Non-Ulrich representation type

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NON-ULRICH REPRESENTATION TYPE

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Abstract. We show that a smooth projective arithmetically Cohen-Macaulay subvariety $X \subset \mathbb{P}^N$ of infinite Cohen-Macaulay type becomes of finite Cohen-Macaulay type by removing Ulrich bundles if and only if $N = 5$ and X is a quartic scroll or $\mathbb{P}^1 \times \mathbb{P}^2$. In turn, we give a complete and explicit classification of ACM bundles over these varieties.

Introduction

Given a smooth projective positive-dimensional subvariety $X \subset \mathbb{P}^N$ over an algebraically closed field $\Bbbk$, we say that X is arithmetically Cohen-Macaulay (ACM) if its homogeneous coordinate $\Bbbk[X]$ ring is Cohen-Macaulay. A locally free sheaf (or *bundle*) $\mathcal{E}$ on X is ACM if the module E of global sections of $\mathcal{E}$ is a maximal Cohen-Macaulay $\Bbbk[X]$-module.

In a few cases, X supports finitely many isomorphism classes of indecomposable ACM bundles (up to twist), so X is of *finite CM representation type*, or *CM-finite*. These varieties are classified in [EH88] and turn out to be: projective spaces, quadrics, rational normal curves, the Veronese surface in $\mathbb{P}^5$ and the rational surface scroll of degree 3 in $\mathbb{P}^4$.

All ACM subvarieties X besides these cases are CM-infinite. A few of them support only 1-dimensional families of isomorphism classes of indecomposable ACM bundles. This happens when X is an elliptic curve (by [Ati57]) or a rational surface scroll of degree 4 in $\mathbb{P}^5$, see [FM17]. In this case, X is of *tame CM representation type*. Otherwise, according to [FPL15], X supports families of pairwise non-isomorphic indecomposable ACM bundles of arbitrarily large dimension, and X is thus called of (geometrically) *wild CM representation type*.

Among ACM bundles, a special role is played by Ulrich bundles. These are characterized by the linearity of the minimal graded free resolution over the polynomial ring of their module of global section. Ulrich bundles, originally studied for computing Chow forms, conjecturally exist over any variety (we refer to [ESW03]). They are important for Boij-Söderberg theory (cf. [ES09, SE10]) and for the determination of the representation type of varieties (according to [FPL15]).

Heuristics about Ulrich bundles point out that, among ACM bundles of a fixed rank, they frequently move in the largest families, i.e. the dimension of their deformation space is maximal among such bundles. For instance, Fano threefolds of

2000 *Mathematics Subject Classification.* Primary 14J60; Secondary 14J45.

F.M. partially supported by GNASA-INdAM. D.F. partially supported by ISITE-BFC project (contract ANR-lS-IDEX-OOOB).

Picard number one and index at least two admit ACM bundles of rank two ; most of them are semistable, and their moduli space has the largest dimension precisely in the case of Ulrich bundles (see [BF11]). This happens also on some Fano threefolds of higher Picard rank (we refer e.g. to [CFM18] and references therein). The above considerations motivate the belief that, when X is CM-wild, there should exist families of pairwise non-isomorphic indecomposable Ulrich bundles of arbitrarily large dimension (so X should be Ulrich wild).

The present paper is devoted to further study the impact of Ulrich bundles on the representation type of a smooth variety. Namely, taking for granted the slogan that Ulrich bundles should move in the largest families, we ask what happens if we exclude them: does the representation type of X change? In particular, can X be downgraded to a CM-finite or CM-tame variety by excluding Ulrich bundles?

Our main contribution is that the answer to this question is negative, except for two CM-tame varieties and for a single CM-wild variety, which is $\mathbb{P}^1 \times \mathbb{P}^2$. More specifically, after excluding Ulrich bundles, the two rational scrolls of degree 4 and $\mathbb{P}^1 \times \mathbb{P}^2$ become of finite CM representation type, while all other varieties keep their representation type unchanged. This is the content of our first theorem.

Theorem A. *Let $X \subset \mathbb{P}^N$ be a smooth projective non-degenerate ACM variety of dimension $n > 0$. Assume that X is CM-infinite.*

Then there exist families of pairwise non-isomorphic indecomposable ACM bundles non-Ulrich on X of arbitrarily large dimension, except if X is:

i) a CM-tame variety, hence a scroll of degree 4 in $\mathbb{P}^5$ or an elliptic curve;
ii) the Segre product $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$.

The exceptional cases above, except for the elliptic curve, support finitely many non-Ulrich ACM bundles.

For the second theorem, we use the structure of the derived category of coherent sheaves over projective bundles to obtain a complete classification of the ACM indecomposable bundles (Ulrich or not) over $\mathbb{P}^1 \times \mathbb{P}^2$ and quartic scrolls. This second case is actually a direct extrapolation from [FM17], so the main point is to treat $\mathbb{P}^1 \times \mathbb{P}^2$, embedded as a degree 3 submanifold of $\mathbb{P}^5$ via the Segre product. To state the result, let us introduce some notation. Consider the projection π from $X = \mathbb{P}^1 \times \mathbb{P}^2$ to $\mathbb{P}^1$ and put F for the divisor class of a fibre of π and L for the pull-back of the class of a line on $\mathbb{P}^2$, also set Ω_π for the cotangent bundle of $\mathbb{P}^2$, pulled-back to X.

Theorem B. *Let $\mathcal{F}$ be an indecomposable ACM sheaf on $\mathbb{P}^1 \times \mathbb{P}^2$, assume $\mathrm{H}^0(\mathcal{F}) = 0$ and $\mathrm{H}^0(\mathcal{F}(1)) \neq 0$. Then $\mathcal{F}$ is:*

i) either an Ulrich bundle of the form:

$$0 \to \mathcal{O}_X(-F)^{\oplus a} \to \mathcal{F} \to \mathcal{O}_X(L-F)^{\oplus b} \to 0, \qquad \text{for some } a, b \in \mathbb{N}, \tag{1}$$

ii) either $\mathcal{O}_X(-1)$ or $\mathcal{O}_X(-L)$ or $\Omega_\pi(L)$.

This has the following surprising corollaries.

Corollary C. *Given a polynomial $p \in \mathbb{Q}[t]$, the moduli space of H-semistable ACM sheaves on X with Hilbert polynomial p is a finite set of points.*

Put $c_0 = 0$, $c_1 = 1$, $c_{k+2} = 3c_{k+1} - c_k$ and $c_{-k} = c_k$ for all $k \geq 0$. The numbers c_k are the odd terms of the Fibonacci sequence.

Corollary D. *For any $k \in \mathbb{Z}$ there is a unique indecomposable sheaf $\mathcal{U}_k$ fitting into:*

$$0 \to \mathcal{O}_X(-F)^{\oplus c_{k-1}} \to \mathcal{U}_k \to \mathcal{O}_X(F-L)^{\oplus c_k} \to 0.$$

The sheaves $\mathcal{U}_k$ are Ulrich and rigid, and satisfy:

$$\mathcal{U}_k^\vee \otimes \omega_X(2) \simeq \mathcal{U}_{1-k}.$$

Up to twist by $\mathcal{O}_X(t)$, any rigid indecomposable ACM bundle on X is isomorphic either to $\mathcal{O}_X$, or to $\Omega_\pi(L)$, or to $\mathcal{U}_k$, for some k.

The paper is organized as follows. We start by recalling some basic notions in §1. In §2 we provide a result ensuring the existence of unbounded families of ACM non-Ulrich sheaves under certain conditions. This is a slight modification of [FPL15, Theorem A]. Sections 3, 4 and 5 are devoted to the proof of Theorem A, with the exception of the statement concerning $\mathbb{P}^1 \times \mathbb{P}^2$. More specifically, §3 proves Theorem A for curves. §5 proves it for varieties of minimal degree (except for $\mathbb{P}^1 \times \mathbb{P}^2$), i.e. non-degenerate integral varieties $X \subset \mathbb{P}^N$ of dimension $n \geq 2$ and degree $d = N - n + 1$. §4.2 proves it when $d > N - n + 1$. Finally, in §6 we analyze ACM bundles on the exceptional case mentioned above, namely the Segre product $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Theorem B is proved in §6.3, cf. in particular Theorem 6.3. The two corollaries above are proved in §6.3.1.

Acknowledgements. We are grateful to Gianfranco Casnati for invaluable help.

1. Background

Let $\Bbbk$ be an algebraically closed field. Given an integer N, set $\mathbb{P}^N$ for the projective space of hyperplanes through the origin of $\Bbbk^{N+1}$.

1.1. Notation and conventions.

Let $X \subset \mathbb{P}^N$ be a closed integral subscheme of dimension n. We assume throughout the paper that X is non-degenerate, namely there is no hyperplane of $\mathbb{P}^N$ that contains X. The variety X is equipped with the very ample line bundle $\mathcal{O}_X(1)$ defined as restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ via the embedding $X \subset \mathbb{P}^N$. We will write H for the divisor class of $\mathcal{O}_X(1)$.

The coordinate ring R of $\mathbb{P}^N$ is the graded polynomial algebra in $N+1$ variables with the standard grading, namely $R = \Bbbk[x_0, \ldots, x_n]$. The homogeneous coordinate ring $\Bbbk[X]$ is the graded algebra $\Bbbk[X] = R/I_X$, where I_X is the homogeneous radical ideal of polynomials vanishing on X.

The degree of X is computed via the Hilbert polynomial of I_X. We will be denoted it by d.

1.2. **Cohen-Macaulay and Ulrich conditions.** Given a coherent sheaf $\mathcal{E}$ on X, the i-th *cohomology module* of $\mathcal{E}$ is the R-module:

$$\mathrm{H}^i_*(\mathcal{E}) = \bigoplus_{k\in\mathbb{Z}} \mathrm{H}^i(X, \mathcal{E}\otimes\mathcal{O}_X(k)).$$

For $i \geq 1$, the R-modules $\mathrm{H}^i_*(\mathcal{E})$ is are artinian. If $\mathcal{E}$ is locally free, then $\mathrm{H}^i_*(\mathcal{E})$ is finitely generated over R.

Definition 1.1. A coherent sheaf $\mathcal{E}$ on X is called *ACM*, standing for *Arithmetically Cohen-Macaulay*, if $\mathcal{E}$ is locally Cohen-Macaulay on X and:

$$\mathrm{H}^i_*(\mathcal{E}) = 0, \qquad \forall i \in \{1, \dots, n-1\}.$$

Equivalently, the minimal graded free resolution of the module of global sections $E = \mathrm{H}^0_*(\mathcal{E})$, seen as R-module, has length $N - n$.

The variety X itself is said to be ACM if X is projectively normal and $\mathcal{O}_X$ is ACM. This is equivalent to ask that $\Bbbk[X]$ is a graded Cohen-Macaulay ring, which in turn amounts to the fact that the minimal graded free resolution of $\Bbbk[X]$ as R-module has length $N - n$. In this case, the line bundles $\mathcal{O}_X(k)$ are ACM.

