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An algorithm to calculate plethysms of Schur functions

Dedicated to Professor Motoyoshi Sakuma on his 70th birthday

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Abstract : We present an algorithm to calculate plethysms of Schur functions which is fitted for computers, and give the decomposition table of plethysms $\{\lambda\}\otimes\{\mu\}$ up to total degree 12.

Key words : Schur function, plethysm, representation, polynomial ring

Introduction

"Plethysm" $\{\lambda\}\otimes\{\mu\}$ is a sort of multiplication of Schur functions, which was first introduced by D.E.Littlewood [13]. This product is needed, for example, to determine the structure of polynomial ring of the irreducible representation space V_λ of $GL(n,\boldsymbol{C})$ corresponding to the partition $\lambda=(\lambda_1,\cdots,\lambda_n)$ $(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n)$. In fact, the character of the space of homogeneous polynomials of degree p on V_λ is given by the plethysm $\{\lambda\}\otimes\{p\}$, and the decomposition of $\{\lambda\}\otimes\{p\}$ gives an information on the $GL(n,\boldsymbol{C})$-irreducible decomposition of $S^p(V_\lambda)$. As another example, the character of $S^p(V_\lambda\otimes W_\mu)$, which is the space of homogeneous polynomials of degree p on the tensor product of two $GL(n,\boldsymbol{C})$- and $GL(m,\boldsymbol{C})$ - irreducible spaces V_λ and W_μ is equal to

$$(\{\lambda\}\ \{\mu\})\otimes\{p\}=\sum_{\nu}(\{\lambda\}\otimes\{\nu\})\,(\{\mu\}\otimes\{\nu\}),$$

where ν runs all over the partitions of p. (See [14; p.331] or [16; p.290].) If we know the decomposition of plethysms, we can explicitly write down the generator of each irreducible component of $S^p(V_\lambda)$, $S^p(V_\lambda\otimes W_\mu)$ by the method stated in [2], which for some cases serves as a useful tool in considering several geometric problems. (For example, in [2], we find a new obstruction of local isometric imbeddings of Riemannian submanifolds with codimension 2 by calculating the plethysm $\{2^2\}\otimes\{3\}$, which is the character of the space of cubic polynomials of curvature tensors. The new obstruction corresponds to the component $\{3^4\}$ in $\{2^2\}\otimes\{3\}$. For another example, see [3].)

Up to the present time, several methods and formulas for calculating plethysm are known [6, 7, 8, 10, 14, 19, 20, 22, 24, 25, 28, 29, 31], etc. (For detailed references, see the bibliography in [12].) But actual calculation for large total degree is hard to perform in practice. In this note we

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present one simple algorithm to calculate plethysms which is fitted for computers, and give a table of plethysms of Schur functions up to total degree 12, by applying this method. In actual calculations, "substitution of polynomials" and "calculation of determinants" are necessary to decompose $\{\lambda\}\otimes\{\mu\}$, and we used the algebraic programming system REDUCE 3.3 to perform these two types of calculations.

1. Notations and formulas

We express the Schur function corresponding to the partition λ as $\{\lambda\}=\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$), where n is the dimension of the representation space, and we put $|\lambda| = \lambda_1 + \cdots + \lambda_n$, which we call the degree of $\{\lambda\}$. (For the definition of Schur functions, see [11], [18].) When the numbers in $\{\lambda\}$ are repeated, we express it in the exponential form such as $\{22211\} = \{2^3 1^2\}$. The Schur function $\{\lambda\}$ is a symmetric polynomial with variables $\varepsilon_1, \cdots, \varepsilon_n$, which express the eigenvalues of a matrix of $GL(n,C)$. For example, we have

$$\{2\} = \sum_i \varepsilon_i^2 + \sum_{i<j} \varepsilon_i \varepsilon_j ,$$

$$\{1^2\} = \sum_{i<j} \varepsilon_i \varepsilon_j .$$

We denote by $\{\lambda\}\otimes\{\mu\}$ the plethysm of two Schur functions $\{\lambda\}$ and $\{\mu\}$, and call $|\lambda||\mu|$ as the total degree of $\{\lambda\}\otimes\{\mu\}$. For the general definition of plethysm, see [12; p.219], [16; p.206, p.289], [26; p.66], etc. There exist several formulas to calculate $\{\lambda\}\otimes\{\mu\}$. For example, the following decompositions are well known:

$$\{m\}\otimes\{2\}=\{2m\}+\{2m-2.2\}+\dots+\begin{cases}\{m^2\}, & m=\text{even}\\ \{m+1.m-1\}, & m=\text{odd}\end{cases},$$

$$\{m\}\otimes\{1^2\}=\{2m-1.1\}+\{2m-3.3\}+\dots+\begin{cases}\{m+1.m-1\}, & m=\text{even}\\ \{m^2\}, & m=\text{odd}\end{cases}.$$

For other formulas, see [5], [6; p.143~], [12; p.222~], [18; p.140~], [19; p.52~]. There exists also a duality formula:

$$(\{\lambda\}\otimes\{\mu\}, \{\nu\}) = \begin{cases}(\{\lambda'\}\otimes\{\mu\}, \{\nu'\}), & |\lambda|=\text{even}\\ (\{\lambda'\}\otimes\{\mu'\}, \{\nu'\}), & |\lambda|=\text{odd}\end{cases},$$

where $(\{\lambda\}\otimes\{\mu\}, \{\nu\})$ is the coefficient of $\{\nu\}$ in $\{\lambda\}\otimes\{\mu\}$, and $\{\lambda'\}$ is the transpose of $\{\lambda\}$. But, unfortunately, general formula for $\{\lambda\}\otimes\{\mu\}$ is not known.

2. An algorithm to calculate the plethysm $\{\lambda\}\otimes\{\mu\}$

Now, we explain an algorithm to calculate the plethysm $\{\lambda\}\otimes\{\mu\}$. First, we put

$$p_k = \{k\} ,$$
$$\sigma_k = \varepsilon_1^k + \cdots + \varepsilon_n^k ,$$

for $k = 1, 2, \cdots$. Then, they are related by the formula:

$$(*) \qquad \sigma_k + \sigma_{k-1} p_1 + \cdots + \sigma_1 p_{k-1} = kp_k \;, \quad k = 1, 2, \cdots .$$

In addition, the Schur function $\{\lambda\}$ can be expressed in terms of p_k by

$$(**) \qquad \{\lambda\} = \begin{vmatrix} p_{\lambda 1} & p_{\lambda 1+1} & p_{\lambda 1+2} & \cdots & p_{\lambda 1+n-1} \\ p_{\lambda 2-1} & p_{\lambda 2} & p_{\lambda 2+1} & \cdots & p_{\lambda 2+n-2} \\ p_{\lambda 3-2} & p_{\lambda 3-1} & p_{\lambda 3} & \cdots & p_{\lambda 3+n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{\lambda n-n+1} & p_{\lambda n-n+2} & p_{\lambda n-n+3} & \cdots & p_{\lambda n} \end{vmatrix}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $p_0 = 1$ and $p_i = 0 \ (i < 0)$. By using the above formula $(*)$ repeatedly, we can express p_k in terms of $\sigma_1 \sim \sigma_k$, and we substitute these equalities into $(**)$. Then, the Schur function $\{\lambda\}$ can be expressed as a polynomial of $\sigma_1 \sim \sigma_{|\lambda|}$:

$$\{\lambda\} = f_\lambda(\sigma_1, \sigma_2, \cdots, \sigma_{|\lambda|}).$$

Similarly, we can express $\{\mu\}$ as

$$\{\mu\} = f_\mu(\sigma_1, \sigma_2, \cdots, \sigma_{|\mu|}).$$

In this situation, we replace σ_k in $f_\mu(\sigma_1, \sigma_2, \cdots, \sigma_{|\mu|})$ by $f_\lambda(\sigma_k, \sigma_{2k}, \sigma_{3k}, \cdots, \sigma_{|\lambda|k})$ $(k = 1, 2, \cdots, |\mu|)$. Then, the resulting polynomial

$$f_\mu\,(f_\lambda(\sigma_1, \sigma_2, \cdots),\ f_\lambda(\sigma_2, \sigma_4, \cdots),\ f_\lambda(\sigma_3, \sigma_6, \cdots), \quad \cdots)$$

gives the desired one which expresses the plethysm $\{\lambda\}\otimes\{\mu\}$.

