Laminations of a graph on a pair of pants

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Abstract

A lamination of a graph embedded on a surface is a collection of pairwise disjoint non-contractible simple closed curves drawn on the graph. In the case when the surface is a sphere with three punctures (a.k.a. a pair of pants), we first identify the lamination space of a graph embedded on that surface as a lattice polytope, then we characterize the polytopes that arise as the lamination space of some graph on a pair of pants. This characterizes the image of a purely topological version of the spectral map for the vector bundle Laplacian for a flat connection on a pair of pants. The proof uses a graph exploration technique akin to the peeling of planar maps.

1 Introduction

The combinatorial study of the determinant of the vector bundle Laplacian on graphs was initiated by Forman [5] followed by Kenyon [7] as a generalization of the classical matrix-tree theorem [12]. While the (reduced) determinant of the usual Laplacian operator on a graph enumerates spanning trees of this graph, the determinant of the vector bundle Laplacian enumerates cycle-rooted spanning forests (CRSFs), which are spanning forests where each connected component is a unicycle (a connected graph with as many vertices as edges). The weight of a CRSF is the product over its cycles of a quantity related to the monodromy of the connection along each cycle.

Of particular interest is the case of a flat $SU(2,\mathbb{C})$ connection on a graph embedded on a surface [7], namely the case when the parallel transports are in $SU(2,\mathbb{C})$ and the monodromy of the connection along each cycle of the graph which is contractible on the surface has to be trivial. In that case, the only CRSFs which contribute to the determinant of the vector bundle Laplacian are those which have no contractible cycles. Such CRSFs are called incompressible CRSFs and the cycles of an incompressible CRSF form a *lamination* of the surface, i.e. a collection of pairwise disjoint non-contractible simple loops. The determinant of the vector bundle Laplacian in the flat connection case can be written as a polynomial in variables of the form $2 - Tr(w)$, where w is the monodromy along a non-contractible cycle on the surface [7]. Moreover these variables are free [3].

The most basic non-simply connected surfaces to consider are the annulus and the torus and was done in [7, 8, 6, 11, 9]. The next simplest case is probably the one of the pair of pants (aka three-holed sphere), briefly mentioned in [7]. It is one of the simplest surfaces for which the fundamental group is non abelian.

A non-contractible cycle on a pair of pants can be of three possible topological types, thus the determinant of the vector-bundle Laplacian associated with a flat $SU(2,\mathbb{C})$ connection on a graph embedded on that surface is a polynomial $P(X,Y,Z)$ of three independent variables.

The map which to a graph on a pair of pants associates the polynomial $P(X,Y,Z)$ is interesting to understand. We shall call it the *spectral map*, extending the terminology of the torus case [8] (this is a slight abuse of terminology, since the image of a graph under the spectral map should be the zero-locus of the polynomial together with a certain divisor on that algebraic variety [6]). Important questions include determining the image of the spectral map as well as the fiber of the spectral map above a given polynomial. This provides information about the probabilistic model associated with the uniform measure on incompressible CRSFs on the graph [7]. The polynomial $P(X,Y,Z)$ also plays an important role in relation with integrable systems, where it serves as the generating function of the integrals of motion [6]. The case of the annulus and the torus have been thoroughly investigated by Kenyon [7, 8]. For a different probabilistic model, the dimer model on bipartite graphs, the spectral map in the torus case is completely understood [10, 6, 4].

To any polynomial in n variables one can associate its Newton polytope, which is the convex hull in $\mathbb{Z}^n$ of the n-tuples of integers $(i_1,\ldots,i_n)$ such that the monomial $X_1^{i_1}\ldots X_n^{i_n}$ has a nonzero coefficient in the polynomial. We define the *topological spectral map*, which to a graph on a surface associates the Newton polytope of the polynomial produced by applying the spectral map. While the image under the spectral map depends on some weights that the edges of the graph may carry, the image under the topological spectral map only depends on the topological graph. The same questions can be asked about the topological spectral map: what is its image and what is the fiber above a given polytope ? These questions were answered in the case of the annulus and the torus [7, 8, 6]. In this article, we characterize the image of the topological spectral map for the pair of pants. The answer is much more involved than the annulus and torus cases. The next step would be to understand the fiber of this topological spectral map above a given polytope. Answering these questions for the spectral map itself in the pair of pants case seems to be much harder.

A monomial $X^iY^jZ^k$ appears in the determinant $P(X,Y,Z)$ of the vector bundle Laplacian of a graph G on a pair of pants if and only if G has a lamination of type (i,j,k), that is a lamination with i cycles around the first hole, j cycles around the second and k cycles around the third. Hence the image under the topological spectral map of G is the lamination space of G, i.e. the set of all (i,j,k) such that G admits a lamination of type (i,j,k). The polytopes that arise in the image of the topological spectral map are exactly those that correspond to the lamination space of some graph on a pair of pants. The remainder of the paper will be formulated only in terms of laminations, no longer in terms of the determinant of the vector-bundle Laplacian, but the reader should keep in mind that the motivation behind this work comes from the spectral map associated with the vector-bundle Laplacian.

Organization of the paper

We introduce the relevant definitions and state our main results in Section 2. In Section 3 we describe an exploration process of a graph on a pair of pants and use it to realize the lamination space of that graph as a polytope. In passing we define three collections of special loops and study their properties. In Section 4 we derive some necessary conditions for the polytopes arising as the lamination space of some graph. We show in Section 5 that these conditions are sufficient by constructing a class of graphs having as a lamination space a given polytope satisfying the aforementioned conditions.

2 Main results

We consider the three-holed sphere Σ obtained by removing from the sphere $\mathbb{S}^2$ three distinct points P_1, P_2 and P_3. Every simple closed curve C on $\mathbb{S}^2$ which does not pass through the points P_i separates $\mathbb{S}^2$ into two hemispheres. For every $1 \leq i \leq 3$, we denote by $H_i(C)$ (resp. $H'_i(C)$) the connected component of $\mathbb{S}^2 \setminus C$ which contains P_i (resp. which does not contain P_i). A simple closed curve C is called *of type* i for $1 \leq i \leq 3$ if one of the hemispheres defined by C contains P_i and the other hemisphere contains the other two points, i.e. if

$$H_i(C) = H'_{i+1}(C) = H'_{i+2}(C).$$

In the previous equalities, as well as in the remainder of this article, the indices $1 \leq i \leq 3$ should be considered modulo 3. We will also denote by $\overline{H_i}(C)$ and $\overline{H'_i}(C)$ the closed hemispheres (containing C this time).

