The Foata correspondence, cycle lengths and anomalies

Sanjay Ramassamy

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Abstract

In their study of the densest jammed configurations for theater models, Krapivsky and Luck observe that two classes of permutations have the same cardinalities and ask for a bijection between them. In this note we show that the Foata correspondence provides the desired bijection.

Krapivsky and Luck (2019) introduced the theater model as a variant of directed random sequential adsorption, where spectators sequentially select a seat in a row of L seats, with the constraint that they cannot go past a cluster of b or more consecutive occupied seats. Configurations where all the seats are eventually occupied are parametrized by permutations σ of $\{1, \ldots, L\}$ such that for any i between 1 and L, one cannot find b consecutive integers $j+1, \ldots, j+b$ with $j + b < i$ and $\sigma(j + k) > \sigma(i)$ for all k between 1 and b. Krapivsky and Luck (2019) showed that the number $D_L^{(b)}$ of such permutations satisfies a linear recurrence relation which implies that they have the same cardinality as the permutations of L elements with cycles of lengths at most b. The authors then asked for a bijective proof of this fact.

The goal of this note is to show that the Foata correspondence provides such a bijection. In Section 1 we recall the Foata correspondence and in Section 2 we show that it provides the desired bijection.

1 The Foata correspondence

Let $\mathcal{S}_L$ be the group of permutations of $\{1, \ldots, L\}$. We will represent permutations in $\mathcal{S}_L$ by words with L distinct letters in $\{1, \ldots, L\}$. For example 1423 denotes the permutation $s \in \mathcal{S}_4$ such that $s(1) = 1$, $s(2) = 4$, $s(3) = 2$ and $s(4) = 3$.

One can associate to every permutation in $\mathcal{S}_L$ its cycle decomposition. Including the fixed points in that decomposition, the above $s \in \mathcal{S}_4$ has cycle decomposition [1][243]. This way of writing is however not unique for two reasons:

- each cycle of length d can be written in d different ways (one can freely choose what element to put first) ;
- if a permutation has k cycles (including singletons corresponding to fixed points) one can have them appear in $k!$ different orders.

The Foata correspondence (Foata (1968); Lothaire (1983)) describes a canonical choice of writing such a cycle decomposition. Firstly we write every cycle by starting by its maximal element. We call the maximal element of a cycle the *cycle head*. For example [243] is written [432]. Secondly, we write the cycles in increasing order of their cycle heads. For example, the cycle head of [1] is 1 and the cycle head of [432] is 4 so we write [1][432]. Removing the brackets, we obtain the word 1432 which can be seen as a permutation. The Foata correspondence associates to any permutation $s \in \mathcal{S}_L$ the permutation $F(s)$ obtained by writing the cycle decomposition of s in the above way and removing the brackets.

Remark 1.1. The largest letter to the left of a given letter a in $F(s)$ corresponds to the cycle head of the cycle to which a belongs in the cycle decomposition of s. This observation will be used later.

2 Cycle lengths and b-anomalies

Definition 2.1. Let $b \geq 1$ be an integer. Let $a_1 \cdots a_L$ denote a permutation in $\mathcal{S}_L$, with $L \geq 1$. A consecutive subword $a_{i+1} \cdots a_{i+b}$ is called a *b-anomaly* if there exists $1 \leq j \leq i$ such that $a_j > \max(a_{i+1}, \ldots, a_{i+b})$.

If we draw the point diagram associated with a permutation (which is just a plot of the graph of the corresponding function from $\{1, \ldots, L\}$ to itself), a b-anomaly corresponds to b points with consecutive abscissae for which one can find a point strictly above and to the left of all the b points.

The following result relates the cycle lengths of a permutation s to the b-anomalies of its image $F(s)$ under the Foata correspondence.

Proposition 2.2. *Let $b \geq 1$, $L \geq 1$ and $s \in \mathcal{S}_L$. Then s has a cycle of length at least $b+1$ if and only if $F(s)$ has a b-anomaly.*

Proof. Assume s has a cycle of length $d \geq b+1$. We write it $[c_1 c_2 \cdots c_d]$ with c_1 being the cycle head, that is, the largest element of the cycle. Then the subword $c_2 \cdots c_d$ of $F(s)$ forms a $(d-1)$-anomaly. Any consecutive subword of length b of this $(d-1)$-anomaly provides a b-anomaly.

Conversely, assume $F(s) = a_1 \cdots a_L$ has a b-anomaly $a_{i+1} \cdots a_{i+b}$. By definition of the b-anomaly, the set

$$X_i^s := \{j \leq i | a_j > \max(a_{i+1}, \ldots, a_{i+b})\}$$

is non-empty, so $\max_{j \in X_i^s} a_j$ is well-defined and equal to some a_h. Then by Remark 1.1, in the cycle decomposition of s, a_h is the head of the cycle to which each a_{i+k} with $1 \leq k \leq b$ belongs, so there are at least $b+1$ elements in that cycle in s. □

As a consequence of Proposition 2.2, in order to obtain the bijection requested by Krapivsky and Luck (2019), it suffices to compose the Foata correspondence with the involution sending every permutation $s \in \mathcal{S}_L$ to $\widetilde{s} \in \mathcal{S}_L$ defined by $\widetilde{s}(i) = L + 1 - s(L + 1 - i)$ for every i, whereby the point diagram is rotated by 180 degrees.

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Département de Mathématiques et Applications, École normale supérieure, CNRS, PSL University, 45 rue d'Ulm, 75005 Paris, France

E-mail address: `sanjay.ramassamy at ens.fr`