This is expressed as a polynomial of σ_k , and we rewrite $\{\lambda\}\otimes\{\mu\}$ in terms of p_k by using the formula $(*)$ repeatedly once again. And next, we rearrange this polynomial with respect to the ordering $p_{|\lambda||\mu|}$, $p_{|\lambda||\mu|-1}$, $\cdots$, p_2, p_1 . As a final step, we rewrite this polynomial as a sum of Schur functions in the following way. If the last term of this polynomial is $p_m{}^a\, p_{m-1}{}^b \cdots p_2{}^c\, p_1{}^d$, then we calculate the Schur function $\{m^a\ m\text{-}1^b \cdots 2^c\ 1^d\}$ by using the formula $(**)$, and rewrite $\{\lambda\}\otimes\{\mu\}$ in the form

$$\{\lambda\}\otimes\{\mu\} = f(p_{|\lambda||\mu|}, \cdots, p_1) + \{m^a\ m\text{-}1^b \cdots 2^c\ 1^d\}.$$

Next, we rewrite the remaining polynomial $f(p_{|\lambda||\mu|}, \cdots, p_1)$ as a sum of a polynomial $g(p_{|\lambda||\mu|}, \cdots, p_1)$ and the Schur function corresponding to the last term of $f(p_{|\lambda||\mu|}, \cdots, p_1)$, in the same way as above. Then repeating this procedure one by one, we finally arrive at the desired decomposition of

$\{\lambda\}\otimes\{\mu\}$.

As one example, we calculate the plethysm $\{21\}\otimes\{1^2\}$ by this method. First, by using the above formula (*), we have

$$\begin{aligned} p_1 &= \sigma_1 , \\ p_2 &= 1/2\cdot(\sigma_2+\sigma_1 p_1) = 1/2\cdot(\sigma_1^2+\sigma_2) , \\ p_3 &= 1/3\cdot(\sigma_3+\sigma_2 p_1+\sigma_1 p_2) = 1/6\cdot(\sigma_1^3+3\sigma_1\sigma_2+2\sigma_3) . \end{aligned}$$

Then, by using (**), we have

$$\begin{aligned} \{1^2\} &= -p_2+p_1^2 \\ &= 1/2\cdot(\sigma_1^2-\sigma_2) , \end{aligned}$$

and

$$\begin{aligned} \{21\} &= -p_3+p_2 p_1 \\ &= 1/3\cdot(\sigma_1^3-\sigma_3) . \end{aligned}$$

We replace σ_1 and σ_2 in $\{1^2\}$ by $1/3\cdot(\sigma_1^3-\sigma_3)$ $(=\{21\})$ and $1/3\cdot(\sigma_2^3-\sigma_6)$, respectively. Then, we have

$$\begin{aligned} \{21\}\otimes\{1^2\} &= 1/2\cdot\{1/9\cdot(\sigma_1^3-\sigma_3)^2-1/3\cdot(\sigma_2^3-\sigma_6)\} \\ &= 1/18\cdot(\sigma_1^6-2\sigma_1^3\sigma_3+\sigma_3^2-3\sigma_2^3+3\sigma_6) . \end{aligned}$$

Next, by using the formula (*) once again, we have

$$\begin{aligned} \sigma_1 &= p_1 , \\ \sigma_2 &= 2p_2-\sigma_1 p_1 \\ &= 2p_2-p_1^2 , \\ \sigma_3 &= 3p_3-\sigma_2 p_1-\sigma_1 p_2 \\ &= 3p_3-3p_2 p_1+p_1^3 , \\ \sigma_4 &= 4p_4-\sigma_3 p_1-\sigma_2 p_2-\sigma_1 p_3 \\ &= 4p_4-4p_3 p_1-2p_2^2+4p_2 p_1^2-p_1^4 , \\ \sigma_5 &= 5p_5-\sigma_4 p_1-\sigma_3 p_2-\sigma_2 p_3-\sigma_1 p_4 \\ &= 5p_5-5p_4 p_1-5p_3 p_2+5p_3 p_1^2+5p_2^2 p_1-5p_2 p_1^3+p_1^5 , \\ \sigma_6 &= 6p_6-\sigma_5 p_1-\sigma_4 p_2-\sigma_3 p_3-\sigma_2 p_4-\sigma_1 p_5 \\ &= 6p_6-6p_5 p_1-6p_4 p_2+6p_4 p_1^2-3p_3^2+12p_3 p_2 p_1-6p_3 p_1^3+2p_2^3-9p_2^2 p_1^2+6p_2 p_1^4-p_1^6 . \end{aligned}$$

We substitute these expressions into the above. Then, we have finally

$$\begin{aligned} \{21\}\otimes\{1^2\} &= p_6-p_5 p_1-p_4 p_2+p_4 p_1^2+p_3 p_2 p_1-p_3 p_1^3-p_2^3+p_2^2 p_1^2 \\ &= p_6-2p_4 p_2+p_3 p_2 p_1+\{2^2 1^2\} \\ &= p_6-p_5 p_1-2p_4 p_2+p_4 p_1^2+p_3^2+\{321\}+\{2^2 1^2\} \end{aligned}$$

$= p_6 - p_5 p_1 - p_4 p_2 + p_4 p_1^2 + \{3^2\} + \{321\} + \{2^2 1^2\}$
$= \{41^2\} + \{3^2\} + \{321\} + \{2^2 1^2\}.$

(In the last modifications, we used the equalities

$$\begin{aligned}
\{41^2\} &= p_6 - p_5 p_1 - p_4 p_2 + p_4 p_1^2, \\
\{3^2\} &= - p_4 p_2 + p_3^2, \\
\{321\} &= p_5 p_1 - p_4 p_1^2 - p_3^2 + p_3 p_2 p_1, \\
\{2^2 1^2\} &= - p_5 p_1 + p_4 p_2 + p_4 p_1^2 - p_3 p_1^3 - p_2^3 + p_2^2 p_1^2,
\end{aligned}$$

which are obtained from the formula (**).)

These calculations are actually hard to perform by hand as the size of the matrix in (**) becomes large. But, with the aid of the computer, we can perform them relatively in a simple way.

3. The table of $\{\lambda\}\otimes\{\mu\}$

Now, we summarize the results in the following. We first exhibit the formulas of $\{\lambda\}$ that are expressed as polynomials of σ_i. These equalities play a fundamental role in calculating $\{\lambda\}\otimes\{\mu\}$ as explained above.

$$\{1\} = \sigma_1$$

$$\begin{aligned}
\{2\} &= 1/2 \cdot (\sigma_1^2 + \sigma_2) \\
\{1^2\} &= 1/2 \cdot (\sigma_1^2 - \sigma_2)
\end{aligned}$$

$$\begin{aligned}
\{3\} &= 1/6 \cdot (\sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3) \\
\{21\} &= 1/3 \cdot (\sigma_1^3 - \sigma_3) \\
\{1^3\} &= 1/6 \cdot (\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3)
\end{aligned}$$

$$\begin{aligned}
\{4\} &= 1/24 \cdot (\sigma_1^4 + 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 + 6\sigma_4) \\
\{31\} &= 1/8 \cdot (\sigma_1^4 + 2\sigma_1^2\sigma_2 - \sigma_2^2 - 2\sigma_4) \\
\{2^2\} &= 1/12 \cdot (\sigma_1^4 - 4\sigma_1\sigma_3 + 3\sigma_2^2) \\
\{21^2\} &= 1/8 \cdot (\sigma_1^4 - 2\sigma_1^2\sigma_2 - \sigma_2^2 + 2\sigma_4) \\
\{1^4\} &= 1/24 \cdot (\sigma_1^4 - 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 - 6\sigma_4)
\end{aligned}$$

$$\begin{aligned}
\{5\} &= 1/120 \cdot (\sigma_1^5 + 10\sigma_1^3\sigma_2 + 20\sigma_1^2\sigma_3 + 15\sigma_1\sigma_2^2 + 30\sigma_1\sigma_4 + 20\sigma_2\sigma_3 + 24\sigma_5) \\
\{41\} &= 1/30 \cdot (\sigma_1^5 + 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3 - 6\sigma_5) \\
\{32\} &= 1/24 \cdot (\sigma_1^5 + 2\sigma_1^3\sigma_2 - 4\sigma_1^2\sigma_3 + 3\sigma_1\sigma_2^2 - 6\sigma_1\sigma_4 + 4\sigma_2\sigma_3) \\
\{31^2\} &= 1/20 \cdot (\sigma_1^5 - 5\sigma_1\sigma_2^2 + 4\sigma_5) \\
\{2^2 1\} &= 1/24 \cdot (\sigma_1^5 - 2\sigma_1^3\sigma_2 - 4\sigma_1^2\sigma_3 + 3\sigma_1\sigma_2^2 + 6\sigma_1\sigma_4 - 4\sigma_2\sigma_3) \\
\{21^3\} &= 1/30 \cdot (\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 + 5\sigma_2\sigma_3 - 6\sigma_5)
\end{aligned}$$

$\{1^5\} = 1/120 \cdot (\sigma_1^5 - 10\sigma_1^3\sigma_2 + 20\sigma_1^2\sigma_3 + 15\sigma_1\sigma_2^2 - 30\sigma_1\sigma_4 - 20\sigma_2\sigma_3 + 24\sigma_5)$

$\{6\} = 1/720 \cdot (\sigma_1^6 + 15\sigma_1^4\sigma_2 + 40\sigma_1^3\sigma_3 + 45\sigma_1^2\sigma_2^2 + 90\sigma_1^2\sigma_4 + 120\sigma_1\sigma_2\sigma_3 + 144\sigma_1\sigma_5 + 15\sigma_2^3 + 90\sigma_2\sigma_4 + 40\sigma_3^2 + 120\sigma_6)$