Let G be a connected nonempty graph embedded in $\mathbb{S}^2$. The connected components of $\mathbb{S}^2 \setminus G$ are topological disks, they are called the faces of G and we denote by $\mathcal{F}$ the set of faces of G. We say that G is a Σ*-graph* if there exist three distinct faces $(F_1, F_2, F_3) \in \mathcal{F}$ (called *marked faces*) such that P_i is in the interior of F_i for all $1 \leq i \leq 3$. A Σ-graph is more than just a graph embedded in Σ because we require that the graph actually separates the three punctures. A *lamination* of the Σ-graph G is a collection L of pairwise disjoint simple loops on G such that each loop in L is non-contractible on Σ. By disjoint we mean having no vertex in common. For any non-negative integers m_1, m_2 and m_3, a lamination is said to be *of type* (m_1, m_2, m_3) if for any $1 \leq i \leq 3$ it contains m_i loops of type i. The *lamination space* $\mathcal{L}(G)$ of a Σ-graph G is defined to be the set of all $(m_1, m_2, m_3) \in (\mathbb{Z}_+)^3$ such that G admits a lamination of type (m_1, m_2, m_3). Below we will describe the lamination space of a given Σ-graph G as the integer points of a lattice polytope defined in terms of some geometric characteristics of G.

In order to simplify the inequalities defining the lamination space, we have allowed a lamination to be empty, in which case its topological type is $(0, 0, 0)$. Note however that the polynomials $P(X, Y, Z)$ arising in the image of the spectral map have no constant term, so the only difference between the image of a graph G under the topological spectral map and the lamination space of G will be the presence or absence of the point $(0, 0, 0)$.

We define a distance function d_G on $\mathcal{F}$ such that any two faces sharing a vertex are at distance 1 for d_G. Let G^* be the dual graph of G (seen as a graph in $\mathbb{S}^2$). Construct $\widetilde{G^*}$ by adding to G^* a dual edge between any two dual vertices such that the corresponding two primal faces share a primal vertex. The distance d_G is defined to be the usual graph distance on the vertex set of $\widetilde{G^*}$, which is canonically in bijection with $\mathcal{F}$. In the special case when all the vertices of G have degree 3, then $\widetilde{G^*} = G^*$ and d_G is the classical distance between two faces corresponding to the graph distance on the dual graph. From now on, whenever we mention the distance between two faces of G, the distance function will implicitly be d_G.

Define $d_1(G) := d_G(F_2, F_3)$, $d_2(G) := d_G(F_1, F_3)$ and $d_3(G) := d_G(F_1, F_2)$. Also, for any $1 \leq i \leq 3$, define $M_i(G)$ to be the maximal number of pairwise disjoint simple loops of type i one can simultaneously draw on G. Given a Σ-graph G, we define the sextuple

$$\sigma(G) := (M_1(G), M_2(G), M_3(G), d_1(G), d_2(G), d_3(G)) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3.$$

See Figure 1 for an example.

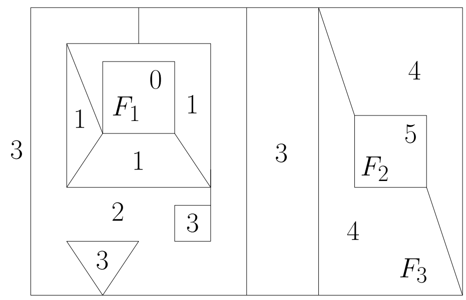


Figure 1: A Σ-graph G with each face labelled by its distance to the marked face F_1. For this graph, $\sigma(G) = (4, 1, 1, 1, 4, 5)$.

Given a sextuple of integers $\tau = (a, b, c, d, e, f) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$, we define the convex lattice polytope $\mathcal{P}_\tau$ by

$$\mathcal{P}_\tau := \left\{(x, y, z) \in (\mathbb{Z}_+)^3 | x \leq a,\ y \leq b,\ z \leq c,\ y + z \leq d,\ x + z \leq e,\ x + y \leq f\right\}.$$

Proposition 2.1. *For any Σ-graph G, its lamination space $\mathcal{L}(G)$ is the polytope $\mathcal{P}_{\sigma(G)}$.*

Proposition 2.1 is proved in Section 3.

Remark 2.2. The inequalities $m_i \leq M_i(G)$ are not redundant with the inequalities $m_i + m_{i+1} \leq d_{i+2}(G)$, as illustrated by Figure 2. On that picture, $d_1(G) = d_2(G) = d_3(G) = 2$ and $M_1(G) = M_2(G) = M_3(G) = 1$. The triple $(m_1, m_2, m_3) = (2, 0, 0)$ verifies the inequalities $m_i + m_{i+1} \leq d_{i+2}$, but that graph has no lamination of type $(2, 0, 0)$. This proposition corrects a statement made in [7], where the inequalities $m_i \leq M_i(G)$ were missing.

We can now characterize all the convex lattice polytopes that arise as the lamination space of some Σ-graph. By the previous proposition, it suffices to characterize the sextuples τ that arise as some $\sigma(G)$.

Theorem 2.3. *Fix $\tau = (\mu_1, \mu_2, \mu_3, \delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$. There exists a Σ-graph G such that $\sigma(G) = \tau$ if and only if the following inequalities hold for all $1 \leq i \leq 3$:*

(T_1) $\max(\mu_{i+1}, \mu_{i+2}) \leq \delta_i \leq \mu_{i+1} + \mu_{i+2}$*;*

(T_2) $\delta_{i+1} + \delta_{i+2} \leq 2\mu_i + \delta_i + 1$.

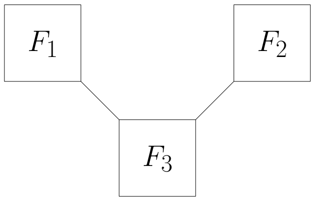


Figure 2: An example of a graph G illustrating the need to require the inequalities $m_i \leq M_i(G)$ in order to characterize the types of laminations that can arise.

The fact that conditions (T_1) and (T_2) are necessary is proved in Section 4, while the fact that they are sufficient is proved in Section 5 by explicitly constructing a Σ-graph G such that $\sigma(G) = \tau$ whenever τ satisfies the two conditions.

Remark 2.4. Setting $\nu_i = \mu_{i+1} + \mu_{i+2} - \delta_i$ for every $1 \leq i \leq 3$, conditions (T_1) and (T_2) can be rewritten in the following compact form: for every $1 \leq i \leq 3$,

$$0 \leq \nu_i \leq \min(\mu_{i+1}, \mu_{i+2}, \nu_{i+1} + \nu_{i+2} + 1). \tag{2.1}$$

The quantity ν_i will acquire a geometric meaning in Section 3, as the depth of intersection between two collections of special loops around P_{i+1} and P_{i+2}.

