THEORY IN ARBITRARY SPACE-TIME DIMENSIONS

In the present section we briefly summarize the characteristic features of four-fermion theories in arbitrary space-time dimensions $2 \leq D < 4$ for vanishing temperature and

chemical potential. We start with the simple Lagrangian

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i\gamma_\mu \partial^\mu \psi_k - \frac{\lambda_0}{2N} \sum_{k=1}^{N} (\bar{\psi}_k \psi_k)^2 \,, \tag{2.1}$$

where index k represents the flavors of the fermion field ψ, N is the number of flavors and λ_0 is a bare coupling constant. In the following, for simplicity, we neglect the flavor index. In two dimensions the theory is nothing but the Gross-Neveu model[3] in which Lagrangian (2.1) is invariant under the discrete chiral transformation,

$$\psi \longrightarrow \gamma_5 \psi \,. \tag{2.2}$$

The above discrete chiral symmetry in two dimensions prevents the Lagrangian to have mass terms. In arbitrary dimensions the transformation (2.2) may be generalized so that $\bar{\psi}\psi \rightarrow -\bar{\psi}\psi$.

The theory has a global $SU(N)$ flavor symmetry : the Lagrangian is invariant under the transformation,

$$\psi \longrightarrow e^{i\Sigma_\alpha g_a T^a} \psi \tag{2.3}$$

where T^a are generators of the $SU(N)$ symmetry. Under the circumstance of the global $SU(N)$ symmetry we may work in the scheme of the $1/N$ expansion.

For practical calculations in four-fermion theories it is convenient to introduce auxiliary field σ and consider the following equivalent Lagrangian

$$\mathcal{L}_y = \bar{\psi} i\gamma_\mu \partial^\mu \psi - \frac{N}{2\lambda_0} \sigma^2 - \bar{\psi}\sigma\psi \,. \tag{2.4}$$

If the non-vanishing vacuum expectation value is assigned to the auxiliary field σ, then there appears a mass term for the fermion field ψ and the discrete chiral symmetry(the Z_2 symmetry in odd dimensions) is eventually broken.

We would like to find a ground state of the system described by four-fermion theories. For this purpose we evaluate an effective potential for composite field $\bar{\psi}\psi$ in the theory described by Eq. (2.1). This effective potential is essentially the same as the one for field σ in the theory described by Eq. (2.4). The effective potential $V_0(\sigma)$ in the leading order of the $1/N$ expansion reads (See Appendix A)

$$V_0(\sigma) = \frac{1}{2\lambda_0} \sigma^2 + i \ln \det(i\gamma_\mu \partial^\mu - \sigma) + \mathrm{O}(1/N) \,, \tag{2.5}$$

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where the suffix 0 for $V_0(\sigma)$ is introduced to keep the memory that $T = \mu = 0$ with T the temperature and μ the chemical potential.

For integral dimensions the effective potential is in general divergent. Performing the integral in the potential (2.5) we obtain

$$V_0(\sigma) = \frac{1}{2\lambda_0}\sigma^2 - \frac{1}{(2\pi)^{D/2}D}\Gamma\left(1 - \frac{D}{2}\right)\sigma^D .\tag{2.6}$$

We clearly see that the effective potential is divergent in two and four dimensions. It happens to be finite in three dimensions in the leading order of the $1/N$ expansion. If the next-to-leading order is taken into account, it may be divergent in three dimensions $D = 3$.

As is well-known, four-fermion theory is renormalizable in two dimensions. Therefore the potential (2.6) is made finite at $D = 2$ by the usual renormalization procedure. For $D = 3$ four-fermion theory is known to be renormalizable in the sense of the $1/N$ expansion.[4] The potential (2.6) is finite by itself and hence we do not need renormalization in the leading order of the $1/N$ expansion at $D = 3$. In four dimensions four-fermion theory is not renormalizable and the finite effective potential can not be defined in four dimensions. We regard the effective potential for $D = 4 - \epsilon$ with ϵ sufficiently small positive as a regularization of the one in four dimensions.

We perform renormalization in two dimensions by imposing the renormalization condition,

$$\left.\frac{\partial^2 V_0(\sigma)}{\partial\sigma^2}\right|_{\sigma=\sigma_0} = \frac{1}{\lambda_r} .\tag{2.7}$$

where σ_0 is the renormalization scale. From this renormalization condition we get

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_r} + \frac{1}{(2\pi)^{D/2}}(D-1)\Gamma\left(1 - \frac{D}{2}\right)\sigma_0^{D-2} .\tag{2.8}$$

The renormalized effective potential reads

$$\begin{aligned}\frac{V_0(\sigma)}{\sigma_0^D} &= \frac{1}{2\lambda}\frac{\sigma^2}{\sigma_0^2} \\ &\quad + \frac{1}{2(2\pi)^{D/2}}(D-1)\Gamma\left(1 - \frac{D}{2}\right)\frac{\sigma^2}{\sigma_0^2} - \frac{1}{(2\pi)^{D/2}D}\Gamma\left(1 - \frac{D}{2}\right)\frac{\sigma^D}{\sigma_0^D}.\end{aligned}\tag{2.9}$$

Here we used the dimensionless coupling constant λ defined by

$$\lambda = \lambda_r\sigma_0^{D-2} .\tag{2.10}$$

The renormalized potential (2.9) is no longer divergent in the whole range of D considered here : $2 \leq D < 4$.

The ground state of the theory is determined by observing the minimum of the effective potential. The necessary condition for the minimum is given by

$$
\frac{\partial V_0(\sigma)}{\partial \sigma}\bigg|_{\sigma=m} = m\sigma_0^{D-2} \left[\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{\Gamma\left(1-\frac{D}{2}\right)}{(2\pi)^{D/2}} \left(\frac{m}{\sigma_0}\right)^{D-2} \right] = 0 \, . \tag{2.11}
$$

where m is the dynamical fermion mass and λ_c is defined by

$$
\lambda_c = (2\pi)^{D/2} \left[(1-D)\Gamma\left(1-\frac{D}{2}\right) \right]^{-1} \, . \tag{2.12}
$$

If the coupling constant λ is no less than a critical value λ_c, the gap equation allows a non-trivial solution which is given by

$$
m = \sigma_0 \left[\frac{(2\pi)^{D/2}}{\Gamma\left(1-\frac{D}{2}\right)} \left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right) \right]^{1/(2-D)} \, . \tag{2.13}
$$

In Fig. 1. we plot the critical coupling constant λ_c as a function of dimension D . The non-trivial solution m of the gap equation (2.11) corresponds to the nonvanishing vacuum expectation value of the composite field σ and is equal to the dynamically generated fermion mass. For some special values of D the solution m simplifies :

$$
m = \sigma_0 \, \mathrm{e}^{1-\pi/\lambda} \qquad ; D = 2 \, , \tag{2.14}
$$

$$
m = \sigma_0 \left(2 - \frac{\sqrt{\pi}}{\lambda} \right) \qquad ; D = 3 \, . \tag{2.15}
$$

and λ_c also simplifies :

$$
\lambda_c = 0 \qquad ; D = 2 \, , \tag{2.16}
$$

$$
\lambda_c = \frac{\pi}{\sqrt{2}} \qquad ; D = 3 \, . \tag{2.17}
$$

As is well-known, the shape of the effective potential $V_0(\sigma)$ is of single-well for $\lambda < \lambda_c$ while it is of double-well for $\lambda > \lambda_c$.
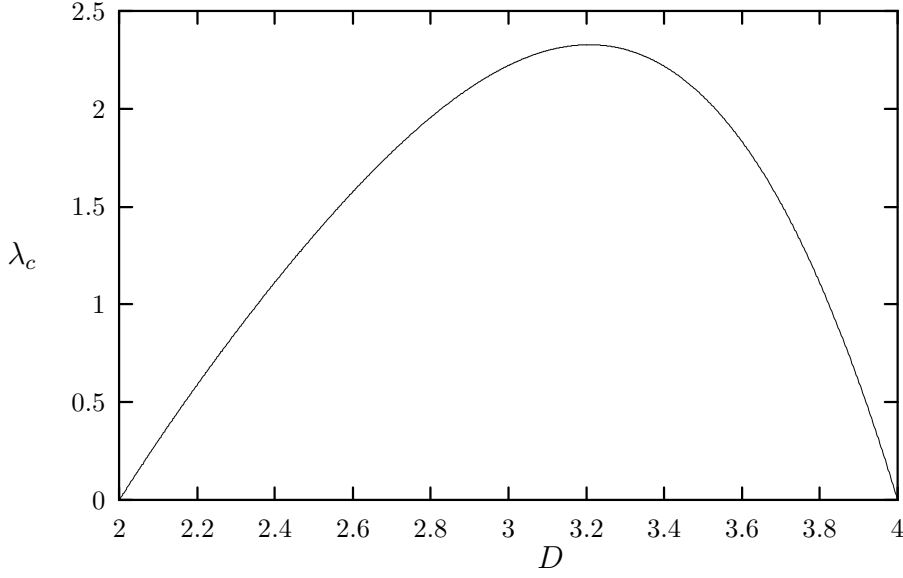
Fig. 1 Critical coupling λ_c of the four-fermion theory as a function of dimension D .

The renormalization group β function is found to be[5]

$$\beta(\lambda) \;=\; \sigma_0 \left.\frac{d\lambda}{d\sigma_0}\right|_{\lambda_0} , \tag{2.18}$$

$$=\; \frac{D-2}{\lambda_c}\lambda(\lambda_c - \lambda) , \tag{2.19}$$

in the leading order of the $1/N$ expansion. It simplifies for some special values of D :

$$\beta(\lambda) \;=\; -\frac{\lambda^2}{\pi} \qquad\qquad ; D = 2 , \tag{2.20}$$

$$=\; -\lambda\left(\frac{\sqrt{2}\lambda}{\pi} - 1\right) \qquad ; D = 3 . \tag{2.21}$$

We find from Eq. (2.19) that $\lambda = \lambda_c$ is the ultraviolet stable fixed point. In particular for $D = 2$ the theory is asymptotically free and for $D = 3$ the theory has a nontrivial ultraviolet fixed point at $\lambda = \dfrac{\pi}{\sqrt{2}}$.

