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CONHARMONIC TRANSFORMATIONS OF TWISTED PRODUCT MANIFOLDS

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ABSTRACT. We study the twisted product manifold with vanishing conharmonic curvature tensor which is invariant under the conharmonic transformation, and characterize the total space, base space and each fibre.

1. INTRODUCTION

For a generalization of the warped product of two Riemannian manifolds introduced by Bishop and O'Neill [2], Chen [3] introduced the twisted product of two Riemannian manifolds in order to construct a large family of totally umbilical submanifolds.

In general, the harmonicity of functions is not preserved by the conformal transformation. In this point of view, Ishi [4] introduced the conharmonic transformation which is a conformal transformation with preserving the harmonicity of a certain function. The conharmonic curvature tensor is invariant under the conharmonic transformation.

In this paper we study the twisted product manifold with vanishing conharmonic curvature tensor and characterize the total space, base space and each fibre.

Key Words : twisted product manifolds, conharmonically flat
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2. PRELIMINARIES

Let (B, g) and $(F, \bar{g})$ be Riemannain manifolds with dimensions n and p respectively, and f a positive differentiable function on $B \times F$. Let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the canonical projections. Then, the twisted product $M = B \times_f F$ is a differentiable manifold $B \times F$, equipped with the metric G defined by

$$G(X, Y) = g(\pi_* X, \pi_* Y) + f^2 \bar{g}(\sigma_* X, \sigma_* Y) \tag{2.1}$$

for any vector fields X and Y on $B \times F$. If the function f depends only on the point of B, then $B \times_f F$ is a warped product. For a local coordinate system (u^a) of B, the metric tensor g has the components (g_{ba}). Similarily, for a local coordinate system (u^x) of F, $\bar{g}$ has the components $(\bar{g}_{yx})$. Then with respect to the local coordinate system (u^a, u^x) of M, G has the components

$$(G_{ji}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & f^2 \bar{g}_{yx} \end{pmatrix}. \tag{2.2}$$

Throughout this paper, the ranges of indices are as follows :

$$i, j, k, \cdots : 1, 2, \cdots, n + p = m$$

$$a, b, c, \cdots : 1, 2, \cdots, n$$

$$x, y, z, \cdots : n + 1, \cdots, n + p$$

unless otherwise stated.

Let ∇_b (resp. ∇_x) be the components of the covariant derivative with respect to g (resp. $\bar{g}$) and $\{{}^{a}_{bc}\}$ (resp. $\{{}^{\bar{x}}_{yz}\}$) the Christoffel symbol of g (resp. $\bar{g}$). Then the Christoffel symbols $\{{}^{\tilde{k}}_{ji}\}$ of G on M are given as follows

(2. 3) $\{\tilde{^a_{bc}}\} = \{^a_{bc}\},$

(2. 4) $\{\tilde{^x_{yz}}\} = \{\bar{^x_{yz}}\} + \frac{1}{f}(f_y\delta^x_z + f_z\delta^x_y - f^x\bar{g}_{yz}),$

(2. 5) $\{\tilde{^x_{ya}}\} = \frac{1}{f}f_a\delta^x_y,$

(2. 6) $\{\tilde{^a_{xy}}\} = -ff^a\bar{g}_{xy}$

and the others are zero, where $f_a = \frac{\partial f}{\partial u^a}$ and $f_y = \frac{\partial f}{\partial u^y}$.

