A Lyapunov Approach for Robust State Feedback Synthesis and Its Application to Robot Control

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Abstract : Robust stabilization for a class of nonlinear dynamical systems with uncertainties is investigated. Based on the stabilizability of a nominal system (i.e. the system in the absence of uncertainty), a class of continuous state feedback controllers for uncertain dynamical systems are presented. Compared with those reported in the control literature, such a class of state feedback controllers are non−saturation type, have a rather simple form, and can guarantee practical stability and asymptotic stability of uncertain dynamical systems in terms of the choice of gain control function. Particularly, no chattering will appear in implementation for the control. A numerical example on the robust stabilization of a simple pendulum is given to demonstrate the utilization of the results.

As an application of the results presented in this paper, the robust control problem of robot manipulators is also discussed, and a simpler robust control law for n−link robot manipulators is derived.

1. Introduction

Robust stabilization of dynamical systems with significant uncertainties has been widely researched over the last decade, and a number of new approaches have been proposed for synthesizing state feedback controllers which lead to some desired performance, e.g. asymptotic stability, ultimate boundedness, etc., of the state of an uncertain dynamical system [1, 2]. For nonlinear and time varying dynamical systems with significant uncertainties described only in terms of bounds on their possible sizes, it seems that the so called *Lyapunov* minimax approach [3] be one of the most effective approaches for stabilizing controller synthesis.

The *Lyapunov* minimax approach is generally based on the stabilizability of a nominal system (i.e. the system in the absence of uncertainty). Roughly speaking, a *Lyapunov* function of the stable nominal system is employed as a *Lyapunov* function candidate for the actual uncertain dynamical systems, and a control law is then chosen such that the *Lyapunov* function decreases along every possible trajectory of uncertain dynamical systems.

By employing the *Lyapunov* minimax approach, some types of state feedback controllers have been proposed to guarantee the desired performance (see, e.g. [3]–[12]). Particularly, in [3] a class of saturation–type state feedback controller has been proposed such that uniform asymptotic stability of uncertain nonlinear systems can be guaranteed. In [4], a continuous state feedback controller guaranteeing uniform ultimate boundedness of uncertain nonlinear systems is presented. In [5], another class of practically stabilizing controllers are also proposed. Moreover, in [6,7] a class of asymptotically stabilizing continuous controllers with a more complicated form are proposed.

In this paper, we discuss the problem of robust stabilization of uncertain nonlinear dynamical systems. Based on the stabilizability of a nominal system, a class of continuous state feedback controllers for uncertain nonlinear dynamical systems are presented. Compared with the state feedback controllers reported in the control literature, such a class of state feedback controllers are non–saturation type, have a rather simple form, and can guarantee the practical stability and asymptotic stability of uncertain nonlinear dynamical systems by choosing different gain control functions. Particularly, becouse of the simplity of the controller proposed in this paper, no chattering will appear in implementation for the control. A numerical example on the robust stability of a simple pendulum is given to demonstrate the synthesis procedure of state feedback controller presented in this paper, and the results of its simulation show that this controller is rather easily implemented, and no chattering will appear for the control.

As an application of the results presented in this paper, we also discuss the problem of robust control of robot manipulators. Based on the development of [16], we derive a simpler robust control law for n–link robot manipulators by making use of the method developed in the paper.

The paper consists of the following parts. In Section 2, the problem to be tackled in this paper is precisely stated and some necessary assumptions are introduced. In Section 3, we propose a class of state feedback controllers and discuss some relative problems. An illustrative example on pendulum is also given in this section. In Section 4, as an application of the results presented in Section 3, we discuss the problem of robust control of robot manipulators. Finally, the paper is concluded in Section 5 with a brief discussion of the results.

2. Problem Formulation

Consider an uncertain nonlinear dynamical system, which satisfies the so–called matching condition, described by the following differential equations:

$$\frac{dx}{dt} = f(x,t) + B(x,t)[u + \xi(x,t)], \quad x(t_0) = x^0 \tag{1}$$

where $t \in R^+$ is the time, $x(t) \in R^n$ is the state vector, $x(t) \in R^m$ is the control vector, $\zeta(x,t)$ represents the system uncertainties and is assumed to be bounded in magnitude, usually in its *Euclidean* norm denoted by $\| \cdot \|$. The corresponding system without uncertainty, called nominal system, is described by

$$\frac{dx}{dt} = f(x,t) + B(x,t)u, \quad x(t_0) = x^0 \tag{2}$$

where $f(\cdot) : R^n \times R^+ \to R^m$ and $B(\cdot) : R^n \times R^+ \to R^{n \times m}$ are known. Moreover, the so called unforced nominal system of dynamical system (2) is defined by

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x^0 \tag{3}$$

Throughout this paper, we will consider the uncertain dynamical system described in (1). When the assumption of matching conditions is not satisfied, some standard techniques have also been developed (see, e.g. [13,14,15]).

Before giving our main results, we introduce for system (1) the following standard assumptions.

Assumption 2.1 The known functions $f(\cdot) : R^n \times R^+ \to R^m$ and $B(\cdot) : R^n \times R^+ \to R^{n \times m}$, as well as the unknown function $\zeta(\cdot) : R^n \times R^+ \to R^m$ are continuous, uniformly bounded with respect to time; and locally uniformly bounded with respect to the state x. $\zeta(\cdot) \in \Sigma_\zeta$, where Σ_ζ is a specified set.

Assumption 2.2 The uncertainty $\zeta(\cdot)$ is bounded in *Euclidean* norm by a known function, i.e. there exists a nonnegative continuous function $p(\cdot) : R^n \times R^+ \to R^+$ such that

$$\|\xi(x,t)\| \le \rho(x,t) \tag{4}$$

for all $(x,t) \in R^n \times R^+$ and for all $\zeta(\cdot) \in \Sigma_\rho$.

In addition, the function $\rho(\cdot) : R^n \times R^+ \to R^+$ is assumed, without loss of generality, to be uniformly bounded with respect to time, and locally uniformly bounded with respect to the state x.

Assumption 2.3 The origin $x = 0$ is uniformly asymptotically stable in the large for the unforced nominal system (3). More specifically, there is a family of *Lyapunov* functions $\mathbb{R}(x, t)$ for dynamical system (3). That is, there are an C^1-function $V(\cdot) \in \mathbb{R}(x, t) : R^n \times R^+ \to R^+$ and strictly increasing continuous scalar functions $\gamma_i(\cdot) R^+ \to R^+ = 1,2,3$, which satisfy

$$\gamma_i(0) = 0, \quad i = 1,2,3$$

$$\lim_{r \to \infty} \gamma_i(0) = \infty, \quad i = 1,2$$

such that for all $(x,t) \in (x, t) : R^n \times R^+$

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|)$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t) \le -\gamma_3(\|x\|)$$

Remark 2.1 Here, *Assumption 2.1* is a technical assumption for mathematical completeness. *Assumption 2.2* defines the uncertainty band (in general state dependent) for $\zeta(x,t)$. *Assumption 2.3* shows that the nominal system must be internally stable in the sense that there exists a *Lyapunov* function. Indeed, in order to guarantee robust stability of uncertain dynamical

systems, their nominal systems must be stabilizable. It is under such an assumption that we can discuss the robust stability of dynamical systems with uncertainties.