We turn our attention to the theory governed by the following Lagrangian,

$$\mathcal{L} = \bar{\psi}i\gamma_\mu\partial^\mu\psi - \frac{\lambda_0}{2N}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] . \tag{2.22}$$

In four dimensions this Lagrangian defines the Nambu-Jona-Lasinio model.[1] The Lagrangian is invariant under the chiral $U(1)$ transformation in even dimensions,

$$\psi \longrightarrow \mathrm{e}^{i\theta\gamma_5}\psi . \tag{2.23}$$

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The chiral $U(1)$ symmetry prevents the Lagrangian to have mass terms. Using the auxiliary field method, we obtain the effective potential

$$V_0(\sigma') = \frac{1}{2\lambda_0}\sigma'^2 + i\ln\det(i\gamma_\mu\partial^\mu - \sigma') + O(1/N)\,, \tag{2.24}$$

$$\sigma' = \sqrt{\sigma^2 + \pi^2}\,. \tag{2.25}$$

Corresponding to the two four-fermion interaction terms in the Lagrangian (2.22), we need two kinds of auxiliary fields σ and π. In two dimensions it is well-known that π loop has an infra-red divergence and we can not neglect the next to leading order terms of the $1/N$ expansion.[6] Therefore our present analysis is not sufficient to deal with the Lagrangian (2.22) in two dimensions.

The effective potentials (2.5) and (2.24) have the same form in the leading order of the $1/N$ expansion. Thus using the effective potential (2.5), we can discuss the phase structure of both theories except for the case in two dimensions.

3 PHASE STRUCTURE AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

3.1 Effective potential and gap equation

As we have seen in the previous section the chiral symmetry is broken above the critical coupling. The symmetry broken spontaneously may be restored in an environment of high temperature and density. To see whether this situation is realized we would like to study the phase structure at finite temperature and chemical potential.

The n-point Green function at finite temperature and chemical potential is defined by

$$G_n^{\beta\mu} = \frac{\sum_\alpha e^{-\beta(E_\alpha - \mu N_\alpha)}\langle\alpha|\mathrm{T}(\phi(x_1)\cdots\phi(x_n))|\alpha\rangle}{\sum_\alpha e^{-\beta(E_\alpha - \mu N_\alpha)}}\,, \tag{3.1}$$

where $\phi(x)$ represents a component of fermion field $\psi(x)$ or $\bar{\psi}(x)$, E_α and N_α are the energy and particle number in the state specified by quantum number α respectively, $\beta = \dfrac{1}{kT}$ with k the Boltzmann Constant and T the temperature and μ is the chemical potential. The generating functional for Green functions (3.1) reads

$$Z^{\beta\mu}[J] = \mathrm{Tr}\left[e^{-\beta(H - \mu N)}\mathrm{T}\exp i\int dx J(x)\phi(x)\right]\,, \tag{3.2}$$

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where H is the Hamiltonian, N is the number operator and $J(x)$ is the source function for field $\phi(x)$.

Following the standard procedure of the Matsubara Green function[7] (See Appendix B) we calculate the effective potential in our theory in the leading order of the $1/N$ expansion as described already in the case of $T = \mu = 0$ in Eq. (2.5). We find

$$V(\sigma) = \frac{1}{2\lambda_0}\sigma^2 + i\frac{1}{\beta}\sum_n \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\ln \det \frac{\not{k} - \sigma}{\not{k}}, \tag{3.3}$$

where the four-momentum k^μ is given by

$$k^\mu = (i\omega_n - \mu, \vec{k}), \tag{3.4}$$

and the discrete variable ω_n given by

$$\omega_n = \frac{(2n+1)\pi}{\beta}, \tag{3.5}$$

according to the anti-periodic boundary condition.

Performing a summation and integrating over angle variables (See Appendix B) we get

$$V(\sigma) = V_0(\sigma) + V^{\beta\mu}(\sigma), \tag{3.6}$$

where $V_0(\sigma)$ is the effective potential for $T = \mu = 0$ shown in Eq. (2.6) and $V^{\beta\mu}(\sigma)$ is given by

$$V^{\beta\mu}(\sigma) = -\frac{1}{\beta}\frac{\sqrt{2}}{(2\pi)^{(D-1)/2}}\frac{1}{\Gamma\left(\frac{D-1}{2}\right)}\sigma_0^D$$

$$\times \int dk k^{D-2}\left\{\ln\frac{1 + e^{-\beta(\sqrt{k^2+\sigma^2}+\mu)}}{1 + e^{-\beta(k+\mu)}} + \ln\frac{1 + e^{-\beta(\sqrt{k^2+\sigma^2}-\mu)}}{1 + e^{-\beta(k-\mu)}}\right\}. \tag{3.7}$$

It is found that $V^{\beta\mu}(\sigma)$ is finite in the space-time dimension $2 \leq D < 4$. Thus we need not renormalize the parts including the effects of finite temperature and density. We apply the same renormalization condition as shown in Eq. (2.7) and obtain the renormalized effective potential as in the $T = \mu = 0$ case. The renormalized effective potential $V(\sigma)$ is given by replacing V_0 in Eq. (3.6) by the one in Eq. (2.9).

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If we perform the integration first in Eq. (3.3) and leave the infinite sum we find

$$
V(\sigma) = \frac{\sigma^2}{2\lambda_0} + \frac{2^{D/2-1}}{\beta} \frac{\Gamma\left(\frac{1-D}{2}\right)}{(4\pi)^{(D-1)/2}}
$$

$$
\times \sum_{n=-\infty}^{\infty} \left[\{(\omega_n + i\mu)^2 + \sigma^2\}^{(D-1)/2} - \{(\omega_n + i\mu)^2\}^{(D-1)/2} \right] . \qquad (3.8)
$$

Although the above expression is essentially equivalent to the one in Eq. (3.7), we shall often use the expression (3.8) in the later argument of the gap equation.

The gap equation is obtained through the stationary condition

$$
\left. \frac{\partial V(\sigma)}{\partial \sigma} \right|_{\sigma=m_{\beta\mu}} = 0 . \qquad (3.9)
$$

where $m_{\beta\mu}$ is a possible dynamical mass of the fermion. If we apply the expression (3.7) in Eq. (3.9), we find

$$
\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{1}{(2\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \left(\frac{m_{\beta\mu}}{\sigma_0}\right)^{D-2}
$$

$$
+ \frac{\sqrt{2}}{\Gamma((D-1)/2)(2\pi)^{(D-1)/2}} \sigma_0^{2-D}
$$

$$
\times \int_0^{\infty} dk\, k^{D-2} \frac{1}{\sqrt{k^2 + m_{\beta\mu}^2}} \left\{ \frac{1}{1 + e^{\beta(\sqrt{k^2+m_{\beta\mu}^2}+\mu)}} + \frac{1}{1 + e^{\beta(\sqrt{k^2+m_{\beta\mu}^2}-\mu)}} \right\}
$$

$$
= 0 . \qquad (3.10)
$$

In Eq. (3.10) we neglected the trivial solution $m_{\beta\mu} = 0$. If Eq. (3.8) is employed, the following expression is obtained,

$$
\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{2^{D/2}}{\beta\sigma_0} \frac{\Gamma\left(\frac{3-D}{2}\right)}{(4\pi)^{(D-1)/2}} \sum_{n=-\infty}^{\infty} \left\{ \frac{(\omega_n + i\mu)^2 + m_{\beta\mu}^2}{\sigma_0^2} \right\}^{(D-3)/2} = 0 . \qquad (3.11)
$$

3.2 Phase boundary

The expression (3.7) is useful to calculate the effective potential numerically as a function of the composite field σ . Using Eq. (3.7) we calculate the effective potential for $D = 2$ to find the behavior of the effective potential[8] as shown in Fig. 2.
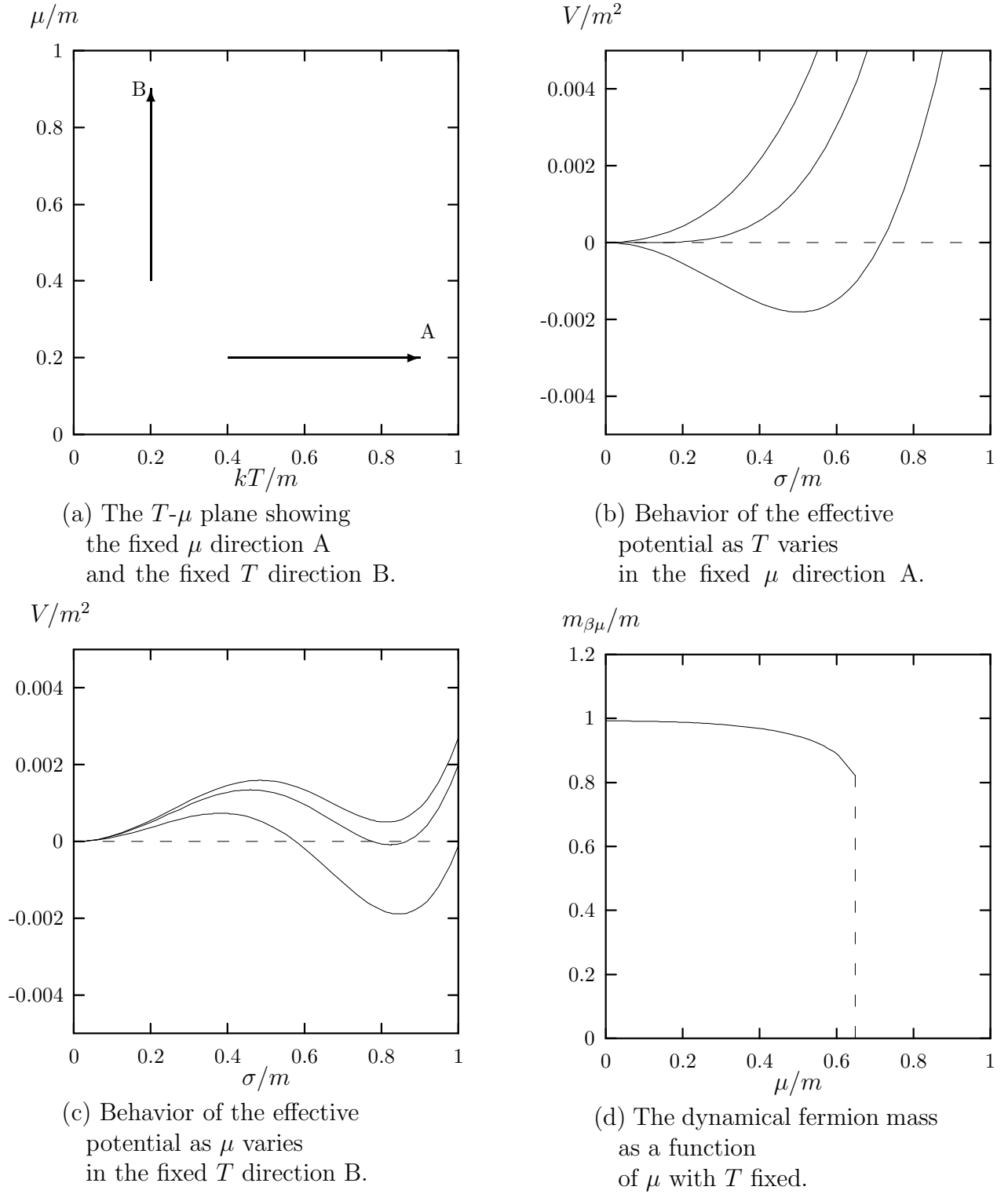
As is seen in Fig. 2, we observe that

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μ/m

V/m^2

V/m^2

$m_{\beta\mu}/m$

(a) The T-μ plane showing
the fixed μ direction A
and the fixed T direction B.