$\{51\} = 1/144 \cdot (\sigma_1^6 + 9\sigma_1^4\sigma_2 + 16\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 + 18\sigma_1^2\sigma_4 - 3\sigma_2^3 - 18\sigma_2\sigma_4 - 8\sigma_3^2 - 24\sigma_6)$

$\{42\} = 1/80 \cdot (\sigma_1^6 + 5\sigma_1^4\sigma_2 + 5\sigma_1^2\sigma_2^2 - 10\sigma_1^2\sigma_4 - 16\sigma_1\sigma_5 + 5\sigma_2^3 + 10\sigma_2\sigma_4)$

$\{41^2\} = 1/72 \cdot (\sigma_1^6 + 3\sigma_1^4\sigma_2 + 4\sigma_1^3\sigma_3 - 9\sigma_1^2\sigma_2^2 - 12\sigma_1\sigma_2\sigma_3 - 3\sigma_2^3 + 4\sigma_3^2 + 12\sigma_6)$

$\{3^2\} = 1/144 \cdot (\sigma_1^6 + 3\sigma_1^4\sigma_2 - 8\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 - 18\sigma_1^2\sigma_4 + 24\sigma_1\sigma_2\sigma_3 - 9\sigma_2^3 - 18\sigma_2\sigma_4 + 16\sigma_3^2)$

$\{321\} = 1/45 \cdot (\sigma_1^6 - 5\sigma_1^3\sigma_3 + 9\sigma_1\sigma_5 - 5\sigma_3^2)$

$\{31^3\} = 1/72 \cdot (\sigma_1^6 - 3\sigma_1^4\sigma_2 + 4\sigma_1^3\sigma_3 - 9\sigma_1^2\sigma_2^2 + 12\sigma_1\sigma_2\sigma_3 + 3\sigma_2^3 + 4\sigma_3^2 - 12\sigma_6)$

$\{2^3\} = 1/144 \cdot (\sigma_1^6 - 3\sigma_1^4\sigma_2 - 8\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 + 18\sigma_1^2\sigma_4 - 24\sigma_1\sigma_2\sigma_3 + 9\sigma_2^3 - 18\sigma_2\sigma_4 + 16\sigma_3^2)$

$\{2^21^2\} = 1/80 \cdot (\sigma_1^6 - 5\sigma_1^4\sigma_2 + 5\sigma_1^2\sigma_2^2 + 10\sigma_1^2\sigma_4 - 16\sigma_1\sigma_5 - 5\sigma_2^3 + 10\sigma_2\sigma_4)$

$\{21^4\} = 1/144 \cdot (\sigma_1^6 - 9\sigma_1^4\sigma_2 + 16\sigma_1^3\sigma_3 + 9\sigma_1^2\sigma_2^2 - 18\sigma_1^2\sigma_4 + 3\sigma_2^3 - 18\sigma_2\sigma_4 - 8\sigma_3^2 + 24\sigma_6)$

$\{1^6\} = 1/720 \cdot (\sigma_1^6 - 15\sigma_1^4\sigma_2 + 40\sigma_1^3\sigma_3 + 45\sigma_1^2\sigma_2^2 - 90\sigma_1^2\sigma_4 - 120\sigma_1\sigma_2\sigma_3 + 144\sigma_1\sigma_5 - 15\sigma_2^3 + 90\sigma_2\sigma_4 + 40\sigma_3^2 - 120\sigma_6)$

Next, we give the table of plethysms up to total degree 12. Note that the results on $\{21^3\}\otimes\{2\}$ ([28; p.383]), $\{2\}\otimes\{4\}$ ([25;p.341]), $\{2^2\}\otimes\{4\}$ ([14; p.359]) contain some mistakes and we correct the first two plethysms in the following table. As for the correct decomposition of $\{2^2\}\otimes\{4\}$, see [2; p.130～].

Total degree = 4

$\{2\}\otimes\{2\} = \{4\} + \{2^2\}$.

$\{2\}\otimes\{1^2\} = \{31\}$.

$\{1^2\}\otimes\{2\} = \{2^2\} + \{1^4\}$.

$\{1^2\}\otimes\{1^2\} = \{21^2\}$.

Total degree = 6

$\{3\}\otimes\{2\} = \{6\} + \{42\}$.

$\{3\}\otimes\{1^2\} = \{51\} + \{3^2\}$.

$\{21\}\otimes\{2\} = \{42\} + \{321\} + \{31^3\} + \{2^3\}$.

$\{21\}\otimes\{1^2\} = \{41^2\} + \{3^2\} + \{321\} + \{2^21^2\}$.

$\{2\}\otimes\{3\} = \{6\} + \{42\} + \{2^3\}$.

$\{2\}\otimes\{21\} = \{51\} + \{42\} + \{321\}$.

$\{2\}\otimes\{1^3\} = \{41^2\} + \{3^2\}.$

$\{1^3\}\otimes\{2\} = \{2^3\} + \{21^4\}.$

$\{1^3\}\otimes\{1^2\} = \{2^21^2\} + \{1^6\}.$

$\{1^2\}\otimes\{3\} = \{3^2\} + \{2^21^2\} + \{1^6\}.$

$\{1^2\}\otimes\{21\} = \{321\} + \{2^21^2\} + \{21^4\}.$

$\{1^2\}\otimes\{1^3\} = \{31^3\} + \{2^3\}.$

Total degree = 8

$\{4\}\otimes\{2\} = \{8\} + \{62\} + \{4^2\}.$

$\{4\}\otimes\{1^2\} = \{71\} + \{53\}.$

$\{31\}\otimes\{2\} = \{62\} + \{521\} + \{51^3\} + \{4^2\} + \{431\} + \{42^2\} + \{3^21^2\}.$

$\{31\}\otimes\{1^2\} = \{61^2\} + \{53\} + \{521\} + \{431\} + \{421^2\} + \{3^22\}.$

$\{2^2\}\otimes\{2\} = \{4^2\} + \{42^2\} + \{3^21^2\} + \{2^4\}.$

$\{2^2\}\otimes\{1^2\} = \{431\} + \{32^21\}.$

$\{21^2\}\otimes\{2\} = \{42^2\} + \{41^4\} + \{3^21^2\} + \{32^21\} + \{321^3\} + \{2^4\} + \{2^21^4\}.$

$\{21^2\}\otimes\{1^2\} = \{421^2\} + \{3^22\} + \{32^21\} + \{321^3\} + \{31^5\} + \{2^31^2\}.$

$\{2\}\otimes\{4\} = \{8\} + \{62\} + \{4^2\} + \{42^2\} + \{2^4\}.$

$\{2\}\otimes\{31\} = \{71\} + \{62\} + \{53\} + \{521\} + \{431\} + \{42^2\} + \{32^21\}.$

$\{2\}\otimes\{2^2\} = \{62\} + \{521\} + \{4^2\} + \{42^2\} + \{3^21^2\}.$

$\{2\}\otimes\{21^2\} = \{61^2\} + \{53\} + \{521\} + \{431\} + \{421^2\} + \{3^22\}.$

$\{2\}\otimes\{1^4\} = \{51^3\} + \{431\}.$

$\{1^4\}\otimes\{2\} = \{2^4\} + \{2^21^4\} + \{1^8\}.$

$\{1^4\}\otimes\{1^2\} = \{2^31^2\} + \{21^6\}.$

$\{1^2\}\otimes\{4\} = \{4^2\} + \{3^21^2\} + \{2^4\} + \{2^21^4\} + \{1^8\}.$

$\{1^2\}\otimes\{31\} = \{431\} + \{3^21^2\} + \{32^21\} + \{321^3\} + \{2^31^2\} + \{2^21^4\} + \{21^6\}.$

$\{1^2\}\otimes\{2^2\} = \{42^2\} + \{3^21^2\} + \{321^3\} + \{2^4\} + \{2^21^4\}.$

$\{1^2\}\otimes\{21^2\} = \{421^2\} + \{3^22\} + \{32^21\} + \{321^3\} + \{31^5\} + \{2^31^2\}.$

$\{1^2\}\otimes\{1^4\} = \{41^4\} + \{32^21\}$.

Total degree = 9

$\{3\}\otimes\{3\} = \{9\} + \{72\} + \{63\} + \{52^2\} + \{4^21\}$.

$\{3\}\otimes\{21\} = \{81\} + \{72\} + \{63\} + \{621\} + \{54\} + \{531\} + \{432\}$.

$\{3\}\otimes\{1^3\} = \{71^2\} + \{63\} + \{531\} + \{3^3\}$.

$\{21\}\otimes\{3\} = \{63\} + \{531\} + \{52^2\} + \{521^2\} + \{4^21\} + \{432\} + \{431^2\} + 2\{42^21\} + \{421^3\} + \{41^5\} + \{3^3\} + \{3^221\} + \{3^21^3\} + \{32^3\} + \{32^21^2\}$.