Now, the question is to synthesize a class of state feedback controllers $u(t)$ that can guarantee the stability of nonlinear dynamical system (1) in the presence of the uncertainty $\zeta(x,t) \in \Sigma_\zeta$.

In the next section, we will concern the concept of practical stability. Therefore, we introduce the following definition [5].

Definition 2.1   The uncertain dynamical system described by (1) is said to be practically stabilizable, if, given any $\underline{d} > 0$ there is a control law $p(\cdot) : R^n \times R^+ \rightarrow R^m$ for which given any initial time $t_0 \in R^+$, any initial state $x^0 \in R^n$ and any uncertainty $\zeta \in \Sigma_p$, the following conditions hold:

(i) *Existence of solution.*

$$\frac{dx}{dt} = f(x,t) + B(x,t)[p(x,t) + \xi(x,t)]$$ (5)

possesses a solution $x(\cdot) : [t_0, t_1] \rightarrow R^n, x\ (t_0) = x^0, t_1 > t_0$

(ii) *Uniform boundedness.*   Given any $r > 0$ and any solution $x\ (\cdot)$: $[t_0, t_1] \rightarrow R^n, x\ (t_0) = x^0$,of (5) with $\|x^0\| \le r$, there is a constant $d\ (r) > 0$ such that

$$\|x(t)\| \le d(r), \quad \text{for all } t \in [t_0, t_1]$$

(iii) *Extension of solution.*   Every solution $x\ (\cdot) : [t_0, t_1] \rightarrow R^n, x\ (t_0) = x^0$of (5) can be continued onto $[t_0, \infty)$.

(iv) *Uniform ultimate boundedness.*   Given any $\bar{d} \ge \underline{d}$, any $r > 0$, and any solution $x(\cdot)$: $[t_0, \infty) \rightarrow R^n, x\ (t_0) = x^0$, of (5) with $\|x^0\| \le r$, there exists a finite time $T(\bar{d}, r) < \infty$, possibly dependent on r but not on t_0, such that

$$\|x(t)\| \le \bar{d}, \quad \text{for all } t \ge t_0 + T(\bar{d}, r)$$

(v) *Uniform stability.*   Given any $\bar{d} \ge \underline{d}$, and any solution $x\ (\cdot)$: $[t_0, \infty) \rightarrow R^n, x\ (t_0) = x^0$, of (5), there is a constant $\delta\ (\bar{d}) > 0$, such that $\|x^0\| \le \delta\ (\bar{d})$ implies that

$$\|x(t)\| \le \bar{d}, \quad \text{for all } t \ge t_0$$

In the rest of this section, we introduce three lemmas which will be used in the proofs of our main results.

Lemma 2.1.   *Let* $V(\cdot):R^n \times R^+ \rightarrow R^m$*be a* *Lyapunov* function candidate for a given continuous dynamical system

$$\frac{dx(t)}{dt} = F(x(t),t)$$ (6)

with the following properties:

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|)$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) F(x,t) \le -\gamma_3(\|x\|) + 2\varepsilon$$

where $\varepsilon > 0$ is a constant, the functions γ_i, $i = 1,2,3$, are defined in *Assumption 2.3*. Then, if

$$2\varepsilon < \liminf_{r \to \infty} \gamma_3(r) := l \tag{7}$$

every solution $x(t; x^0, t_0)$ of dynamical system (6) is both uniformly bounded and uniformly ultimately bounded. More precisely, one has the following results.

(i) *Uniform boundedness.* If $x(\cdot)$: $[t_0, t_1] \to R^n$, $x(t_0) = x^0$ is a solution of the system, then

$$\|x(t_0)\| \le r \quad \Rightarrow \quad \|x(t)\| \le d(r), \qquad \forall t \in [t_0, t_1]$$

where

$$d(r) = \begin{cases} (\gamma_1^{-1} \circ \gamma_2)(R), & \text{if } r \le R \\ (\gamma_1^{-1} \circ \gamma_2)(r), & \text{if } r > R \end{cases}$$

and

$$R = \gamma_3^{-1}(2\varepsilon)$$

Furthermore, the solution has a continuation over $[t_0, \infty)$.

(ii) *Uniform ultimate boundedness*: If $x(\cdot)$: $[t_0, \infty) \to R^n$ $x(t_0) = x^0$, is a solution of the system, then for given $\bar{d} > (\gamma_1^{-1} \circ \gamma_2)(R)$

$$\|x(t)\| \le \bar{d}, \qquad \forall t \ge t_0 + T(\bar{d}, r)$$

where

$$T(\bar{d}, r) = \begin{cases} 0, & \text{if } r \le \bar{R} \\ [\gamma_2(r) - \gamma_1(\bar{R})] / [\gamma_3(\bar{R}) - 2\varepsilon], & \text{if } r > \bar{R} \end{cases}$$

and

$$\bar{R} = (\gamma_2^{-1} \circ \gamma_1)(\bar{d})$$

Proof: This lemma is a compact form of the theorem proved in [4].

Lemma 2.2. *Let* $V(\cdot, \cdot)$ $R^n \times R^+ \to R^+$ *be a* Lyapunov *function candidate for the given*

continuous dynamical system described by (6), and with the following properties:

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|)$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^\tau V(x,t) F(x,t) \le -\gamma_3(\|x\|) + \psi(t)$$

where the function γ_i, $i = 1,2,3$, are defined in *Assumption 2.3*, and $\Psi(\cdot)$ is a continuous function satisfying

$$\lim_{t \to \infty} \int_{t_0}^{t} \psi(\tau) d\tau \le \overline{\Psi} < \infty$$

where $\overline{\Psi}$ is any constant. Then, the dynamical system described by (6) is uniformly asymptotically stable. That is, for any solution $x(t) := x(t; t_0, x^0)$ of (6),

$$\lim_{t \to \infty} \|x(t)\| = 0$$

Proof: (see Appendix A).

Lemma 2.3. *Let* $V(\cdot) R^n \times R^+ \to R^+$ *be a Lyapunov* function candidate for dynamical system (6) with the following properties:

$$(\lambda_{\min} \|x\|)^* \le V(x,t) \le (\lambda_{\max} \|x\|)^*$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^\tau V(x,t) F(x,t) \le -2\alpha V(x,t) + 2\tilde{\varepsilon} \|x\|^2$$

where $\lambda_{\min}$, $\lambda_{\max}$, α, and ε are positive scalar constants satisfying

$$\alpha > \tilde{\varepsilon} / \lambda_{\min}^2 \tag{8}$$

Then, the solution $x(\cdot):[t_0, \infty) \to R^n$, $x(t_0) = x^0$ of dynamical system (6) is globally exponentially stable in the sense that

$$\|x(t)\| \le \frac{\lambda_{\max}}{\lambda_{\min}} \|x^0\| \exp\left\{ -\left(\alpha - \frac{\tilde{\varepsilon}}{\lambda_{\min}^2} \right)(t - t_0) \right\} \tag{9}$$

Proof: (see Appendix B).

