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Abstract

This note contains some proofs for the paper "Discretization of Homogeneous Systems Using Euler Method with a State-Dependent Step" of the same authors.

Key words: Homogeneous systems; Discretization; Euler method.

1 Proof of Theorem 9

Let us take a discretization step $h > 0$ and consider the behavior of a $\mathbf{r}$–homogeneous Lyapunov function V (satisfying (7)) on the sequence generated by (2). For this purpose define $x_i = \Lambda_r(\|x_i\|_r)y_i$ for some $y_i \in S_r(1)$:

$$V(x_{i+1}) - V(x_i) = V(x_i + \|x_i\|_r^{-\nu} hf(x_i)) - V(x_i) = \|x_i\|_r^{\mu}[V(y_i + hf(y_i)) - V(y_i)].$$

Since V is twice continuously differentiable, then by the Taylor expansion theorem with Lagrange remainder [1] there is $\theta \in (0, 1)$ such that

$$V(y_i + hf(y_i)) = V(y_i) + \left.\frac{\partial V(\xi)}{\partial \xi}\right|_{\xi=y_i} hf(y_i) + \frac{h^2}{2} f^\top(y_i) \left.\frac{\partial^2 V(\xi)}{\partial \xi^2}\right|_{\xi=y_i+\theta hf(y_i)} f(y_i),$$

then

$$V(x_{i+1}) - V(x_i) = \|x_i\|_r^{\mu} \left.\frac{\partial V(\xi)}{\partial \xi}\right|_{\xi=y_i} hf(y_i) + \|x_i\|_r^{\mu}\frac{h^2}{2} f^\top(y_i) \left.\frac{\partial^2 V(\xi)}{\partial \xi^2}\right|_{\xi=y_i+\theta hf(y_i)} f(y_i).$$

Note that from (7),

$$\left.\frac{\partial V(\xi)}{\partial \xi}\right|_{\xi=y_i} hf(y_i) \le -ha,$$

and there is $v \in (0, +\infty)$ such that

$$\sup_{y\in S_r(1)} \sup_{\theta\in(0,1)} f^\top(y) \left.\frac{\partial^2 V(\xi)}{\partial \xi^2}\right|_{\xi=y+\theta f(y)} f(y) \le v.$$

Therefore,

$$V(x_{i+1}) - V(x_i) \le h\|x_i\|_r^{\mu} \left(\frac{h}{2}v - a\right)$$

and for all $h \in (0, h_0]$ with $h_0 = \min\{1, \frac{a}{v}\}$ we obtain:

$$V(x_{i+1}) - V(x_i) \le -\frac{a}{2}h\|x_i\|_r^{\mu} \le -\alpha hV(x_i),$$

where $\alpha = \frac{a}{2c_2}$ and the property given in (7) was used on the last step. Assume that actually $h_0 = \min\{1, \frac{a}{v}, 2\frac{c_2}{a}\}$,

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then

$$V(x_{i+1}) \leq (1-\alpha h)V(x_i)$$

for all $i = 0, 1, \ldots$ and the values of $V(x_i)$ are monotonously decreasing, hence the sequence $\{x_i\}_{i=0}^{\infty}$ is converging (the property (b) is substantiated). Global boundedness of $\{x_i\}_{i=0}^{\infty}$ (*i.e.* the property (a)) follows from the properties of the function V for $\gamma = (c_1^{-1}c_2)^{1/\mu}$.

Let us evaluate the rate of convergence for different signs of the degree of homogeneity ν of (1) stated in the property (c). Let $\nu = 0$, then trivially $t_{i+1} = t_i + h$ and an exponential rate of convergence is recovered. From (2) we have:

$$t_{i+1} - t_i = \frac{h}{\|x_i\|_r^{\nu}} \leq \frac{h}{V^{\frac{\nu}{\mu}}(x_i)} \begin{cases} c_1^{\nu/\mu} & \nu < 0 \\ c_2^{\nu/\mu} & \nu > 0 \end{cases}.$$

As it has been proven above

$$V(x_i) \leq (1-\alpha h)^i V(x_0)$$

for all $i = 0, 1, \ldots$, then for $\nu < 0$ and for any $k \geq 0$ an estimate can be derived:

$$t_{k+1} \leq \sum_{i=0}^{k} \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_i)} \leq \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_0)} \sum_{i=0}^{k} (1-\alpha h)^{-i\frac{\nu}{\mu}}.$$

Since $|1-\alpha h| < 1$ for the selected h_0, we get

$$t_{+\infty} \leq \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_0)} \sum_{i=0}^{+\infty} (1-\alpha h)^{-i\frac{\nu}{\mu}} \leq \frac{c_1^{\nu/\mu} h V^{-\frac{\nu}{\mu}}(x_0)}{1-(1-\alpha h)^{-\frac{\nu}{\mu}}} < +\infty$$

for all $x_0 \neq 0$, which implies a finite-time convergence to the origin of any sequence $\{x_i\}_{i=0}^{\infty}$ in (2) but for $i \to +\infty$. For $\nu > 0$ we are interesting in the time of convergence from an infinite initial condition to $B_r(1)$, then it is necessary to repeat all above arguments in the inverse time, which leads to exactly the same estimate of such a time.