In order to prove Proposition 2.1 and the necessity of the conditions (T_1) and (T_2) in Theorem 2.3, we will explore any Σ-graph G starting from the face F_1, discover a first layer consisting of the faces at distance 1 from F_1, then a second layer consisting of the faces at distance 2, etc. We will perform the same exploration starting from the faces F_2 and F_3 and understand how the boundaries of the layers arising in each of these three explorations interact with each other. In the case of simple triangulations, our construction is very similar to the layer decomposition of Krikun [13]. More generally, this construction resembles the peeling process for planar maps (see for example [2]). The difference is that here we use a distance which differs slightly from the graph distance on the dual graph. Instead of peeling an edge by discovering the face on the other side of the edge, we are peeling a vertex, by discovering all the unknown faces containing a vertex which is on the boundary of what we have already explored.

3 Special loops around a puncture

In this section we first describe an exploration process of a Σ-graph G starting from a marked face, which will trace out a collection of special loops on G centered around a marked face. Then we will study how two collections of special loops intersect each other and deduce from this a proof of Proposition 2.1.

3.1 A collection of special loops around a puncture

We start by an elementary observation, which we will be using several times. Let G be a connected planar graph and $\widetilde{G}$ be a subgraph of G. One defines the distance function $d_{\widetilde{G}}$ on the set of the connected components of $\mathbb{S}^2 \setminus \widetilde{G}$ in exactly the same way as the distance d_G was defined on the faces of G. Note that the connected components of $\mathbb{S}^2 \setminus \widetilde{G}$ do not have to be topological disks, they may be disks with multiple punctures or even the whole sphere is $\widetilde{G}$ is empty. Then we have the following result, the proof of which is easy and omitted.

Lemma 3.1. *Let G be a connected planar graph and $\widetilde{G}$ be a subgraph of G. Let F and F' be two faces of G and let $\widetilde{F}$ and $\widetilde{F'}$ be the two connected components of $\mathbb{S}^2 \setminus \widetilde{G}$ containing respectively F and F'. Then $d_{\widetilde{G}}(\widetilde{F}, \widetilde{F'}) \leq d_G(F, F')$.*

Let G be a Σ-graph. For any $k \geq 0$ and $1 \leq i \leq 3$, define

$$A_i^k = \{F \in \mathcal{F} | d_G(F, F_i) = k\}. \tag{3.1}$$

For any $k \geq 1$ and $1 \leq i \leq 3$ such that A_i^k is nonempty, define B_i^k to be the boundary of the set $\bigcup_{j=0}^{k-1} A_i^j$ of faces that are at distance less than k to F_i. Each B_i^k is the union of simple loops that are pairwise edge-disjoint but not necessarily pairwise vertex-disjoint. The case when B_i^k consists in the union of several loops corresponds to a branching event in the peeling terminology, see e.g. [1]. The following lemma describes structural properties of the simple loops in B_i^k and will be used repeatedly in the remainder of the article.

Lemma 3.2. *Let G be a Σ-graph, let $1 \leq i \leq 3$ and let $k \geq 1$ be such that B_i^k is defined. Then, we have*

$$\bigcup_{j \geq k} A_i^j = \bigcup_{\substack{C \subset B_i^k \\ C \text{ simple loop}}} \overline{H_i'}(C). \tag{3.2}$$

Furthermore, if C and C' are two distinct simple loops contained in B_i^k, then $H_i'(C) \cap H_i'(C') = \emptyset$. Finally, if $C \subset B_i^k$ is a simple loop, then the faces in $\overline{H_i}(C)$ sharing an edge with C are in A_i^{k-1}.

Proof. Let $C \subset B_i^k$ be a simple loop and assume there is a face $F \subset \overline{H_i'}(C)$ such that $d_G(F_i, F) \leq k-1$. Then one can find a sequence $(F^0, \ldots, F^n)$ of faces such that $n = d_G(F_i, F)$, $F^0 = F_i$, $F^n = F$ and for every $1 \leq j \leq n$, the faces F^{j-1} and F^j share a vertex. By construction we have that for every $1 \leq j \leq n$, $d_G(F_i, F^j) = j$. Denoting by k' the largest j such that $F^j \subset \overline{H_i}(C)$, we have that $d_G(F_i, F^{k'}) \leq k-2$ and by connectedness of the path of faces from the hemisphere $H_i(C)$ to the hemisphere $H_i'(C)$, $F^{k'}$ shares a vertex v with C. This yields a contradiction because all the faces containing the vertex v are at distance at most $k-1$ of F_i, hence B_i^k cannot pass through v so v cannot lie on C. We deduce from this that

$$\bigcup_{\substack{C \subset B_i^k \\ C \text{ simple loop}}} \overline{H_i'}(C) \subset \bigcup_{j \geq k} A_i^j.$$

Furthermore, by construction, every edge in B_i^k has on one side a face in A_i^{k-1} and on the other side a face in A_i^k. This implies that all the faces in $\overline{H_i}(C)$ that contain an edge in C must be in A_i^{k-1}.

If C and C' are two distinct simple loops contained in B_i^k and $H_i'(C) \cap H_i'(C') \neq \emptyset$ then we can find a face F which satisfies one of the following two conditions:

1. $F \subset \overline{H_i}(C) \cap \overline{H_i'}(C')$ and F shares an edge with C ;

2. $F \subset \overline{H_i}(C') \cap \overline{H_i'}(C)$ and F shares an edge with C'.

This yields a contradiction because it implies on the one hand that $d_G(F, F_i) = k-1$ and on the other hand that $d_G(F, F_i) \geq k$. Thus $H_i'(C)$ and $H_i'(C')$ must be disjoint.