(b) Behavior of the effective
potential as T varies
in the fixed μ direction A.

(c) Behavior of the effective
potential as μ varies
in the fixed T direction B.

(d) The dynamical fermion mass
as a function
of μ with T fixed.

Fig. 2 Behaviors of the effective potential and the dynamical fermion mass for $D = 2$

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1. There is a second-order phase transition as T is increased with μ kept small fixed.

2. There is a first-order phase transition as μ is increased with T kept small fixed.

Thus we expect that the first- and second-order phase transition coexist on the $T - \mu$ plane. For D just above 2 this property is expected to persist. On the other hand at $D = 3$ we observe only the second-order phase transition for varying T and μ.[9] This situation is illustrated in Fig. 3.
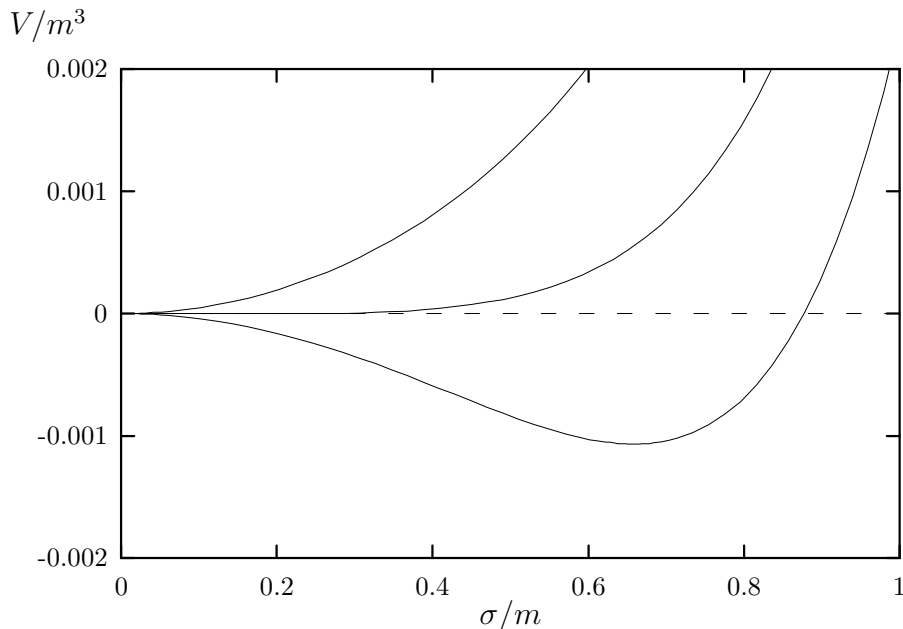
Fig. 3 Behavior of the effective potential at $D = 3$ along the B line in Fig. 2 (a).

The above preliminary analysis suggests that the first-order phase transition observed at $D = 2$ disappears at some critical dimension $D_c \leq 3$.

In order to see the situation more precisely we would like to perform a rigorous analysis on the critical temperature T_c and chemical potential μ_c. We would like to find the critical line on the $T - \mu$ plane in an analytic form if possible. Let us first assume that the phase transition is of the second order. Since for the second-order phase transition the nontrivial dynamical fermion mass develops continuously at the critical temperature and chemical potential as illustrated in Fig. 4, the critical temperature and chemical potential are determined by letting $m_{\beta\mu} \to 0$ in the gap equation. Thus we set $m_{\beta\mu} = 0$
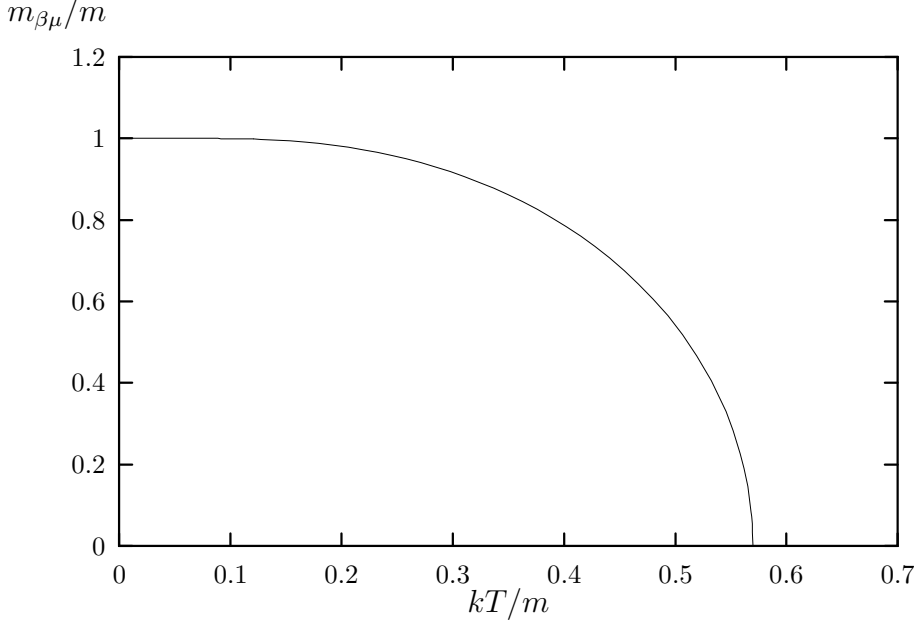
Fig. 4 Behavior of the dynamical fermion mass $m_{\beta\mu}$ in the case of the second-order phase transition as T varies with μ fixed .

in Eq. (3.11) and obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{2(2\pi)^{D/2-2}}{\sqrt{\pi}}\Gamma\left(\frac{3-D}{2}\right)(\sigma_0\beta)^{2-D}\mathrm{Re}\ \zeta\left(3-D,\ \frac{1}{2}+i\frac{\beta\mu}{2\pi}\right) = 0 \,. \qquad (3.12)$$

where we used the following relation

$$\sum_{n=-\infty}^{\infty}\left\{(\omega_n+i\mu)^2\right\}^{-\nu} = \left(\frac{\beta}{2\pi}\right)^{2\nu}\left\{\zeta\left(2\nu,\ \frac{1}{2}-i\frac{\beta\mu}{2\pi}\right)+\zeta\left(2\nu,\ \frac{1}{2}+i\frac{\beta\mu}{2\pi}\right)\right\} \,, (3.13)$$

with the definition of the generalized zeta function $\zeta(z, a)$,[10]

$$\zeta(z, a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^z} \quad ;\mathrm{Re}\,z > 1 \,, \qquad (3.14)$$

Eq. (3.12) provides us with the relation between the critical temperature T_c and chemical potential μ_c, i. e. the critical curve on the $T-\mu$ plane if the transition is of the second-order. It should be noted that the relation (3.13) holds only for $D < 2$ according to the condition $\mathrm{Re}\,(2\nu) > 1$ as given in Eq. (3.14). In Eq. (3.12), however, we analytically continue the variable D to the region $\mathrm{Re}\,D \geq 2$. As a typical example the second-order critical curve at $D = 2.5$ is presented in Fig. 5 using Eq. (3.12).

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(a) Second-order critical curve
at $D = 2.5$

(b) Second-order critical curves
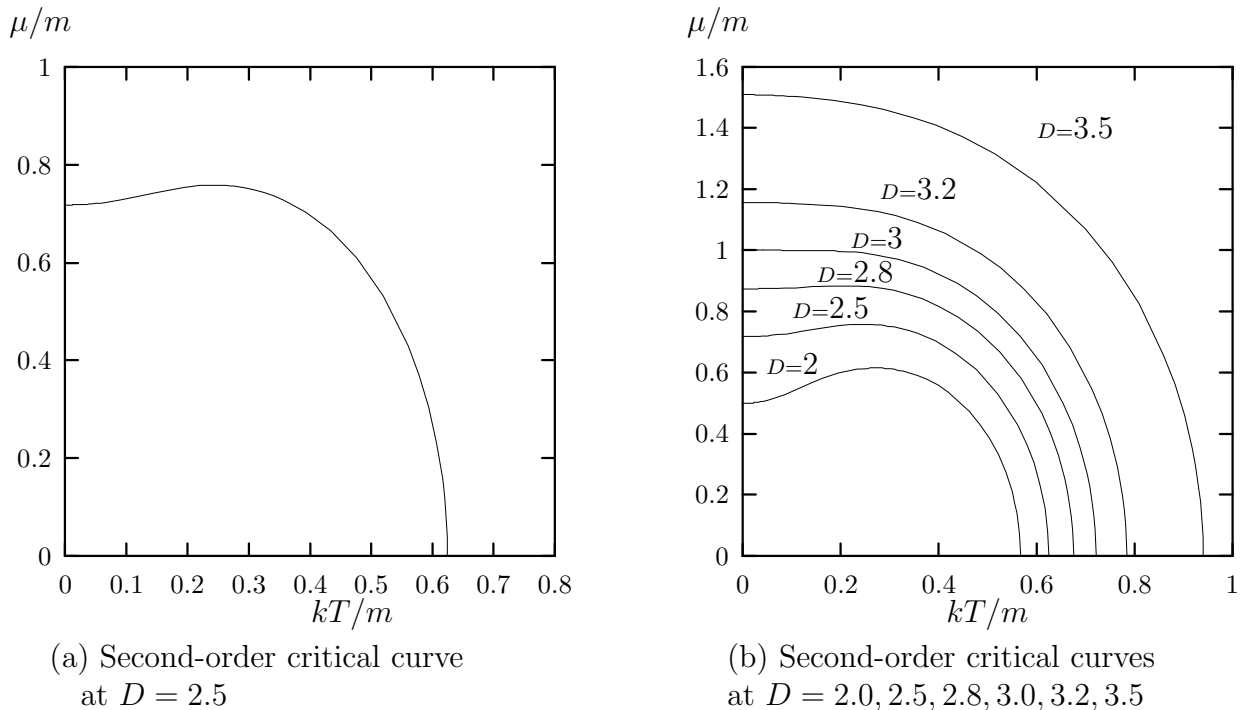at $D = 2.0, 2.5, 2.8, 3.0, 3.2, 3.5$