$\{21\}\otimes\{21\} = \{621\} + \{54\} + 2\{531\} + \{52^2\} + 2\{521^2\} + \{51^4\} + \{4^21\} + 3\{432\} + 3\{431^2\} + 3\{42^21\} + 2\{421^3\} + 3\{3^221\} + \{3^21^3\} + \{32^3\} + 2\{32^21^2\} + \{321^4\} + \{2^41\}$.

$\{21\}\otimes\{1^3\} = \{61^3\} + \{531\} + \{52^2\} + \{521^2\} + \{4^21\} + \{432\} + 2\{431^2\} + \{42^21\} + \{421^3\} + \{3^3\} + \{3^221\} + \{3^21^3\} + \{32^3\} + \{32^21^2\} + \{2^31^3\}$.

$\{1^3\}\otimes\{3\} = \{3^3\} + \{32^21^2\} + \{31^6\} + \{2^31^3\}$.

$\{1^3\}\otimes\{21\} = \{3^221\} + \{32^21^2\} + \{321^4\} + \{2^41\} + \{2^31^3\} + \{2^21^5\} + \{21^7\}$.

$\{1^3\}\otimes\{1^3\} = \{3^21^3\} + \{32^3\} + \{2^31^3\} + \{2^21^5\} + \{1^9\}$.

Total degree = 10

$\{5\}\otimes\{2\} = \{10\} + \{82\} + \{64\}$.

$\{5\}\otimes\{1^2\} = \{91\} + \{73\} + \{5^2\}$.

$\{41\}\otimes\{2\} = \{82\} + \{721\} + \{71^3\} + \{64\} + \{631\} + \{62^2\} + \{541\} + \{531^2\} + \{4^22\}$.

$\{41\}\otimes\{1^2\} = \{81^2\} + \{73\} + \{721\} + \{631\} + \{621^2\} + \{5^2\} + \{541\} + \{532\} + \{4^21^2\}$.

$\{32\}\otimes\{2\} = \{64\} + \{62^2\} + \{541\} + \{532\} + \{531^2\} + \{4^22\} + \{4321\} + \{42^3\} + \{3^31\}$.

$\{32\}\otimes\{1^2\} = \{631\} + \{5^2\} + \{541\} + \{532\} + \{52^21\} + \{4^21^2\} + \{43^2\} + \{4321\} + \{3^22^2\}$.

$\{31^2\}\otimes\{2\} = \{62^2\} + \{61^4\} + \{531^2\} + \{52^21\} + \{521^3\} + \{4^22\} + \{4321\} + \{431^3\} + \{42^3\} + \{421^4\} + \{3^221^2\}$.

$\{31^2\}\otimes\{1^2\} = \{621^2\} + \{532\} + \{52^21\} + \{521^3\} + \{51^5\} + \{4^21^2\} + \{4321\} + \{431^3\} + \{42^21^2\} + \{3^22^2\} + \{3^2 1^4\}$.

$\{2^21\}\otimes\{2\} = \{4^22\} + \{4321\} + \{431^3\} + \{42^3\} + \{3^31\} + \{3^221^2\} + \{32^31\} + \{32^21^3\} + \{2^5\}$.

$\{2^21\}\otimes\{1^2\} = \{4^21^2\} + \{43^2\} + \{4321\} + \{42^21^2\} + \{3^22^2\} + \{3^221^2\} + \{3^21^4\} + \{32^31\} + \{2^41^2\}$.

$\{21^3\}\otimes\{2\} = \{42^3\} + \{421^4\} + \{3^221^2\} + \{32^31\} + \{32^21^3\} + \{321^5\} + \{31^7\} + \{2^5\} + \{2^31^4\}.$

$\{21^3\}\otimes\{1^2\} = \{42^21^2\} + \{41^6\} + \{3^22^2\} + \{3^21^4\} + \{32^31\} + \{32^21^3\} + \{321^5\} + \{2^41^2\} + \{2^21^6\}.$

$\{2\}\otimes\{5\} = \{10\} + \{82\} + \{64\} + \{62^2\} + \{4^22\} + \{42^3\} + \{2^5\}.$

$\{2\}\otimes\{41\} = \{91\} + \{82\} + \{73\} + \{721\} + \{64\} + \{631\} + \{62^2\} + \{541\} + \{532\} + \{52^21\} + \{4^22\} + \{4321\} + \{42^3\} + \{32^31\}.$

$\{2\}\otimes\{32\} = \{82\} + \{73\} + \{721\} + \{64\} + \{631\} + 2\{62^2\} + \{541\} + \{532\} + \{531^2\} + \{52^21\} + \{4^22\} + \{4321\} + \{42^3\} + \{3^221^2\}.$

$\{2\}\otimes\{31^2\} = \{81^2\} + \{73\} + \{721\} + 2\{631\} + \{621^2\} + \{5^2\} + \{541\} + 2\{532\} + \{531^2\} + \{52^21\} + \{4^21^2\} + \{43^2\} + \{4321\} + \{42^21^2\} + \{3^22^2\}.$

$\{2\}\otimes\{2^21\} = \{721\} + \{64\} + \{631\} + \{62^2\} + \{621^2\} + \{541\} + \{532\} + \{531^2\} + \{52^21\} + \{4^22\} + \{4321\} + \{431^3\} + \{3^31\}.$

$\{2\}\otimes\{21^3\} = \{71^3\} + \{631\} + \{621^2\} + \{541\} + \{532\} + \{531^2\} + \{521^3\} + \{4^21^2\} + \{43^2\} + \{4321\}.$

$\{2\}\otimes\{1^5\} = \{61^4\} + \{531^2\} + \{4^22\}.$

$\{1^5\}\otimes\{2\} = \{2^5\} + \{2^31^4\} + \{21^8\}.$

$\{1^5\}\otimes\{1^2\} = \{2^41^2\} + \{2^21^6\} + \{1^{10}\}.$

$\{1^2\}\otimes\{5\} = \{5^2\} + \{4^21^2\} + \{3^22^2\} + \{3^21^4\} + \{2^41^2\} + \{2^21^6\} + \{1^{10}\}.$

$\{1^2\}\otimes\{41\} = \{541\} + \{4^21^2\} + \{4321\} + \{431^3\} + \{3^22^2\} + \{3^221^2\} + \{3^21^4\} + \{32^31\} + \{32^21^3\} + \{321^5\} + \{2^41^2\} + \{2^31^4\} + \{2^21^6\} + \{21^8\}.$

$\{1^2\}\otimes\{32\} = \{532\} + \{4^21^2\} + \{4321\} + \{431^3\} + \{42^21^2\} + \{3^22^2\} + \{3^221^2\} + 2\{3^21^4\} + \{32^31\} + \{32^21^3\} + \{321^5\} + \{2^41^2\} + \{2^31^4\} + \{2^21^6\}.$

$\{1^2\}\otimes\{31^2\} = \{531^2\} + \{4^22\} + \{4321\} + \{431^3\} + \{42^3\} + \{42^21^2\} + \{421^4\} + \{3^31\} + 2\{3^221^2\} + \{32^31\} + 2\{32^21^3\} + \{321^5\} + \{31^7\} + \{2^5\} + \{2^31^4\}.$

$\{1^2\}\otimes\{2^21\} = \{52^21\} + \{43^2\} + \{4321\} + \{431^3\} + \{42^21^2\} + \{421^4\} + \{3^22^2\} + \{3^221^2\} + \{3^21^4\} + \{32^31\} + \{32^21^3\} + \{321^5\} + \{2^41^2\}.$

$\{1^2\}\otimes\{21^3\} = \{521^3\} + \{4321\} + \{42^3\} + \{42^21^2\} + \{421^4\} + \{41^6\} + \{3^31\} + \{3^221^2\} + \{32^31\} + \{32^21^3\}.$

$\{1^2\}\otimes\{1^5\} = \{51^5\} + \{42^21^2\} + \{3^22^2\}.$

Total degree = 12

$\{6\}\otimes\{2\} = \{12\} + \{10.2\} + \{84\} + \{6^2\}.$

$\{6\}\otimes\{1^2\} = \{11.1\} + \{93\} + \{75\}.$

$\{51\}\otimes\{2\} = \{10.2\} + \{921\} + \{91^3\} + \{84\} + \{831\} + \{82^2\} + \{741\} + \{731^2\} + \{6^2\} + \{651\} + \{642\} + \{5^21^2\}.$

$\{51\}\otimes\{1^2\} = \{10.1^2\} + \{93\} + \{921\} + \{831\} + \{821^2\} + \{75\} + \{741\} + \{732\} + \{651\} + \{641^2\} + \{5^22\}.$

$\{42\}\otimes\{2\} = \{84\} + \{82^2\} + \{741\} + \{732\} + \{731^2\} + \{6^2\} + \{651\} + 2\{642\} + \{6321\} + \{62^3\} + \{5^21^2\} + \{543\} + \{5421\} + \{53^21\} + \{4^3\} + \{4^22^2\}.$

$\{42\}\otimes\{1^2\} = \{831\} + \{75\} + \{741\} + \{732\} + \{72^21\} + \{651\} + \{642\} + \{641^2\} + \{63^2\} + \{6321\} + \{5^22\} + \{543\} + \{5421\} + \{532^2\} + \{4^231\}.$

