Differential positivity with respect to cones of rank $k \geq 2$

C. Mostajeran * **R. Sepulchre** *

* *Department of Engineering, University of Cambridge, UK; (csm54@cam.ac.uk, r.sepulchre@eng.cam.ac.uk).*

Abstract: We consider a generalized notion of differential positivity of a dynamical system with respect to cone fields generated by cones of rank $k \geq 2$. The property refers to the contraction of such cone fields by the linearization of the flow along trajectories. It provides the basis for a generalization of differential Perron-Frobenius theory, whereby the Perron-Frobenius vector field which shapes the one-dimensional attractors of a differentially positive system is replaced by a distribution of rank k that results in k-dimensional integral submanifold attractors instead. We further develop the theory in the context of invariant cone fields and invariant differential positivity on Lie groups and illustrate the key ideas with an extended example involving consensus on the space of rotation matrices $SO(3)$.

Keywords: Positivity, Differential Analysis, Monotone Systems, Manifolds, Lie Groups, Consensus, Synchronization

1. INTRODUCTION

Differential analysis provides a framework for studying a nonlinear dynamical system by analyzing the linearization of the system at every point in the state space. The motivation is that the local behavior of a system often has strong implications for the global nonlinear behavior. A classical result in differential analysis is that of contraction analysis whereby a local measure of contraction is used to establish stability without the explicit construction of a distance function between converging trajectories. See Lohmiller and Slotine (1998) for details. For more recent developments in differential analysis, see Forni and Sepulchre (2014); van der Schaft (2013); Simpson-Porco and Bullo (2014).

A recent contribution to differential analysis is the notion of differentially positive systems introduced in Forni and Sepulchre (2015) as systems whose linearizations along trajectories are positive. Differential positivity is a natural extension of linear positivity theory to nonlinear systems. Recall that positive systems are linear systems that leave a cone invariant Bushell (1973). Positive systems find many applications in control engineering, including in stabilization Farina and Rinaldi (2000), observer design Bonnabel et al. (2011), and distributed control Moreau (2004). Perron-Frobenius theory illustrates how positivity restricts the asymptotic behavior of a system: if the system is strictly positive in the sense that it maps the boundary of a cone $\mathcal{K}$ into its interior, then the trajectories asymptotically align with the one-dimensional subspace spanned by the dominant eigenvector of the system.

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In this paper, we consider systems that are positive with respect to so-called cones of rank $k \geq 2$. These structures are a generalization of classical solid convex cones and lead to a weakened notion of monotonicity. For linear systems that are strictly positive with respect to a cone of rank $k \geq 2$, the one-dimensional dominant eigenspace of Perron-Frobenius theory is replaced by an eigenspace of dimension k; see Fusco and Oliva (1991). Cones of rank $k \geq 2$ have been used to study the existence of periodic orbits via a Poincare-Bendixson property for a new class of monotone systems (as in Smith (1980); Sanchez (2009, 2010)), including for cyclic feedback systems Mallet-Paret and Smith (1990); Mallet-Paret and Sell (1996). Here we develop a differential analytic formulation of positivity with respect to a cone field of rank $k \geq 2$, which is applicable to the study of nonlinear systems. We discuss how the property leads to a generalization of differential Perron-Frobenius theory, whereby the attractors of the system are shaped by a distribution of rank k.

The study of differentially positive systems requires the construction of a cone field, which assigns a cone to the tangent space at each point of the state space. In many applications, the state space is not a vector space, but is a nonlinear manifold. In particular, nonlinear spaces that are homogeneous and highly symmetric are ubiquitous in applications. An important feature of such spaces is that the basic structures used in the local analysis can be made invariant with respect to the symmetries of the space. There is a strong advantage to be gained from making the differential analysis invariant by including the symmetries of the state space in the analysis as much as possible. It is within this context that we discuss invariant differential positivity on Lie groups, which form an important class of homogeneous spaces. One motivation for the development of a theory of invariant differential positivity is to arrive at a theory of consensus on nonlinear spaces Sepulchre et al. (2008) which is based on the ideas of order and positivity. In many applications, the agents evolve on nonlinear manifolds that are Lie groups Sarlette et al. (2010); Tron et al. (2012) or homogeneous spaces Sarlette and Sepulchre (2009). For instance, oscillators evolve on the circle $\mathbb{S}^1$, satellite attitudes on $SO(3)$, and vehicles move in the Euclidean group $SE(3)$. Exploiting suitable notions of order and positivity on homogeneous spaces offers the possibility of extending existing successful positivity results in linear consensus to consensus theory on nonlinear spaces.