Fig. 5 Second-order critical curves

Eq. (3.12) simplifies when $D = 2$ and $D = 3$. We note that the generalized zeta function $\zeta(z, a)$ has a pole at $z = 1$ and its behavior near $z = 1$ is given by

$$\zeta(z, a) \rightarrow \frac{1}{z - 1} - \psi(a) \qquad ; z \rightarrow 1 \,, \tag{3.15}$$

where $\psi(a)$ is the digamma function. Using Eq. (3.15) for $z = 3 - D$ and applying some formulae for the digamma function we find for $D = 2$

$$2 \operatorname{Re} \psi \left(1 + i \frac{\beta \mu}{\pi} \right) - \operatorname{Re} \psi \left(1 + i \frac{\beta \mu}{2\pi} \right) = \ln \frac{\beta m}{\pi} \,, \tag{3.16}$$

where we employed Eq. (2.13) with m the dynamical fermion mass for $T = \mu = 0$. Eq. (3.16) exactly reproduces the result obtained in Ref. 11. To derive the corresponding formula for $D = 3$ we employ the following expansion formula for the generalized zeta function:

$$\zeta(x, a) = \frac{1}{2} - a + x \left\{ \ln \Gamma(a) - \frac{1}{2} \ln 2\pi \right\} + O(x^2) \,. \tag{3.17}$$

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Using Eq. (3.17) with $x = 3 - D$ and $a = \dfrac{1}{2} + i\dfrac{\beta\mu}{2\pi}$ we rewrite Eq. (3.12) as follows,

$$\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{\sqrt{2}}{\pi\beta\sigma_0} \ln \frac{2\left|\Gamma\left(i\dfrac{\beta\mu}{\pi}\right)\right|^2}{\left|\Gamma\left(i\dfrac{\beta\mu}{2\pi}\right)\right|^2} = 0 \,, \tag{3.18}$$

which reduces to the simple formula

$$e^{\beta m/2} = 2\cosh\frac{\beta\mu}{2} \,. \tag{3.19}$$

Eq. (3.19) exactly agrees with the one obtained in Ref. 9.

Our formula (3.12) gives a correct critical curve on the $T - \mu$ plane as far as the transition is of the second order. If the transition is of the first order, the curve given by the formula (3.12) corresponds to the local maximum (first extremum) of the effective potential and does not give a phase boundary. In such a case we have to find the critical curve corresponding to the true minimum (second extremum) of the effective potential. The critical curve for the first-order phase transition is obtained by eliminating the variable σ in the following two equations:

$$V(\sigma) = 0 \,, \qquad \frac{\partial V(\sigma)}{\partial \sigma} = 0 \,, \tag{3.20}$$

where we assume that $\sigma \neq 0$.

As we have seen before there exists a first-order phase transition for $2 \leq D < 3$. Here we have to perform an analysis based on Eq. (3.20). Unfortunately we are unable to solve this problem analytically. We then employ expression (3.6) with (3.7) and (3.10) to find the critical T and μ numerically through the use of MATHEMATICA on our workstation. For example by numerically calculating the effective potential at $D = 2.5$ for $\beta m = 10$ we obtain a figure of the effective potential as shown in Fig. 6.

As can be seen in Fig. 6 the critical chemical potential is given by $\dfrac{\mu}{m} = 0.81$ at $\dfrac{\sigma}{m} = 0.96$. Repeating this kind of analysis many times we find the critical curve for the first-order phase transition at $D = 2.5$. In Fig. 7 is shown the first-order critical curve by the dashed line while the second-order critical curve is given in the full line. We performed the full analysis of the above type for $2 \leq D < 3$ and obtained the critical
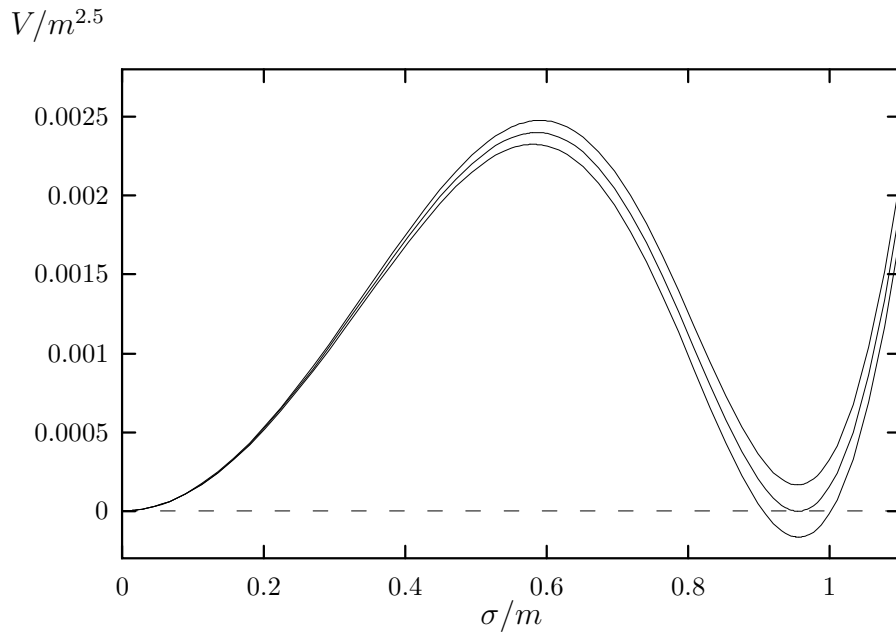
15

$V/m^{2.5}$

Fig. 6 Searching for the first order critical point by directly observing the behavior of the effective potential.
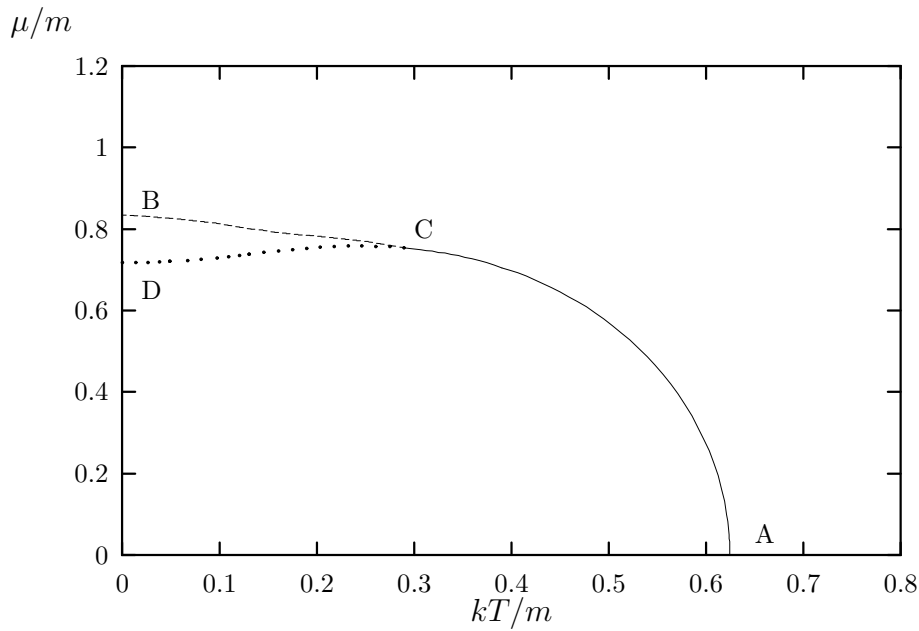
μ/m

Fig. 7 Critical curve in $D = 2.5$. The dashed line represents the first-order phase transition while the full line represents the second-order phase transition line. The dotted line signals an appearance of the first extremum (local maximum) in the effective potential and has nothing to do with the phase transition.
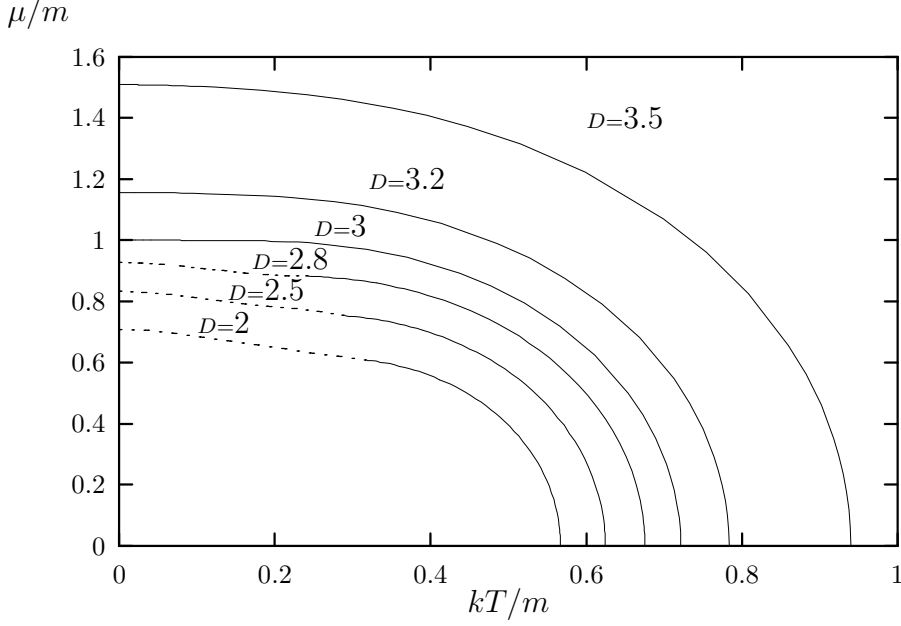
16

Fig. 8 Critical curves for $2 \leq D < 4$.

curves as shown in Fig. 8. In Fig. 8 we clearly observe that the first-order critical line given in the dashed line smoothly disappears at $D = 3$. We in fact checked that the phase transition is always of second order for $3 \leq D < 4$.