Finally, let F be a face in A_i^j with $j \geq k$. We construct F^c the connected component of $\bigcup_{j\geq k} A_i^j$ containing F as follows. We say that two faces in $\bigcup_{j\geq k} A_i^j$ are neighbors if they share an edge (not just a vertex) and F^c is the set of all faces in $\bigcup_{j\geq k} A_i^j$ that are reachable from F by walking across neighboring faces (these intermediate faces on the path must also lie in $\bigcup_{j\geq k} A_i^j$). Then F^c is a connected set, with boundary denoted by B^c. There exists a simple loop $C \subset B^c$ such that $F^c \subset \overline{H_i'}(C)$. By construction of B^c, for every edge e of C, the face adjacent to e in the hemisphere $\overline{H_i}(C)$ is in A_i^{k-1} and the face adjacent to e in the hemisphere $\overline{H_i'}(C)$ is in A_i^k. Hence $C \subset B_i^k$. We conclude that

$$\bigcup_{j\geq k} A_i^j \subset \bigcup_{\substack{C\subset B_i^k \\ C \text{ simple loop}}} \overline{H_i'}(C).$$

□

Fix $1 \leq i \leq 3$ and recall that the indices $i+1$ and $i+2$ are considered modulo 3. Since $d_G(F_i, F_{i+1}) = d_{i+2}(G)$, Lemma 3.2 implies for any $1 \leq k \leq d_{i+2}(G)$ the existence of a unique simple loop $C_{i,i+1}^k \subset B_i^k$ such that $F_{i+1} \subset \overline{H_i'}(C_{i,i+1}^k)$. Similarly, for any $1 \leq k \leq d_{i+1}(G)$ there exists a unique simple loop $C_{i,i+2}^k \subset B_i^k$ such that $F_{i+2} \subset \overline{H_i'}(C_{i,i+2}^k)$. For $1 \leq k \leq \min(d_{i+1}(G), d_{i+2}(G)) - 1$, if $C_{i,i+1}^k \neq C_{i,i+2}^k$, then $C_{i,i+1}^{k+1} \neq C_{i,i+2}^{k+1}$ by Lemma 3.2. Thus there exists a unique integer $\widetilde{M_i} \geq 0$ such that for any $1 \leq k \leq \widetilde{M_i}$ we have $C_{i,i+1}^k = C_{i,i+2}^k$ and for any $\widetilde{M_i}+1 \leq k \leq \min(d_{i+1}(G), d_{i+2}(G))$ we have $C_{i,i+1}^k \neq C_{i,i+2}^k$. If $1 \leq k \leq \widetilde{M_i}$ we denote simply by C_i^k the simple loop $C_{i,i+1}^k = C_{i,i+2}^k$. The following lemma gives the value of $\widetilde{M_i}$.

Lemma 3.3. *Let G be a Σ-graph. For any $1 \leq i \leq 3$, we have $\widetilde{M_i} = M_i(G)$. Furthermore, for any fixed $1 \leq i \leq 3$, the loops $(C_i^k)_{1\leq k\leq M_i(G)}$ are pairwise disjoint.*

Proof. Fix $1 \leq i \leq 3$. Consider C_i^k and $C_i^{k'}$ for some $1 \leq k < k' \leq \widetilde{M_i}$. First observe that these loops are nested, i.e. $H_i(C_i^k) \subset H_i(C_i^{k'})$. If there exists a vertex v lying on both these loops, denoting by e an edge of C_i^k containing v and by F the face containing e and lying in $\overline{H_i}(C_i^k)$, it follows from Lemma 3.2 that $d_G(F, F_i) = k-1$, which contradicts the fact that F shares the vertex v with some face F' lying in $\overline{H_i'}(C_i^{k'})$ because $d_G(F', F_i) \geq k' > k$ again by Lemma 3.2. So the loops $(C_i^k)_{1\leq k\leq \widetilde{M_i}}$ are pairwise disjoint.

It remains to prove that $\widetilde{M_i} = M_i(G)$. Since the simple loops C_i^k of type i are pairwise disjoint for $1 \leq k \leq \widetilde{M_i}$, their union constitutes a lamination with $\widetilde{M_i}$ loops of type i hence $\widetilde{M_i} \leq M_i(G)$.

Let L be a lamination consisting in $M_i(G)$ simple loops of type i denoted by C^k, $1 \leq k \leq M_i(G)$, which are nested in such a way that for any $1 \leq k \leq$

$M_i(G)-1$, $H_i(C^k) \subset H_i(C^{k+1})$. If F is a face in $\overline{H_i'}(C^k)$ for some $1 \le k \le M_i(G)$, then by Lemma 3.1, we have that $d_G(F_i, F) \ge k$ hence

$$\overline{H_i'}(C^k) \subset \bigcup_{j \ge k} A_i^j. \tag{3.3}$$

This implies that

$$\bigcup_{j=0}^{k-1} A_i^j \subset \overline{H_i}(C^k).$$

Recalling that B_i^k is defined as the boundary of $\bigcup_{j=0}^{k-1} A_i^j$, we deduce that B_i^k is well-defined for all $1 \le k \le M_i(G)$ and that $B_i^k \subset \overline{H_i}(C^k)$. Since C^k is of type i, this implies that any simple loop contained in B_i^k is either contractible or of type i. So $k \le \widetilde{M}_i$. This statement holds for every $1 \le k \le M_i(G)$ so $M_i(G) \le \widetilde{M}_i$. □

Remark 3.4. For any $1 \le k \le M_i(G)$, the simple loop C_i^k is of type i, for any $M_i(G)+1 \le k \le d_{i+1}(G)$ the simple loop $C_{i,i+1}^k$ is of type $i+1$ and for any $M_i(G)+1 \le k \le d_{i+2}(G)$ the simple loop $C_{i,i+2}^k$ is of type $i+2$.

The loops C_i^k are called *special loops of type i*. These special loops are optimal if one wants to pack as many disjoint simple loops of a given type as possible. For example C_i^1 is the "tightest" simple loop of type i one can draw, C_i^2 is the "tightest" simple loop of type i one can draw which would be disjoint from C_i^1, etc. See Figure 3 for an illustration.

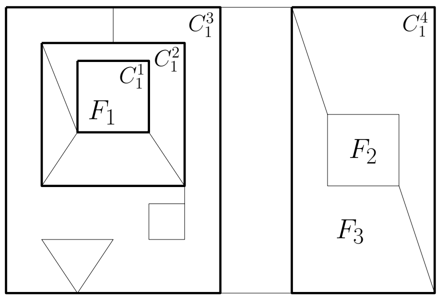


Figure 3: Representation in bold of the special loops C_1^k for the graph G of Figure 1.

3.2 Intersection of two collections of special loops

We will now describe how two collections of special loops of two different types intersect each other.