Remark 2.2 It is worth noting that if (8) is satisfied, there may also be some estimate inequalities with other forms on $\|x(t)\|$. But in this paper, we will use the estimate inequality (9).

In addition, corresponding to *Lemma 2.1* we also introduce the following definition.

Definition 2.2 If the scalar function $\gamma_3 : (\cdot) : R^+ \to R^+$, defined in *Assumption 2.3* is bounded, then we define a constant ℓ satisfying

$$\liminf_{r \to \infty} \gamma_3(r) := \tilde{l} < \infty \tag{10}$$

3   Robust State Feedback Synthesis

3.1 State feedback controller guaranteeing practical stability

Based on the *Lyapunov* stability theory, we propose the following state feedback controller:

$$u(t) = p(x,t) \tag{11a}$$

where

$$p(x,t) = -k(t)\rho^2(x,t)B^{\mathrm{T}}(x,t)\nabla_x V(x,t) \tag{11b}$$

where $k(t) > 0$ is a chosen continuous function and called the control gain function.

Remark 3.1   It is obvious from *Assumptions 2.1* to *2.3* that the state feedback controller proposed in (11) is uniformly continuous. It is worth noting that compared with the controllers reported in the control literature, the controller (11) possesses a rather simple form. Therefore, in practical control problems, it is easier to be implemented. Since the proposed controller is not saturation-type, no chattering will appear in implementation for the control.

Remark 3.2   It is worth noting that the similar state feedback controllers have been proposed in [5] and [22]. The controllers proposed in [5] and [22] can only guarantee uniform ultimate boundedness (or practical stability) of uncertain dynamical systems. But, for state feedback controller (11) proposed in this paper, we introduce a control gain function $k(t)$, and by appropriately selecting it, one may obtain better transient behavior for closed–loop dynamical system (1), (11), i.e. one can guarantee not only uniform ultimate boundedness but also asymptotic stability of uncertain dynamical systems which will further be shown in the following illustrative example.

Thus, we have the following theorem which shows the practical stability and asymptotic stability of closed-loop dynamical system (1), (11).

Theorem 3.1   Consider dynamical system (1) satisfying *Assumptions 2.1* to *2.3*. Then, for every $\zeta(\cdot) \in \Sigma_\zeta$, the closed-loop nonlinear dynamical system described by (1) and (11) has the following properties:

(i)If the control gain function k (t) is chosen such that k (t) is finite for all time, and satisfies

$$\frac{1}{4k(t)} := 2\varepsilon < \tilde{\ell} \tag{12}$$

where ε is a constant and $\tilde{\ell}$ is defined in *Definition 2.2*, then, closed–loop dynamical system (1), (11) is practically stable.

(ii)If the control gain function k (t) is chosen such that

$$\lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{4k(\tau)} d\tau \leq \tilde{K} < \infty \tag{13}$$

where $\tilde{K}$ is any constant, then, closed-loop dynamical system (1),(11) is uniformly asymptotically stable.

Proof: Let $x(t) := x(t; t_0, x^0)$ be a solution of closed-loop dynamical system (1), (11). Furthermore, employing the same *Lyapunov* function as given in *Assumption 2.3*, we have the *Lyapunov* derivative $\pounds(\cdot):R^n \times R^+ \to R$ for closed-loop dynamical system (1), (11) given by

$$
\begin{aligned}
\pounds(x,t) &:= \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\dot{x} \\
&= \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\{f(x,t) + B(x,t)[p(x,t) + \xi(x,t)]\}
\end{aligned}
\tag{14}
$$

From *Assumptions 2.2* and *2.3* and (11b) we further have

$$
\begin{aligned}
\pounds(x,t) &\leq -\gamma_3(\|x\|) - k(t)\rho^2(x,t)\|B^T(x,t)\nabla_x V(x,t)\|^2 \\
&\quad + \rho(x,t)\|B^T(x,t)\nabla_x V(x,t)\| \\
&= -\gamma_3(\|x\|) - \left[\sqrt{k(t)}\rho(x,t)\|B^T(x,t)\nabla_x V(x,t)\| - \frac{1}{2\sqrt{k(t)}}\right]^2 + \frac{1}{4k(t)} \\
&\leq -\gamma_3(\|x\|) + \frac{1}{4k(t)}
\end{aligned}
\tag{15}
$$

Therefore, from (15) we can obtain the following two analysis results:

(a) If $k(t)$ is selected such that inequality (12) is satisfied, one can apply directly the results of *Lemma 2.1*. From *Assumption 2.1* and *Lemma 2.1*, it is obvious that the requirements on (i) existence of solution, (ii) uniform boundedness, (iii) extension of solution, and (iv) uniform ultimate boundedness (see *Definition 2.1*) are satisfied, and one has the qualitative estimate results similar to ones given in *Lemma 2.1* on uniform boundedness and uniform ultimate boundedness.

Finally, for uniform stability, we easily prove it by making use of a proof method similar to that of *Corollary* presented in [4].

Therefore, we can conclude that closed-loop nonlinear dynamical system (1), (11) is practically stable for every $\zeta(\cdot) \in \Sigma_\zeta$.

(b) If $k(t)$ is chosen such that (13) is satisfied, then it follows from *Lemma 2.2* that closed-loop dynamical system (1), (11) is uniformly asymptotically stable.   ■

As a special case, we consider the problem of robust stabilization of uncertain dynamical systems with linear nominal part described by

$$\frac{dx(t)}{dt} = Ax(t) + B[u(t) + \xi(x,t)] , \qquad x(t_0) = x^0 \tag{16}$$

where $A \in R^{n \times n}$ and $B \in R^{n \times m}$ are constant matrices.

For dynamical system (16), we introduce the following standard assumption.

Assumption 3.1   The pair (A, B) given in (16) is completely controllable.

It follows from *Assumption 3.1* that for any symmetric positive definite matrix $Q \in R^{n \times n}$ the algebraic Riccati quation of the form

$$A^T P + PA - PBB^T P = -Q \tag{17}$$

has a solution $P \in R^{n \times n}$, which is also symmetric positive definite.