3.3 Specific points on the critical curve

It is interesting to note that we are able to find analytically some specific points on the critical curve. Those points are shown in Fig. 7 by A, B, C and D.

Let us first consider point A. Since we know that at this point the transition is of second order, we are free to use the formula (3.12) for determining point A. We set $\mu = 0$ in Eq. (3.12) to obtain

$$\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{2(2\pi)^{D/2-2}}{\sqrt{\pi}} \Gamma\left(\frac{3-D}{2}\right) (\beta\sigma_0)^{2-D}(2^{3-D} - 1)\zeta(3 - D) = 0 \,, \qquad (3.21)$$

where $\zeta(z)$ is the zeta function. Using Eq. (2.13) we rewrite Eq. (3.21) in the following form:

$$\beta m = 2\pi \left[\frac{2\Gamma\left(\dfrac{3-D}{2}\right)}{\sqrt{\pi}\Gamma\left(\dfrac{2-D}{2}\right)} (2^{3-D} - 1)\zeta(3 - D) \right]^{1/(D-2)} . \qquad (3.22)$$

Eq. (3.22) gives the critical temperature.

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For $D = 2$ Eq. (3.22) reduces to the well-known formula for the critical temperature,[12]

$$\beta m = \pi e^{-\gamma} \,. \tag{3.23}$$

To see this we employ the following expansion formulae in Eq. (3.22),

$$
\begin{aligned}
\Gamma(\varepsilon) &= \frac{1}{\varepsilon} - \gamma + O(\varepsilon^2) \,, \\
\Gamma\left(\frac{1}{2} + \varepsilon\right) &= \sqrt{\pi} \left\{ 1 - \varepsilon(2\ln 2 + \gamma) + O(\varepsilon^2) \right\} \,, \\
\zeta(1 + 2\varepsilon) &= \frac{1}{2\varepsilon} + \gamma + O(\varepsilon) \,,
\end{aligned}
\tag{3.24}
$$

with γ the Euler constant and to obtain

$$\beta m = \lim_{\varepsilon \to 0} 2\pi \left\{ 1 + 2\varepsilon(\ln 2 + \gamma) \right\}^{-1/2\varepsilon} \,, \tag{3.25}$$

which clearly reproduces Eq. (3.23). For $D = 3$ Eq. (3.22) again reproduces the well-known formula[9]

$$\beta m = 2\ln 2 \,. \tag{3.26}$$

To derive Eq. (3.26) from Eq. (3.22) we only need the following expansion formula near $D = 3$,

$$\zeta(3 - D) = -\frac{1}{2} - \frac{1}{2}(3 - D)\ln 2\pi + O((3 - D)^2) \,. \tag{3.27}$$

In Fig. 9 the critical temperature is plotted as a function of dimension D by the use of Eq. (3.22).

We next consider point B. On the μ-axis for which $T = 0 \, (\beta \to \infty)$ the gap equation (3.10) reads for $\mu < m_{\beta\mu}$

$$\Gamma\left(\frac{2 - D}{2}\right)(m^{D-2} - m_{\beta\mu}^{D-2}) = 0 \,, \tag{3.28}$$

and for $\mu \geq m_{\beta\mu}$

$$\Gamma\left(\frac{2 - D}{2}\right)(m^{D-2} - m_{\beta\mu}^{D-2}) + \frac{2\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)} \int_0^{\sqrt{\mu^2 - m_{\beta\mu}^2}} dk \frac{k^{D-2}}{\sqrt{k^2 + m_{\beta\mu}^2}} = 0 \,. \tag{3.29}$$

Eq. (3.28) gives a simple solution

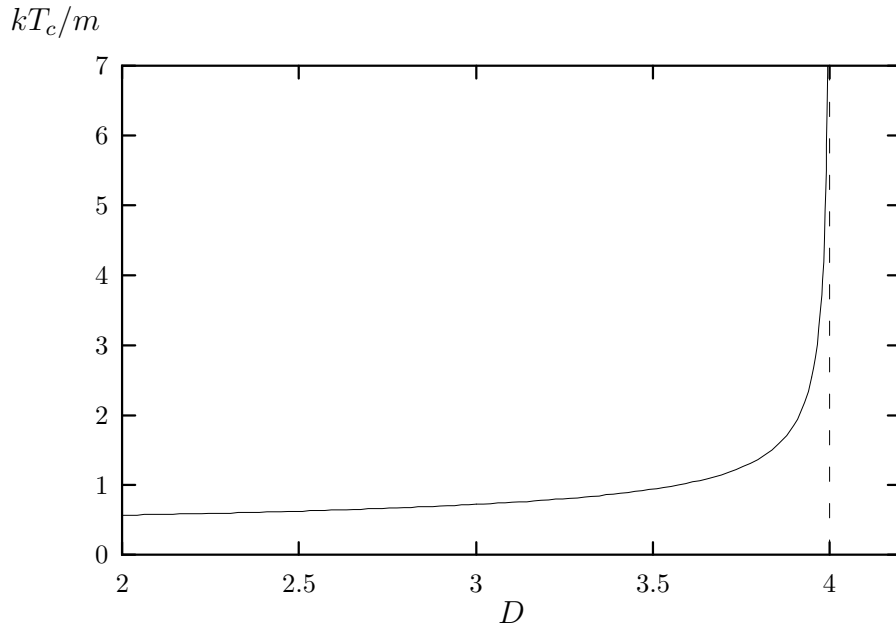$$m_{\beta\mu} = m \,, \tag{3.30}$$

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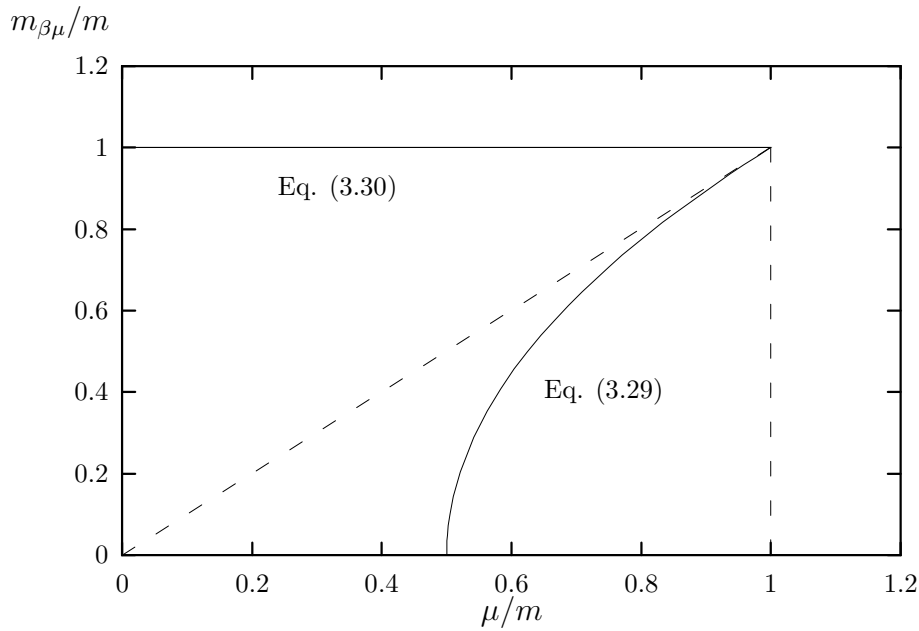Fig. 9 critical temperature for $\mu = 0$ as a function of dimension D .

Fig. 10 Dynamical fermion mass as a function of μ with $T = 0$.

which is typically of the first-order type since it has a gap at $\mu = \mu_c$ as is seen in Fig. 2(d). On the other hand Eq. (3.29) gives a more complicated solution which is shown in Fig. 10. This solution is typically of the second-order type.

For $2 \leq D < 3$ the phase transition along the μ-axis is of first order as we have seen before. Hence the solution (3.30) gives a true vacuum and the solution of Eq. (3.29) corresponds to the first extremum of the effective potential. We adopt the solution (3.30) to study the condition,

$$\lim_{\beta \to \infty} V(\sigma = m) = 0 . \tag{3.31}$$

Using Eq. (3.6) with Eq. (3.7) we rewrite the condition (3.31) as follows,

$$\frac{1}{D}\Gamma\left(\frac{4-D}{2}\right)m^D + \frac{2\sqrt{\pi}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^\mu dk\, k^{D-2}(k-\mu) = 0 , \tag{3.32}$$

which provides us with the critical chemical potential for $2 \leq D < 3$,

$$\mu = m\left\{\frac{3}{4}\mathrm{B}\left(\frac{4-D}{2}, \frac{D+1}{2}\right)\right\}^{1/D} . \tag{3.33}$$

It is easy to check that Eq. (3.33) satisfies the condition . For $D = 2$ Eq. (3.33) simplifies to[8,13]

$$\mu = \frac{m}{\sqrt{2}} . \tag{3.34}$$

For $3 \leq D < 4$ the phase transition along the μ-axis is of second order and hence we have to adopt the case of Eq. (3.29). Letting $m_{\beta\mu} \to 0$ in Eq. (3.29) and solving for μ we find

$$\mu = m\left\{\frac{1}{2}\mathrm{B}\left(\frac{4-D}{2}, \frac{D-1}{2}\right)\right\}^{1/(D-2)} . \tag{3.35}$$

Obviously Eq. (3.35) satisfies the condition $\mu \geq m_{\beta\mu}$. For $D = 3$ Eq. (3.35) simplifies to

$$\mu = m , \tag{3.36}$$

which agrees with the known result.[9] In Fig. 11 the the critical chemical potential is plotted as a function of dimension D by using Eqs. (3.33) and (3.35).