Lemma 3.5. *Let G be a Σ-graph and let $1 \le i \le 3$. Fix two integers $1 \le j \le d_i(G)$ and $1 \le k \le d_i(G)$. Then $C_{i+1}^k \cap C_{i+2}^j = \emptyset$ if and only if $j + k \le d_i(G)$. Furthermore, $H_{i+1}(C_{i+1}^k) \cap H_{i+2}(C_{i+2}^{d_i(G)+1-k}) = \emptyset$.*

Proof. Assume that $C_{i+1}^k \cap C_{i+2}^j \neq \emptyset$. Let v be a vertex in $C_{i+1}^k \cap C_{i+2}^j$, e be an edge in C_{i+1}^k containing v, e' be an edge in C_{i+2}^j, F be the face in

$\overline{H_{i+1}}(C_{i+1}^k)$ containing e and F' be the face in $\overline{H_{i+2}}(C_{i+2}^j)$ containing e'. Then by Lemma 3.2, $d_G(F_{i+1}, F) = k - 1$ and $d_G(F_{i+2}, F') = j - 1$. Since F and F' share the vertex v, we also have $d_G(F, F') = 1$. By the triangle inequality, we conclude that $d_G(F_{i+1}, F_{i+2}) \leq j + k - 1$. Thus $j + k > d_i(G)$.

Conversely, assume that $C_{i+1}^k \cap C_{i+2}^j = \emptyset$. Then either $C_{i+2}^j \subset H_{i+2}(C_{i+1}^k)$ or $C_{i+2}^j \subset H_{i+1}(C_{i+1}^k)$. The latter alternative cannot be true, otherwise we would have $F_i \subset H_i(C_{i+1}^k) = H_{i+2}(C_{i+1}^k) \subset H_{i+2}(C_{i+2}^j)$, which would entail that C_{i+2}^j is either contractible or of type $i+1$. Hence $C_{i+2}^j \subset H_{i+2}(C_{i+1}^k)$. Furthermore, as observed in the proof of Lemma 3.3, the special loops of a given type are nested and disjoint, which implies that the loops $C_{i+1}^1, \ldots, C_{i+1}^k, C_{i+2}^1, \ldots, C_{i+2}^j)$ are pairwise disjoint, thus they form a lamination L. Let F'_{i+1} (resp. F'_{i+2}) denote the connected component of $\mathbb{S}^2 \setminus L$ containing P_{i+1} (resp. P_{i+2}) Then by Lemma 3.1, since L is a subgraph of G, we have $d_i(G) = d_G(F_{i+1}, F_{i+2}) \geq d_L(F'_{i+1}, F'_{i+2}) = j + k$.

Finally, assume that $H_{i+1}(C_{i+1}^k) \cap H_{i+2}(C_{i+2}^{d_i(G)+1-k}) \neq \emptyset$. Then we can find a face $F \subset \overline{H_{i+1}}(C_{i+1}^k) \cap \overline{H_{i+2}}(C_{i+2}^{d_i(G)+1-k})$ which has at least one edge in common with C_{i+2}^j. By Lemma 3.2, we have $F \in A_{i+2}^{d_i(G)-k}$ so $B_{i+2}^{d_i(G)-k}$ intersects F (this intersection may be just a single vertex). On the other hand, it follows from Lemma 3.2 that $A_{i+2}^{d_i(G)-k-1} \subset H_{i+2}(C_{i+2}^{d_i(G)-k})$ so $B_{i+2}^{d_i(G)-k} \subset \overline{H_{i+2}}(C_{i+2}^{d_i(G)-k})$. Thus we obtain that $\overline{H_{i+1}}(C_{i+1}^k) \cap \overline{H_{i+2}}(C_{i+2}^{d_i(G)-k}) \neq \emptyset$, and the previous paragraph would entail that $k + (d_i(G) - k) > d_i(G)$. This is the desired contradiction, hence $H_{i+1}(C_{i+1}^k) \cap H_{i+2}(C_{i+2}^{d_i(G)+1-k}) = \emptyset$. $\square$

It follows from Lemma 3.5 that the nonnegative integer $n_i(G) := M_{i+1}(G) + M_{i+2}(G) - d_i(G)$ counts the number of special loops of type $i+1$ (resp. $i+2$) which intersect some special loop of type $i+2$ (resp. $i+1$). For every $1 \leq i \leq 3$, we will call this integer $n_i(G)$ the *depth of intersection* of the special loops of types $i+1$ and $i+2$.

We use the properties of these special loops to prove Proposition 2.1.

Proof of Proposition 2.1. Assume G has a lamination L of type (m_1, m_2, m_3). Let $1 \leq i \leq 3$. Then $m_i \leq M_i(G)$ by definition of $M_i(G)$. Furthermore, as in the proof of Lemma 3.5, we have $m_{i+1} + m_{i+2} \leq d_i(G)$. Thus $(m_1, m_2, m_3) \in \mathcal{P}_{\sigma(G)}$

Conversely, assume that we have a triple of integers (m_1, m_2, m_3) satisfying the six inequalities defining $\mathcal{P}_{\sigma(G)}$. Set

$$L = \left\{C_1^1, \ldots, C_1^{m_1}, C_2^1, \ldots, C_2^{m_2}, C_3^1, \ldots, C_3^{m_3}\right\}.$$

Observe that for every $1 \leq i \leq 3$, C_i^k is well-defined because $k \leq m_i \leq M_i(G)$. By Lemma 3.5, the fact that $m_{i+1} + m_{i+2} \leq d_i(G)$ for every i implies that these loops are pairwise disjoint. So L is a lamination and its type is (m_1, m_2, m_3) by construction. $\square$

4 Necessity of conditions (T_1) and (T_2)

In this section, we prove one direction of Theorem 2.3. Let G be a Σ-graph. In order to alleviate notation, we will drop the dependency of M_i, d_i and n_i on G in this section. We will show that the six components of $\sigma(G)$ satisfy the inequalities (T_1) and (T_2) of Theorem 2.3. By symmetry it suffices to consider the case $i = 1$.

4.1 Inequalities (T_1) are verified

The inequalities $M_2 \leq d_1$ and $M_3 \leq d_1$ follow from Lemma 3.3, thus

$$\max(M_2, M_3) \leq d_1.$$

To prove the other inequality, we distinguish several cases.

Case when $M_2 \geq 1$, $M_3 \geq 1$ and $C_2^{M_2} \cap C_3^{M_3} \neq \emptyset$.

Let v be a vertex in that intersection. Then one can find two faces F and F' containing v and such that $F \subset \overline{H_2}(C_2^{M_2})$, F shares an edge with $C_2^{M_2}$, $F' \subset \overline{H_3}(C_3^{M_3})$ and F' shares an edge with $C_3^{M_3}$. By the triangle inequality and Lemma 3.2 we have

$$d_G(F_2, F_3) \leq d_G(F_2, F) + d_G(F, F') + d_G(F', F_3) \leq (M_2 - 1) + 1 + (M_3 - 1),$$

thus, $d_1 \leq M_2 + M_3 - 1$ in that case.