Here,we propose the following state feedback controller for dynamical system(16)

$$u(t) = p_1(x,t) + p_2(x,t) \tag{18a}$$

where

$$p_1(x,t) = -\frac{1}{2} B^T P \, x(t) \tag{18b}$$

$$p_1(x,t) = -k(t)\rho^2(x,t)B^T P \, x(t) \tag{18c}$$

Remark 3.3   The control (18) consists of two parts p_1 ($\cdot$) and p_2 ($\cdot$). Here, p_1 ($\cdot$) is linear state feedback controller which stabilizes the nominal system, and p_2 ($\cdot$) is continuous nonlinear state feedback controller which is used to compensate for the nonlinear system uncertainties in dynamical system (16) to produce some types of stability results. It is interesting to note that if system uncertainties are bounded by a constant, i.e. $\rho \equiv constant$, the compensation control (18c) is reduced to a linear form.

Thus, from controller (18), we have the following corollary which shows practical stability and asymptotic stability of dynamical system (16).

Corollary 3.1   Consider dynamical system (16) satisfying *Assumption 2.2* and *Assumption 3.1*. Then, closed-loop dynamical system (16), (18) has the following properties:
(i) If the control gain function k (t) is chosen such that $1/k$ (t) is finite for all time, and satisfies (12), then closed−loop dynamical system (16), (18) is practically stable.
(ii) If the control gain function k (t) is chosen such that inequality (13) is satisfied, then the closed−loop dynamical system (16), (18) is uniformly asymptotically stable.

Proof: Applying the state feedback controller (18) to (16) yields

$$\frac{dx(t)}{dt} = \left(A - \frac{1}{2} BB^T P \right) x(t) + B[p_2(x,t) + \xi(x,t)] \tag{19}$$

For the nominal dynamical system of (19), that is,

$$\frac{dx(t)}{dt} = \left(A - \frac{1}{2} BB^\mathsf{T} P \right) x(t) \tag{20}$$

we define a scalar function $V(\cdot)$: $R^n \to R^+$ as follws.

$$V(x) = x^\mathsf{T} P x$$

where P is the solution of the algebraic *Riccati* equation (17).

Firstly, by the *Rayleigh* principle [17], we have

$$\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2 \tag{21}$$

Secondly, for nominal dynamical system (20), we have

$$\frac{dV(x(t))}{dt} = -x^\mathsf{T} Q x \le -\lambda_{\min}(Q)\|x\|^2 \tag{22}$$

It is clear from (21) and (22) that one can take the bounding functions γ_i ($\cdot$) to be

$$\gamma_1(r) := \lambda_{\min}(P) r^2 \tag{23a}$$

$$\gamma_2(r) := \lambda_{\max}(P) r^2 \tag{23b}$$

$$\gamma_3(r) := \lambda_{\min}(Q) r^2 \tag{23c}$$

where $\lambda_{\max}$ ($\cdot$) and $\lambda_{\min}$ ($\cdot$) denote the maximum and minimum eigenvalues of the matrix ($\cdot$) , respectively.

Therefore, by making use of a method similar to that in the proof of *Theorem 3.1*, from (23) we can obtain the results given in this corollary. ∎

3.2 State feedback controller guaranteeing exponential stability

In this subsection, we consider a class of uncertain nonlinear dynamical systems whose unforced nominal system is exponentially stable, and whose uncertain perturbations satisfy the so-called cone condition. For such a class of uncertain nonlinear dynamical systems, we can synthesize a class of state feedback controllers with a simple form such that the exponential stability of the dynamical system can be guaranteed.

Assumption 3.2　The origin $x = 0$ is an exponentially stable equilibrium point of unforced nominal system (3). More specifically, there exists a C^1-function $V(\cdot):R^n \times R^+ \to R^+$ which satisfies

$$\left(\lambda_{\min}\|x\|\right)^2 \le V(x,t) \le \left(\lambda_{\max}\|x\|\right)^2$$

$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^\mathsf{T} V(x,t) f(x,t) \le -2\alpha V(x,t)$$

for all (x,t) $R^n \times R$, where $\lambda_{\min}$, $\lambda_{\max}$, and α are positive scalar constants.

Assumption 3.3 The uncertainty $\zeta(\cdot) : R^n \times R \rightarrow R^m$ is bounded in *Euclidean* norm by a known function, and satisfies

$$\|\xi(x,t)\| \le \beta\|x\| \tag{24}$$

where β is a nonnegative constant.

Under *Assumptions 3.2* and *3.3*, we propose the following state feedback controller:

$$u(t) = \tilde{p}(x,t) \tag{25a}$$

where

$$\tilde{p}(x,t) = -\tilde{k}B^{\mathsf{T}}(x,t)\nabla_x V(x,t) \tag{25b}$$

where k is a positive constant and selected such that the following relation is satisfied:

$$\tilde{\alpha} := \alpha - \frac{\beta^2}{4\tilde{k}\lambda_{\min}^2} > 0 \tag{26}$$

Thus, we have the following theorem which shows exponential stability of closed-loop dynamical system (1), (25).

Theorem 3.2 Consider dynamical system (1) satisfying *Assumptions 2.1, 3.2*, and *3.3*. Then, closed-loop nonlinear dynamical system (1), (25) is exponentially stable. More precisely, we have

$$\|x(t)\| \le \frac{\lambda_{\max}}{\lambda_{\min}} \|x(t_0)\| \exp\{-\tilde{\alpha}(t - t_0)\} \tag{27}$$

Proof: Let $x(t)$ be the solution of closed-loop dynamical system (1), (25). Then, the *Lyapunov* derivative $\pounds(\cdot):R^n \times R^+ \rightarrow R$ for the solution $x(t)$ is given by

$$\pounds(x,t) := \frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathsf{T}}V(x,t)\dot{x}$$

$$= \frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathsf{T}}V(x,t)\{f(x,t) + B(x,t)[\tilde{p}(x,t) + \xi(x,t)]\} \tag{28}$$

From *Assumptions 3.2* and *3.3*, we further have

$$\pounds(x,t) \le -2\alpha V(x,t) - \tilde{k}\|B^{\mathsf{T}}(x,t)\nabla_x V(x,t)\|^2 + \beta\|B^{\mathsf{T}}(x,t)\nabla_x V(x,t)\|\|x\|$$

$$= -2\alpha V(x,t) - \left[\sqrt{\tilde{k}}\|B^{\mathsf{T}}(x,t)\nabla_x V(x,t)\| - \frac{\beta}{2\sqrt{\tilde{k}}}\|x\|\right]^2 + \frac{\beta^2}{4\tilde{k}}\|x\|^2$$

$$\le -2\alpha V(x,t) + 2\tilde{\varepsilon}\|x\|^2 \tag{29}$$

where

$$2\tilde{\varepsilon} := \frac{\beta^2}{4\tilde{k}}$$

By making use of *Lemma 2.2*, from (29) we obtain

$$\|x(t)\| \le \frac{\lambda_{max}}{\lambda_{min}} \|x(t_0)\| \exp\left\{-\left(\alpha - \frac{\tilde{\varepsilon}}{\lambda_{min}^2}\right)(t - t_0)\right\}$$ (30)