Definition 1.2. Let d be the degree of the embedded variety $X \subset \mathbb{P}^N$. A rank-r ACM sheaf $\mathcal{E}$ on X is said to be *Ulrich* if there is $t \in \mathbb{Z}$ such that $\mathrm{H}^0(X, \mathcal{E}(t-1)) = 0$ and $h^0(X, \mathcal{E}(t)) = rd$. We say that $\mathcal{E}$ is *initialized by* t (we omit "by t" if $t = 0$).

Given a coherent sheaf $\mathcal{E}$ on X, asking that $\mathcal{E}$ is initialized and Ulrich is tantamount to the condition $\mathrm{H}^i(X, \mathcal{E}(-j)) = 0$ for all integers i and all $1 \leq j \leq n$, cf. [ES09, Proposition 2.1].

Remark 1.3. We should warn the reader that the usual definition of Ulrich sheaf in the literature is equivalent to our definition of initialized Ulrich sheaf. We adopted this slightly different definition in order to work with sheaves which are Ulrich up to a twist.

1.3. **Semistability.** Let $X \subset \mathbb{P}^N$ be a closed subscheme of dimension $n > 0$ emdedded by the very ample divisor H. Stability of sheaves on X will always mean Gieseker stability of pure n-dimensional sheaves with respect to the polarisation H.

The Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on X, computed with respect to H, is denoted by $P(\mathcal{E}, t)$. The rank of $\mathcal{E}$ is defined as the element $r \in \mathbb{Q}$ such that the leading term of $P(\mathcal{E}, t)$ equals $rd/n!$. For $r \neq 0$, we write $\mathrm{p}(\mathcal{E}, t) := P(\mathcal{E}, t)/r$ for the *reduced Hilbert polynomial* of $\mathcal{E}$.

Given polynomials $p, q \in \mathbb{Q}[t]$, we write $p \preceq q$ if $p(t) \leq q(t)$ for $t \gg 0$. A coherent sheaf $\mathcal{E}$ of rank $r \neq 0$ is semistable if it is pure (i.e. all its subsheaves have support of dimension n) and, for any non-zero subsheaves $\mathcal{F} \subsetneq \mathcal{E}$, we have $\mathrm{p}(\mathcal{F}, t) \preceq \mathrm{p}(\mathcal{E}, t)$. Stability is defined by strict inequalities.

A coherent sheaf $\mathcal{E}$ on X is simple if $\mathrm{Hom}_X(\mathcal{E}, \mathcal{E})$ is generated by $\mathrm{id}_{\mathcal{E}}$.

2. Constructing unbounded families of non-Ulrich bundles

Let $X \subset \mathbb{P}^N$ be a non-degenerate closed subscheme of dimension $n > 0$. We propose here a criterion for existence of unbounded families of non-Ulrich bundles on X. Since this does not really depend on X being smooth or ACM we formulate it in a more general setting than what is actually needed to prove Theorem A. The result is a slight modification of [FPL15, Theorem A].

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be simple semistable ACM sheaves such that* $\mathrm{p}(\mathcal{B}) \prec \mathrm{p}(\mathcal{A})$ *and assume* $\dim_{\mathbb{k}} \mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A}) \geq 3$*. Then the following holds:*

i) the subscheme X is CM-wild;
ii) if $n \geq 2$ and $\mathcal{A}$ and $\mathcal{B}$ are not Ulrich initialized by the same integer, then X supports families of pairwise non-isomorphic non-Ulrich indecomposable ACM sheaves of arbitrarily large dimension;
iii) the same conclusion as in ii) holds also for $n = 1$ if there is no $t \in \mathbb{Z}$ such that $\mathrm{H}^0(X, \mathcal{A}(t)) = \mathrm{H}^1(X, \mathcal{B}(t)) = 0$.

Proof. We use the setting and notation of [FPL15, Theorem A]. To be in position of applying that result, we should verify that any non-zero morphism $\mathcal{A} \to \mathcal{B}$ is an isomorphism. But this is obvious, since $\mathrm{p}(\mathcal{B}) \prec \mathrm{p}(\mathcal{A})$ and $\mathcal{A}$ and $\mathcal{B}$ are semistable, so any morphism $\mathcal{A} \to \mathcal{B}$ is actually zero, so i) is clear.

Therefore X supports families of pairwise non-isomorphic indecomposable ACM sheaves of arbitrarily large dimension. Assume now that no integer t turns $\mathcal{A}(t)$ and $\mathcal{B}(t)$ into initialized Ulrich sheaves. Recall that, by construction, the sheaves appearing in the families provided by [FPL15, Theorem A] are extensions of copies of $\mathcal{A}$ and $\mathcal{B}$. If a sheaf $\mathcal{E}$ is an extension of say a copies of $\mathcal{A}$ and b copies of $\mathcal{B}$, it suffices to prove that $\mathcal{E}$ is actually non-Ulrich, as soon as a, b are both non-zero.

To check this, by contradiction we let t be an integer that initializes $\mathcal{E}$ as Ulrich sheaf, i.e. such that $\mathrm{H}^i(X, \mathcal{E}(t-j)) = 0$ for all $i \in \mathbb{N}$ and $1 \leq j \leq n$. Since $\mathcal{A}$ and $\mathcal{B}$ are ACM by assumption, we have the vanishing $\mathrm{H}^i(X, \mathcal{A}(t-j))) = \mathrm{H}^i(X, \mathcal{B}(t-j)))$ for $1 \leq i \leq n-1$ and for all $j \in \mathbb{Z}$.

By definition, $\mathcal{E}$ fits into an exact sequence of the form:

$$0 \to \mathcal{A}^{\oplus a} \to \mathcal{E} \to \mathcal{B}^{\oplus b} \to 0,$$

where we may assume $a \neq 0 \neq b$. Therefore, from the vanishing $\mathrm{H}^i(X, \mathcal{E}(t-j)) = 0$ we deduce $\mathrm{H}^0(X, \mathcal{A}(t-j)) = 0 = \mathrm{H}^n(X, \mathcal{B}(t-j))$ for $1 \leq j \leq n$.

Now, if $n \geq 2$, because $\mathcal{A}$ is ACM, the vanishing $\mathrm{H}^1(X, \mathcal{A}(t-j)))$ takes place for all $j \in \mathbb{Z}$ so we see that $\mathrm{H}^0(X, \mathcal{E}(t-j)) = 0$ implies $\mathrm{H}^0(X, \mathcal{B}(t-j)) = 0$ for $1 \leq j \leq n$. This implies that $\mathcal{B}$ is Ulrich, initialized by t, and similarly we get that the holds true for $\mathcal{A}$. But this is excluded, and we conclude that ii) holds.

With the same setup we can prove also iii). Indeed, when X is a curve, a coherent sheaf $\mathcal{E}$ is Ulrich if and only if there is $t \in \mathbb{Z}$ such that $\mathrm{H}^i(X, \mathcal{E}(t)) = 0$ for all i, which implies $\mathrm{H}^0(X, \mathcal{A}(t)) = 0$ and $\mathrm{H}^1(X, \mathcal{B}(t)) = 0$. But our assumption implies that there is no $t \in \mathbb{Z}$ such that $\mathrm{H}^0(X, \mathcal{A}(t))$ and $\mathrm{H}^1(X, \mathcal{B}(t))$ vanish together, so $\mathcal{E}$ is not Ulrich. □

3. Curves

We now prove Theorem A for curves, i.e. when $n = 1$. So let $X \subset \mathbb{P}^N$ be a smooth, non-degenerate projective curve of genus $g \geq 1$ and degree d, embedded by the complete linear series $|\mathcal{O}_X(H)|$.

If $g = 1$, then the statement of Theorem A is that X supports infinitely many non-Ulrich bundles $\mathcal{E}$. This is clear as one can see for instance by choosing $\mathcal{E}$ to have rank 1 and degree $d-1$, so that no twist $\mathcal{E}(t)$ is an initialized Ulrich sheaf.

So we can assume $g \geq 2$. Then, we take $\mathcal{A}_1$ and $\mathcal{A}_2$ to be a generic non-isomorphic line bundles of degree $d+g-1$. The genericity condition here is defined by asking $\mathrm{H}^i(X, \mathcal{A}_j(-H)) = 0$ for all i, j. Next, we choose a pair of line bundles $\mathcal{B}_1$ and $\mathcal{B}_2$ on X where $\mathcal{B}_1$ is again a generic line bundle of degree $d+g-1$, not isomorphic to $\mathcal{A}_1$ or $\mathcal{A}_2$, and $\mathcal{B}_2$ is a such that $\mathcal{B}_2(-H)$ is a general point in Riemann's Theta divisor of X, by which we mean that $\mathcal{B}_2$ is a line bundle of degree $d+g-1$ with $\mathrm{H}^i(X, \mathcal{B}_2(-1)) \cong \Bbbk$ for $i = 0, 1$.

Further, since X is irreducible and $\mathcal{A}_1$, $\mathcal{A}_2$ are non-isomorphic line bundles of the same degree, we have $\mathrm{Hom}_X(\mathcal{A}_i, \mathcal{A}_j) = 0$. By Riemann-Roch we deduce that $\mathrm{Ext}^1_X(\mathcal{A}_1, \mathcal{A}_2) \neq 0$, as the dimension of this space is at least $-\chi(\mathcal{A}_1^\vee \otimes \mathcal{A}_2) = g-1 \geq 1$. Likewise we have $\mathrm{Ext}^1_X(\mathcal{B}_1, \mathcal{B}_2) \neq 0$. Then, we can choose non-trivial elements in these extension spaces and define two vector bundles of rank 2, say $\mathcal{A}$ and $\mathcal{B}$, fitting into non-trivial extensions of the form:

$$0 \to \mathcal{A}_1 \to \mathcal{A} \to \mathcal{A}_2 \to 0, \qquad 0 \to \mathcal{B}_1 \to \mathcal{B} \to \mathcal{B}_2 \to 0.$$

Clearly, $\mathcal{A}$ and $\mathcal{B}$ are locally free sheaves of rank 2. Also, again the fact that the lines bundles $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ are pairwise non-isomorphic and of the same degree implies $\mathrm{Hom}_X(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_X(\mathcal{B}, \mathcal{A}) = 0$. Therefore, Riemann-Roch gives $\dim_{\Bbbk} \mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A}) = 4(g-1) \geq 4$.