$\{41^2\}\otimes\{2\} = \{82^2\} + \{81^4\} + \{731^2\} + \{72^21\} + \{721^3\} + \{642\} + \{6321\} + \{631^3\} + \{62^3\} + \{621^4\} + \{5^21^2\} + \{5421\} + \{541^3\} + \{5321^2\} + \{4^22^2\} + \{4^21^4\}.$

$\{41^2\}\otimes\{1^2\} = \{821^2\} + \{732\} + \{72^21\} + \{721^3\} + \{71^5\} + \{641^2\} + \{6321\} + \{631^3\} + \{62^21^2\} + \{5^22\} + \{5421\} + \{541^3\} + \{532^2\} + \{531^4\} + \{4^221^2\}.$

$\{4\}\otimes\{3\} = \{12\} + \{10.2\} + \{93\} + \{84\} + \{82^2\} + \{741\} + \{6^2\} + \{642\} + \{4^3\}.$

$\{4\}\otimes\{21\} = \{11.1\} + \{10.2\} + \{93\} + \{921\} + 2\{84\} + \{831\} + \{75\} + \{741\} + \{732\} + \{651\} + \{642\} + \{543\}.$

$\{4\}\otimes\{1^3\} = \{10.1^2\} + \{93\} + \{831\} + \{75\} + \{741\} + \{63^2\} + \{5^22\}.$

$\{3^2\}\otimes\{2\} = \{6^2\} + \{642\} + \{5^21^2\} + \{53^21\} + \{4^22^2\} + \{3^4\}.$

$\{3^2\}\otimes\{1^2\} = \{651\} + \{63^2\} + \{5421\} + \{43^22\}.$

$\{321\}\otimes\{2\} = \{642\} + \{6321\} + \{631^3\} + \{62^3\} + \{5^21^2\} + \{543\} + 2\{5421\} + \{541^3\} + 2\{53^21\} + \{532^2\} + 2\{5321^2\} + \{52^31\} + \{52^21^3\} + \{4^3\} + \{4^231\} + 2\{4^22^2\} + \{4^221^2\} + \{4^21^4\} + \{43^22\} + 2\{43^21^2\} + 2\{432^21\} + \{4321^3\} + \{42^4\} + \{3^4\} + \{3^321\} + \{3^22^21^2\}.$

$\{321\}\otimes\{1^2\} = \{641^2\} + \{63^2\} + \{6321\} + \{62^21^2\} + \{5^22\} + \{543\} + 2\{5421\} + \{541^3\} + \{53^21\} + 2\{532^2\} + 2\{5321^2\} + \{531^4\} + \{52^31\} + 2\{4^231\} + 2\{4^221^2\} + 2\{43^22\} + \{43^21^2\} + 2\{432^21\} + \{4321^3\} + \{42^31^2\} + \{3^321\} + \{3^31^3\} + \{3^22^3\}.$

$\{31^3\}\otimes\{2\} = \{62^3\} + \{621^4\} + \{5321^2\} + \{52^31\} + \{52^21^3\} + \{521^5\} + \{51^7\} + \{4^22^2\} + \{4^21^4\} + \{432^21\} + \{4321^3\} + \{431^5\} + \{42^4\} + \{42^21^4\} + \{3^22^21^2\} + \{3^21^6\}.$

$\{31^3\}\otimes\{1^2\} = \{62^21^2\} + \{61^6\} + \{532^2\} + \{531^4\} + \{52^31\} + \{52^21^3\} + \{521^5\} + \{4^221^2\}$

$+ \{432^21\} + \{4321^3\} + \{431^5\} + \{42^31^2\} + \{421^6\} + \{3^22^3\} + \{3^221^4\}.$

$\{31\}\otimes\{3\} = \{93\} + \{831\} + \{82^2\} + \{821^2\} + \{75\} + 2\{741\} + 2\{732\} + \{731^2\} + 2\{72^21\} + \{721^3\} + \{71^5\} + \{651\} + 2\{642\} + 3\{641^2\} + 2\{63^2\} + 3\{6321\} + 2\{631^3\} + \{62^3\} + \{62^21^2\} + 2\{5^22\} + \{543\} + 4\{5421\} + \{541^3\} + \{53^21\} + 2\{532^2\} + 2\{5321^2\} + \{531^4\} + \{4^3\} + 2\{4^231\} + \{4^221^2\} + \{43^22\} + \{43^21^2\} + \{432^21\} + \{3^31^3\}.$

$\{31\}\otimes\{21\} = \{921\} + \{84\} + 2\{831\} + \{82^2\} + 2\{821^2\} + \{81^4\} + \{75\} + 3\{741\} + 4\{732\} + 4\{731^2\} + 3\{72^21\} + 2\{721^3\} + 3\{651\} + 5\{642\} + 5\{641^2\} + 2\{63^2\} + 7\{6321\} + 3\{631^3\} + \{62^3\} + 2\{62^21^2\} + \{621^4\} + 2\{5^22\} + 2\{5^21^2\} + 4\{543\} + 6\{5421\} + 3\{541^3\} + 4\{53^21\} + 3\{532^2\} + 4\{5321^2\} + \{531^4\} + \{52^31\} + 3\{4^231\} + 2\{4^22^2\} + 2\{4^221^2\} + 2\{43^22\} + 2\{43^21^2\} + \{432^21\} + \{4321^3\} + \{3^321\}.$

$\{31\}\otimes\{1^3\} = \{91^3\} + \{831\} + \{82^2\} + \{821^2\} + 2\{741\} + \{732\} + 3\{731^2\} + \{72^21\} + \{721^3\} + \{6^2\} + \{651\} + 3\{642\} + 2\{641^2\} + 2\{63^2\} + 3\{6321\} + 2\{631^3\} + \{62^3\} + \{62^21^2\} + \{5^22\} + 2\{5^21^2\} + \{543\} + 3\{5421\} + \{541^3\} + 3\{53^21\} + \{532^2\} + 2\{5321^2\} + \{52^21^3\} + \{4^3\} + \{4^231\} + \{4^22^2\} + \{4^221^2\} + \{4^21^4\} + \{43^22\} + \{43^21^2\} + \{432^21\} + \{3^4\}.$

$\{3\}\otimes\{4\} = \{12\} + \{10.2\} + \{93\} + \{84\} + \{82^2\} + \{741\} + \{732\} + \{6^2\} + \{642\} + \{62^3\} + \{5421\} + \{4^3\}.$

$\{3\}\otimes\{31\} = \{11.1\} + \{10.2\} + 2\{93\} + \{921\} + \{84\} + 2\{831\} + \{82^2\} + 2\{75\} + 2\{741\} + 2\{732\} + \{72^21\} + \{651\} + 2\{642\} + \{641^2\} + \{63^2\} + \{6321\} + \{5^22\} + \{543\} + \{5421\} + \{532^2\} + \{4^231\}.$

$\{3\}\otimes\{2^2\} = \{10.2\} + \{921\} + 2\{84\} + \{831\} + \{82^2\} + \{741\} + \{732\} + \{731^2\} + \{6^2\} + \{651\} + 2\{642\} + \{6321\} + \{5^21^2\} + \{543\} + \{53^21\} + \{4^22^2\}.$

$\{3\}\otimes\{21^2\} = \{10.1^2\} + \{93\} + \{921\} + \{84\} + 2\{831\} + \{821^2\} + \{75\} + 2\{741\} + 2\{732\} + \{731^2\} + 2\{651\} + \{642\} + \{641^2\} + 2\{63^2\} + \{6321\} + \{5^22\} + \{543\} + \{5421\} + \{53^21\} + \{43^22\}.$

$\{3\}\otimes\{1^4\} = \{91^3\} + \{831\} + \{741\} + \{731^2\} + \{6^2\} + \{642\} + \{63^2\} + \{5^21^2\} + \{53^21\} + \{3^4\}.$

$\{2^3\}\otimes\{2\} = \{4^3\} + \{4^22^2\} + \{43^21^2\} + \{42^4\} + \{3^22^21^2\} + \{2^6\}.$

$\{2^3\}\otimes\{1^2\} = \{4^231\} + \{432^21\} + \{3^31^3\} + \{32^41\}.$

$\{2^21^2\}\otimes\{2\} = \{4^22^2\} + \{4^21^4\} + \{43^21^2\} + \{432^21\} + \{4321^3\} + \{42^4\} + \{42^21^4\} + \{3^4\} + \{3^321\} + 2\{3^22^21^2\} + \{3^221^4\} + \{3^21^6\} + \{32^41\} + \{32^31^3\} + \{2^6\} + \{2^41^4\}.$