It is obvious from (30) that if we select a control gain parameter k such that

$$\tilde{\alpha} = \alpha - \frac{\beta^2}{4\tilde{k}\lambda_{min}^2} > 0$$

then, closed-loop dynamical system (1), (25) is exponentially stable.   ∎

3.3 An illustrative example

In order to demonstrate the utilization of the approach, we consider a simple pendulum which was analyzed in [4,5]. However, it will be shown in this paper that the desired asymptotic stability can actually be guaranteed by using a simpler state feedback controller. A pendulum of length ℓ is subjected to a control moment u (per unit mass). The point of support is subject to an uncertain acceleration υ, with $|\upsilon(t)| \le \hat{\rho}\ell = constant$ (see Fig.1).
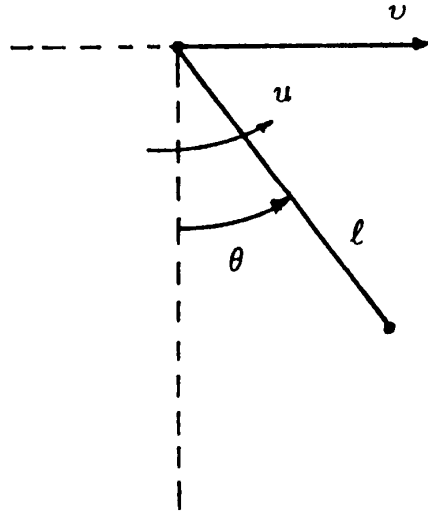

Fig.1   Illustrative exampie: Pendulum

Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, one obtains the state equations

$$\frac{dx_1(t)}{dt} = x_2(t)$$ (31a)

$$\frac{dx_2(t)}{dt} = -a\sin(x_1(t)) + u(t) - \upsilon(t)\frac{\cos(x_1(t))}{\ell}$$ (31b)

where $\alpha > 0$ is a given constant. In order to satisfy *Assumtion 2.3*, by employing a linear state

feedback controller one can create a new unforced nominal system for which the zero state is the only uniformly asymptotically stable equilibrium state. .Thus, we let

$$u(t) = -bx_1(t) - cx_2(t) + p(x(t),t) \tag{32}$$

where b and c are positive constants, and $p(\cdot)$ is a control of type (11). The system equation is then

$$\frac{dx(t)}{dt} = f(x(t)) + B[p(x(t),t) + \xi(x(t),t)]$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -bx_1 - cx_2 - a\sin(x_1) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}^\mathsf{T}$$

$$\xi(x,t) = -\upsilon(t)\frac{\cos(x_1)}{\ell}$$

The new unforced nominal system has $x = 0$ as its only equilibrium state. It is readily seen that the function given by

$$V(x,t) = \left(b + \frac{1}{2}c^2\right)x_1^2 + cx_1x_2 + x_2^2 + 2a(1 - \cos(x_1))$$

is a *Lyapunov* function with

$$\gamma_1(\|x\|) = \lambda_{\min}(P)\|x\|^2$$

$$\gamma_2(\|x\|) = \begin{cases} \lambda_{\max}(P)\|x\|^2 + 2a(1 - \cos(\|x\|)), & \|x\| \le \pi \\ \lambda_{\max}(P)\|x\|^2 + 4a, & \|x\| > \pi \end{cases}$$

$$\gamma_3(\|x\|) = \eta\|x\|^2$$

where

$$P := \begin{bmatrix} b + \frac{1}{2}c^2 & \frac{1}{2}c \\ \frac{1}{2}c & 1 \end{bmatrix}$$

$$\eta := \min\{\bar{b}c, c\}$$

$$\bar{b} := b + a\min_{x_1}\left(\frac{\sin(x_1)}{x_1}\right) > 0$$

It is obvious that

$$\|\xi(x,t)\| = \left|-\upsilon(t)\frac{\cos(x_1)}{\ell}\right| \le \hat{\rho}|\cos(x_1)| \tag{33}$$

Here, we select a control gain function as

$$k(t) = \bar{k}\exp\left[0.05(t - t_0)\right] \tag{34}$$

where $\bar{k}$ is any positive constant. It is obvious that the control gain function given in (34) satisfies condition (13). Therefore, it follows from *Theorem 3.1* that

$$p(x,t) = -\bar{k}\exp\left[0.05(t - t_0)\right]\left(\hat{\rho}\left|\cos(x_1)\right|\right)^2(cx_1 + 2x_2) \tag{35}$$

Controller (35) will guarantee the asymptotic stability of dynamical system (32). In order to illustrate this, a simulation is given with the following parameters

$$\ell = 3.0, \quad \hat{\rho} = 5.0,$$
$$a = 1.0, \quad b = 2.0, \quad c = 2.0$$

For this simulation, we also assume that the uncertainty υ is represented by

$$\upsilon = \ell\,\hat{\rho}\,\sin(-x_1)$$

The results of simulation of this example for the different parameter $\bar{k}$ are depicted in Fig.2. It is shown from Fig.2 that closed-loop nonlinear system (32), (35) is indeed uniformly asymptotically stable and no chattering will appear for the control. In addition, the results of our simulation also show that uncertain system (32) without controller (35) is not asymptotically stable.

From Fig.2(c), we can observe that for a smaller $\bar{k}$, the control has its lower magnitude. Thus, one can select an appropriate parameter of control gain function to prevent control saturation for practical control problems. Of course, as a trade-off, the system will have a slower rate of convergence to its equilibrium state (see Fig.2(a)and Fig.2(b)).
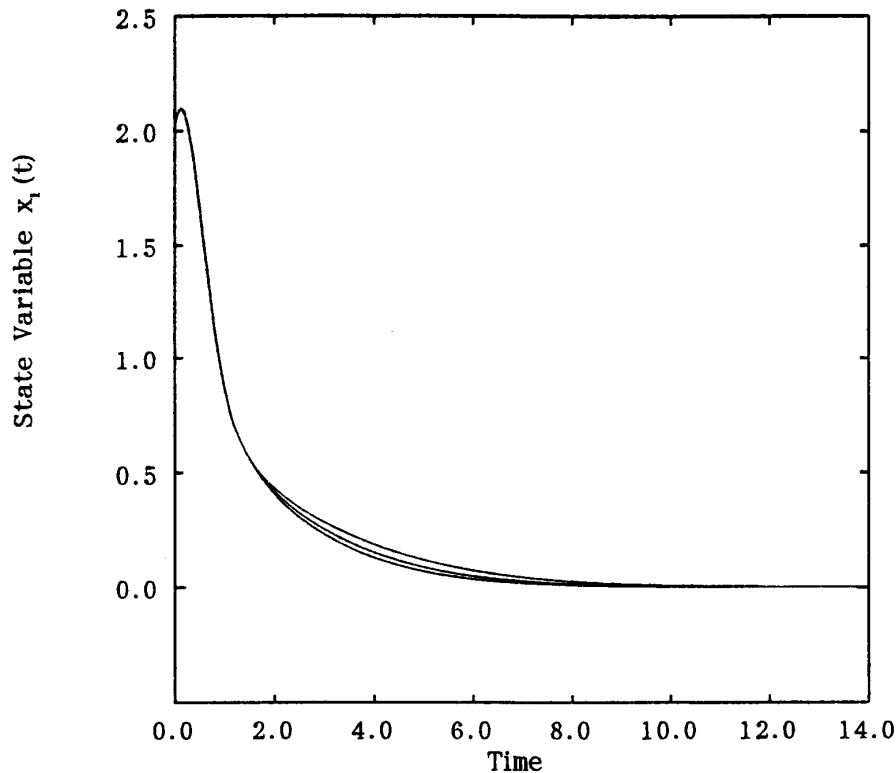