Let $\tilde{R}$, R and $\bar{R}$ be the curvature tensors of M, B and F respectively. Then we have

(2. 7) $\tilde{R}_{dcb}{}^a = R_{dcb}{}^a,$

(2. 8) $\tilde{R}_{dxy}{}^z = \frac{1}{f}(\partial_d f_y)\delta^z_x - \frac{1}{f}(\partial_d f^z)\bar{g}_{xy} - \frac{1}{f^2}f_d f_y\delta^z_x + \frac{1}{f^2}f_d f^z\bar{g}_{xy},$

(2. 9) $\tilde{R}_{dxb}{}^z = \frac{1}{f}(\nabla_d f_b)\delta^z_x, \quad \tilde{R}_{dxy}{}^a = -f(\nabla_d f^a)\bar{g}_{xy},$

(2.10) $\tilde{R}_{xyz}{}^a = -f(\partial_x f^a)\bar{g}_{yz} + f(\partial_y f^a)\bar{g}_{xz} - f^a f_y\bar{g}_{xz} + f^a f_x\bar{g}_{yz},$

(2.11)
$$\begin{aligned}\tilde{R}_{xyz}{}^w = \bar{R}_{xyz}{}^w &+ \tfrac{1}{f}\{(\nabla_x f_z)\delta^w_y + (\nabla_y f^w)\bar{g}_{xz} - (\nabla_x f^w)\bar{g}_{yz} - (\nabla_y f_z)\delta^w_x\} \\ &- \tfrac{2}{f^2}(f_x f_z\delta^w_y + f_y f^w\bar{g}_{xz} - f_x f^w\bar{g}_{yz} - f_y f_z\delta^w_x) \\ &+ \|f_e\|^2(\bar{g}_{xz}\delta^w_y - \bar{g}_{yz}\delta^w_x) - \tfrac{\|f_v\|^2}{f^2}(\bar{g}_{yz}\delta^w_x - \bar{g}_{xz}\delta^w_y),\end{aligned}$$

and the others are zero.

The components of Ricci tensors are given by

(2.12) $\tilde{S}_{ab} = S_{ab} - \frac{p}{f}(\nabla_a f_b),$

(2.13) $\tilde{S}_{ax} = -(p-1)(\frac{1}{f}\partial_a f_x - \frac{1}{f^2}f_a f_x),$

(2.14)
$$\begin{aligned}\tilde{S}_{yx} = \bar{S}_{yx} &- f(\triangle f)\bar{g}_{yx} - \tfrac{1}{f}(\bar{\triangle} f)\bar{g}_{yx} - \tfrac{(p-2)}{f}\nabla_y f_x \\ &+ \tfrac{2(p-2)}{f^2}f_y f_x - (p-1)\|f_e\|^2\bar{g}_{yx} - \tfrac{(p-3)}{f^2}\|f_w\|^2\bar{g}_{yx},\end{aligned}$$

where $\triangle f = \nabla_e f^e, \bar{\triangle} f = \nabla_x f^x$ and $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively.

Let $\tilde{K}$, K and $\bar{K}$ be the scalar curvatures of M, B and F respectively. Then we have

$$(2.15) \qquad \tilde{K} = K + \tfrac{1}{f^2}\bar{K} - \tfrac{2p}{f}(\Delta f) - \tfrac{2(p-1)}{f^3}(\bar{\Delta} f) - \tfrac{p(p-1)}{f^2}\|f_e\|^2 - \tfrac{(p-1)(p-4)}{f^4}\|f_x\|^2.$$

3. CONHARMONICALLY FLAT TWISTED PRODUCT MANIFOLD

A harmonic function w is defined as a function whose Laplacian vanishes. Let ρ be a positive function on M. Then $(M, \bar{g} = e^{2\rho}g)$ is conformally diffeomorphic to (M, g) and the conformal diffeomorphism $\phi : (M, g) \to (M, \bar{g})$ is called a conformal transformation. A harmonic function is not in general transformed into a harmonic function by the conformal transformation. For the harmonic function w, $\bar{w} = e^{2\alpha\rho}w$ may become a harmonic function for a suitable constant α. Then we obtain [4]

$$(3.1) \qquad \Delta_{\bar{g}}\bar{w} = \nabla e^{2(\alpha-1)\rho}\{\Delta_g w + 2\alpha(\Delta_g\rho)w + (4\alpha + m - 2)\rho^i(\partial_i w) + 2\alpha(2\alpha + m - 2)\|\rho_i\|^2 w\}.$$