To conclude the proof, we show that $\mathrm{H}^0(X, \mathcal{A}(t)) = 0$ implies $\mathrm{H}^1(X, \mathcal{B}(t)) \neq 0$. To do this, note that $\mathcal{A}$ is an initialized Ulrich bundle. This follows easily from the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are initialized Ulrich bundles, which in turn is given by our condition $\mathrm{H}^i(X, \mathcal{A}_j(-1)) = 0$ for all i, j. Then, $\mathrm{H}^0(X, \mathcal{A}(t)) = 0$ forces $t < 0$. But for $t < 0$ we get:

$$h^1(X, \mathcal{B}(t)) \geq h^1(X, \mathcal{B}_2(t)) = h^0(X, \mathcal{B}_2^\vee \otimes \omega_X(-t)) \geq h^0(X, \mathcal{B}_2^\vee \otimes \omega_X) = 1.$$

Therefore, case iii) of Theorem 2.1 implies our statement. Theorem A is thus proved for $n = 1$.

4. Higher dimension and degree

Let $X \subset \mathbb{P}^N$ be a smooth ACM subvariety of dimension $n \geq 2$ and put d for the degree of X. We assume that X is not of minimal degree, which is to say we suppose $d \geq N - n + 2$, and we check Theorem A. In this range we prove that X supports families of arbitrarily large dimension of pairwise non-isomorphic (ACM) indecomposable non-Ulrich bundles. The case that X is a del Pezzo surface is treated in the next paragraph, while the remaining cases are basically already in [FPL15] as we will see in §4.2.

4.1. **Del Pezzo surfaces.** For this subsection, our variety $X \subset \mathbb{P}^N$ is a smooth, anticanonically embedded del Pezzo surface. In particular $n = 2$ and $d = N$. Recall that X is either a blow-up of $\mathbb{P}^2$ at $9 - d$ points, or the product variety $\mathbb{P}^1 \times \mathbb{P}^1$. We construct ACM bundles (Ulrich or not) on X with the same methods in both cases, only with a slightly different choice of the invariants.

If X is a blow-up of $\mathbb{P}^2$, we fix a birational surjective morphism $\pi : X \to \mathbb{P}^2$ and let $\mathcal{O}_X(L) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $M = 2L$. Given $(a, b) \in \mathbb{N}^2$, with $a \geq 2$, we put $D(a, b) = 3ab - a^2 - b^2 + 1$ and $b_a = 2a$. In the second case we set π_1 and π_2 to be the projection maps onto the two $\mathbb{P}^1$ factors and let $\mathcal{O}_X(L) = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathcal{O}_X(F) = \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$. This time we take $(a, b) \in \mathbb{N}^2$ with $a \geq 1$ and we put $D(a, b) = 4ab - a^2 - b^2 + 1$, $b_a = 3a$, $M = 2L + F$.

Proposition 4.1. *Choose* (a, b) *so that* $D(a, b) > 0$ *and* $b \geq b_a$. *Then, for* f *general enough in* $\mathrm{Hom}_X(\mathcal{O}_X(L)^{\oplus b}, \mathcal{O}_X(M)^{\oplus a})$*, the sheaf* $\mathcal{E} = \ker(f)$ *is simple, locally free and ACM, with* $\dim \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E}) = D(a, b)$ *and* $\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) = 0$*;* $\mathcal{E}$ *is not Ulrich when* $b > b_a$.

Proof. Since the locally free sheaf $\mathcal{H} = \mathcal{H}om_X(\mathcal{O}_X(L)^{\oplus b}, \mathcal{O}_X(M)^{\oplus a}) \simeq \mathcal{O}_X(M - L)^{\oplus ba}$ is globally generated, for a general choice of $f \in \mathrm{H}^0(\mathcal{H})$ we have that the associated map $f : \mathcal{O}_X(L)^{\oplus b} \to \mathcal{O}_X(M)^{\oplus 2a}$ is surjective. Then, the sheaf $\mathcal{E} = \ker(f)$ is locally free of rank $b - a \geq b_a - a \geq 2$. We write down the exact sequence:

$$0 \to \mathcal{E} \to \mathcal{O}_X(L)^{\oplus b} \to \mathcal{O}_X(M)^{\oplus a} \to 0. \qquad (2)$$

Next, observe that the $\Bbbk$-vector space $\mathrm{Hom}_X(\mathcal{O}_X(L), \mathcal{O}_X(M))$ has dimension 3 or 4 depending on whether X is a blow-up of $\mathbb{P}^2$ or $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In both cases, the assumption $D(a, b) > 0$ ensures that [Kac80, Theorem 4] applies (cf. the argument of [CMRPL12, Proposition 3.5 (i)]) and shows that $\mathcal{E}$ is simple if f is general enough. The same argument proves $\dim \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E}) = D(a, b)$ and $\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}) = 0$.

Next, we show that $\mathcal{E}$ is ACM. Note that $\mathcal{O}_X(L)$ is ACM for the polarization H and that $\mathrm{H}^0(\mathcal{O}_X(M + tH)) = 0$ for any integer $t \leq -1$, so (2) gives $\mathrm{H}^1(\mathcal{E}(tH)) = 0$ for all $t \leq -1$. Also, Serre duality gives $\mathrm{H}^k(\mathcal{O}_X(L - H)) = \mathrm{H}^k(\mathcal{O}_X(M - H)) = 0$ for all k. So once we make sure that $\mathrm{H}^1(\mathcal{E}) = 0$, we will get that $\mathcal{E}(H)$ is H-regular and hence $\mathrm{H}^1(\mathcal{E}(tH)) = 0$ for $t \geq 0$, so that $\mathcal{E}$ will be proved to be ACM.

So let us prove $\mathrm{H}^1(\mathcal{E}) = 0$. If X is a blow-up of $\mathbb{P}^2$, this follows from [EH92, Propositions 1.1 and 4.1] in view of the assumption $b \geq b_a$. When $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, first we note that the condition $\mathrm{H}^1(\mathcal{E}) = 0$ is open on flat families, so to get the statement for general f it suffices to prove it for one choice of $f_0 \in \mathrm{H}^0(\mathcal{H})$, provided that the associated $f_0 : \mathcal{O}_X(L)^{\oplus b} \to \mathcal{O}_X(M)^{\oplus a}$ is surjective. Then we choose f_0 so that $\mathcal{E} \simeq \mathcal{O}_X(L)^{b-b_a} \oplus \mathcal{E}_0^{\oplus a}$, with $\mathcal{E}_0$ fitting into:

$$0 \to \mathcal{E}_0 \to \mathcal{O}_X(L)^{\oplus 3} \to \mathcal{O}_X(M) \to 0. \qquad (3)$$

Then, we use an argument analogous to [ESW03, Proposition 5.9] to show that $\mathcal{E}_0$ is Ulrich on (X, H), where $H = 2L + 2F$. Indeed, $c_1(\mathcal{E}_0) = L - F$ so $\mathcal{E}_0 \simeq \mathcal{E}_0^\vee(L - F)$ and the dual of (3) yields the exact sequence:

$$0 \to \mathcal{O}_X(-L - 2F) \to \mathcal{O}_X(-L)^{\oplus 3} \to \mathcal{E}_0 \to 0.$$

This implies immediately $\mathrm{H}^*(\mathcal{E}_0) = 0$. Also, (3) gives:

$$0 \to \mathcal{E}_0(-H) \to \mathcal{O}_X(-L-2F)^{\oplus 3} \to \mathcal{O}_X(-F) \to 0.$$

which implies $\mathrm{H}^*(\mathcal{E}_0(-H)) = 0$, so that $\mathcal{E}_0$ is Ulrich.

We have thus proved that $\mathcal{E}$ is ACM. Finally, $\mathcal{E}$ is not Ulrich as soon as $b > b_a$. Indeed, from (2) we have $\mathrm{H}^0(\mathcal{E}(-H)) = 0$ so we have that $\mathcal{E}$ is not Ulrich as soon as we show:

$$0 < \chi(\mathcal{E}) < d(b-a).$$

Now on one hand the assumption $b > b_a$ guarantees $\chi(\mathcal{E}) > 0$. On the other hand, when X is a blow-up of $\mathbb{P}^2$ we get $\chi(\mathcal{E}) = 3(b-2a) < 3(b-a) \leq d(b-a)$, while for $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (hence $d = 8$) we have $\chi(\mathcal{E}) = 2(b-3a) < 2(b-a) \leq 8(b-a)$. In both cases the desired equality holds and the statement is proved. □

Remark 4.2. The previous proof actually implies that, for $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ embedded by $2L + 2F$ and $b = 4a$, the sheaf $\mathcal{E}$ obtained by a general f as in the previous lemma is a simple Ulrich bundle of rank $2a$.

The previous proposition shows Theorem A when X is a del Pezzo surface. Indeed, choosing for instance $b = b_a + 1$ provides families of non-Ulrich pairwise non-isomorphic indecomposable bundles whose dimension a positive quadratic function of a.

4.2. **The higher range.** We justify here that Theorem A holds for all reduced closed ACM subschemes $X \subset \mathbb{P}^N$ of dimension n and degree d as soon as $d \geq N - n + 3$, or $d = N - n + 2$ and $n \geq 3$. We essentially extract this from [FPL15] up to the lemma, proved below, that syzygies of Ulrich sheaves are never Ulrich.

Indeed [FPL15, Theorem 4.2] asserts that, when a subscheme X as above is not of minimal degree, then it is of wild CM-type. The proof proceeds by reduction to a transverse linear section Y of X of dimension 1 in case $d \geq N - n + 3$, or of dimension two in case $d = N - n + 2$. Namely, setting c for the codimension of Y in X, one first constructs families of arbitrarily large dimension of indecomposable pairwise non-isomorphic Ulrich sheaves $\mathcal{F}$ on Y. Then, one defines ACM sheaves $\mathcal{E}$ on X by considering the sheafified syzygy of order c of $F = \mathrm{H}^0_*(\mathcal{F})$ as $\Bbbk[X]$-module. Then, one shows via [FPL15, Theorem B] that families of sheaves $\mathcal{E}$ constructed this way are still made-up of indecomposable pairwise non-isomorphic ACM sheaves.

The range of (N, n, d) which we are considering here corresponds to the assumption that X is not of minimal degree, i.e. $d \geq N - n + 2$, and that the linear section Y supporting unbounded families of Ulrich sheaves is of codimension $c \geq 1$. So we have to prove the following result.