Fig.2   (a)   Responses of state variable $x_1(t)$ with $\bar{k} = 0.08, 0.10, 0.12$ and initial state $x_1(0) = 2.0$.
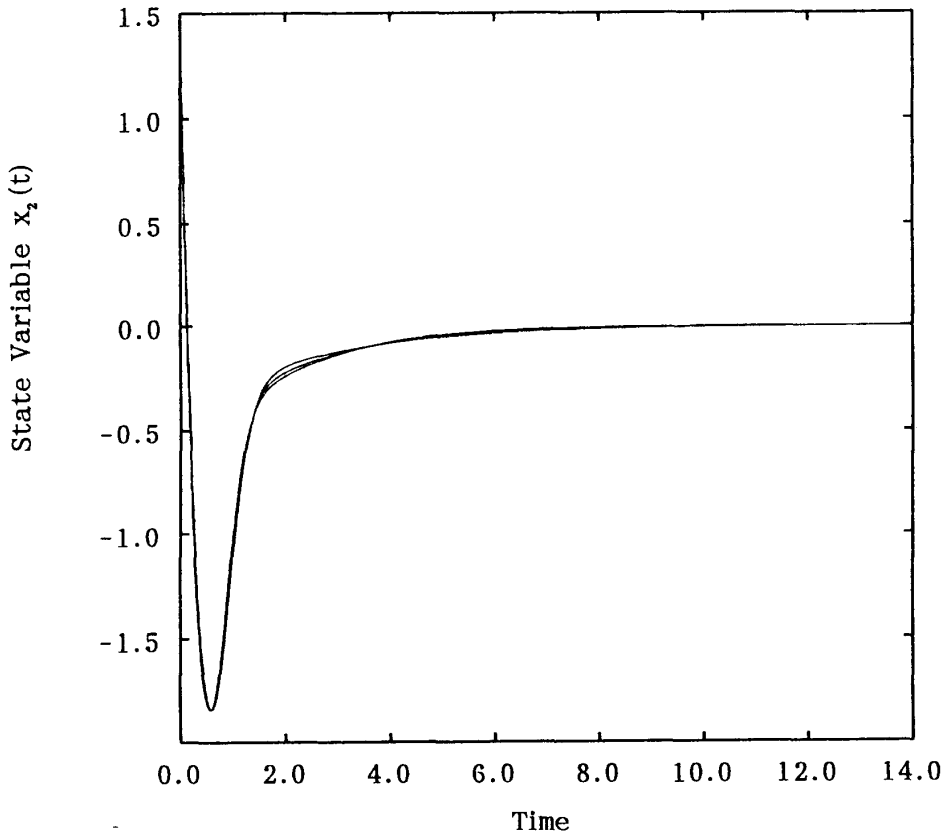
Fig.2   (b) Responses of state variable $x_2(t)$ with $\overline{k} = 0.08, 0.10, 0.12$ and initial state $x_2(0)=1.5$.
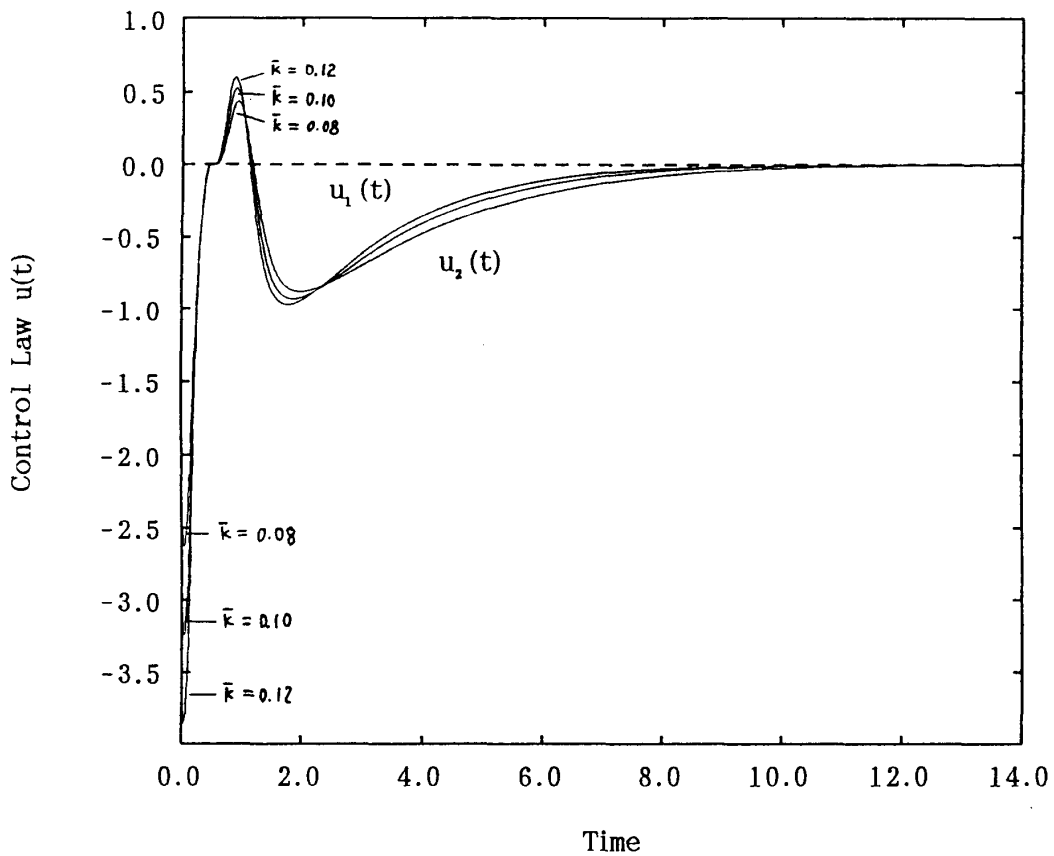

Fig.2   (c) The time response of the control law (40) for $\overline{k} = 0.08, 0.10, 0.12$

4 Application to Robust Control of Robot Manipulators

Consider the following *Euler-Lagrange* dynamical equations for an n-link robot [18].