It is worth noting here that, if we would have used the second- order result (3.35) for $2 \leq D < 3$, we would have obtained point D shown in Fig. 7. The value of μ at point D is given by Eq. (3.35) with $2 \leq D < 3$. For $D = 2$ we find
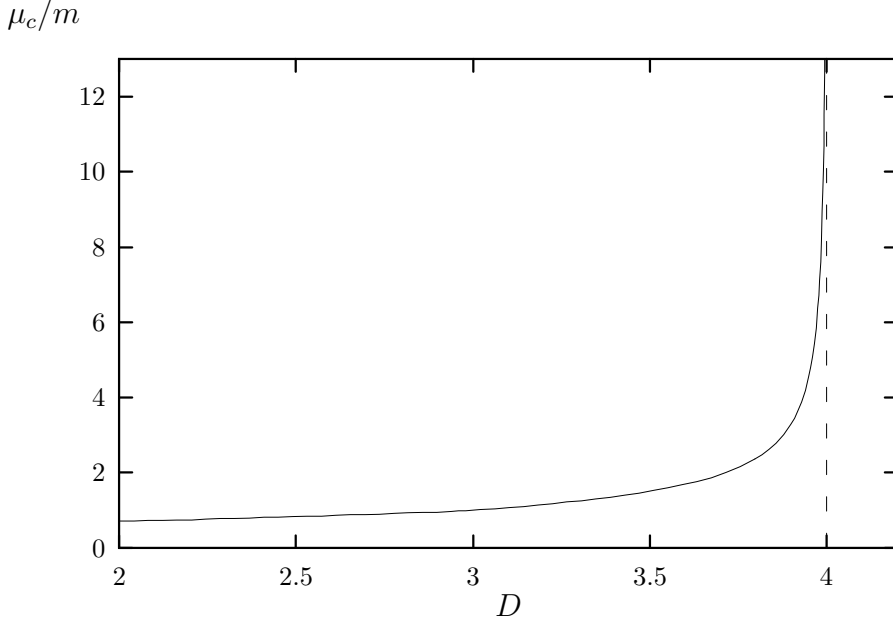
$$\mu = \frac{m}{2} , \tag{3.37}$$

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Fig. 11 Critical chemical potential for $T = 0$ as a function of dimension D.

instead of Eq. (3.34). This, however, corresponds to the first extremum of the effective potential and has nothing to do with the phase transition.[14]

Finally we consider point C on the critical curve. This point appears only when $2 \leq D < 3$. At point C the first-order critical curve meets with the second-order critical curve. To find out the value of T and μ at point C we proceed as follows: Let us consider the gap equation

$$\frac{\partial V}{\partial \sigma} \equiv \sigma f(\sigma) = 0 . \tag{3.38}$$

We fix the dynamical mass at the first nonvanishing extremum . Thus we have

$$f(\sigma) = 0 . \tag{3.39}$$

By fixing σ at σ_m temperature T is related to chemical potential μ through Eq. (3.39). As shown in Fig. 12 the relation between T and μ is plotted in the form of the curve G just as we have done for the critical curve. Since the dynamical fermion mass is fixed to σ_m on this curve G, the behavior of the effective potential along the curve G looks as in Fig. 13. Obviously the derivative of the effective potential, i. e. $f(\sigma)$, changes sign when the curve G crosses the critical curve. On the critical curve the effective potential is flat in the region $\sigma \leq \sigma_m$ as seen in Fig. 13. Thus at the crossing E of the curve G
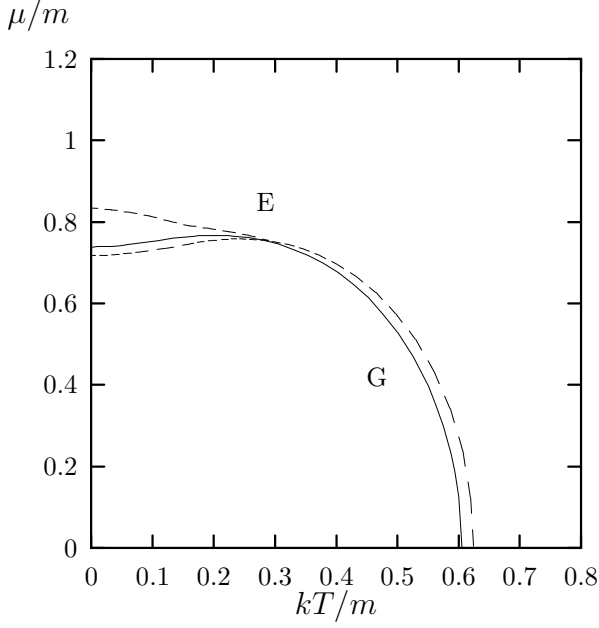
21

Fig. 12 The T-μ plot as given by
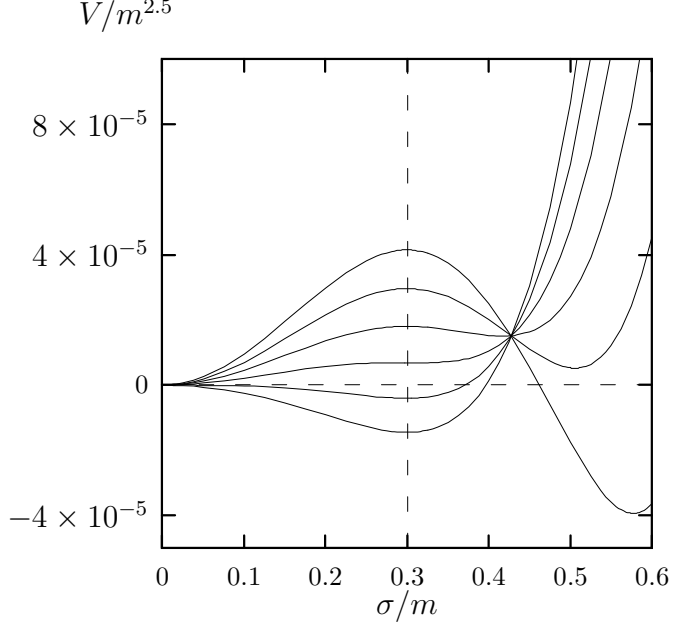Eq. (3.39) when σ is fixed at σ_m .

Fig. 13 Behavior of the effective
potential as T and μ vary on the
curve G ($\sigma_m = 0.3$).

and the critical curve the second derivative of the effective potential has to vanish,

$$\frac{\partial^2 V}{\partial \sigma^2} = f(\sigma) + \sigma \frac{\partial f}{\partial \sigma} = 0 \,. \tag{3.40}$$

Combining Eq. (3.39) with Eq. (3.40) we have a condition for the point E,

$$\frac{\partial f}{\partial \sigma} = 0 \,. \tag{3.41}$$

The point E divides the second-order region from the first-order region along the curve
G. By letting $\sigma \to 0$ in Eq. (3.40) we reach to the point C in Fig. 7. Hence the condition
to get the point C is given by

$$\left. \frac{\partial f}{\partial \sigma} \right|_{\sigma=0} = 0 \,. \tag{3.42}$$

Inserting Eq. (3.11) into Eq. (3.42) we finally obtain the condition for point C,

$$\mathrm{Re}\, \zeta \left(5 - D, \frac{1}{2} + i \frac{\beta \mu}{2\pi} \right) = 0 \,. \tag{3.43}$$

By solving Eq. (3.43) for $\beta \mu$ we have a curve in the T-μ plane. Looking for the crossing
of this curve with the critical line we find the value of β and μ at point C. In Fig. 14 the
value of $\beta \mu$ at point C is plotted as a function of dimension D by using Eq. (3.43).
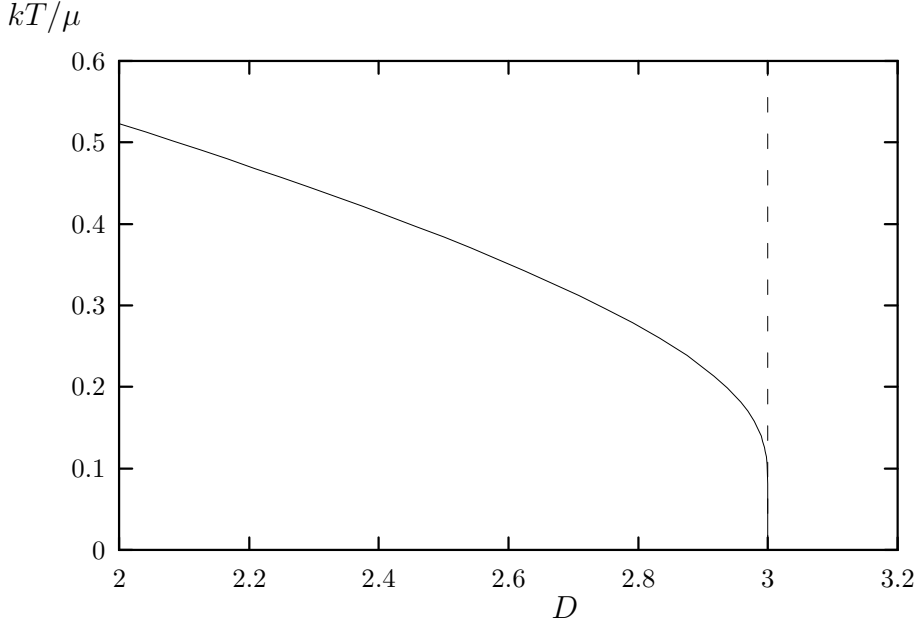
22

Fig. 14 The value of kT/μ at point C as a function of dimension D .

3.4 Dynamical fermion mass at finite T and μ

We have seen that the broken chiral symmetry at low temperature and chemical potential is restored at certain critical temperature and chemical potential. The symmetry restoration is of second order for $3 \leq D < 4$. For $2 \leq D < 3$ two types of the phase transition, first and second order, coexist.

The dynamically generated fermion mass at low temperature and chemical potential disappears above certain critical temperature and chemical potential. In the following we shall examine the behavior of the dynamical fermion mass as a function of the temperature and chemical potential by numerically solving the gap equation. As typical dimensions we choose $D = 2.5, 3.0, 3.5$.