Lemma 4.3. *Assume that X is not of minimal degree take a transverse linear section Y of of codimension $c \geq 1$. Let $\mathcal{F}$ be an Ulrich sheaf on Y. Then the sheafified syzygy $\mathcal{E}$ of order c of $\mathcal{F}$ is not Ulrich.*

Proof. First recall what we mean by sheafified syzygies. Put ι for the embedding of Y in X. We may harmlessly suppose that $\mathcal{F}$ is initialized. Let again $F = \mathrm{H}^0_*(\mathcal{F})$ be considered as a finitely generated graded $\Bbbk[X]$-module and take its minimal graded resolution over $\Bbbk[X]$. This is an exact complex of free $\Bbbk[X]$-modules:

$$0 \leftarrow F \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{\ell-1} \stackrel{d_\ell}{\longleftarrow} F_\ell \leftarrow \cdots$$

The ACM sheaf $\mathcal{E}$ is then defined as the sheafification of the image of d_c, so setting $\mathcal{F}_i$ for the sheafification of F_i we get a long exact sequence of coherent $\mathcal{O}_X$-modules:

$$0 \leftarrow \iota_* \mathcal{F} \leftarrow \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_{c-1} \leftarrow \mathcal{E} \leftarrow 0.$$

Here, the sheaves $\mathcal{F}_i$ are of the form:

$$\mathcal{F}_i = \bigoplus_{j \geq i} \mathcal{O}_X(-i-j)^{\oplus a_{i,j}},$$

for some integers $a_{i,j}$, only finitely many of which are non-zero. Since $\mathcal{F}$ is Ulrich, the minimal graded free resolution of F over S is linear, hence the linear syzygies of F appear among the F_i, which implies $a_{i,i} \neq 0$ for all i and $a_{0,j} = 0$ for $j \neq 0$.

We use a sheafified version of [FPL15, Lemma 3.2 and sequence (3.2)] to get $\mathrm{H}^0(Y, \mathcal{F}^\vee(1)) = 0$ and an exact sequence:

$$0 \leftarrow \iota_* \mathcal{F}^\vee(c) \leftarrow \mathcal{E}^\vee \leftarrow \mathcal{F}_{c-1}^\vee \leftarrow \cdots \leftarrow \mathcal{F}_1^\vee \leftarrow \mathcal{F}_0^\vee \leftarrow 0. \tag{4}$$

Having set up all this, we prove that no integer t turns $\mathcal{E}$ into an initialized Ulrich sheaf. Indeed, let t be such an integer so that $\mathrm{H}^i(X, \mathcal{E}(t-j)) = 0$ for all i and $1 \leq j \leq n$. First note that $\mathcal{F}_c$ maps non-trivially to $\mathcal{E}$ so the fact that $a_{c,c} \neq 0$ gives $\mathrm{H}^0(X, \mathcal{E}(c)) \neq 0$. Therefore $t \leq -c$. So, it suffices to prove that $\mathrm{H}^n(X, \mathcal{E}(c-n)) \neq 0$, as then we would get $t \geq 1-c$, which is a contradiction.

Our goal to finish the proof is thus to check $\mathrm{H}^n(X, \mathcal{E}(c-n)) \neq 0$. Assume on the contrary that $\mathrm{H}^n(X, \mathcal{E}(c-n)) = 0$. Via Serre duality, this gives $\mathrm{H}^0(X, \mathcal{E}^\vee \otimes \omega_X(n-c)) = 0$. By [FPL15, Lemma 3.1] we have, since X is not of minimal degree, that $\mathrm{H}^0(X, \omega_X(n-1)) \neq 0$, so there is an injective map $\mathcal{O}_X \to \omega_X(n-1)$. Tensoring $\mathcal{E}^\vee(1-c)$ with this map, we see that the vanishing $\mathrm{H}^0(X, \mathcal{E}^\vee \otimes \omega_X(n-c)) = 0$ implies $\mathrm{H}^0(X, \mathcal{E}^\vee(1-c)) = 0$.

Finally we use (4). Put $\mathcal{Q}_i$ for the cokernel image of the map $\mathcal{F}_{i-1}^\vee \to \mathcal{F}_i^\vee$ in that sequence. By the $\mathrm{H}^0(Y, \mathcal{F}^\vee(1)) = 0$ we obtain from $\mathrm{H}^0(X, \mathcal{E}^\vee(1-c)) = 0$ that $\mathrm{H}^0(X, \mathcal{Q}_{c-1}(1-c)) = 0$. One easily checks that, since A is ACM, the vanishing $\mathrm{H}^1(X, \mathcal{Q}_{c-2}(1-c)) = 0$ holds. So we finally get $\mathrm{H}^0(X, \mathcal{F}_{c-1}^\vee(1-c)) = 0$, which is absurd because $a_{c-1,c-1} \neq 0$. This concludes the proof. □

5. Varieties of minimal degree

Given $n \geq 2$ and a non-decreasing sequence $a = (a_1, \ldots, a_n)$ of integers $1 \leq a_1 \leq \cdots \leq a_n$ put $d = \sum_{i=1}^n a_i$ and $N = d+n-1$. We denote by $S(a) = S(a_1, \ldots, a_n)$ the rational normal scroll defined as the projectivization of $\oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, embedded as a submanifold of degree d in $\mathbb{P}^{d+n-1}$ by the tautological relatively ample line bundle. We set H for the hyperplane class and F for the class of a fibre of the projection $S(a) \to \mathbb{P}^1$. Let $\mathcal{L} = \mathcal{O}_X((d-1)F - H)$.

We know by [MR13] that the rational normal scroll $X = S(a)$ is Ulrich-wild except when $a = (1, a_2)$ with $a_2 \leq 3$ or $a = (2,2)$. However, this is because X supports unbounded families of Ulrich bundles, which appear as extensions of copies $\mathcal{O}_X(-F)$ and $\mathcal{L}$.

If we seek unbounded families of non-Ulrich bundles, we should be a bit more careful. We choose the sequence of integers $(a_1,\dots,a_n)$ in a way that the scroll $X = S(a)$ is not CM-finite, i.e. $n \ge 2$ and actually $n \ge 3$ or else $n = 2$ and $d \ge 4$. We start by recalling from [FM17] the construction of rigid Ulrich bundles on X. Put $\ell = (n-1)d - n$, so that $h^i(\mathcal{O}_X(H - dF)) = 0$ for all $i \ne 1$ and hence by Riemann-Roch:

$$\ell = -\chi(\mathcal{L}^\vee(-F)) = h^1(\mathcal{O}_X(H - dF)) \ge 2,$$

in our range for (d,n). Define recursively the Fibonacci-like numbers $a_{\ell,k} \in \mathbb{N}$ by:

$$a_{\ell,0} = 0, \qquad a_{\ell,1} = 1, \qquad a_{\ell,k+2} = \ell a_{\ell,k+1} - a_{\ell,k}, \qquad \forall k \in \mathbb{N}.$$

Since $\ell \ge 2$, the sequence $(a_{\ell,k})$ is strictly increasing along k.

Recall the notion of exceptional sheaf $\mathcal{E}$ on X, namely $\mathcal{E}$ is a simple coherent sheaf such that $\mathrm{Ext}^i_X(\mathcal{E},\mathcal{E}) = 0$ for $i > 0$. Recall also that two exceptional sheaves $(\mathcal{E},\mathcal{F})$ form an exceptional pair if $\mathrm{Ext}^i_X(\mathcal{F},\mathcal{E}) = 0$ for all i. The pair $(\mathcal{L},\mathcal{O}_X(-F))$ is exceptional. We mentioned that $h^1(\mathcal{L}^\vee(-F)) = \ell$ and $h^i(\mathcal{L}^\vee(-F)) = 0$ for $i \ne 1$,

Now, we refer again to [FM17, §2] to recall that, having set up this, we get that for each $k \ge 0$ there is a unique exceptional sheaf $\mathcal{U}_k$ which fits into:

$$0 \to \mathcal{O}_X(-F)^{\oplus a_{\ell,k}} \to \mathcal{U}_k \to \mathcal{L}^{\oplus a_{\ell,k+1}} \to 0. \tag{5}$$

Theorem A will be proved for X if check the following result.

Lemma 5.1. *The sheaf $\mathcal{B} = \mathcal{U}_k$ and $\mathcal{A} = \mathcal{O}_X$ satisfy the assumptions of Theorem 2.1 as soon as we take $k = 0$ for $n \ge 4$, $k = 1$ for $n = 3$ and $d \ge 4$, or $k = 4$ for $n = 2$ and $d \ge 5$.*

Proof. We know from [FM17, §2] that $\mathcal{U}_k$ is an exceptional Ulrich bundle which is actually initialized by $t = 1$. As a consequence, $\mathcal{U}_k$ is (strictly) semistable simple sheaf with:

$$\mathrm{p}(\mathcal{U}_k) = d\binom{t+n-1}{n} \prec \mathrm{p}(\mathcal{O}_X).$$

Since $\mathcal{O}_X$ is not Ulrich and is obviously stable, our task consists in checking the condition on the dimension of the extension space. One checks $h^i(\mathcal{L}^\vee) = 0$ for $i \ge 2$ so by Riemann-Roch:

$$\chi(\mathcal{L}^\vee) = 2n + (1-n)d.$$

Looking at (5), we deduce $h^i(\mathcal{U}_k^\vee) = 0$ for $i \ge 2$ so:

$$\begin{aligned} \dim_{\mathbb{k}} \mathrm{Ext}^1_X(\mathcal{U}_k,\mathcal{O}_X) &= h^1(\mathcal{U}_k^\vee) \ge -\chi(\mathcal{U}_k^\vee) = \\ &= a_{\ell,k+1}\chi(\mathcal{L}^\vee) + 2a_{\ell,k} = \\ &= a_{\ell,k+1}(2n + (1-n)d) + 2a_{\ell,k}. \end{aligned} \tag{6}$$

For $n \ge 4$, we get $d \ge 4$. Here $k = 0$ so we take $\mathcal{U}_0 = \mathcal{L}$ and check $h^1(\mathcal{L}^\vee) = -\chi(\mathcal{L}^\vee) \ge 4$ because $(n-1)d - 2n \ge 3d - 8 \ge 4$, so our assertion is proved.

For $n = 3$ and $k = 1$, formula (6) gives dimension at least $4d^2 - 18d + 16$, which is greater or equal 8 as soon as $d \ge 4$.

For $n = 2$, formula (6) gives dimension at least $d^5 - 12d^4 + 51d^3 - 92d^2 + 65d - 12$ when we take $k = 4$. This takes values at least 8 as soon as $d \geq 5$. $\square$

6. The Segre product of a line and a plane

Let us now turn to the analysis of the Segre product $X = \mathbb{P}^1 \times \mathbb{P}^2$, which we consider as a smooth submanifold of $\mathbb{P}^5$. In other words, X is the rational normal scroll $X = S(1,1,1)$ of degree $d = 3$ embedded by the tautological relatively ample divisor H, hence X has minimal degree. A smooth hyperplane section of X is the CM-finite cubic scroll $S(1,2)$.

Our goal here is to classify all ACM indecomposable bundles on X. Of course, this is not quite possible, since Ulrich bundles form a wild class in terms of representation theory, so we focus on non-Ulrich bundles and we classify all those.