$\{2^21^2\}\otimes\{1^2\} = \{4^221^2\} + \{43^22\} + \{432^21\} + \{4321^3\} + \{431^5\} + \{42^31^2\} + \{3^321\} + \{3^31^3\} + \{3^22^3\} + \{3^22^21^2\} + \{3^221^4\} + \{32^41\} + \{32^31^3\} + \{32^21^5\} + \{2^51^2\}.$

$$\begin{aligned}\{2^2\}\otimes\{3\} &= \{6^2\} + \{642\} + \{62^3\} + \{5^21^2\} + \{5421\} + \{53^21\} + \{5321^2\} + \{4^3\} + 2\{4^22^2\} \\ &\quad + \{4^21^4\} + \{43^21^2\} + \{432^21\} + \{42^4\} + \{3^4\} + \{3^22^21^2\} + \{2^6\}.\end{aligned}$$

$$\begin{aligned}\{2^2\}\otimes\{21\} &= \{651\} + \{642\} + \{6321\} + \{5^21^2\} + \{543\} + 2\{5421\} + \{541^3\} + \{53^21\} \\ &\quad + \{532^2\} + \{5321^2\} + \{52^31\} + \{4^231\} + 2\{4^22^2\} + \{4^221^2\} + \{43^22\} + \{43^21^2\} \\ &\quad + 2\{432^21\} + \{4321^3\} + \{42^4\} + \{3^321\} + \{3^22^21^2\} + \{32^41\}.\end{aligned}$$

$$\begin{aligned}\{2^2\}\otimes\{1^3\} &= \{641^2\} + \{63^2\} + \{5^22\} + \{5421\} + \{532^2\} + \{5321^2\} + \{4^231\} + \{4^221^2\} \\ &\quad + \{43^22\} + \{432^21\} + \{42^31^2\} + \{3^31^3\} + \{3^22^3\}.\end{aligned}$$

$$\begin{aligned}\{21^4\}\otimes\{2\} &= \{42^4\} + \{42^21^4\} + \{41^8\} + \{3^22^21^2\} + \{3^21^6\} + \{32^41\} + \{32^31^3\} + \{32^21^5\} \\ &\quad + \{321^7\} + \{2^6\} + \{2^41^4\} + \{2^21^8\}.\end{aligned}$$

$$\begin{aligned}\{21^4\}\otimes\{1^2\} &= \{42^31^2\} + \{421^6\} + \{3^22^3\} + \{3^221^4\} + \{32^41\} + \{32^31^3\} + \{32^21^5\} + \{321^7\} \\ &\quad + \{31^9\} + \{2^51^2\} + \{2^31^6\}.\end{aligned}$$

$$\begin{aligned}\{21^2\}\otimes\{3\} &= \{63^2\} + \{62^21^2\} + \{61^6\} + \{5421\} + \{53^21\} + \{532^2\} + 2\{5321^2\} + \{531^4\} \\ &\quad + \{52^31\} + 2\{52^21^3\} + \{521^5\} + \{4^231\} + 2\{4^221^2\} + \{4^21^4\} + 2\{43^22\} \\ &\quad + \{43^21^2\} + 4\{432^21\} + 3\{4321^3\} + 2\{431^5\} + 3\{42^31^2\} + \{42^21^4\} + \{421^6\} \\ &\quad + \{3^4\} + \{3^321\} + 2\{3^31^3\} + 2\{3^22^3\} + 2\{3^22^21^2\} + 2\{3^221^4\} + \{3^21^6\} \\ &\quad + \{32^41\} + 2\{32^31^3\} + \{32^21^5\} + \{2^51^2\} + \{2^31^6\}.\end{aligned}$$

$$\begin{aligned}\{21^2\}\otimes\{21\} &= \{6321\} + \{62^21^2\} + \{621^4\} + \{543\} + \{5421\} + \{541^3\} + 2\{53^21\} + 2\{532^2\} \\ &\quad + 4\{5321^2\} + 2\{531^4\} + 3\{52^31\} + 3\{52^21^3\} + 2\{521^5\} + \{51^7\} + 2\{4^231\} \\ &\quad + 2\{4^22^2\} + 3\{4^221^2\} + \{4^21^4\} + 3\{43^22\} + 4\{43^21^2\} + 6\{432^21\} + 7\{4321^3\} \\ &\quad + 3\{431^5\} + 2\{42^4\} + 5\{42^31^2\} + 4\{42^21^4\} + 2\{421^6\} + 4\{3^321\} + 2\{3^31^3\} \\ &\quad + 2\{3^22^3\} + 5\{3^22^21^2\} + 4\{3^221^4\} + \{3^21^6\} + 3\{32^41\} + 3\{32^31^3\} + 2\{32^21^5\} \\ &\quad + \{321^7\} + \{2^51^2\} + \{2^41^4\}.\end{aligned}$$

$$\begin{aligned}\{21^2\}\otimes\{1^3\} &= \{631^3\} + \{62^3\} + \{5421\} + \{53^21\} + \{532^2\} + 2\{5321^2\} + \{531^4\} + \{52^31\} \\ &\quad + 2\{52^21^3\} + \{521^5\} + \{4^3\} + \{4^231\} + \{4^22^2\} + \{4^221^2\} + \{4^21^4\} + \{43^22\} \\ &\quad + 3\{43^21^2\} + 3\{432^21\} + 3\{4321^3\} + \{431^5\} + 2\{42^4\} + 2\{42^31^2\} + 3\{42^21^4\} \\ &\quad + \{421^6\} + \{41^8\} + \{3^4\} + \{3^321\} + 2\{3^31^3\} + \{3^22^3\} + 3\{3^22^21^2\} + \{3^221^4\} \\ &\quad + \{3^21^6\} + \{32^41\} + 2\{32^31^3\} + \{32^21^5\} + \{2^6\}.\end{aligned}$$

$$\begin{aligned}\{21\}\otimes\{4\} &= \{84\} + \{741\} + \{732\} + \{731^2\} + \{651\} + 2\{642\} + \{641^2\} + 3\{6321\} \\ &\quad + \{631^3\} + 2\{62^3\} + \{62^21^2\} + \{621^4\} + \{5^21^2\} + 2\{543\} + 3\{5421\} \\ &\quad + 2\{541^3\} + 3\{53^21\} + 2\{532^2\} + 4\{5321^2\} + \{531^4\} + 3\{52^31\} + 2\{52^21^3\} \\ &\quad + \{521^5\} + \{51^7\} + \{4^3\} + \{4^231\} + 3\{4^22^2\} + 2\{4^221^2\} + \{4^21^4\} + 2\{43^22\} \\ &\quad + 3\{43^21^2\} + 3\{432^21\} + 3\{4321^3\} + \{431^5\} + 2\{42^4\} + \{42^31^2\} + \{42^21^4\} \\ &\quad + 2\{3^321\} + 2\{3^22^21^2\}.\end{aligned}$$

$$\begin{aligned}\{21\}\otimes\{31\} &= \{831\} + \{75\} + 2\{741\} + 2\{732\} + 2\{731^2\} + 2\{72^21\} + \{721^3\} + 2\{651\} \\ &\quad + 4\{642\} + 5\{641^2\} + 3\{63^2\} + 8\{6321\} + 4\{631^3\} + 2\{62^3\} + 5\{62^21^2\} \\ &\quad + 2\{621^4\} + \{61^6\} + 3\{5^22\} + 2\{5^21^2\} + 4\{543\} + 10\{5421\} + 5\{541^3\}\end{aligned}$$

$+ 7\{53^21\} + 9\{532^2\} + 12\{5321^2\} + 5\{531^4\} + 6\{52^31\} + 5\{52^21^3\}$
$+ 2\{521^5\} + 7\{4^231\} + 4\{4^22^2\} + 8\{4^221^2\} + 2\{4^21^4\} + 6\{43^22\} + 7\{43^21^2\}$
$+ 11\{432^21\} + 8\{4321^3\} + 2\{431^5\} + 2\{42^4\} + 5\{42^31^2\} + 2\{42^21^4\} + \{421^6\}$
$+ \{3^4\} + 4\{3^321\} + 3\{3^31^3\} + 3\{3^22^3\} + 3\{3^22^21^2\} + 2\{3^221^4\} + 2\{32^41\}$
$+ \{32^31^3\}.$

$\{21\}\otimes\{2^2\} = \{82^2\} + \{741\} + \{732\} + 2\{731^2\} + \{72^21\} + \{721^3\} + \{6^2\} + \{651\}$
$+ 4\{642\} + 2\{641^2\} + \{63^2\} + 5\{6321\} + 4\{631^3\} + 3\{62^3\} + 2\{62^21^2\}$
$+ 2\{621^4\} + \{5^22\} + 3\{5^21^2\} + 2\{543\} + 7\{5421\} + 3\{541^3\} + 6\{53^21\}$
$+ 4\{532^2\} + 9\{5321^2\} + 2\{531^4\} + 3\{52^31\} + 4\{52^21^3\} + \{521^5\} + 2\{4^3\}$
$+ 3\{4^231\} + 6\{4^22^2\} + 4\{4^221^2\} + 3\{4^21^4\} + 3\{43^22\} + 6\{43^21^2\} + 7\{432^21\}$
$+ 5\{4321^3\} + \{431^5\} + 3\{42^4\} + 2\{42^31^2\} + 2\{42^21^4\} + 2\{3^4\} + 2\{3^321\}$
$+ \{3^31^3\} + \{3^22^3\} + 4\{3^22^21^2\} + \{3^221^4\} + \{3^21^6\} + \{32^41\} + \{32^31^3\} + \{2^6\}.$