In Fig. 15 we show the behavior of the dynamical fermion mass as a function of the chemical potential for some fixed temperatures. We clearly observe that for $D = 2.5$ the mass gap appears at the critical chemical potential as T increases while no mass gap shows up above $D = 3$. This behavior at $D = 2.5$ is a reflection of the smooth transition from the first-order to second-order phase transition. It should be noted here that, while at $D = 3$ the phase transition is of second order, there is an exceptional case at $T = 0$ where the mass gap exists at the critical chemical potential $\mu = m$. In Fig. 16 we present

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the behavior of the dynamical fermion mass as a function of the temperature for some fixed chemical potentials. Here for $D = 2.5$ the mass gap observed at low temperature disappears for higher temperatures while no mass gap is observed above $D = 3$.
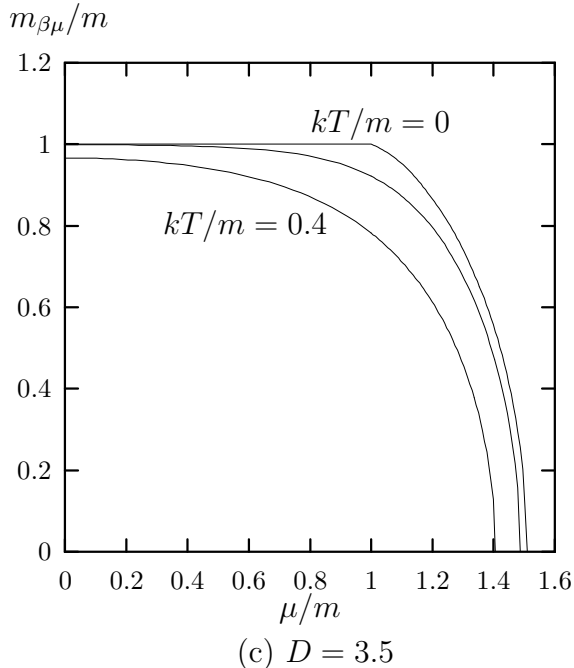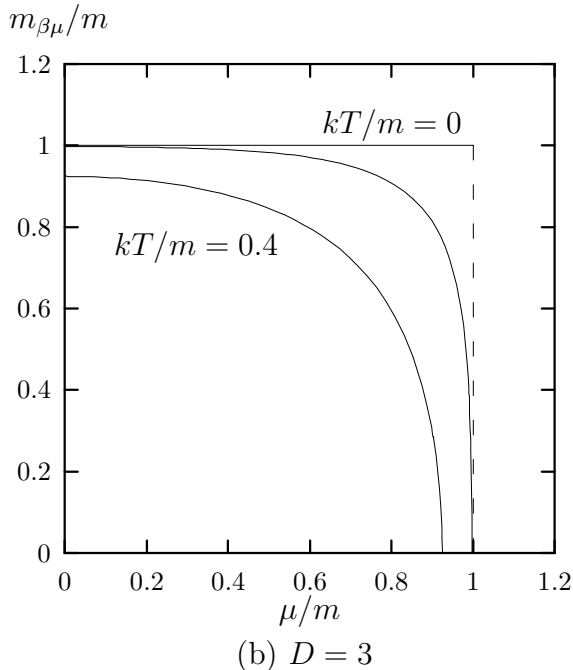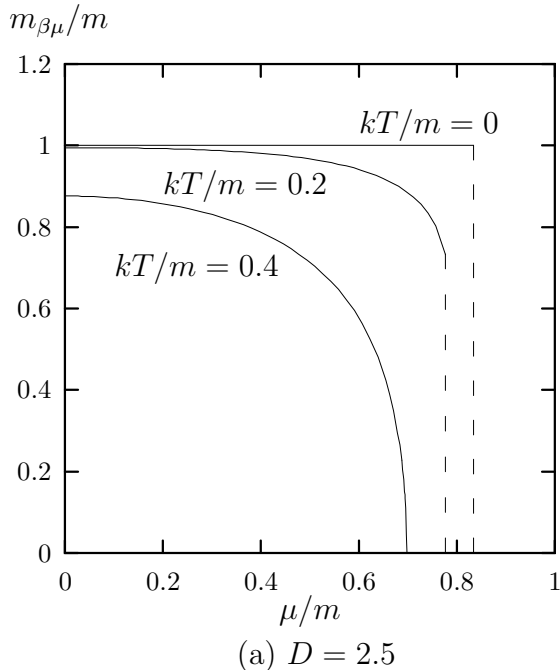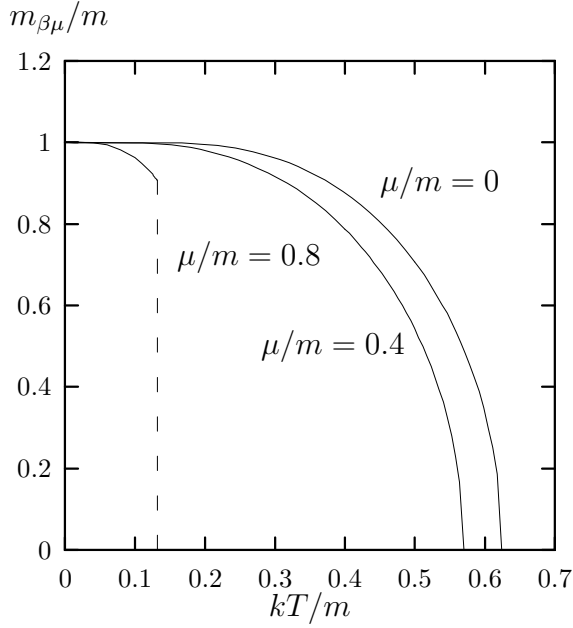

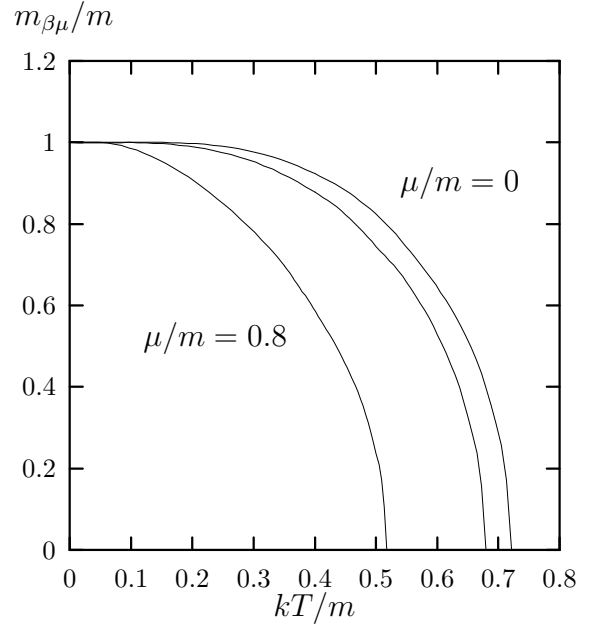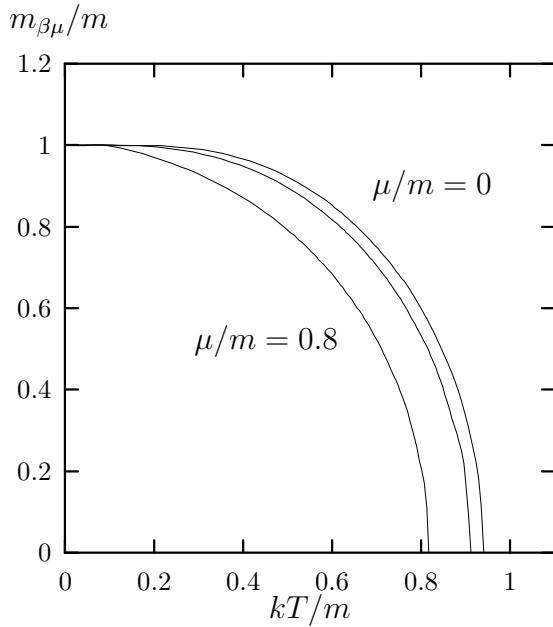(a) $D = 2.5$

(b) $D = 3$

(c) $D = 3.5$

Fig. 15 Dynamical fermion mas $m_{\beta\mu}$ as a function of the chemical potential μ with temperature kT/m fixed at 0,0.2,0.4.

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(a) $D = 2.5$

(b) $D = 3$

(c) $D = 3.5$

Fig. 16 Dynamical fermion mass $m_{\beta\mu}$ as a function of the temperature T with the chemical potential μ/m fixed at 0,0.4,0.8.

4   CONCLUSION

We have investigated the phase structure of four-fermion theories at finite temperature T and chemical potential μ in arbitrary dimensions by using the effective potential and gap equation in the leading order of the $1/N$ expansion. The theory under consideration

is renormalizable below 4 dimensions and give an insight into the phase structure of the theory in 4 dimensions.

Starting from the theory with broken chiral symmetry at vanishing T and μ we calculated the renormalized effective potential for finite T and μ in the leading order of the $1/N$ expansion in arbitrary dimensions $2 \leq D < 4$. We found that the broken chiral symmetry was restored at a certain critical temperature and chemical potential. The phase transition from the broken phase to the symmetric phase is either of first order or of second order for $2 \leq D < 3$ and is of second order for $3 \leq D < 4$. We found the boundary curve dividing the symmetric phase and broken phase in the T-μ plane. The dynamical mass generated in the broken phase is studied as a function of the temperature and chemical potential.

At $D = 2$ and $D = 3$ formulae derived in the present investigation mostly reduce to the known results in the preceding works. Some results obtained in the present work are not known previously. On the critical curve we found analytic expressions for some specific points.

Although the present work is restricted mostly to the analysis of the mathematical properties of the four fermion theory, we are interested in applying our results to physical problems in the early stage of the Universe. We will continue our work further and hope to publish reports on these problems.

ACKNOWLEDGMENTS

APPENDIX A

EFFECTIVE POTENTIAL FOR $T = \mu = 0$

In this appendix we present details of the calculation of the effective potential given in Eq. (2.6). After the Fourier transformation Eq. (2.5) becomes

$$V_0(\sigma) = \frac{1}{2\lambda_0}\sigma^2 + i\int \frac{d^D k}{(2\pi)^D}\ln \det\left(\frac{\not{k} - \sigma}{\not{k}}\right). \tag{A.1}$$

It should be noted that the effective potential is normalized so that $V_0(0) = 0$. The second term of the right-hand side of Eq. (A.1) is rewritten as.

$$i\int \frac{d^D k}{(2\pi)^D}\ln \det\left(\frac{\not{k} - \sigma}{\not{k}}\right)$$

$$= -\frac{1}{2}\int \frac{d^D k}{(2\pi)^D i}\left[\ln \det\left(\frac{\not{k} - \sigma}{\not{k}}\right) + \ln \det\left(\frac{-\not{k} - \sigma}{-\not{k}}\right)\right]$$

$$= -\frac{1}{2}\text{tr}\int \frac{d^D k}{(2\pi)^D i}\ln\left(\frac{-k^2 + \sigma^2}{-k^2}\right). \tag{A.2}$$

Performing the Wick rotation $k^0 \to iK^0$ we get

$$= -\frac{1}{2}\text{tr}\int \frac{d^D K}{(2\pi)^D}\ln\left(1 + \frac{\sigma^2}{K^2}\right)$$

$$= -\frac{1}{2}\text{tr}\int \frac{d^D K}{(2\pi)^D}\int_0^1 dx\frac{1}{x + K^2/\sigma^2}$$

$$= -\frac{1}{(2\pi)^{D/2}D}\Gamma\left(1 - \frac{D}{2}\right)\sigma^D. \tag{A.3}$$

Inserting Eq. (A.3) into Eq. (3.3) the effective potential (2.6) is obtained.