6.1. **A first classification result via homological non-vanishing.**

Let us first give the basic ACM bundles that will be the output of the classification. Put π for the projection $X \to \mathbb{P}^1$ and Ω_π for the relative cotangent bundle. Here X is a product so Ω_π is the pull-back of the cotangent bundle of $\mathbb{P}^2$ via the projection $\sigma : X \to \mathbb{P}^2$. Set $L = H - F$, so $\mathcal{O}_X(L) = \sigma^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Recall:

$$\omega_X \simeq \mathcal{O}_X(-2F - 3L).$$

We easily that $\Omega_\pi(L)$ and $\mathcal{O}_X(L)$ are ACM. We start with a lemma, inspired on [BM11], that classifies these sheaves as bundles with a specific non-vanishing.

Lemma 6.1. *Let $\mathcal{E}$ be a locally free sheaf on X. Then $\mathcal{E} \simeq \Omega_\pi(L)$ if and only if $\mathcal{E}$ is indecomposable and:*

$$\mathrm{H}^1(\mathcal{E}) = \mathrm{H}^1(\mathcal{E}(-1)) = \mathrm{H}^2(\mathcal{E}(-2)) = 0, \qquad \mathrm{H}^1(\mathcal{E}(-L)) \neq 0. \tag{7}$$

Proof. One implication is clear, so we assume that $\mathcal{E}$ is an indecomposable locally free sheaf satisfying (7) and we prove $\mathcal{E} \simeq \Omega_\pi(L)$. Recall the standard isomorphism $\mathrm{Ext}^1_X(\mathcal{O}_X(L), \mathcal{E}) \simeq \mathrm{H}^1(\mathcal{E}(-L))$. Then, write the vertical Euler sequence:

$$0 \to \Omega_\pi(L) \to \mathcal{O}_X^{\oplus 3} \to \mathcal{O}_X(L) \to 0, \tag{8}$$

and apply $\mathrm{Hom}_X(-, \mathcal{E})$ to it. Since $\mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{E}) = \mathrm{H}^1(\mathcal{E}) = 0$, we get a surjection:

$$\mathrm{Hom}_X(\Omega_\pi(L), \mathcal{E}) \twoheadrightarrow \mathrm{Ext}^1_X(\mathcal{O}_X(L), \mathcal{E}) \simeq \mathrm{H}^1(\mathcal{E}(-L)).$$

Take $e \in \mathrm{H}^1(\mathcal{E}(-L)) \setminus \{0\}$ and consider a map $f : \Omega_\pi(L) \to \mathcal{E}$ lying in the preimage of e under the above surjection.

Further, we consider the dual vertical Euler sequence, written in the form:

$$0 \to \mathcal{O}_X(-2L) \to \mathcal{O}_X(-L)^{\oplus 3} \to \Omega_\pi(L) \to 0. \tag{9}$$

Note that, by Serre duality our assumption gives:

$$\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{O}_X(-L)) \simeq \mathrm{H}^2(\mathcal{E}(-2))^\vee = 0.$$

Next, we write the horizontal Euler sequence in the form:

$$0 \to \mathcal{O}_X(-2L - 2F) \to \mathcal{O}_X(-2L - F)^{\oplus 2} \to \mathcal{O}_X(-2L) \to 0. \tag{10}$$

Again our assumption gives, via Serre duality:

$$\mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2L-F)) \simeq \mathrm{H}^1(\mathcal{E}(-1))^\vee = 0.$$

We have thus a surjection as composition of surjections:

$$\mathrm{Hom}_X(\mathcal{E},\Omega_\pi(L)) \twoheadrightarrow \mathrm{Ext}^1_X(\mathcal{E},\mathcal{O}_X(-2L)) \twoheadrightarrow \mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2)). \tag{11}$$

Choose a generator k of the vector space $\mathrm{H}^3(\omega_X)$ and $h \in \mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2))$ such that the Yoneda product

$$\mathrm{H}^1(\mathcal{E}(-L)) \otimes \mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2)) \to \mathrm{H}^3(\mathcal{O}_X(-2F-3L)) \simeq \mathrm{H}^3(\omega_X)$$

sends $e \otimes h$ to k. Choose then $g : \mathcal{E} \to \Omega_\pi(L)$ lying in the preimage of h under the surjection (11).

It is well-known that $\Omega_{\mathbb{P}^2}$ is a simple sheaf, and by the Künneth formula the same holds for $\Omega_\pi(L)$. Therefore, as soon as the map $g \circ f$ is non-zero it must be a non-zero multiple of the identity. This implies immediately that $\Omega_\pi(L)$ is a direct summand of $\mathcal{E}$, which forces $\mathcal{E} \simeq \Omega_\pi(L)$ because $\mathcal{E}$ is indecomposable.

It remains to check that $g \circ f \neq 0$. To do this, we consider the following commutative diagram of Yoneda maps.

$$\begin{array}{ccc}
\mathrm{Hom}_X(\Omega_\pi(L),\mathcal{E}) \otimes \mathrm{Hom}_X(\mathcal{E},\Omega_\pi(L)) & \longrightarrow & \mathrm{Hom}_X(\Omega_\pi(L),\Omega_\pi(L)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(\Omega_\pi(L),\mathcal{E}) \otimes \mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2)) & \longrightarrow & \mathrm{Ext}^2_X(\Omega_\pi(L),\mathcal{O}_X(-2)) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_X(\mathcal{O}_X(L),\mathcal{E}) \otimes \mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2)) & \longrightarrow & \mathrm{H}^3(\omega_X)
\end{array} \tag{12}$$

Our goal is to prove that the map appearing in the top row sends $f \otimes g$ to a non-zero element. The upper map in the left column sends $f \otimes g$ to $f \otimes h$, so it suffices to check that the map in the middle row sends $f \otimes h$ to a non-zero element. In turn, the lower map in the left column sends $f \otimes h$ to $e \otimes h$, so it is enough to show that the map in the bottom row sends $e \otimes h$ to a non-zero element. But this last map sends $e \otimes h$ to k, hence we are done. □

In a similar vein we show the following.

Lemma 6.2. *Let $\mathcal{E}$ be an indecomposable locally free sheaf on X. Then:*

i) there is an isomorphism $\mathcal{E} \simeq \mathcal{O}_X(-L)$ if and only if:

$$\mathrm{H}^0(\mathcal{E}) = \mathrm{H}^1(\mathcal{E}(-L)) = \mathrm{H}^2(\mathcal{E}(-F-2L)) = 0, \qquad \mathrm{H}^0(\mathcal{E}(L)) \neq 0. \tag{13}$$

ii) there is an isomorphism $\mathcal{E} \simeq \mathcal{O}_X(-1)$ if and only if:

$$\mathrm{H}^0(\mathcal{E}(L)) = \mathrm{H}^1(\mathcal{E}(-F)) = \mathrm{H}^2(\mathcal{E}(-1)) = 0, \qquad \mathrm{H}^0(\mathcal{E}(1)) \neq 0. \tag{14}$$

Proof. Both items have an obvious implication, what we have to prove is that $\mathcal{E}$ is isomorphic to the desired sheaf after assuming the cohomological conditions.

Let us prove i). Choose a non-zero element f of $\mathrm{H}^0(\mathcal{E}(L)) \simeq \mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E})$. Next, we choose a generator k of $\mathrm{H}^3(\omega_X)$ and note that by Serre duality there exists $h \in \mathrm{Ext}^3_X(\mathcal{E},\mathcal{O}_X(-2F-4L))$ such that the Yoneda pairing

$$\mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E}) \otimes \mathrm{Ext}^3_X(\mathcal{E},\mathcal{O}_X(-2F-4L)) \to \mathrm{H}^3(\omega_X)$$

sends $f \otimes h$ to k.

Next, write the following again the exact sequences (8), (9) and (10), twisted by lines bundles on X so that they take the following form:

$$0 \to \mathcal{O}_X(-2F-4L) \to \mathcal{O}_X(-2F-3L)^{\oplus 3} \to \Omega_\pi(-2F-L) \to 0,$$
$$0 \to \Omega_\pi(-2F-L) \to \mathcal{O}_X(-2)^{\oplus 3} \to \mathcal{O}_X(-2F-L) \to 0,$$
$$0 \to \mathcal{O}_X(-2F-L) \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{O}_L(-L) \to 0.$$

We remark that the vanishing assumptions of i) and Serre duality imply:

$$\mathrm{Ext}^3_X(\mathcal{E},\mathcal{O}_X(-2F-3L)) \simeq \mathrm{H}^0(\mathcal{E})^* = 0,$$
$$\mathrm{Ext}^2_X(\mathcal{E},\mathcal{O}_X(-2)) \simeq \mathrm{H}^1(\mathcal{E}(-L))^* = 0,$$
$$\mathrm{Ext}^1_X(\mathcal{E},\mathcal{O}_X(-1)) \simeq \mathrm{H}^2(\mathcal{E}(-F-2L))^* = 0.$$

Therefore, applying $\mathrm{Hom}_X(\mathcal{E},-)$ to the three sequences above we get a surjection:

$$\mathrm{Hom}_X(\mathcal{E},\mathcal{O}_X(-L)) \twoheadrightarrow \mathrm{Ext}^3_X(\mathcal{E},\mathcal{O}_X(2F-4L)). \tag{15}$$

We choose now $g \in \mathrm{Hom}_X(\mathcal{E},\mathcal{O}_X(-L))$ in the preimage of h.

Therefore we have a commutative diagram of the form:

$$\begin{array}{ccc}
\mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E}) \otimes \mathrm{Hom}_X(\mathcal{E},\mathcal{O}_X(-L)) & \longrightarrow & \mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{O}_X(-L)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E}) \otimes \mathrm{Ext}^1_X(\mathcal{E},\mathcal{O}_X(-2F-L)) & \to & \mathrm{Ext}^1_X(\mathcal{O}_X(-L),\mathcal{O}_X(-2F-L)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E}) \otimes \mathrm{Ext}^2_X(\mathcal{E},\Omega_\pi(-2F-L)) & \longrightarrow & \mathrm{Ext}^2_X(\mathcal{O}_X(-L),\Omega_\pi(-2F)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(\mathcal{O}_X(-L),\mathcal{E}) \otimes \mathrm{Ext}^3_X(\mathcal{E},\mathcal{O}_X(2F-4L)) & \to & \mathrm{Ext}^3(\mathcal{O}_X(-L),\mathcal{O}_X(-2F-4L))
\end{array}$$

where the horizontal maps are given by the Yoneda pairing, the left vertical ones are given by the factorization of the map (15) while the maps in the right vertical column are obtained by applying $\mathrm{Hom}_X(\mathcal{O}_X(-L),-)$ to the three exact sequences above. Since the identity map of $\mathcal{O}_X(-L)$ is sent to ω_X via the composition of vertical maps by construction, if follows that $g \circ f$ is sent to the identity of $\mathcal{O}_X(-L)$ via the top horizontal map. This says that $\mathcal{O}_X(-L)$ is a direct summand of $\mathcal{E}$, and therefore proves $\mathcal{E} \simeq \mathcal{O}_X(-L)$ by indecomposability of $\mathcal{E}$.