The paper is organized as follows. We begin with a review of the structure of cones of rank $k \geq 2$ and discuss the generation of invariant cone fields on Lie groups, with a focus on left-invariant cone fields as an example. In Section 3, we introduce the notion of differential positivity with respect to higher rank cone fields. In Section 4, we review invariant differential positivity on Lie groups (Mostajeran and Sepulchre (2016)) and provide a theorem which generalizes differential Perron-Frobenius theory to systems that are differentially positive with respect to invariant higher rank cone fields on Lie groups. In Section 5, we study an extended example involving consensus of N agents on $SO(3)$ through differential positivity with respect to a left-invariant cone field of rank $k = 3$ on $SO(3)^N$.

2. INVARIANT CONE FIELDS

2.1 Cones of rank $k \geq 2$

Throughout this paper, a cone is a closed set $\mathcal{K}$ in a vector space $\mathcal{V}$ that satisfies the following: (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for all $\lambda \in \mathbb{R}_{\geq 0}$, and (iii) $\mathcal{K} \cap -\mathcal{K} = \{0\}$. That is, it is assumed that the cone is pointed, closed, and convex. Furthermore, we assume that the cone is solid in the sense that it contains $n := \dim \mathcal{V}$ linearly independent vectors. We now generalize the notion of cones to cones of rank k, the simplest case of which is a cone of rank 1. Given a convex cone $\mathcal{K}$, the set $\mathcal{C} = \mathcal{K} \cup -\mathcal{K}$ defines a generalized cone of rank 1. The full generalization is given by the following definition.

Definition 1. A closed set $\mathcal{C}$ in a vector space $\mathcal{V}$ is said to be a cone of rank k if

(i) $x \in \mathcal{K}$, $\alpha \in \mathbb{R} \Rightarrow \alpha x \in \mathcal{K}$,
(ii) $\max\{\dim W : W \text{ a subspace of } \mathcal{V},\ W \subset \mathcal{K}\} = k$.

Note that if $\mathcal{K}$ is a convex cone, $\mathcal{C} = \mathcal{K} \cup -\mathcal{K}$ satisfies the above conditions for $k = 1$. Given a cone $\mathcal{C}$ of rank k, the closure of the set $\mathcal{V}\backslash\mathcal{C}$ is a cone of rank $n-k$, and is known as the complementary cone $\mathcal{C}^c$ of $\mathcal{C}$.

A polyhedral convex cone in a vector space $\mathcal{V}$ of dimension n endowed with an inner product $\langle \cdot, \cdot \rangle$ can be specified by a collection of inequalities of the form

$$x \in \mathcal{V} : \quad \langle n_i, x \rangle \geq 0, \tag{1}$$

where $\{n_1, \cdots, n_m\}$ is a set of $m \geq n$ vectors in $\mathcal{V}$ that span $\mathcal{V}$. For each i, the equation (1) defines a halfspace defined by the normal vector $n_i \in \mathcal{V}$. If we relax the requirement that $\{n_i\}$ span $\mathcal{V}$ and instead require that $\dim \text{span}\{n_i\} = l \leq n$, the collection of inequalities (1) define a convex set $\tilde{\mathcal{K}}$ of rank $k = n - l + 1$, such that the set $\mathcal{C} := \tilde{\mathcal{K}} \cup -\tilde{\mathcal{K}}$ is a generalized cone of rank $k = n-l+1$. In particular, $n - k + 1$ inequalities of the form (1), with linearly independent n_i, can be used to define a cone of rank k. A very simple example of a class of polyhedral cones of rank k is obtained from the positive orthant $\mathcal{K} = \mathbb{R}^n_+ = \{x_i : x_i \geq 0\}$ in $\mathbb{R}^n$ by eliminating $k - 1$ of the inequalities $x_i \geq 0$ and retaining the remaining ones. The resulting set $\tilde{\mathcal{K}}$ can be used to generate a generalized cone of rank k as $\mathcal{C} = \tilde{\mathcal{K}} \cup -\tilde{\mathcal{K}}$.