$\{21\}\otimes\{21^2\} = \{821^2\} + \{741\} + 2\{732\} + 2\{731^2\} + 2\{72^21\} + 2\{721^3\} + \{71^5\}$
$+ 2\{651\} + 3\{642\} + 5\{641^2\} + 3\{63^2\} + 8\{6321\} + 5\{631^3\} + 2\{62^3\}$
$+ 5\{62^21^2\} + 2\{621^4\} + 3\{5^22\} + 2\{5^21^2\} + 4\{543\} + 11\{5421\} + 6\{541^3\}$
$+ 7\{53^21\} + 8\{532^2\} + 12\{5321^2\} + 5\{531^4\} + 5\{52^31\} + 4\{52^21^3\} + \{521^5\}$
$+ \{4^3\} + 6\{4^231\} + 4\{4^22^2\} + 9\{4^221^2\} + 2\{4^21^4\} + 7\{43^22\} + 7\{43^21^2\}$
$+ 10\{432^21\} + 8\{4321^3\} + 2\{431^5\} + 2\{42^4\} + 5\{42^31^2\} + 2\{42^21^4\}$
$+ 4\{3^321\} + 3\{3^31^3\} + 3\{3^22^3\} + 4\{3^22^21^2\} + 2\{3^221^4\} + 2\{32^41\}$
$+ 2\{32^31^3\} + \{32^21^5\} + \{2^51^2\}.$

$\{21\}\otimes\{1^4\} = \{81^4\} + \{731^2\} + \{72^21\} + \{721^3\} + 2\{642\} + \{641^2\} + 3\{6321\} + 2\{631^3\}$
$+ \{62^3\} + \{62^21^2\} + \{621^4\} + 2\{5^21^2\} + 2\{543\} + 3\{5421\} + 3\{541^3\}$
$+ 3\{53^21\} + 2\{532^2\} + 4\{5321^2\} + \{531^4\} + 2\{52^31\} + \{52^21^3\} + 2\{4^231\}$
$+ 3\{4^22^2\} + 2\{4^221^2\} + 2\{4^21^4\} + \{43^22\} + 3\{43^21^2\} + 3\{432^21\} + 3\{4321^3\}$
$+ \{42^4\} + \{42^31^2\} + \{42^21^4\} + \{3^4\} + 2\{3^321\} + 2\{3^22^21^2\} + \{3^221^4\}$
$+ \{32^41\} + \{32^31^3\} + \{2^41^4\}.$

$\{2\}\otimes\{6\} = \{12\} + \{10.2\} + \{84\} + \{82^2\} + \{6^2\} + \{642\} + \{62^3\} + \{4^3\} + \{4^22^2\} + \{42^4\}$
$+ \{2^6\}.$

$\{2\}\otimes\{51\} = \{11.1\} + \{10.2\} + \{93\} + \{921\} + \{84\} + \{831\} + \{82^2\} + \{75\} + \{741\}$
$+ \{732\} + \{72^21\} + \{651\} + 2\{642\} + \{6321\} + \{62^3\} + \{543\} + \{5421\}$
$+ \{532^2\} + \{52^31\} + \{4^231\} + \{4^22^2\} + \{432^21\} + \{42^4\} + \{32^41\}.$

$\{2\}\otimes\{42\} = \{10.2\} + \{93\} + \{921\} + 2\{84\} + \{831\} + 2\{82^2\} + 2\{741\} + 2\{732\}$
$+ \{731^2\} + \{72^21\} + \{6^2\} + \{651\} + 3\{642\} + 2\{6321\} + 2\{62^3\} + \{5^21^2\}$
$+ \{543\} + 2\{5421\} + \{53^21\} + \{532^2\} + \{5321^2\} + \{52^31\} + \{4^3\} + 2\{4^22^2\}$
$+ \{43^21^2\} + \{432^21\} + \{42^4\} + \{3^22^21^2\}.$

$\{2\}\otimes\{41^2\} = \{10.1^2\} + \{93\} + \{921\} + 2\{831\} + \{821^2\} + \{75\} + 2\{741\} + 2\{732\}$
$+ \{731^2\} + \{72^21\} + \{651\} + \{642\} + 2\{641^2\} + 2\{63^2\} + 2\{6321\}$

$$\begin{aligned}&+ \{62^21^2\} + 2\{5^22\} + \{543\} + 2\{5421\} + \{53^21\} + 2\{532^2\} + \{5321^2\}\\&+ \{52^31\} + \{4^231\} + \{4^221^2\} + \{43^22\} + \{432^21\} + \{42^31^2\} + \{3^22^3\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{3^2\} &= \{93\} + \{831\} + \{82^2\} + \{75\} + \{741\} + \{732\} + \{72^21\} + \{642\} + \{641^2\}\\&+ \{63^2\} + \{6321\} + \{62^3\} + \{5^22\} + \{5421\} + \{532^2\} + \{5321^2\} + \{4^231\}\\&+ \{432^21\} + \{3^31^3\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{321\} &= \{921\} + \{84\} + 2\{831\} + \{82^2\} + \{821^2\} + \{75\} + 2\{741\} + 3\{732\}\\&+ 2\{731^2\} + 2\{72^21\} + 2\{651\} + 3\{642\} + 2\{641^2\} + \{63^2\} + 4\{6321\}\\&+ \{631^3\} + \{62^3\} + \{62^21^2\} + \{5^22\} + \{5^21^2\} + 2\{543\} + 3\{5421\} + \{541^3\}\\&+ 2\{53^21\} + 2\{532^2\} + 2\{5321^2\} + \{52^31\} + \{4^231\} + \{4^22^2\} + \{4^221^2\}\\&+ \{43^22\} + \{43^21^2\} + \{432^21\} + \{4321^3\} + \{3^321\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{31^3\} &= \{91^3\} + \{831\} + \{821^2\} + \{741\} + \{732\} + 2\{731^2\} + \{721^3\} + \{651\}\\&+ \{642\} + 2\{641^2\} + 2\{63^2\} + 2\{6321\} + \{631^3\} + \{62^21^2\} + \{5^22\} + \{5^21^2\}\\&+ \{543\} + 2\{5421\} + \{541^3\} + 2\{53^21\} + \{532^2\} + \{5321^2\} + \{52^21^3\}\\&+ \{4^231\} + \{4^221^2\} + \{43^22\} + \{432^21\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{2^3\} &= \{82^2\} + \{741\} + \{731^2\} + \{72^21\} + \{6^2\} + 2\{642\} + \{6321\} + \{631^3\} + \{62^3\}\\&+ \{5^21^2\} + \{5421\} + \{53^21\} + \{5321^2\} + \{4^3\} + \{4^22^2\} + \{4^21^4\} + \{43^21^2\} + \{3^4\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{2^21^2\} &= \{821^2\} + \{741\} + \{732\} + \{731^2\} + \{72^21\} + \{721^3\} + \{651\} + \{642\}\\&+ 2\{641^2\} + \{63^2\} + 2\{6321\} + \{631^3\} + \{62^21^2\} + \{5^22\} + \{543\}\\&+ 2\{5421\} + \{541^3\} + \{53^21\} + \{532^2\} + \{5321^2\} + \{531^4\} + \{4^231\}\\&+ \{4^221^2\} + \{43^22\} + \{43^21^2\}.\end{aligned}$$

$$\begin{aligned}\{2\}\otimes\{21^4\} &= \{81^4\} + \{731^2\} + \{721^3\} + \{642\} + \{641^2\} + \{6321\} + \{631^3\} + \{621^4\}\\&+ \{5^21^2\} + \{543\} + \{5421\} + \{541^3\} + \{53^21\} + \{5321^2\} + \{4^231\} + \{4^22^2\}.\end{aligned}$$

$$\{2\}\otimes\{1^6\} = \{71^5\} + \{631^3\} + \{5421\} + \{4^3\}.$$

$$\{1^6\}\otimes\{2\} = \{2^6\} + \{2^41^4\} + \{2^21^8\} + \{1^{12}\}.$$

$$\{1^6\}\otimes\{1^2\} = \{2^51^2\} + \{2^31^6\} + \{21^{10}\}.$$

$$\{1^4\}\otimes\{3\} = \{3^4\} + \{3^22^21^2\} + \{3^21^6\} + \{32^31^3\} + \{2^6\} + \{2^41^4\} + \{2^31^6\} + \{2^21^8\} + \{1^{12}\}.$$

$$\begin{aligned}\{1^4\}\otimes\{21\} &= \{3^321\} + \{3^22^21^2\} + \{3^221^4\} + \{32^41\} + \{32^31^3\} + \{32^21^5\} + \{321^7\} + \{2^51^2\}\\&+ 2\{2^41^4\} + \{2^31^6\} + \{2^21^8\} + \{21^{10}\}.\end{aligned}$$

$$\{1^4\}\otimes\{1^3\} = \{3^31^3\} + \{3^22^3\} + \{32^31^3\} + \{32^21^5\} + \{31^9\} + \{2^51^2\} + \{2^31^6\}.$$

$$\begin{aligned}\{1^3\}\otimes\{4\} &= \{4^3\} + \{43^21^2\} + \{42^4\} + \{42^21^4\} + \{41^8\} + \{3^31^3\} + \{3^22^21^2\} + \{32^31^3\}\\&+ \{32^21^5\} + \{2^6\}.\end{aligned}$$

$$\begin{aligned}\{1^3\}\otimes\{31\} &= \{4^231\} + \{43^21^2\} + \{432^21\} + \{4321^3\} + \{42^31^2\} + \{42^21^4\} + \{421^6\} + \{3^321\}\\&+ 2\{3^31^3\} + \{3^22^3\} + \{3^22^21^2\} + 2\{3^221^4\} + 2\{32^41\} + 2\{32^31^3\} + 2\{32^21^5\}\\&+ \{321^7\} + \{31^9\} + \{2^51^2\} + \{2^41^4\} + \{2^31^6\}.\end{aligned}$$