The proof of ii) is similar, so we only sketch the argument. The strategy this time is to apply $\mathrm{Hom}_X(\mathcal{E}, -)$ to the exact sequences:

$$0 \to \mathcal{O}_X(-3F-4L) \to \mathcal{O}_X(-2F-4L)^{\oplus 2} \to \mathcal{O}_X(-F-4L) \to 0,$$
$$0 \to \mathcal{O}_X(-F-4L) \to \mathcal{O}_X(-F-3L)^{\oplus 3} \to \Omega_\pi(-1) \to 0,$$
$$0 \to \Omega_\pi(-1) \to \mathcal{O}_X(-F-2L)^{\oplus 3} \to \mathcal{O}_L(-1) \to 0,$$

and to use Serre duality which gives, via the assumption of ii):

$$\mathrm{Ext}^3_X(\mathcal{E}, \mathcal{O}_X(-3F-4L)) \simeq \mathrm{H}^0(\mathcal{E}(L))^* = 0,$$
$$\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{O}_X(-F-3L)) \simeq \mathrm{H}^1(\mathcal{E}(-F))^* = 0,$$
$$\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{O}_X(-F-2L)) \simeq \mathrm{H}^2(\mathcal{E}(-1))^* = 0.$$

The rest of the proof follows the same pattern as in i). □

6.2. **Beilinson-type spectral sequence.** We use the derived category $\mathrm{D}(X)$ of bounded complexes of coherent sheaves over the smooth projective variety X, in order to write the Beilinson-type spectral sequence associated with a coherent sheaf $\mathcal{E}$ on X after fixing a convenient full exceptional sequence in $\mathrm{D}(X)$. Indeed, the point is that the terms of this spectral sequence take a special form when $\mathcal{E}$ is ACM, and this will be our basic tool to classify such sheaves.

6.2.1. *Background on exceptional objects and mutations.* Let us first recall some terminology. An object $\mathcal{E}$ of $\mathrm{D}(X)$ is called *exceptional* if $\mathrm{Ext}^\bullet_X(\mathcal{E}, \mathcal{E}) = \Bbbk$, concentrated in degree zero. An ordered set of exceptional objects $(\mathcal{E}_0, \ldots, \mathcal{E}_s)$ is called an *exceptional collection* if $\mathrm{Ext}^\bullet_X(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i > j$. An exceptional collection is *full* when $\mathrm{Ext}^\bullet_X(\mathcal{E}_i, \mathcal{F}) = 0$ for all i implies $\mathcal{F} = 0$. Equivalently, the collection is full when $\mathrm{Ext}^\bullet_X(\mathcal{F}, \mathcal{E}_i) = 0$ implies $\mathcal{F} = 0$.

Exceptional collections can be *mutated*, let us recall what that means. Let $\mathcal{E}$ be an exceptional object in $\mathrm{D}(X)$. Then there are endofunctors $\mathrm{L}_\mathcal{E}$ and $\mathrm{R}_\mathcal{E}$ of $\mathrm{D}(X)$, called *right* and *left mutation functors* such that, for all $\mathcal{F}$ in $\mathrm{D}(X)$ there are functorial distinguished triangles:

$$\mathrm{L}_\mathcal{E}(\mathcal{F}) \to \mathrm{Ext}^\bullet_X(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \to \mathrm{L}_\mathcal{E}(\mathcal{F})[1],$$
$$\mathrm{R}_\mathcal{E}(\mathcal{F})[-1] \to \mathcal{F} \to \mathrm{Ext}^\bullet_X(\mathcal{F}, \mathcal{E})^\vee \otimes \mathcal{E} \to \mathrm{R}_\mathcal{E}(\mathcal{F}).$$

For all $i = 0, \ldots, s$ we define the *right* and *left dual* objects:

$$\mathcal{E}_i^{\mathrm{d}} = \mathrm{L}_{\mathcal{E}_0}\mathrm{L}_{\mathcal{E}_1} \cdots \mathrm{L}_{\mathcal{E}_{s-i}}\mathcal{E}_{s-i},$$
$${}^{\mathrm{d}}\mathcal{E}_i = \mathrm{R}_{\mathcal{E}_s}\mathrm{R}_{\mathcal{E}_{s-1}} \cdots \mathrm{R}_{\mathcal{E}_{s-i+1}}\mathcal{E}_{s-i}.$$

It turns out that, if $\mathcal{E}_\bullet = (\mathcal{E}_0, \ldots, \mathcal{E}_s)$ is a full exceptional collection, then both $(\mathcal{E}_0^{\mathrm{d}}, \ldots, \mathcal{E}_s^{\mathrm{d}})$ and ${}^{\mathrm{d}}\mathcal{E}_\bullet = ({}^{\mathrm{d}}\mathcal{E}_0, \ldots, {}^{\mathrm{d}}\mathcal{E}_s)$ also are full exceptional collections, called the *left* and *right dual* collections of $(\mathcal{E}_0, \ldots, \mathcal{E}_s)$. We refer to [GK04, §2.6]. The dual collections are characterized by the following property:

$$\mathrm{Ext}^\ell_X({}^{\mathrm{d}}\mathcal{E}_i, \mathcal{E}_j) \simeq \mathrm{Ext}^\ell_X(\mathcal{E}_i, \mathcal{E}_j^{\mathrm{d}}) \simeq \begin{cases} \Bbbk, & \text{if } i+j=s \text{ and } i=\ell, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Given an object $\mathcal{F}$ of $\mathrm{D}(X)$ and a full exceptional collection $(\mathcal{E}_0,\ldots,\mathcal{E}_s)$, there is a spectral sequence:

$$\bigoplus_{r+t=q} \mathrm{Ext}^r_X({}^{\mathrm{d}}\mathcal{E}_{s-p},\mathcal{F}) \otimes \mathcal{H}^t(\mathcal{E}_p) = E_1^{p,q} \Rightarrow \mathcal{H}^{p+q-s}(\mathcal{F}),$$

where $\mathcal{H}^i$ denotes the i-th homology sheaf of $\mathcal{F}$. This means that, for all (p,q) such that $p+q \neq s$ we have $E_\infty^{p,q}=0$, while:

$$\bigoplus_{p+q=s} E_\infty^{p,q} \simeq \mathrm{gr}(\mathcal{F}),$$

where $\mathrm{gr}(\mathcal{F})$ denotes the graded object with respect to a filtration of $\mathcal{F}$ of the form:

$$\mathcal{F}=\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_s \supset \mathcal{F}_{s+1}=0, \qquad \text{with:} \qquad \mathcal{F}_j/\mathcal{F}_{j+1} \simeq E_\infty^{j,s-j}.$$

The r-th differential of the spectral sequence reads $\delta_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

6.2.2. *An exceptional collection adapted to ACM sheaves.* This is indeed a full exceptional collection for $\mathrm{D}(X)$ as it follows easily from the fact that the standard Beilinson collection $(\mathcal{O}_{\mathbb{P}^2}(-2),\mathcal{O}_{\mathbb{P}^2}(-1),\mathcal{O}_{\mathbb{P}^2})$ is full exceptional for $\mathrm{D}(\mathbb{P}^2)$ and that $\mathrm{D}(X)$ is generated by $\pi^*\mathrm{D}(X) \otimes \mathcal{O}_X(-F)$ and $\pi^*\mathrm{D}(X)$. Thus we may choose the following full exceptional collection for X.

$$(\mathcal{O}_X(-L-F),\mathcal{O}_X(-L),\mathcal{O}_X(-F),\mathcal{O}_X,\mathcal{O}_X(L-F),\mathcal{O}_X(L)).$$

By a right mutation given by the Euler sequences on $\mathbb{P}^1$, this is replaced by:

$$(\mathcal{O}_X(-L),\mathcal{O}_X(F-L),\mathcal{O}_X(-F),\mathcal{O}_X,\mathcal{O}_X(L-F),\mathcal{O}_X(L)).$$

Since $\mathcal{O}_X$ and $\mathcal{O}_X(L-F)$ are mutually orthogonal, mutation gives:

$$(\mathcal{O}_X(-L),\mathcal{O}_X(F-L),\mathcal{O}_X(-F),\mathcal{O}_X(L-F),\mathcal{O}_X,\mathcal{O}_X(L)).$$

Finally, a right mutation given by the Euler sequences on $\mathbb{P}^2$ gives the following full exceptional collection of vector bundles over X.

$$\mathcal{E}_\bullet = (\mathcal{O}_X(-L),\mathcal{O}_X(F-L),\mathcal{O}_X(-F),\mathcal{O}_X(L-F),\Omega_\pi(L),\mathcal{O}_X).$$

Setting $\mathcal{T}_\pi = \Omega_\pi^\vee \simeq \Omega_\pi(3L)$, we write the left dual of this collection as:

$$(17) \qquad {}^{\mathrm{d}}\mathcal{E}_\bullet = (\mathcal{O}_X,\mathcal{O}_X(L),\mathcal{O}_X(1)[1],\mathcal{T}_\pi(F)[1],\mathcal{O}_X(F+2L)[2],\mathcal{O}_X(2)[2]),$$

Note that $\mathcal{E}_1 = \mathcal{O}_X(F-L)$ is the Ulrich line bundle $\mathcal{L}$ from §5. By Künneth's formula one gets another special feature of this collection, namely that:

$$(18) \qquad \mathrm{Hom}_X(\mathcal{E}_0,\mathcal{E}_3) = \mathrm{Hom}_X(\mathcal{E}_0,\mathcal{E}_2) = \mathrm{Hom}_X(\mathcal{E}_1,\mathcal{E}_2) = \mathrm{Hom}_X(\mathcal{E}_1,\mathcal{E}_4) = 0.$$

6.3. **Beilinson resolution of non-Ulrich sheaves.** Our goal for this section is to prove the next result.

Theorem 6.3. *Up to twist by $\mathcal{O}_X(t)$, an indecomposable ACM bundle $\mathcal{F}$ on X is either isomorphic to $\mathcal{O}_X(-1)$, or to $\mathcal{O}_X(-L)$ or to $\Omega_\pi(L)$, or to an Ulrich bundle $\mathcal{U}$ fitting into:*

$$0 \to \mathcal{O}_X(-F)^{\oplus a} \to \mathcal{U} \to \mathcal{O}_X(F-L)^{\oplus b} \to 0, \qquad \textit{for some } (a,b) \in \mathbb{N}^2.$$

The words "up to a twist" have the following more precise meaning: up to replacing $\mathcal{F}$ with $\mathcal{F}(t)$ we may assume that $h^0(\mathcal{F}) = 0$ and $h^0(\mathcal{F}(1)) \neq 0$. Then $\mathcal{F}$ is exactly one of the sheaves appearing in the statement of Theorem 6.3. In other words, Theorem 6.3 proves Theorem B from the introduction.