2 Proof of Proposition 11

Let $x_i \in S_r(1)$ be an arbitrary fixed vector, then the equation (3) can be rewritten as

$$\Delta = hz(x_i + \Delta)$$

for $\Delta = x_{i+1} - x_i$, where $z : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function defined as $z(x) = \frac{1}{\|x\|_r^{\nu}} f(x)$ with $z(0) = 0$. Indeed, $z(\Lambda_r(\lambda)x) = \Lambda_r(\lambda)z(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$, *i.e.* z is $\mathbf{r}$–homogeneous vector field of degree 0 that is also continuous on $S_r(1)$. Denote $B(1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ as the unit ball in $\mathbb{R}^n$, if

$$h < \inf_{x_i \in S_r(1), \Delta \in B(1)} \|z(x_i + \Delta)\|^{-1}$$

then the function $hz(x_i + \cdot) : B(1) \to B(1)$ is continuous on the convex compact set $B(1)$. Hence, using the Brouwer fixed-point theorem [2] we conclude that the last equation has a solution with respect to $\Delta \in B(1)$ for any $x_i \in S_r(1)$. The conclusion for any $x_i \in \mathbb{R}^n$ follows from Corollary 6 or Proposition 5.

3 Proof of Theorem 12

Defining $x_{i+1} = \Lambda_r(\|x_{i+1}\|_r)y_{i+1}$ with $y_{i+1} \in S_r(1)$, we obtain:

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= V(x_{i+1}) - V(x_{i+1} - \|x_{i+1}\|_r^{-\nu} hf(x_{i+1})) \\ &= \|x_{i+1}\|_r^{\mu} [V(y_{i+1}) - V(y_{i+1} - hf(y_{i+1}))] \\ &= \|x_{i+1}\|_r^{\mu} \left.\frac{\partial V(\xi)}{\partial \xi}\right|_{\xi = y_{i+1}} hf(y_{i+1}) \\ &-\|x_{i+1}\|_r^{\mu} \frac{h^2}{2} f^{\top}(y_{i+1}) \left.\frac{\partial^2 V(\xi)}{\partial \xi^2}\right|_{\xi = y_{i+1} - \theta hf(y_{i+1})} f(y_{i+1}) \end{aligned}$$

with application of the Taylor expansion theorem with Lagrange remainder [1] on the last step. Next, similarly

$$\left.\frac{\partial V(\xi)}{\partial \xi}\right|_{\xi = y_{i+1}} hf(y_{i+1}) \leq -ha$$

from (7), and if the matrix $\frac{\partial^2 V(\xi)}{\partial \xi^2}$ is nonnegative definite for all $\xi \in \mathbb{R}^n$, then

$$V(x_{i+1}) - V(x_i) \leq -ah\|x_{i+1}\|_r^{\mu}$$

for any $h > 0$. If this is not the case, then there is $w \in (0, +\infty)$ such that

$$\sup_{y \in S_r(1)} \sup_{\theta \in (0,1)} f^{\top}(y) \left.\frac{\partial^2 V(\xi)}{\partial \xi^2}\right|_{\xi = y - \theta f(y)} f(y) \leq w,$$

which results in

$$V(x_{i+1}) - V(x_i) \leq h\|x_{i+1}\|_r^{\mu} \left(\frac{h}{2}w - a\right)$$

and for all $h \in (0, h_0]$ with $h_0 = \min\{1, \frac{a}{w}\}$ we obtain:

$$V(x_{i+1}) - V(x_i) \leq -\frac{a}{2}h\|x_{i+1}\|_r^{\mu} \leq -\alpha h V(x_{i+1}),$$

where again $\alpha = \frac{a}{2c_2}$ and the property given in (7) was used on the last step. Finally,

$$V(x_{i+1}) \leq (1+\alpha h)^{-1} V(x_i)$$

for all $i = 0, 1, \ldots$ and the values of $V(x_i)$ are monotonously decreasing, hence the sequence $\{x_i\}_{i=0}^{\infty}$ is converging (the property (b) is substantiated). Global boundedness of $\{x_i\}_{i=0}^{\infty}$ (*i.e.* the property (a)) follows from the properties of the function V for $\gamma = (c_1^{-1}c_2)^{1/\mu}$.

The property (c) can be proven applying the same arguments as in the proof of Theorem 9.

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