Case when $M_2 \geq 1$, $M_3 \geq 1$ and $C_2^{M_2} \cap C_3^{M_3} = \emptyset$.

In that case $d_G(\overline{H_2}(C_2^{M_2}), \overline{H_3}(C_3^{M_3})) \geq 1$, and since by Lemma 3.2 we have that

$$\bigcup_{j \leq M_3 - 1} A_3^j \subset \overline{H_3}(C_3^{M_3}),$$

we deduce that $d_G(\overline{H_2}(C_2^{M_2}), \bigcup_{j \leq M_3 - 1} A_3^j) \geq 1$. So

$$\bigcup_{j \leq M_3} A_3^j \subset \overline{H_2'}(C_2^{M_2}) \tag{4.1}$$

and since $M_2 \geq 1$, we have that $B_3^{M_3+1}$ is non-empty. By Lemma 3.2 there exists a simple loop $C \subset B_3^{M_3+1}$ such that $F_2 \subset \overline{H_3'}(C)$. Thus $\overline{H_3'}(C) = \overline{H_2}(C)$, and relation (4.1) implies that $\overline{H_2}(C_2^{M_2}) \subset \overline{H_2}(C)$. Since C is disjoint from all the C_3^k with $1 \leq k \leq M_3$, it cannot be of type 3 (this would contradict the fact that M_3 is the maximal number of disjoint simple loops of type 3), thus $F_1 \subset \overline{H_3}(C)$. So C is of type 2, hence has to intersect $C_2^{M_2}$, otherwise this would contradict the fact that M_2 is the maximal number of disjoint simple loops of type 2. Considering the two non-disjoint simple loops $C \subset B_3^{M_3+1}$ and $C_2^{M_2} \subset B_2^{M_2}$, one concludes by selecting two appropriate faces F and F' as in the previous case and applying Lemma 3.2, which yields $d_1 \leq M_2 + M_3$.

Case when $M_2 = 0$ or $M_3 = 0$.

We first show that M_2 and M_3 cannot be both zero.

Lemma 4.1. *If $M_2 = 0$ then $M_1 \geq 1$ and $M_3 \geq 1$.*

Proof. Assume that $M_2 = 0$. The boundary B_2^1 of F_2 is nonempty even though it contains no simple loop of type 2. Since $d_G(F_3, F_2) \geq 1$ and $d_G(F_1, F_2) \geq 1$, by Lemma 3.2, there exist two simple loops C and C' contained in B_2^1 such that $F_3 \subset \overline{H_2'}(C)$ and $F_1 \subset \overline{H_2'}(C')$. Furthermore, $C \neq C'$ otherwise C would be of type 2. By Lemma 3.2, this implies that $H_2'(C) \cap H_2'(C') = \emptyset$, so C is a simple loop of type 3 and C' is a simple loop of type 1. Thus $M_3 \geq 1$ and $M_1 \geq 1$. $\square$

In the remainder of the proof we assume that $M_2 = 0$. As in the proof of Lemma 4.1, pick $C \subset B_2^1$ a simple loop of type 3. Since $C \subset \overline{F_2} \subset \overline{H_3'}(C_3^{M_3})$, the special loop $C_3^{M_3}$ must intersect C in at least a vertex v, otherwise C would be an (M_3+1)-st simple loop of type 3 which is disjoint from all the special loops C_3^k with $1 \leq k \leq M_3$. Let e be an edge of $C_3^{M_3}$ containing the vertex v and let F be the face in $\overline{H_3}(C_3^{M_3})$ containing e. Then $d_G(F_3, F) = M_3 - 1$ and $d_G(F, F_2) = 1$ so $d_1 = d_G(F_2, F_3) \leq M_3$.

4.2 Inequality (T_2) is verified

By definition of n_3, we have that $d_3+1-M_1 = M_2+1-n_3$. Hence it follows from Lemma 3.5 that $H_1(C_1^{M_1}) \cap H_2(C_2^{M_2+1-n_3}) = \emptyset$. Since both these hemispheres are open, we even have $\overline{H_1}(C_1^{M_1}) \cap H_2(C_2^{M_2+1-n_3}) = \emptyset$, thus $C_1^{M_1}$ is disjoint from $H_2(C_2^{M_2+1-n_3})$. Similarly $C_1^{M_1}$ is disjoint from $H_3(C_3^{M_3+1-n_2})$. The nesting of the special loops implies that $C_2^{M_2-n_3} \cup C_3^{M_3-n_2}$ is contained in $H_1'(C_1^{M_1})$.

Case when $\min(M_2 - n_3, M_3 - n_2) \geq 1$ **and** $\max(M_2 - n_3, M_3 - n_2) \geq 2$**.**

We reason by contradiction and assume that $n_1 > n_2 + n_3 + 1$. Without loss of generality assume that $M_2 - n_3 \geq 1$ and $M_3 - n_2 \geq 2$, hence $M_2 - n_3$ and $M_3 - n_2 - 1$ are both at least 1. By Lemma 3.5, since $M_2 - n_3 + M_3 - n_2 - 1 > d_1$, we have that $C_2^{M_2-n_3} \cap C_3^{M_3-n_2-1} \neq \emptyset$. Thus $C_2^{M_2-n_3} \cap H_3(C_3^{M_3-n_2}) \neq \emptyset$ and we can draw from $C_2^{M_2-n_3}$ and $C_3^{M_3-n_2}$ a simple closed curve of type 1 contained inside $H_1'(C_1^{M_1})$, which produces an (M_1+1)-th disjoint curve of type 1, contradiction.

Case when $M_2 = n_3$ **or** $M_3 = n_2$**.**

Without loss of generality assume that $M_2 = n_3$. Then

$$n_1 \leq M_2 \leq n_3 \leq n_2 + n_3 + 1.$$

Case when $M_2 - n_3 = M_3 - n_2 = 1$**.**

Then $n_1 = n_3 + 1 + n_2 + 1 - d_1 \leq n_2 + n_3 + 1$.

5 Graphs achieving any $\sigma(G)$

In this section, given $\tau = (\mu_1, \mu_2, \mu_3, \delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$ satisfying inequalities (T_1) and (T_2), we construct a graph G such that $\sigma(G) = \tau$. In the generic case, the graphs G will be constructed by gluing together several building blocks, most of which will be Young diagrams. Recall that the Young diagram $Y_{(\lambda_1,\dots,\lambda_n)}$ associated with the partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 1$ is (in French notation) the diagram consisting in n rows of left-aligned square boxes where the i-th row counted from the bottom contains λ_i boxes.