We will prove the theorem through several claims. The very first argument is to use Lemma 6.1. Note that the vanishing conditions appearing in that lemma are verified for any twist of $\mathcal{F}$ since $\mathcal{F}$ is ACM, so if there is a twist $t \in \mathbb{Z}$ such that $\mathrm{H}^1(\mathcal{F}(tH-L)) \neq 0$, we will have $\mathcal{F}(t) \simeq \Omega_\pi(L)$. Theorem 6.3 is proved in this case.

Therefore, from now on we may assume $\mathrm{H}^1(\mathcal{F}(tH-L)) = 0$ for all $t \in \mathbb{Z}$. The next step is to observe that, since $\mathcal{F}$ locally free and $\mathcal{O}_X(1)$ is very ample, there is a unique $t_0 \in \mathbb{Z}$ such that $\mathcal{F}(t_0)$ satisfies $h^0(\mathcal{F}(t_0)) = 0$ and $h^0(\mathcal{F}(t_0+1)) \neq 0$. We implicitly replace $\mathcal{F}$ with $\mathcal{F}(t_0)$ from now on. In particular we have $\mathrm{H}^0(X, \mathcal{F}) = 0$. We put:

$$a_{i,j} = \dim_{\Bbbk} \mathrm{Ext}^i_X({}^{\mathrm{d}}\mathcal{E}_j, \mathcal{F}).$$

Claim 6.4. *Let $\mathcal{F}$ be as above. Then $a_{1,3} = a_{2,4} = a_{2,3} = a_{3,4} = 0$.*

Proof. Recall that, in view of Lemma 6.1 we may assume $\mathrm{H}^1(\mathcal{F}(-L)) = 0$. Let us summarize the vanishing conditions we have so far by writing down the matrix $(a_{i,j})$. Traditionally one rather writes the table $(b_{i,j}) = (a_{5-i,5-j})$:

$\mathcal{F}(-2)[-2]$	$\mathcal{F}(-F-2L)[-2]$	$\mathcal{F} \otimes \Omega_\pi(-F)[-1]$	$\mathcal{F}(-1)[-1]$	$\mathcal{F}(-L)$	$\mathcal{F}$
$a_{5,5}$	$a_{5,4}$	0	0	0	0
0	$a_{4,4}$	$a_{4,3}$	$a_{4,2}$	0	0
0	$a_{3,4}$	$a_{3,3}$	0	$a_{3,1}$	$a_{3,0}$
0	$a_{2,4}$	$a_{2,3}$	0	$a_{2,1}$	0
0	0	$a_{1,3}$	0	0	0
0	0	0	0	0	0
$\mathcal{O}_X(-L)$	$\mathcal{O}_X(F-L)$	$\mathcal{O}_X(-F)$	$\mathcal{O}_X(L-F)$	$\Omega_\pi(L)$	$\mathcal{O}_X$

This table means that the (p,q)-th term of $E_1^{p,q}$ is the direct sum of as many copies of $\mathcal{E}_i$ as the coefficient $(b_{i,j})$ appearing the above table. Also, the coefficients above are obtained by computing the dimension of the cohomology of the bundle appearing on the i-th column of the first row, reading cohomological degree from bottom to top, with a shift indicated by the brackets.

Let us focus on the summand $\mathcal{O}_X(L-F)^{\oplus a_{2,4}} = E_1^{1,2}$. By (18) we have $\delta_1^{1,2} = 0$. Obviously $\delta_r^{1,2} = 0$ for $r \geq 2$. Also, $E_1^{p,q} = 0$ for $p+q \leq 2$, so $\mathcal{O}_X(L-F)^{\oplus a_{2,4}}$

survives to $E_\infty^{1,2}$, which in turn is zero because $E_\infty^{p,q}$ is concentrated at $p+q=5$. Therefore $a_{2,4}=0$. By the same reason we get $a_{1,3}=0$. Summing up, $E_1^{p,q}=0$ for $p+q\le 3$.

Let us now look at the summand $\mathcal{O}_X(-F)^{\oplus a_{2,3}}=E_1^{2,2}$. The map $\delta_r^{2,2}$ is clearly zero for all $r\ge 1$, and since we proved $E_1^{p,q}=0$ for $p+q\le 3$ we get $a_{2,3}=0$ again because $E_\infty^{p,q}$ is concentrated at $p+q=5$. The last vanishing $a_{3,4}=0$ follows a similar pattern. □

In terms of the Beilinson spectral sequence, the previous claim shows $E_1^{p,q}=0$ for $p+q\le 4$. Because of (18), we have $\delta_r^{2,3}=0$ for all $r\ge 1$, so the vanishing of $E_1^{p,q}$ with $p+q\le 4$ implies that the term $\mathcal{O}_X(-F)^{\oplus a_{3,3}}$ survives at $E_\infty^{2,3}$ and is thus a direct summand of $\mathrm{gr}(\mathcal{F})$. By the same reason, $\mathcal{O}_X(F-L)^{\oplus a_{4,4}}$ survives at $E_\infty^{1,4}$. This means that the filtration of $\mathcal{F}$ induced by the Beilinson-type spectral sequence takes the form:

$$0=\mathcal{F}_6\subset\mathcal{F}_5\subset\cdots\subset\mathcal{F}_0=\mathcal{F},\qquad\text{with:}\qquad \begin{aligned}&\mathcal{F}_5=\mathcal{F}_4=\mathcal{F}_3=0,\\&\mathcal{F}_2\simeq\mathcal{O}_X(-F)^{\oplus a_{3,3}},\\&\mathcal{F}_1/\mathcal{F}_2\simeq\mathcal{O}_X(F-L)^{\oplus a_{4,4}},\\&\mathcal{F}/\mathcal{F}_1\simeq E_\infty^{0,5}.\end{aligned}\tag{19}$$

Our next goal is to compute $E_\infty^{0,5}$.

Lemma 6.5. *There is an exact sequences:*

$$0\to E_\infty^{0,5}\to\mathcal{G}\to\Omega_\pi(L)^{\oplus a_{2,1}}\to 0,\tag{20}$$

where $\mathcal{G}$ is a coherent sheaf on X fitting into a long exact sequence:

$$\begin{aligned}0\to\mathcal{G}\to\mathrm{Ker}(\delta_1^{0,5})\to\mathrm{Ker}(\delta_1^{2,4})\to\mathrm{Ker}(\delta_1^{4,3})\to&\\ \to\mathrm{Coker}(\delta_1^{0,5})\to\mathrm{Coker}(\delta_1^{2,4})\to\mathrm{Coker}(\delta_1^{4,3})\to 0.&\end{aligned}\tag{21}$$

Before going into the proof, let us display the maps $\delta_1^{p,q}$ we are interested in:

$$\delta_1^{0,5}:\mathcal{O}_X(-L)^{\oplus a_{5,5}}\to\mathcal{O}_X(F-L)^{\oplus a_{5,4}},\tag{22}$$

$$\delta_1^{2,4}:\mathcal{O}_X(-F)^{\oplus a_{4,3}}\to\mathcal{O}_X(L-F)^{\oplus a_{4,2}},\tag{23}$$

$$\delta_1^{4,3}:\Omega_\pi(L)^{\oplus a_{3,1}}\to\mathcal{O}_X^{\oplus a_{3,0}}.\tag{24}$$

Proof. We rewrite the cohomology table $(b_{i,j})$ in view of the vanishing proved in the previous claim and after removing $a_{3,3}$ and $a_{4,4}$ which do not contribute to $E_\infty^{0,5}$ as we have just seen.

$\mathcal{F}(-2)[-2]$	$\mathcal{F}(-F-2L)[-2]$	$\mathcal{F}\otimes\Omega_\pi(-F)[-1]$	$\mathcal{F}(-1)[-1]$	$\mathcal{F}(-L)$	$\mathcal{F}$
$a_{5,5}$	$a_{5,4}$	0	0	0	0
0	0	$a_{4,3}$	$a_{4,2}$	0	0
0	0	0	0	$a_{3,1}$	$a_{3,0}$
0	0	0	0	$a_{2,1}$	0
0	0	0	0	0	0
0	0	0	0	0	0
$\mathcal{O}_X(-L)$	$\mathcal{O}_X(F-L)$	$\mathcal{O}_X(-F)$	$\mathcal{O}_X(L-F)$	$\Omega_\pi(L)$	$\mathcal{O}_X$

In view of this table, we see that the differential δ_1 has only three possibly non-zero terms, namely $\delta_1^{0,5}$, $\delta_1^{2,4}$ and $\delta_1^{4,3}$. So $E_2^{p,q}$ differs from $E_1^{p,q}$ only when (p,q) equals $(0,5)$, $(1,5)$, $(2,4)$, $(3,4)$, $(4,3)$ and $(5,3)$, and we get:

$$\begin{aligned} E_2^{0,5} &\simeq \ker(\delta_1^{0,5}), & E_2^{1,5} &\simeq \mathrm{Coker}(\delta_1^{0,5}), \\ E_2^{2,4} &\simeq \ker(\delta_1^{2,4}), & E_2^{3,4} &\simeq \mathrm{Coker}(\delta_1^{2,4}), \\ E_2^{4,3} &\simeq \ker(\delta_1^{4,3}), & E_2^{5,3} &\simeq \mathrm{Coker}(\delta_1^{4,3}). \end{aligned}$$

Now, since $E_\infty^{p,q}$ is concentrated at $p+q=5$, we realize that actually $E_3^{5,3}=0$ so the map $\delta_2^{3,4}: E_2^{3,4}\to E_2^{5,3}$ is surjective and actually also $E_3^{3,4}=0$ so the kernel of $\delta_2^{3,4}$ is the image of $\delta_2^{0,5}$. We have thus proved the second line of (21). By the same reason we have exactness of the sequence:

$$\mathrm{Ker}(\delta_1^{0,5})\to\mathrm{Ker}(\delta_1^{2,4})\to\mathrm{Ker}(\delta_1^{4,3}), \tag{25}$$

where the maps are just $\delta_2^{0,5}$ and $\delta_2^{2,4}$.

This completes the analysis of the second page. We turn now to E_3. Note that $E_3^{1,5}\simeq\mathrm{Ker}(\delta_2^{1,5})$ is the kernel of the map $\delta_2^{1,5}:\mathrm{Coker}(\delta_1^{0,5})\to\mathrm{Coker}(\delta_1^{2,4})$ appearing in (21). Similarly $E_3^{4,3}\simeq\mathrm{Coker}(\delta_2^{2,4})$ is the cokernel of $\delta_2^{2,4}:\mathrm{Ker}(\delta_1^{2,4})\to\mathrm{Ker}(\delta_1^{4,3})$ in (25). Since $E_4^{1,5}=E_4^{4,3}=0$, $\delta_3^{1,5}$ gives an isomorphism of $E_3^{1,5}$ to $E_3^{4,3}$, hence exactness of (21) is proved at the connecting map between the two rows.