A second class of cones of rank k can be defined using quadratic forms. These cones are known as quadratic cones of rank k and are a generalization of the idea of quadratic cones of rank 1. Let P be a symmetric invertible $n \times n$ matrix with k positive eigenvalues and $n - k$ negative eigenvalues. Then the set

$$\mathcal{C}(P) = \{x \in \mathcal{V} : \langle x, Px \rangle \geq 0\}, \tag{2}$$

can be shown to define a cone of rank k. In particular, if P_1 is a $k \times k$ symmetric positive definite matrix and P_2 is an $(n-k) \times (n-k)$ symmetric positive definite matrix, the block diagonal matrix $P = \text{diag}(P_1, -P_2)$ has k positive eigenvalues and $n - k$ negative eigenvalues and defines a cone of rank k in $\mathbb{R}^n$ via the inequality

$$\begin{aligned} x^T P x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & -P_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1^T P_1 x_1 - x_2^T P_2 x_2 \geq 0, \end{aligned} \tag{3}$$

where $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^{n-k}$.

2.2 Invariant cone fields on a Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g} \cong T_eG$, where e denotes the identity element of G, and fix $a \in G$. The left and right translation maps $L_a, R_g : G \to G$ are defined by $L_a(g) = ag$ and $R_a(g) = ga$, respectively. For any point $g_0 \in G$ the diffeomorphism L_g induces a vector space isomorphism $dL_g|_{g_0} : T_{g_0}G \to T_{gg_0}G$. A vector field X on G is said to be left-invariant if $X_{ag} = dL_a|_g X_g$, for each $a, g \in G$. Note that a left-invariant vector field X can be uniquely generated by the vector X_e at identity, since $X_g = dL_g|_e X_e$. Similarly, one can use the vector space isomorphisms $dL_g|_e : T_eG \to T_gG$ to generate corresponding fields over the tangent bundle TG given objects in the tangent space T_eG at the identity element.

A pointed convex cone field $\mathcal{K}_G$ on G smoothly assigns a cone $\mathcal{K}_G(g) \subset T_gG$ to each point $g \in G$. This idea can clearly be generalized to cone fields of rank k for any $1 \leq k \leq n-1$, where $n = \dim G$, so that a smooth cone field $\mathcal{C}_G$ of rank k smoothly assigns a cone $\mathcal{C}_G(g)$ of fixed rank k to each point $g \in G$.

Definition 2. A cone field $\mathcal{C}_G$ of rank k on a Lie group G is said to be *left-invariant* if

$$\mathcal{C}_G(g_1 g_2) = dL_{g_1}\big|_{g_2} \mathcal{C}_G(g_2), \tag{4}$$

for all $g_1, g_2 \in G$.

Clearly a left-invariant cone field of any rank is fully characterized by the cone in the tangent space at identity $T_eG = \mathfrak{g}$. Specifically, given a cone $\mathcal{C}$ in $\mathfrak{g}$, the corresponding left-invariant cone field is described by $\mathcal{C}_G(g) = dL_g|_e \mathcal{C}$. For example, if we are given a polyhedral cone $\mathcal{C}$ of rank k in T_eG that is specified via a collection of inequalities

$\langle n_i, x\rangle_e \geq 0$ for $x \in T_eG$, where $\{n_1, \cdots, n_m\}$ is a set of m vectors in T_eG with $\dim \operatorname{span}\{n_i\} = n - k + 1 \leq n$, and $\langle \cdot, \cdot \rangle_e$ is an inner product on T_eG, then the corresponding left-invariant cone field of rank k can be defined by the collection of inequalities

$$\delta g \in T_gG : \quad \langle dL_g\big|_e n_i, \delta g\rangle_g \geq 0, \tag{5}$$

for all $g \in G$, where $\langle \cdot, \cdot \rangle_g$ is the unique left-invariant Riemannian metric corresponding to the inner-product $\langle \cdot, \cdot \rangle_e$ in T_eG.

Similarly, given a quadratic cone $\mathcal{C}$ of rank k in T_eG defined by $\langle x, Px\rangle_e \geq 0$ for $x \in T_eG$, where P is a symmetric invertible $n \times n$ matrix with k positive eigenvalues and $n-k$ negative eigenvalues, the corresponding left-invariant cone field is given by

$$\delta g \in T_gG : \quad \langle \delta g, dL_g\big|_e P dL_g^{-1}\big|_g \delta g\rangle_g \geq 0, \tag{6}$$

for all $g \in G$. Similarly, one can also define notions of right invariant cone fields of any rank k using the vector space isomorphisms $dR_g|_{g_0}$ induced by right translations on G.