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{36}$$

where

$$q \in R^n, \quad \tau \in R^n$$

Furthermore, following the development in [16], when the *Lagrangian* dynamical equations are linearly parameterizable, one can have

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Y(q,\dot{q},\ddot{q})\theta = \tau \tag{37}$$

where $\theta \in R^p$ is the unknown parameter vector whose deviation from a known nominal θ_0 is given by

$$\left\| \tilde{\theta} \right\| := \left\| \theta - \theta_0 \right\| \le \rho \tag{38}$$

where $\rho \in R^+$ is a known positive constant.

The nominal control vector τ_0 is defined by

$$\begin{aligned} \tau_0 &= M_0(q)a + C_0(q,\dot{q})v + g_0(q) - Kr \\ &= Y(q,\dot{q},v,a)\theta_0 - Kr \end{aligned} \tag{39}$$

where the quantities v, a, and r are given by

$$\begin{aligned} v &= \dot{q}^d - \Lambda\tilde{q}, & a &= \dot{v} \\ r &= \dot{\tilde{q}} + \Lambda\tilde{q}, & \tilde{q} &= q - q^d \end{aligned} \tag{40}$$

where q^d is a given twice continuously differentiable reference trajectory, and the gain matrices K and Λ are positive definite (diagonal) matrices.

The control input τ in terms of the nominal control vector τ_0 is given by

$$\tau = \tau_0 + Y(q,\dot{q},v,a)u \tag{41}$$

where u is an additional control input that will be synthesized for robustification to the parameter uncertainty represented by $\tilde{\theta}$. Then, substituting (41) into (36) yields

$$M(q)\dot{r} + C(q,\dot{q})r + Kr = Y(q,\dot{q},v,a)\left(\tilde{\theta} + u\right) \tag{42}$$

For system (42), following [4], Spong [16] proposed a continuous state feedback controller of the form

$$u(t) = \begin{cases} -\rho Y^\mathsf{T} r / \left\| Y^\mathsf{T} r \right\|, & \text{if} \left\| Y^\mathsf{T} r \right\| > \varepsilon \\ -\rho Y^\mathsf{T} r / \varepsilon, & \text{if} \left\| Y^\mathsf{T} r \right\| \le \varepsilon \end{cases} \tag{43}$$

which guarantees uniform ultimate boundedness for system (42).

On the other hand, following [6], Yaz [19] proposed another continuous state feedback controller of the form

$$u(t) = -\frac{Y^\mathsf{T} r \rho}{\left\| Y^\mathsf{T} r \right\| + \gamma_1 \exp(-\gamma_2 t)}, \quad \gamma_1 > 0, \quad \gamma_2 > 0 \tag{44}$$

which results in the exponential stability for system (42). It is obvious that controller (43) is saturation-type, and can only guarantee uniform ultimate boundedness. Though controller (44) can guarantee exponential stability, but it has a complicated structure. In this paper, following the approach developed in Section 3, we will propose the following continuous state feedback controller guaranteeing some types of stability results of system (42).

$$u(t) = -k(t)\rho^2 Y^\mathsf{T} r \tag{45}$$

where $k(t)$ is the control gain function.

Thus, we have the following theorem which shows that some types of stability results of system (42) can be acheived under a simple continuous state feedback controller (45).

Theorem 4.1   Consider system (42). Then, for any θ satisfying (38), closed-loop system (42), (45) has the following properties:

(i) If the control gain function $k(t)$ is chosen such that $1/k(t)$ is finite for all time, and satisfies (12), then closed-loop dynamical system (42), (45) is practically stable.

(ii) If the control gain function $k(t)$ is chosen such that inequality (13) is satisfied, then closed-loop dynamical system (42), (45) is uniformly asymptotically stable.

Proof: The proof is similar to that of *Theorem 3.1*, so we will merely sketch the argument.

Following the same discussion as that given in [16], we define a *Lyapunov* function candidate for system (42) as

$$V = \frac{1}{2} r^\mathsf{T} M(q) r + \tilde{q} \Lambda K \tilde{q} \tag{46}$$

A simple calculation shows that along solution trajectories of system (42),

$$\frac{dV}{dt} = -r^\mathsf{T} K r + 2\tilde{q}^\mathsf{T} \Lambda K r - 2\tilde{q}^\mathsf{T} \Lambda K \Lambda \tilde{q} + r^\mathsf{T} Y \left\{ \tilde{\theta} + u \right\}$$

$$= -x^\mathsf{T} Q x + r^\mathsf{T} Y \left\{ \tilde{\theta} + u \right\} \tag{47}$$

where

$$x := \begin{bmatrix} r \\ \tilde{q} \end{bmatrix}, \quad Q := \begin{bmatrix} K & K\Lambda \\ \Lambda K & 2\Lambda K\Lambda \end{bmatrix}$$

The first term on the right-hand side of (47) is negative definite since $\Lambda K \Lambda > 0$, and $K - K \Lambda (2 \Lambda K \Lambda)^{-1} \Lambda K = (1/2) K > 0$[16] as is also shown in [20]. Furthermore, from (38) and (45) we have

$$\frac{dV}{dt} \le -\lambda_{\min}(Q)\|x\|^2 + \|Y^\mathsf{T} r\| \rho - k(t)\rho^2 \|Y^\mathsf{T} r\|^2$$

$$\le -\lambda_{\min}(Q)\|x\|^2 + \frac{1}{4k(t)} \tag{48}$$

To complete the proof, it suffices to notice the following (see [16]). With class-K functions $\alpha(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(\|x\|)I \le M(q) \le \alpha_2(\|x\|)I$$

it can be shown that there exist class-K functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|) \tag{49}$$

Thus, from (48) and (49) we have known that *Assumption 2.3* is satisfied with

$$\liminf_{s \to \infty} \gamma_3(s) = \infty \tag{50}$$

where

$$\gamma_3(s) := \lambda_{\min}(Q)s^2$$

Then, from *Theorem 3.1* one can easily obtain the results given in this theorem. ∎

Remark 4.1   State feedback controller (45) proposed in this paper is continuous, and can guarantee practical stability and asymptotic stability of robot system (42) by choosing different gain functions. It is obvious that controller (45) has a rather simpler structure than (43) and (44), and is more easily implemented. In addition, for controller (45), we may select an appropriate control gain function $k(t)$ such that the closed-loop system has better transient behavior.

5   Conclusion

In this paper, the problem of robust stabilization for a class of nonlinear dynamical systems with uncertainties has been discussed. By *Lyapunov* approach, we propose a class of state feedback controllers with a rather simple structure, which can guarantee practical stability and asymptotic stability of uncertain dynamical systems by choosing different control gain functions. Such a class of state feedback controllers are easily implemented, and no chattering will appear in implementation for the control. As an application of the results developed in this paper, we also discuss the problem of robust control of robot manipulators and propose a simpler robust control

law for n-link robot manipulators.