Immediately we have

Proposition 3.1. *Let w be a harmonic function on (M, g) and $\alpha = \frac{2-m}{4}$. Then the function $\bar{w}$ defined by (3.1) is harmonic for $\bar{g}$ if and only if*

$$(3.2) \qquad \Delta_g\rho + \tfrac{m-2}{2}\|\rho_k\|^2 = 0, \text{ where } m > 2.$$

In this point of view, Y. Ishi [4] called it a conharmonic transformation which is the conformal transformation ϕ satisfying the equation (3.2). It is well known that the conharmonic curvature tensor T defined by

$$(3.3) \qquad T_{kji}{}^h = \tilde{R}_{kji}{}^h + \tfrac{1}{m-2}(\tilde{S}_{ik}\delta^h_j + \tilde{S}^h_j\tilde{g}_{ik} - \tilde{S}_{ij}\delta^h_k - \tilde{S}^h_k\tilde{g}_{ij})$$

is invariant under the conharmonic transformation [4]. If T vanishes identically on M, then we call M a conharmonically flat manifold. It is well known that the conformally flat manifold is conharmonically flat if and only if the scalar curvature vanishes.

Let $M = B\times_f F$ be a conharmonically flat twisted product manifold and $m > 2$. Then, the identites (2.7)–(2.14) and (3.3) imply

$$\begin{aligned} R_{dcb}{}^{a} &= \tfrac{1}{m-2}(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^a g_{cb} - S_c{}^a g_{db}) \\ &\quad -\tfrac{p}{(m-2)f}(\delta_d^a\nabla_c f_b - \delta_c^a\nabla_d f_b + g_{cb}\nabla_d f^a - g_{db}\nabla_c f^a), \end{aligned} \tag{3.4}$$

$$\tfrac{n-1}{f}(\partial_d f_y)\delta_x^z - \tfrac{n-1}{f}(\partial_d f^z)\bar{g}_{xy} - \tfrac{n-1}{f^2} f_d f_y\delta_x^z + \tfrac{n-1}{f^2} f_d f^z\bar{g}_{xy} = 0, \tag{3.5}$$

$$\begin{aligned} &\tfrac{n-2}{f}(\nabla_d f_b)\delta_x^z + S_{bd}\delta_x^z + \tfrac{1}{f^2}\{\bar{S}_x^z - f(\triangle f)\delta_x^z - \tfrac{1}{f}(\bar{\triangle} f)\delta_x^z - \tfrac{p-2}{f}\nabla_x f^z \\ &\quad + \tfrac{2(p-2)}{f^2} f_x f^z - (p-1)\|f_e\|^2\delta_x^z - \tfrac{p-3}{f^2}\|f_y\|^2\delta_x^z\}g_{bd} = 0, \end{aligned} \tag{3.6}$$

$$(n-1)\{f(\partial_x f^a)\bar{g}_{yz} - f(\partial_y f^a)\bar{g}_{xz} + f^a f_y\bar{g}_{xz} - f^a f_x\bar{g}_{yz}\} = 0 \tag{3.7}$$

and

$$\begin{aligned} \bar{R}_{xyz}{}^{w} &= \tfrac{1}{m-2}(\bar{S}_{zy}\delta_x^w + \bar{S}_x^w\bar{g}_{zy} - \bar{S}_{zx}\delta_y^w - \bar{S}_y^w\bar{g}_{zx}) \\ &\quad +\tfrac{n}{(m-2)f}\{(\nabla_y f_z)\delta_x^w + (\nabla_x f^w)\bar{g}_{yz} - (\nabla_x f_z)\delta_y^w - (\nabla_y f^w)\bar{g}_{xz}\} \\ &\quad -\tfrac{2n}{(m-2)f^2}(f_y f_z\delta_x^w + f_x f^w\bar{g}_{yz} - f_x f_z\delta_y^w - f_y f^w\bar{g}_{xz}) \\ &\quad +\tfrac{n-p}{m-2}\|f_e\|^2(\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w) + \tfrac{n-p+4}{(m-2)f^2}\|f_w\|^2(\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w) \\ &\quad -\tfrac{2}{m-2}f(\triangle f)(\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w) - \tfrac{2}{(m-2)f}(\bar{\triangle} f)(\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w). \end{aligned} \tag{3.8}$$