Next we consider the two, three and four dimensional limit of the effective potential. Taking the two dimensional limit $D \to 2$, we get

$$\frac{V_0(\sigma)}{\sigma_0^2} = \frac{1}{2\lambda}\frac{\sigma^2}{\sigma_0{}^2} + \frac{1}{4\pi}\frac{\sigma^2}{\sigma_0{}^2}\left(-3 + \ln\frac{\sigma^2}{\sigma_0{}^2}\right). \tag{A.4}$$

where we use the renormalized coupling constant λ defined in Eq. (2.10) with (2.8) . Taking the three dimensional limit $D \to 3$, we get

$$\frac{V_0(\sigma)}{\sigma_0^3} = \frac{1}{2\lambda}\frac{\sigma^2}{\sigma_0{}^2} - \frac{1}{\sqrt{2\pi}}\left(\frac{\sigma^2}{\sigma_0{}^2} - \frac{1}{3}\frac{\sigma^3}{\sigma_0{}^3}\right). \tag{A.5}$$

If we take the four dimensional limit $D \to 4$, we find

$$\frac{V_0(\sigma)}{\sigma_0^D} = \frac{1}{2\lambda} \frac{\sigma^2}{\sigma_0^2}$$
$$- \frac{1}{(4\pi)^2} \left[6 \left(\frac{1}{\epsilon_4} - \gamma + \ln 2\pi + \frac{1}{3} \right) \frac{\sigma^2}{\sigma_0^2} - \left(\frac{1}{\epsilon_4} - \gamma + \ln 2\pi + \frac{3}{2} - \ln \frac{\sigma^2}{\sigma_0^2} \right) \frac{\sigma^4}{\sigma_0^4} \right]$$
$$+ O(\epsilon_4), \tag{A.6}$$

where

$$\epsilon_4 = \frac{4 - D}{2}. \tag{A.7}$$

To consider the theory in the case of $D = 4$ as a low energy effective theory stemming from the more fundamental theory we calculate the effective potential using the cut-off regularization in the case of $D = 4$ and compare it with the one obtained by the dimensional regularization. We use the renormalization condition (2.7) and calculate the effective potential (3.3) by the cut-off regularization

$$\frac{V_0(\sigma)}{\sigma_0^4} = \frac{1}{2\lambda} \frac{\sigma^2}{\sigma_0^2} - \frac{1}{(4\pi)^2} \left[6 \left(\ln \frac{\Lambda^2}{\sigma_0^2} - \frac{2}{3} \right) \frac{\sigma^2}{\sigma_0^2} - \left(\ln \frac{\Lambda^2}{\sigma_0^2} + \frac{1}{2} - \ln \frac{\sigma^2}{\sigma_0^2} \right) \frac{\sigma^4}{\sigma_0^4} \right]$$
$$+ O \left(\frac{\sigma^2}{\Lambda^2}, \frac{\sigma_0^2}{\Lambda^2} \right), \tag{A.8}$$

where Λ is the cut-off parameter, which is assumed to be larger than σ and σ_0. This result show us that there is a correspondence between ϵ_4 and Λ in the leading order of the $1/N$ expansion,

$$\frac{1}{\epsilon_4} - \gamma + \ln 2\pi + 1 \leftrightarrow \ln \frac{\Lambda^2}{\sigma_0^2}. \tag{A.9}$$

APPENDIX B

EFFECTIVE POTENTIAL AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

We shall show two types of representations for the effective potential and the gap equation at finite temperature and density in this appendix. One represented by a momentum integration is convenient for numerical calculations and the other represented by a summation is convenient for analytical calculations. In the Feynman functional-integral

formalism, thermal Green function is obtained by the following way.

$$\langle \alpha \mid e^{-\beta(H-\mu N)} \mid \alpha \rangle = \langle \alpha, -i\beta \mid \alpha, 0 \rangle$$
$$= C \int [d\psi][d\bar{\psi}] \exp \left(i \int_0^{-i\beta} dt \int d^{D-1}x (\mathcal{L} + \mu N) \right) , \qquad (B.1)$$

where the number operator N is expressed

$$N = \int d^{D-1}x \, \bar{\psi}\gamma^0\psi . \qquad (B.2)$$

Following Eq. (3.2), we obtain the generating functional at finite temperature and density.

$$Z^{\beta\mu}[0] = \int [d\psi][d\bar{\psi}] \exp \left(i \int_0^{-i\beta} dt \int d^{D-1}x (\mathcal{L} + \mu N) \right) . \qquad (B.3)$$

Thus the effective potential (3.3) is modified as

$$V(\sigma) = \frac{1}{2\lambda_0}\sigma^2 - \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln \det \frac{(-iw_n\gamma^0 - ik_i\gamma^i - \sigma + \mu\gamma^0)}{(-iw_n\gamma^0 - ik_i\gamma^i + \mu\gamma^0)} , \qquad (B.4)$$

where the roman index i is taken over the space components ($i = 1 \sim 3$). We calculate the integrand in the second term.

$$\ln \det \frac{[-i(w_n + i\mu)\gamma^0 - ik_i\gamma^i - \sigma]}{[-i(w_n + i\mu)\gamma^0 - ik_i\gamma^i]} = \text{tr} \int_0^{\sigma} ds \frac{1}{i(w_n + i\mu)\gamma^0 + ik_i\gamma^i + s}$$
$$= \text{tr} \int_0^{\sigma} ds \frac{s}{(w_n + i\mu)^2 + k_ik^i + s^2}$$
$$= \frac{1}{2} \text{tr} \ln \frac{(w_n + i\mu)^2 + k_ik^i + \sigma^2}{(w_n + i\mu)^2 + k_ik^i} . \qquad (B.5)$$

To obtain the second line we use the well-known property that the gamma matrix is traceless. Hence we get the effective potential at finite temperature and density.

$$V(\sigma) = \frac{1}{2\lambda_0}\sigma^2 - \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \text{tr} \ln \frac{(w_n + i\mu)^2 + k_ik^i + \sigma^2}{(w_n + i\mu)^2 + k_ik^i} . \qquad (B.6)$$

Next we explain how to deal with the summation and integration in the effective potential (B.6). We can perform a summation in the following way. We divide the summation into two parts.

$$\sum_{n=-\infty}^{\infty} \ln \left[\left(\frac{2n+1}{\beta} + i\mu \right)^2 + b^2 \right]$$

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$$= \sum_{n=0}^{\infty} \ln\left[\left(\frac{2n+1}{\beta} + i\mu\right)^2 + b^2\right] + \sum_{n=0}^{\infty} \ln\left[\left(\frac{-2n-1}{\beta} + i\mu\right)^2 + b^2\right]$$

$$= \sum_{n=0}^{\infty} \ln\left[\left(\left(\frac{2n+1}{\beta}\right)^2 + (b-\mu)^2\right)\left(\left(\frac{2n+1}{\beta}\right)^2 + (b+\mu)^2\right)\right]. \qquad (B.7)$$

Using the formula

$$\sum_{n=0}^{\infty} \ln\left(a^2(2n+1)^2 + b^2\right) = \ln\cosh\frac{\pi b}{2a}, \qquad (B.8)$$

we can perform the summation and get

$$V(\sigma) = \frac{1}{2\lambda_0}\sigma^2 - 2^{D/2-1}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\left(\sqrt{k^i k_i + \sigma^2} - |\vec{k}|\right)$$

$$-2^{D/2-1}\frac{1}{\beta}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\left[\ln\frac{1 + e^{-\beta(\sqrt{k^i k_i + \sigma^2} + \mu)}}{1 + e^{-\beta(|\vec{k}| + \mu)}} + \ln\frac{1 + e^{-\beta(\sqrt{k^i k_i + \sigma^2} - \mu)}}{1 + e^{-\beta(|\vec{k}| - \mu)}}\right]. \quad (B.9)$$

We can also integrate over space components in Eq. (B.6)

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\ln\frac{(w_n + i\mu)^2 + k_i k^i + \sigma^2}{(w_n + i\mu)^2 + k_i k^i}$$

$$= \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\int_0^1 dx\frac{\sigma^2}{\sigma^2 x + (w_n + i\mu)^2 + k_i k^i}$$

$$= -\frac{1}{(4\pi)^{(D-1)/2}}\Gamma\left(\frac{1-D}{2}\right)$$

$$\times \left[\{(w_n + i\mu)^2 + \sigma^2\}^{(D-1)/2} - \{(w_n + i\mu)^2\}^{(D-1)/2}\right]. \qquad (B.10)$$

Hence we get

$$V(\sigma) = \frac{1}{2\lambda_0}\sigma^2 \;+\; \frac{1}{\beta}\frac{1}{\sqrt{2}(2\pi)^{(D-1)/2}}\Gamma\left(\frac{1-D}{2}\right)$$

$$\times \sum_{n=-\infty}^{\infty}\left[\{(w_n + i\mu)^2 + \sigma^2\}^{(D-1)/2} - \{(w_n + i\mu)^2\}^{(D-1)/2}\right]. \quad (B.11)$$

From the two representations of the effective potential (B.9) and (B.11) we can write the gap equation in the two forms.

$$\frac{1}{\lambda_0} \;-\; 2^{D/2-2}\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\sqrt{k_i k^i + \sigma^2}}$$

$$\times \left[\frac{1 - e^{-\beta(\sqrt{k^i k_i + \sigma^2} + \mu)}}{1 + e^{-\beta(\sqrt{k^i k_i + \sigma^2} + \mu)}} + \frac{1 - e^{-\beta(\sqrt{k^i k_i + \sigma^2} - \mu)}}{1 + e^{-\beta(\sqrt{k^i k_i + \sigma^2} - \mu)}}\right] = 0, \qquad (B.12)$$

$$\frac{1}{\lambda_0} \;-\; \frac{1}{\beta}\frac{\sqrt{2}}{(2\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right)\sum_{n=-\infty}^{\infty}\{(w_n + i\mu)^2 + \sigma^2\}^{(D-3)/2} = 0. \qquad (B.13)$$

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