5.1 A class of graphs

To any sextuple $t = (l_1, l_2, l_3, n_1, n_2, n_3) \in \mathbb{Z}_+^6$ such that $n_i \leq \min(l_{i+1}, l_{i+2})$ for all $1 \leq i \leq 3$, we will first associate a graph Γ_t which has the topology of the disk. We will then obtain G_t by gluing two identical copies of Γ_t along their boundaries, like the construction of a pillowcase, which is topologically a sphere. However, we will only glue three disjoint arcs of the boundary of one graph with three disjoint arcs of the boundary of the other graph, hence the result will be a three-holed sphere.

We first define the following building blocks:

- the *connector* K, which is a triangle with edges called (in cyclic order) e'_1, e'_2 and e'_3. See Figure 4a.
- for every $1 \leq i \leq 3$, the Young diagram $Y_{(l_i)}$ consisting in a single row, called a *leg*. We denote its vertical left edge by E_i, its vertical right edge by e_i, its bottom (resp. top) horizontal edges from right to left by $f^1_{i,i+1}, \dots, f^{l_i}_{i,i+1}$ (resp. $f^1_{i,i-1}, \dots, f^{l_i}_{i,i-1}$). See Figure 4b.
- for every $1 \leq i \leq 3$, the Young diagram $Y_{(n_i, n_i-1, \dots, 2, 1)}$ consisting in n_i rows, called a *web*. We denote its horizontal edges on the bottom boundary from left to right by ${f'}^1_{i+1,i+2}, \dots, {f'}^{n_i}_{i+1,i+2}$ and its vertical edges on the left boundary from bottom to top by ${f'}^1_{i+2,i+1}, \dots, {f'}^{n_i}_{i+2,i+1}$. See Figure 4c.

Next, for every $1 \leq i \leq 3$, we glue the edges e_i with e'_i and for every $1 \leq k \leq n_i$ we glue $f^k_{i+1,i+2}$ with ${f'}^k_{i+1,i+2}$ and $f^k_{i+2,i+1}$ with ${f'}^k_{i+2,i+1}$. After gluing two edges together, the result is a single edge. See Figure 5 for an example. We call the resulting graph Γ_t. It has the topology of the disk, with three distinguished edges E_1, E_2 and E_3 on its boundary.

Let Γ'_t be an identical copy of Γ_t. Each edge of the boundary of Γ'_t is in canonical correspondence with an edge of the boundary of Γ_t. In particular, Γ'_t has three distinguished edges E'_1, E'_2 and E'_3 on its boundary. We glue together each pair of corresponding edges, except the three pairs containing the distinguished edges. We call the resulting graph G_t. For every $1 \leq i \leq 3$ we denote by F_i the digon with edges E_i and E'_i.

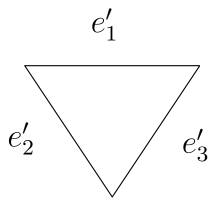
One can compute the components of $\sigma(G_t)$ explicitly.

Lemma 5.1. *Let $t = (l_1, l_2, l_3, n_1, n_2, n_3) \in \mathbb{Z}_+^6$ such that $n_i \leq \min(l_{i+1}, l_{i+2})$ for all $1 \leq i \leq 3$. Then for any $1 \leq i \leq 3$,*

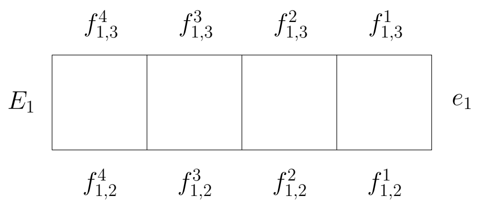
$$M_i(G_t) = 1 + l_i + \max\left(0, \left\lfloor \frac{n_i - \max(n_{i+1}, n_{i+2})}{2} \right\rfloor\right) \tag{5.1}$$

$$d_i(G_t) = 1 + l_{i+1} + l_{i+2} - n_i. \tag{5.2}$$

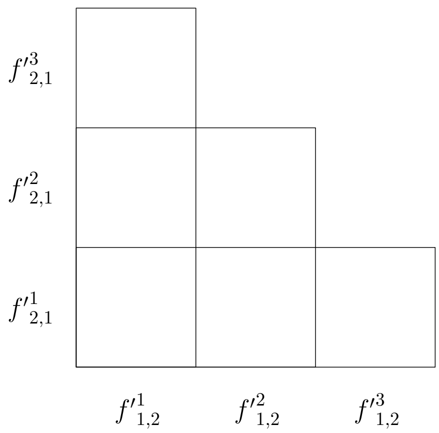
Proof. The proof consists in exploring G_t layer by layer from a face F_i and constructing explicitly the special loops defined in Section 3. Looking at the type of each loop yields the desired conclusion. By symmetry of the graph G_t, it actually suffices to explore the graph Γ_t layer by layer starting from an edge E_i and draw the arcs corresponding to the boundary of each layer. Considering



(a) The connector K.



(b) The leg corresponding to $i = 1$ when $l_1 = 4$.



(c) The web between 1 and 2 when $n_3 = 3$.

Figure 4: The different types of building blocks for the graph Γ_t.

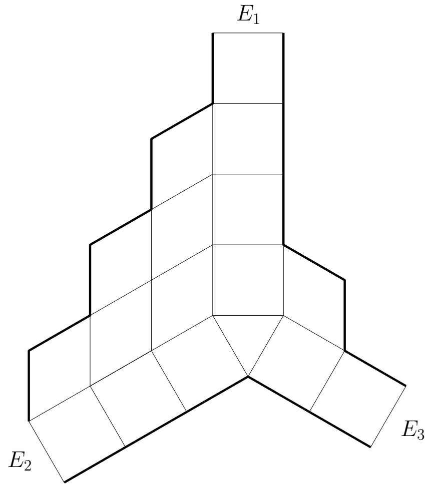


Figure 5: The graph Γ_t for $t = (4,3,2,0,1,3)$. The arcs of the boundary that will be glued to the corresponding arcs of an identical copy appear in bold stroke.

the endpoints of these arcs on the boundary of Γ_t reveals their type when they are glued with a symmetric copy of themselves to form loops in G_t. $\square$

5.2 End of the proof of Theorem 2.3

Fix $\tau = (\mu_1, \mu_2, \mu_3, \delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$ satisfying (T_1) and (T_2). Equivalently one can define $\nu_i := \mu_{i+1} + \mu_{i+2} - \delta_i$, use the μ_i and ν_i as variables and require that they satisfy (2.1). Up to permuting the indices, one may assume that $\nu_3 \leq \nu_2 \leq \nu_1$. We will construct a Σ-graph G such that $\sigma(G) = \tau$. For this we need to distinguish several cases.