Since the scalar curvature tensor vanishes on the conharmonically flat manifold, we obtain

$$(3.9) \qquad K + \frac{1}{f^2}\bar{K} - \frac{2p}{f}(\triangle f) - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}\|f_e\|^2 - \frac{(p-1)(p-4)}{f^4}\|f_x\|^2 = 0$$

using (2.15). We also have

$$(3.10) \qquad (p-1)(n-1)\{\tfrac{1}{f}(\partial_d f_y) - \tfrac{1}{f^2}f_d f_y\} = 0$$

by contracting (3.5) with respect to x and z .

Due to the following equality

$$\tfrac{1}{f}(\partial_d f_y) - \tfrac{1}{f^2}f_d f_y = \partial_d\partial_y(\log f),$$

for $p \neq 1$ and $n \neq 1$, f is a product of certain functions f^* on B and $\bar{f}$ on F. Now Theorem 3.2 can be easily shown if we consider the fibre with the metric $g^*_{xy} = \bar{f}^2\bar{g}_{xy}$.

Theorem 3.2. *If, for $p \neq 1$ and $n \neq 1$, the twisted product manifold $M = B \times_f F$ of the Riemannian manifolds B and F is conharmonically flat, then M is the warped product space $B \times_{f^*} F^*$ of B and F^*.*

Using (3.6), we can get

$$(3.11) \qquad \frac{p(n-2)}{f}\nabla_d f_b + pS_{bd} + \frac{1}{f^2}\{\bar{K} - pf\triangle f - \frac{2(p-1)}{f}(\bar{\triangle} f) - \frac{(p-1)(p-4)}{f^2}\|f_x\|^2 - p(p-1)\|f_e\|^2\}g_{bd} = 0$$

and

$$(3.12) \qquad pK + \frac{n\bar{K}}{f^2} - \frac{2p}{f}(\triangle f) - \frac{2n(p-1)}{f^3}(\bar{\triangle} f) - \frac{n(p-1)(p-4)}{f^4}\|f_x\|^2 - \frac{np(p-1)}{f^2}\|f_e\|^2 = 0.$$

Now

$$(n-p)K = \frac{2p(n-1)}{f}\triangle f \tag{3.13}$$

due to (3.9) and (3.12).

Using (3.4) and (3.13), we obtain

$$pS_{cb} = \frac{m-2}{2(n-1)}Kg_{bc} - \frac{p(n-2)}{f}\nabla_c f_b. \tag{3.14}$$

Hence we can state

Theorem 3.3. *Let $M = B \times_f F$ be the conharmonically twisted product manifold of B and F. If K is constant and $n = 2$, then B is the space of constant curvature.*

If $K = 0$, then the identity (3.14) implies

$$S_{cb} = -\frac{n-2}{f}\nabla_c f_b$$

and that the curvature tensor of B is reduced to

$$R_{dcb}{}^{a} = \frac{1}{n-2}(S_{cb}\delta^a_d + S^a_d g_{cb} - S_{bd}\delta^c_a - S^a_c g_{bd})$$

due to (3.4). Thus we have

Theorem 3.4. *Assume that the twisted product manifold $M = B \times_f F$ of the Riemannian manifolds B and F is conharmonically flat. Then the base space B with $n \geq 3$ is also conharmonically flat if and only if $K = 0$.*