Finally $E_3^{0,5}\simeq\mathrm{Ker}(\delta_2^{0,5})$ is the kernel $\mathcal{G}$ of the first map in (25) and clearly $E_3^{0,5}\simeq E_4^{0,5}$. The map $\delta_4^{0,5}$ thus maps this kernel surjectively onto $E_4^{4,2}\simeq\Omega_\pi(L)^{\oplus a_{2,1}}$, with kernel $E_5^{0,5}\simeq E_\infty^{0,5}$. The lemma is thus proved. □

Lemma 6.6. *In the previous setting, we have:*

$$\mathrm{Ext}^1_X(\mathcal{G},\mathcal{O}_X(F-L))=\mathrm{Ext}^1_X(\mathcal{G},\mathcal{O}_X(-F))=0.$$

Proof. We use the exact sequence (21). Indeed, let $\mathcal{N}$ be one of the two line bundles $\mathcal{O}_X(F-L)$ or $\mathcal{O}_X(-F)$ and apply $\mathrm{Hom}_X(-,\mathcal{N})$ to (21). Set $\mathcal{G}_i$ for the image of the i-th map $\delta_2^{2i-2,6-i}$ of (21). Then our statement is proved if we show that:

$$\mathrm{Ext}^i_X(\mathrm{Ker}(\delta_1^{2i-2,6-i}),\mathcal{N})=0,\qquad \text{for } i=1,2,3. \tag{26}$$

Indeed, this would imply $\mathrm{Ext}^{i+1}_X(\mathcal{G}_i,\mathcal{N})=0$ for $i=1,2$ which in turn would give $\mathrm{Ext}^1_X(\mathcal{G},\mathcal{N})=0$, which is our statement.

To check (26) we look more closely at the defining maps (22), (23) and (24). For $i = 1$, we note that (22) is constant along the factor $\mathbb{P}^2$ of the product $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$ so $\ker(\delta_1^{0,5})$ is the pull-back to X of a torsion-free sheaf on $\mathbb{P}^1$, twisted by $\mathcal{O}_X(-L)$. Such sheaf is then locally free on $\mathbb{P}^1$ and therefore splits as direct sum of line bundles. Actually the form of (22) implies that there are integers c_j, one for each $j \in \mathbb{N}$ (with only finitely many j such that $c_j \neq 0$) such that:

$$\ker(\delta_1^{0,5}) \simeq \bigoplus_{j \in \mathbb{N}} \mathcal{O}_X(-L - jF)^{\oplus c_j}.$$

It follows plainly that $\mathrm{Ext}^1_X(\ker(\delta_1^{0,5}), \mathcal{N}) = 0$ for our choices of $\mathcal{N}$.

For $i = 2$, applying a similar argument to (23) we get that there exists a torsion-free sheaf $\mathcal{V}$ on $\mathbb{P}^2$ such that:

$$\ker(\delta_1^{2,4}) \simeq \sigma^*(\mathcal{V}) \otimes \mathcal{O}_X(-F), \qquad \mathrm{H}^0(\mathbb{P}^2, \mathcal{V}(-1)) = 0. \tag{27}$$

Therefore, by Künneth's formula we have:

$$\mathrm{Ext}^2_X(\ker(\delta_1^{2,4}), \mathcal{O}_X(F - L)) \simeq \mathrm{Ext}^2_{\mathbb{P}^2}(\mathcal{V}, \mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)),$$

which vanishes because Serre duality give and (27) give:

$$\mathrm{Ext}^2_{\mathbb{P}^2}(\mathcal{V}, \mathcal{O}_{\mathbb{P}^2}(-1)) \simeq \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{V}(-2))^\vee = 0.$$

The vanishing for $\mathcal{N} = \mathcal{O}_X(-F)$ is clear.

For $i = 3$, again looking at (24) we get a torsion-free sheaf $\mathcal{W}$ on $\mathbb{P}^2$ such that:

$$\ker(\delta_1^{4,3}) \simeq \sigma^*(\mathcal{W}). \tag{28}$$

This time Künneth's formula provides $\mathrm{Ext}^3_X(\ker(\delta_1^{4,3}), \mathcal{N}) = 0$ immediately. □

Lemma 6.7. *For any sheaf $\mathcal{U}$ which is an extension of copies of $\mathcal{O}_X(F - L)$ and $\mathcal{O}_X(-F)$, we have $\mathrm{Ext}^1_X(E_\infty^{0,5}, \mathcal{U}) = 0$.*

Proof. Clearly, it suffices to check that $\mathrm{Ext}^1_X(E_\infty^{0,5}, \mathcal{N}) = 0$, with $\mathcal{N} = \mathcal{O}_X(-F)$ and $\mathcal{N} = \mathcal{O}_X(F - L)$. According to Lemma 6.5, we need to check $\mathrm{Ext}^1_X(\mathcal{G}, \mathcal{N}) = 0$ and $\mathrm{Ext}^2_X(\Omega_\pi(L), \mathcal{N}) = 0$. The first vanishing comes from the previous lemma and the second one is straightforward. □

Now comes the key point. Indeed, the sheaf $\mathcal{F}_1$ taken from the filtration (19) is an Ulrich sheaf of the form $\mathcal{U}$ as in the previous lemma. Therefore, $\mathcal{F}$ is the direct sum of $E_\infty^{0,5}$ and $\mathcal{F}_1$. But $\mathcal{F}$ is indecomposable, hence either $\mathcal{F}_1 = 0$ and $\mathcal{F} \simeq E_\infty^{0,5}$, or $\mathcal{F} \simeq \mathcal{U}$. In the latter case Theorem 6.3 is proved, so it remains to analyze the former one. So we assume from now on $\mathcal{F} \simeq E_\infty^{0,5}$.

Lemma 6.8. *The sheaf $\mathcal{F} \simeq E_\infty^{0,5}$ is isomorphic to $\mathcal{O}_X(-L)$ or $\mathcal{O}_X(-1)$.*

Proof. Since $\mathcal{F} \simeq E_\infty^{0,5}$ we have $a_{3,3} = a_{4,4} = 0$ so the cohomology table $(b_{i,j})$ looks as in the proof of Lemma 6.5. We argue now on whether $\mathrm{H}^0(\mathcal{F}(L))$ is zero or not.

If $\mathrm{H}^0(\mathcal{F}(L)) \neq 0$, looking at the cohomology table of $\mathcal{F}$ we see that $\mathrm{H}^1(\mathcal{F}(-L)) = \mathrm{H}^2(\mathcal{F}(-F - 2L)) = 0$, and because we are assuming $\mathrm{H}^0(\mathcal{F}) = 0$, we have that item i) of Lemma 6.2 applies to give $\mathcal{F} \simeq \mathcal{O}_X(-L)$.

If $\mathrm{H}^0(\mathcal{F}(L)) = 0$, we use once more the vertical Euler sequence, in the form:

$$0 \to \Omega_\pi(-F) \to \mathcal{O}_X(-1)^{\oplus 3} \to \mathcal{O}_X(-F) \to 0.$$

We tensor this sequence with $\mathcal{F}$ and take cohomology. From the cohomology table of $\mathcal{F}$ we extract $\mathrm{H}^2(\mathcal{F} \otimes \Omega_\pi(-F)) = 0$, which combined with the fact that $\mathcal{F}$ is ACM gives $\mathrm{H}^1(\mathcal{F}(-F)) = 0$. Also, of course $\mathrm{H}^2(\mathcal{F}(-1)) = 0$, while $\mathrm{H}^0(\mathcal{F}(1)) \neq 0$ by assumption. Therefore, item ii) of Lemma 6.2 applies and shows $\mathcal{F} \simeq \mathcal{O}_X(-1)$. □

This completes the proof of Theorem 6.3.

6.3.1. *Proof of Corollaries C and D.* The proof of Corollary C goes as follows. Let $\mathcal{F}$ be a semistable ACM bundle on X. By Theorem B, $\mathcal{F}$ is a direct sum of twists of $\mathcal{O}_X(-1)$, $\mathcal{O}_X(-L)$, and Ulrich bundles.

Since the reduced Hilbert polynomials of any twist of these three types of bundles are pairwise distinct, we have that only one type and a single twist can actually occur as direct summand of $\mathcal{F}$. Now, Ulrich bundles must be copies $\Omega_\pi(L)$ plus extensions as displayed in Theorem B, so the S-class of $\mathcal{F}$ must be represented by a direct sum of line bundles, all having the same Hilbert polynomial, plus copies of twist of $\Omega_\pi(L)$.

Since there are finitely many ways to choose these bundles, the moduli space of semistable ACM bundles with fixed Hilbert polynomial is a finite set. Corollary C is proved.

For the proof of Corollary D, we construct the bundles $\mathcal{U}_k$ by mutation. Put:

$$\begin{aligned}
&\mathcal{U}_{-1} = \mathcal{O}_X(-F), \\
&\mathcal{U}_0 = \mathcal{O}_X(F-L), \\
&\mathcal{U}_1 = \mathrm{L}_{\mathcal{U}_0}\mathcal{U}_{-1}[1], && \mathcal{U}_{k+1} = \mathrm{L}_{\mathcal{U}_k}\mathcal{U}_{k-1}, && \text{for } k \geq 1, \\
&\mathcal{U}_{-2} = \mathrm{R}_{\mathcal{U}_{-1}}\mathcal{U}_0[-1], && \mathcal{U}_{-k-2} = \mathrm{R}_{\mathcal{U}_{-k-1}}\mathcal{U}_{-k} && \text{for } k \geq 1.
\end{aligned}$$

The fact that the objects $\mathcal{U}_k$ are exceptional sheaves having a resolution of the desired form follows as in [FM17, Theorem B].

By Theorem B, any indecomposable rigid ACM bundle on X must be, up to a twist, isomorphic to $\mathcal{O}_X(-1)$, $\mathcal{O}_X(-L)$ or $\Omega_\pi(L)$ or a rigid Ulrich bundle of the form (1). In turn, again as in [FM17, Theorem B] we have that a rigid sheaf appearing as middle term of (1) must be isomorphic to $\mathcal{U}_k$ for some $k \in \mathbb{Z}$, with $(a,b) = (c_{k-1}, c_k)$. Moreover the equality $(a,b) = (c_{k-1}, c_k)$ determines the rigid bundle $\mathcal{U}_k$ uniquely.

Finally, given $k \in \mathbb{Z}$, since $\mathcal{U}_k^\vee \otimes \omega_X(2)$ is a rigid Ulrich bundle which fits as middle term of an extension of the form (1) with the same values of a and b as $\mathcal{U}_{1-k}$, by the uniqueness argument for the rigid bundles $\mathcal{U}_k$ we must have $\mathcal{U}_{1-k} \simeq \mathcal{U}_k^\vee \otimes \omega_X(2)$. This concludes the proof of Corollary D.

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