Case when $\nu_2 < \nu_1 \leq \mu_1 + \nu_2$.

Define $t = (l_1, l_2, l_3, n_1, n_2, n_3)$ by:

$$l_1 = \nu_2 - \nu_1 + \mu_1 \tag{5.3}$$
$$l_2 = \mu_2 - 1 \tag{5.4}$$
$$l_3 = \mu_3 - 1 \tag{5.5}$$
$$n_1 = \nu_1 - 1 \tag{5.6}$$
$$n_2 = 2\nu_2 - \nu_1 \tag{5.7}$$
$$n_3 = \nu_2 + \nu_3 - \nu_1. \tag{5.8}$$

Then $t \in \mathbb{Z}_+^6$ and $n_i \leq \max(l_{i+1}, l_{i+2})$ for all $1 \leq i \leq 3$. Furthermore, by Lemma 5.1, $\sigma(G_t) = \tau$.

Case when $\nu_1 = \nu_2$ and $\nu_3 \geq 1$.

Define $t = (l_1, l_2, l_3, n_1, n_2, n_3)$ by $l_i = \mu_i - 1$ and $n_i = \nu_i - 1$ for all $1 \leq i \leq 3$. Then $t \in \mathbb{Z}_+^6$ and $n_i \leq \max(l_{i+1}, l_{i+2})$ for all $1 \leq i \leq 3$. Furthermore, by Lemma 5.1, $\sigma(G_t) = \tau$.

Case when $\nu_1 = \nu_2$, $\nu_3 = 0$ and $\mu_3 \geq 1$.

We construct G as on Figure 6 by drawing μ_i nested loops around each hole i such that:

- the loops around hole 1 are disjoint from the loops around hole 2 ;
- the outermost ν_1 (resp. ν_2) loops around hole 2 (resp. around hole 1) intersect the outermost ν_1 (resp. ν_2) loops around hole 3.

We also add line segments to make the graph G connected. Then $\sigma(G) = \tau$.

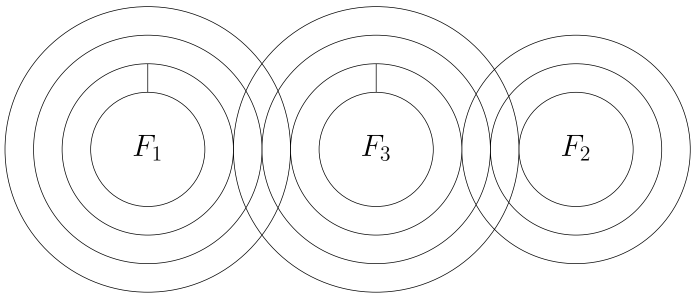


Figure 6: The graph G achieving $\tau = (4, 3, 4, 4, 5, 7)$. Here $(\nu_1, \nu_2, \nu_3) = (3, 3, 0)$.

Case when $\nu_1 = \nu_2$, $\nu_3 = 0$ and $\mu_3 = 0$.

In that case, by the inequalities (2.1), we have that $\nu_1 = \nu_2 = \nu_3 = \mu_3 = 0$. We construct G by drawing μ_1 nested loops around hole 1 and μ_2 nested loops around hole 2 such that the two collections of loops are disjoint and we add a segment to each collection of nested loops to make them connected. Finally we add a single loop surrounding each collection and touching the outermost loop of each collection at a single point, see Figure 7.

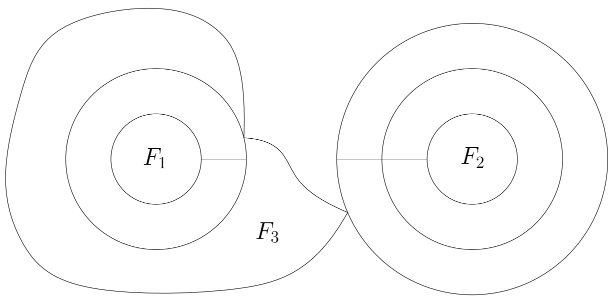


Figure 7: The graph G achieving $\tau = (2, 3, 0, 3, 2, 5)$. Here $\nu_1 = \nu_2 = \nu_3 = 0$.

Case when $\nu_1 > \mu_1 + \nu_2$.

It follows from (2.1) that

$$\nu_2 + \mu_1 - \nu_1 \geq \mu_1 - \nu_3 - 1 \geq -1.$$

Hence in the present case, we have $\nu_1 = \mu_1 + \nu_2 + 1$ and $\nu_3 = \mu_1$. It also follows from (2.1) and the fact that $\nu_2 \geq \nu_3$ that $\nu_2 = \mu_1$. So $\nu_1 = 2\mu_1 + 1$ and $\nu_2 = \nu_3 = \mu_1$.

We construct G by drawing two collections of μ_2 nested loops around hole 2 and μ_3 nested loops around hole 3 with intersection depth equal to ν_1 and adding two line segments to make the graph connected. See Figure 8 for an illustration. Then we have $\sigma(G) = \tau$.

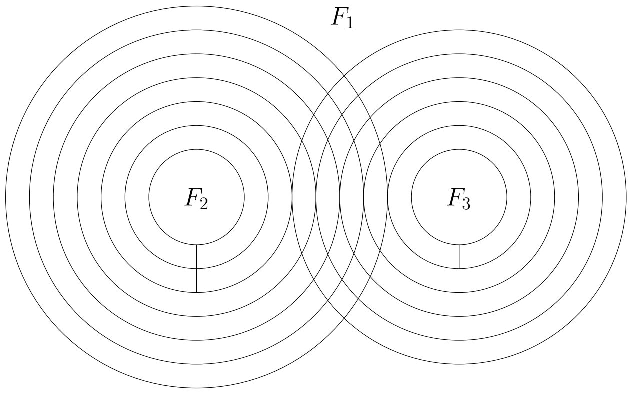


Figure 8: The graph G achieving $\tau = (2, 7, 6, 8, 6, 7)$. Here $(\nu_1, \nu_2, \nu_3) = (5, 2, 2)$.

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