Now, if we contract (3.8) with respect to x and w, then we get

$$
\begin{aligned}
n\bar{S}_{yz} = \bar{K}\bar{g}_{yz} + \tfrac{n(p-2)}{f}\nabla_y f_z + \tfrac{(n-2p+2)}{f}(\bar{\triangle} f)\bar{g}_{yz} \\
-\tfrac{2n(p-2)}{f^2} f_y f_z + \tfrac{n(p-3)-(p-1)(p-4)}{f^2}\|f_x\|^2\bar{g}_{yz} \\
+(n-p)(p-1)\|f_e\|^2\bar{g}_{yz} - 2(p-1)f(\triangle f)\bar{g}_{yz},
\end{aligned} \tag{3.15}
$$

and (3.15) implies

$$
\bar{K} = \tfrac{2(p-1)}{f}(\bar{\triangle} f) + \tfrac{(p-1)(p-4)}{f^2}\|f_x\|^2 + p(p-1)\|f_e\|^2 \tag{3.16}
$$

for $n - p \neq 0$ where $\bar{K}$ is the scalar curvature of F.

Using (3.16), (3.15) is reduced to

$$
\begin{aligned}
\bar{S}_{yz} = \tfrac{(n-2)}{f}\nabla_y f_z + \tfrac{(\bar{\triangle} f)}{f}\bar{g}_{yz} - \tfrac{2(p-2)}{f^2} f_y f_z \\
+\tfrac{(n-3)}{f^2}\|f_x\|^2\bar{g}_{yz} + (p-1)\|f_e\|^2\bar{g}_{yz}.
\end{aligned} \tag{3.17}
$$

If we consider the case of $p = 4$, then

$$
\bar{K} = \tfrac{6}{f}(\bar{\triangle} f) + 12\|f_e\|^2. \tag{3.18}
$$

The condition $\bar{K} = 0$ leads the identity (3.18) to

$$
\bar{\triangle} f + 2f\|f_e\|^2 = 0. \tag{3.19}
$$

Integrating (3.19) on $B \times F$, we have

$$
\textstyle\int_{B\times F} f^*(\bar{\triangle}\bar{f})dV + \int_{B\times F} 2f\|f_e\|^2 dV = 0, \tag{3.20}
$$

where $f = f^*\bar{f}$ was introduced in Theorem 3.2.

Theorem 3.5. (*Green,* [5]) *On a compact orientable Riemannian space M, we have*

$$\int_M (G^{ji}\nabla_j\nabla_i f)d\sigma = 0$$

for any scalar field f, where $d\sigma$ is the volume element.

If we assume that F is compact, then

$$\int_{B\times F} f^*(\bar{\triangle}\bar{f})dV = \int_B f^* dV_B \int_F(\bar{\triangle}\bar{f})dV_F = 0 \tag{3.21}$$

using Theorems 3.2 and 3.5. Therefore, from the equation (3.20), we see that $f\|f_e\|^2 = 0$, and from the equation (3.19), we see that $\bar{\triangle}f = 0$. Using the identities (2.2), (2.4) and (2.6), the Laplacian $\tilde{\triangle}f$ becomes

$$\tilde{\triangle}f = G^{kj}\tilde{\nabla}_k\tilde{\nabla}_j f = \triangle f + \frac{\bar{\triangle}f}{f^2} + \frac{p}{f^2}\|f_a\|^2 + \frac{p-2}{f^3}\|f_x\|^2. \tag{3.22}$$

Since $\bar{\triangle}f = 0$ and $f_e = 0$, we obtain

$$\tilde{\triangle}f = \frac{p-2}{f^3}\|f_x\|^2. \tag{3.23}$$

The following is the well known lemma by Hopf.

Lemma 3.6. (*Hopf,* [5]) *Let M be a compact Riemannian manifold. If f is a function on M such that $\triangle f \geq 0$ everywhere (or $\triangle f \leq 0$ everywhere), then f is a constant function.*

The following theorem can be deduced from the equation (3.22) and Hopf's Lemma.

Theorem 3.7. *Let $M = B \times_f F$ be the conharmonically flat twisted product manifold of compact Riemannian manifolds B and F with $\bar{K} = 0$. If $p = 4$ and $n \neq 1, 4$, then M is the Riemannian product of B and F.*

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