$\{1^3\}\otimes\{2^2\} = \{4^22^2\} + \{43^21^2\} + \{4321^3\} + \{42^4\} + \{42^21^4\} + \{3^321\} + 2\{3^22^21^2\} + \{3^221^4\}$
$+ \{3^21^6\} + \{32^41\} + \{32^31^3\} + \{32^21^5\} + \{321^7\} + \{2^6\} + 2\{2^41^4\} + \{2^21^8\}.$

$\{1^3\}\otimes\{21^2\} = \{4^221^2\} + \{43^22\} + \{432^21\} + \{4321^3\} + \{431^5\} + \{42^31^2\} + \{3^321\} + \{3^31^3\}$
$+ \{3^22^3\} + 2\{3^22^21^2\} + 2\{3^221^4\} + \{3^21^6\} + \{32^41\} + 2\{32^31^3\} + 2\{32^21^5\}$
$+ \{321^7\} + 2\{2^51^2\} + \{2^41^4\} + 2\{2^31^6\} + \{2^21^8\} + \{21^{10}\}.$

$\{1^3\}\otimes\{1^4\} = \{4^21^4\} + \{432^21\} + \{3^4\} + \{3^22^21^2\} + \{3^221^4\} + \{3^21^6\} + \{32^31^3\} + \{2^6\}$
$+ \{2^41^4\} + \{2^31^6\} + \{2^21^8\} + \{1^{12}\}.$

$\{1^2\}\otimes\{6\} = \{6^2\} + \{5^21^2\} + \{4^22^2\} + \{4^21^4\} + \{3^4\} + \{3^22^21^2\} + \{3^21^6\} + \{2^6\} + \{2^41^4\}$
$+ \{2^21^8\} + \{1^{12}\}.$

$\{1^2\}\otimes\{51\} = \{651\} + \{5^21^2\} + \{5421\} + \{541^3\} + \{4^22^2\} + \{4^221^2\} + \{4^21^4\} + \{43^22\}$
$+ \{432^21\} + \{4321^3\} + \{431^5\} + \{3^321\} + 2\{3^22^21^2\} + \{3^221^4\} + \{3^21^6\}$
$+ \{32^41\} + \{32^31^3\} + \{32^21^5\} + \{321^7\} + \{2^51^2\} + \{2^41^4\} + \{2^31^6\} + \{2^21^8\}$
$+ \{21^{10}\}.$

$\{1^2\}\otimes\{42\} = \{642\} + \{5^21^2\} + \{5421\} + \{541^3\} + \{53^21\} + \{5321^2\} + 2\{4^22^2\} + \{4^221^2\}$
$+ 2\{4^21^4\} + \{43^21^2\} + 2\{432^21\} + 2\{4321^3\} + \{431^5\} + \{42^4\} + \{42^21^4\}$
$+ \{3^4\} + \{3^321\} + 3\{3^22^21^2\} + 2\{3^221^4\} + 2\{3^21^6\} + \{32^41\} + 2\{32^31^3\}$
$+ \{32^21^5\} + \{321^7\} + \{2^6\} + 2\{2^41^4\} + \{2^31^6\} + \{2^21^8\}.$

$\{1^2\}\otimes\{41^2\} = \{641^2\} + \{5^22\} + \{5421\} + \{541^3\} + \{532^2\} + \{5321^2\} + \{531^4\} + \{4^231\}$
$+ 2\{4^221^2\} + \{43^22\} + \{43^21^2\} + 2\{432^21\} + 2\{4321^3\} + \{431^5\} + 2\{42^31^2\}$
$+ \{42^21^4\} + \{421^6\} + \{3^321\} + 2\{3^31^3\} + 2\{3^22^3\} + \{3^22^21^2\} + 2\{3^221^4\}$
$+ \{32^41\} + 2\{32^31^3\} + 2\{32^21^5\} + \{321^7\} + \{31^9\} + \{2^51^2\} + \{2^31^6\}.$

$\{1^2\}\otimes\{3^2\} = \{63^2\} + \{5421\} + \{5321^2\} + \{4^221^2\} + \{4^21^4\} + \{43^22\} + \{432^21\} + \{4321^3\}$
$+ \{431^5\} + \{42^31^2\} + \{3^31^3\} + \{3^22^3\} + \{3^22^21^2\} + \{3^221^4\} + \{3^21^6\} + \{32^31^3\}$
$+ \{32^21^5\} + \{2^51^2\} + \{2^31^6\}.$

$\{1^2\}\otimes\{321\} = \{6321\} + \{543\} + \{5421\} + \{541^3\} + \{53^21\} + \{532^2\} + 2\{5321^2\} + \{531^4\}$
$+ \{52^31\} + \{52^21^3\} + \{4^231\} + \{4^22^2\} + 2\{4^221^2\} + \{4^21^4\} + \{43^22\}$
$+ 2\{43^21^2\} + 3\{432^21\} + 4\{4321^3\} + 2\{431^5\} + \{42^4\} + 2\{42^31^2\} + 2\{42^21^4\}$
$+ \{421^6\} + 2\{3^321\} + \{3^31^3\} + \{3^22^3\} + 3\{3^22^21^2\} + 3\{3^221^4\} + \{3^21^6\} + 2\{32^41\}$
$+ 2\{32^31^3\} + 2\{32^21^5\} + \{321^7\} + \{2^51^2\} + \{2^41^4\}.$

$\{1^2\}\otimes\{31^3\} = \{631^3\} + \{5421\} + \{532^2\} + \{5321^2\} + \{531^4\} + \{52^31\} + \{52^21^3\} + \{521^5\}$
$+ \{4^231\} + \{4^221^2\} + \{43^22\} + 2\{43^21^2\} + 2\{432^21\} + 2\{4321^3\} + \{42^4\}$
$+ 2\{42^31^2\} + 2\{42^21^4\} + \{421^6\} + \{41^8\} + \{3^321\} + 2\{3^31^3\} + \{3^22^3\}$
$+ \{3^22^21^2\} + \{3^221^4\} + \{32^41\} + \{32^31^3\} + \{32^21^5\}.$

$\{1^2\}\otimes\{2^3\} = \{62^3\} + \{53^21\} + \{5321^2\} + \{52^21^3\} + \{4^3\} + \{4^22^2\} + \{4^21^4\} + \{43^21^2\}$
$+ \{432^21\} + \{4321^3\} + \{431^5\} + \{42^4\} + \{42^21^4\} + \{3^4\} + 2\{3^22^21^2\} + \{3^21^6\}$
$+ \{32^31^3\} + \{2^6\}.$

$$\begin{aligned}\{1^2\}\otimes\{2^21^2\} = {} & \{62^21^2\} + \{53^21\} + \{532^2\} + \{5321^2\} + \{531^4\} + \{52^31\} + \{52^21^3\} + \{521^5\} \\ & + \{4^231\} + \{4^221^2\} + \{43^22\} + \{43^21^2\} + 2\{432^21\} + 2\{4321^3\} + \{431^5\} \\ & + 2\{42^31^2\} + \{42^21^4\} + \{421^6\} + \{3^321\} + \{3^31^3\} + \{3^22^3\} + \{3^22^21^2\} + \{3^221^4\} \\ & + \{32^41\} + \{32^31^3\}.\end{aligned}$$

$$\begin{aligned}\{1^2\}\otimes\{21^4\} = {} & \{621^4\} + \{5321^2\} + \{52^31\} + \{52^21^3\} + \{521^5\} + \{51^7\} + \{4^22^2\} + \{43^22\} + \{43^21^2\} \\ & + \{432^21\} + \{4321^3\} + \{42^4\} + \{42^31^2\} + \{42^21^4\} + \{3^321\} + \{3^22^21^2\}.\end{aligned}$$

$$\{1^2\}\otimes\{1^6\} = \{61^6\} + \{52^21^3\} + \{432^21\} + \{3^4\}.$$

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