A numerical example of robust stability of a simple pendulum is given to demonstrate the utilization of the results. It is shown from this numerical example that the results obtained in this paper are effective for a class of uncertain nonlinear dynamical systems, and can be expected to have some further applications.

Appendix A: *Proof of Lemma 2.2*

By the continuity of dynamical system (16), it is obvious that dynamical system (6) has a continuous solution x (t; : t_0, x^0).

It follows from the properties of the *Lyapunov* function that

$$
\begin{aligned}
0 \le{}& \gamma_1\big(\|x(t)\|\big) \\
\le{}& V\big(x(t),t\big) \\
={}& V\big(x^0,t_0\big) + \int_{t_0}^{t} \dot{V}\big(x(\tau),\tau\big)d\tau \\
\le{}& \gamma_2\big(\|x^0\|\big) - \int_{t_0}^{t} \gamma_3\big(\|x(\tau)\|\big)d\tau + \int_{t_0}^{t} \psi(\tau)d\tau
\end{aligned}
\tag{A1}
$$

Therefore, from (A1) we can obtain the following two results. First, taking the limit as t approaches infinity on both sides of the inequality (A1), we have

$$
0 \le \gamma_2\big(\|x^0\|\big) - \lim_{t\to\infty}\int_{t_0}^{t} \gamma_3\big(\|x(\tau)\|\big)d\tau + \lim_{t\to\infty}\int_{t_0}^{t} \psi(\tau)d\tau
\tag{A2}
$$

It follows from (A2) and the property of $\psi(t)$ that

$$
0 \le \gamma_2\big(\|x^0\|\big) - \lim_{t\to\infty}\int_{t_0}^{t} \gamma_3\big(\|x(\tau)\|\big)d\tau + \overline{\psi}
$$

i.e.

$$
\lim_{t\to\infty}\int_{t_0}^{t} \gamma_3\big(\|x(\tau)\|\big)d\tau \le \gamma_2\big(\|x^0\|\big) + \overline{\psi}
\tag{A3}
$$

On the other hand, from (A1) we also have

$$
0 \le \gamma_1\big(\|x(t)\|\big) \le \gamma_2\big(\|x^0\|\big) + \int_{t_0}^{t} \psi(\tau)d\tau
\tag{A4}
$$

Since the function $\psi(\cdot)$ is continuous and satisfies the inequality of the lemma, we can define a constant as follows:

$$
\Psi := \sup_{t\in[t_0,\infty)}\left|\int_{t_0}^{t} \psi(\tau)d\tau\right|
\tag{A5}
$$

It follows from (A4) and (A5) that

$$
0 \le \gamma_1\big(\|x(t)\|\big) \le \gamma_2\big(\|x^0\|\big) + \Psi
\tag{A6}
$$

which implies that $x(\cdot)$ is uniformly bounded. Since $x(\cdot)$ has been shown to be continuous, it follows from (6) that $x(\cdot)$ is uniformly continuous. Therefore, $\gamma_3 (\|x\|)$ is uniformly continuous. Applying the *Barbalat* lemma [21] to the inequality (A3) yields

$$\lim_{t \to \infty} \gamma_3(\|x\|) = 0 \tag{A7}$$

Since $\gamma_3(\cdot)$ is a positive definite scalar function, it is obvious from (A7) that we can have

$$\lim_{t \to \infty} \|x(t)\| = 0$$

That is, dynamical system (6) is uniformly asymptotically stable.   ■

Appendix B: *Proof of Lemma 2.3*

Let $x(t)$ be the solution of system (6), and $V(t)$ denote $V(x(t),t)$. Then we have

$$\frac{dV(t)}{dt} := \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t)$$

$$\leq -2\alpha V(t) + 2\tilde{\varepsilon}\|x\|^2 \tag{A8}$$

From (A8) we can obtain the inequality on $V(t)$ as follows.

$$V(t) = \left(\lambda_{\max}\|x(t_0)\|\right)^2 \exp\{-2\alpha(t - t_0)\}$$

$$+ \int_{t_0}^t 2\tilde{\varepsilon}\exp\{-2\alpha(t - \tau)\}\|x(\tau)\|^2 d\tau \tag{A9}$$

Here, we define an auxiliary function $S(t)$ as follows.

$$S(t) = \left\{\left(\lambda_{\max}\|x(t_0)\|\right)^2 \exp\{-2\alpha(t - t_0)\}\right.$$

$$\left. + \int_{t_0}^t 2\tilde{\varepsilon}\exp\{-2\alpha(t - \tau)\}\|x(\tau)\|^2 d\tau\right\}^{\frac{1}{2}} \tag{A10}$$

Then, comparing (A9) with (A10) we have

$$S(t) \geq \sqrt{V(t)} \geq \lambda_{\min}\|x(t)\| \tag{A11}$$

It follows from (A11) that we have

$$\frac{\|x(t)\|}{S(t)} \leq \frac{1}{\lambda_{\min}} \tag{A12}$$

Differentiating (A10) yields

$$\frac{dS(t)}{dt} = -\alpha S(t) + \frac{\|x(t)\|}{S(t)}\tilde{\varepsilon}\|x(t)\|$$

$$\leq -\alpha S(t) + \frac{\tilde{\varepsilon}}{\lambda_{\min}}\|x(t)\| \tag{A13}$$

Moreover, from (A13) we obtain an inequality on $S(t)$ as follows.

$$S(t) \leq \lambda_{\min} \|x(t_0)\| \exp\{-\alpha(t - t_0)\}$$

$$+ \frac{\tilde{\varepsilon}}{\lambda_{\min}} \int_{t_0}^{t} \exp\{-\alpha(t - \tau)\} \|x(\tau)\| d\tau \qquad (A14)$$

It follows from (A11) and (A14) that

$$\|x(t)\| \leq \frac{\lambda_{\max}}{\lambda_{\min}} \|x(t_0)\| \exp\{-\alpha(t - t_0)\}$$

$$+ \frac{\tilde{\varepsilon}}{\lambda_{\min}^2} \int_{t_0}^{t} \exp\{-\alpha(t - \tau)\} \|x(\tau)\| d\tau \qquad (A15)$$

In order to make use of *Bellman-Gronwall's* inequality, we rewrite (A15) as follows.

$$\|x(t)\| \exp\{\alpha(t - t_0)\} \leq \frac{\lambda_{\max}}{\lambda_{\min}} \|x(t_0)\| + \frac{\tilde{\varepsilon}}{\lambda_{\min}^2} \int_{t_0}^{t} \exp\{\alpha(t - \tau)\} \|x(\tau)\| d\tau$$

Then, by *Bellman-Gronwall's* inequality, we obtain the inequality (9). It follows from (9) that system (16) is exponentially stable. ∎

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