3. DIFFERENTIAL POSITIVITY

3.1 Differential positivity with respect to pointed convex cone fields

Let $\mathcal{M}$ be a smooth manifold endowed with a smooth cone field $\mathcal{K}$ such that for each $x \in \mathcal{M}$, $\mathcal{K}(x) \subset T_x\mathcal{M}$ is a closed, convex, pointed cone. For a continuous-time dynamical system Σ on $\mathcal{M}$ governed by $\dot{x} = f(x)$, where f is a smooth vector field on $\mathcal{M}$, the system is said to be differentially positive with respect to the cone field $\mathcal{K}$ if $d\psi_t\big|_x \mathcal{K}(x) \subseteq \mathcal{K}(\psi_t(x))$ for all $x \in \mathcal{M}$ and $t \geq 0$, where $\psi_t(x)$ denotes the flow of Σ at time t starting at x. Here we have assumed that the system is forward complete so that any trajectory $\psi_{(\cdot)}(x_0)$ is defined on a time domain of the form $[t_0, \infty)$.

A cone field $\mathcal{K}$ on a manifold $\mathcal{M}$ induces a conal order $\prec$ on $\mathcal{M}$ and locally partially orders the manifold. A continuous piecewise smooth curve $\gamma : [t_0, t_1] \to \mathcal{M}$ is called a conal curve if $\gamma'(t) \in \mathcal{K}(\gamma(t))$, whenever the derivative exists. We write $x_1 \prec x_2$ for points $x_1, x_2 \in \mathcal{M}$ if there exists a conal curve starting at x_1 that ends at x_2. If the conal order is also antisymmetric, then it defines a partial order on $\mathcal{M}$, although this is generally not the case globally. In the case where $\mathcal{M} = \mathcal{V}$ is a vector space and $\mathcal{K}$ is a constant cone field, the conal order does indeed define a global partial order. A dynamical system defined on a vector space endowed with a partial order $\preceq$ is said to be monotone if for any $x_1, x_2 \in \mathcal{V}$, the trajectories ψ_t satisfy

$$x_1 \preceq x_2 \quad \Rightarrow \quad \psi_t(x_1) \preceq \psi_t(x_2). \tag{7}$$

It is a key observation that differential positivity in a linear space with respect to a constant cone field is precisely the local characterization of monotonicity.

A main contribution of Forni and Sepulchre (2015) is the generalization of Perron-Frobenius theory to the differential framework, whereby the the Perron-Frobenius eigenvector of linear positivity theory is replaced by a Perron-Frobenius vector field $w(x)$ whose integral curves shape the attractors of the system. Differential positivity can be particularly powerful on bounded forward invariant sets, where the existence of an integral curve of w whose image is an attractor for all the trajectories from S can be established under mild technical assumptions.

3.2 Differential positivity with respect to higher rank cones

A pointed convex cone $\mathcal{K}$ in $\mathbb{R}^n$ induces an order in $\mathbb{R}^n$ such that for any pair $x_1, x_2 \in \mathbb{R}^n$, we write $x_1 \preceq x_2$ if $x_2 - x_1 \in \mathcal{K}$. It is of course the convexity of $\mathcal{K}$ and the condition $\mathcal{K} \cap -\mathcal{K} = \{0\}$ that ensure that $\preceq$ defines a global partial order on $\mathbb{R}^n$. As a generalized cone $\mathcal{C}$ of rank k in $\mathbb{R}^n$ does not satisfy these conditions, it does not induce a partial order. Nonetheless, we can still define a weakened generalized notion of an order relation between two points $x_1, x_2 \in \mathbb{R}^n$ with respect to $\mathcal{C}$ in a similar fashion. Specifically, we say that x_1 and x_2 are ordered or related with respect to $\mathcal{C}$ and write $x_1 \sim x_2$ if $x_2 - x_1 \in \mathcal{C}$. Clearly, $x_1 \sim x_2$ if and only if $x_2 \sim x_1$. We say that x_1 and x_2 are strongly ordered and write $x_1 \approx x_2$ if $x_2 - x_1 \in \operatorname{int} \mathcal{C}$. One can now consider monotonicity of systems with respect to these weaker generalized order relations induced by cones of rank $k \geq 2$. Given a cone field $\mathcal{C}$ of rank k on a manifold $\mathcal{M}$, we say that two points $x_1, x_2 \in \mathcal{M}$ are ordered with respect to $\mathcal{C}$ and write $x_1 \sim x_2$ if there exists a conal curve γ connecting x_1 to x_2, so that $\gamma'(t) \in \mathcal{C}(\gamma(t))$, at all points along the curve. Indeed, if $\mathcal{C}$ is a cone field of rank k on a manifold $\mathcal{M}$, the continuous-time system $\dot{x} = f(x)$ on $\mathcal{M}$ with semiflow $\psi_t(x)$ is said to be *monotone* with respect to $\mathcal{C}$ if

$$x_1 \sim x_2 \quad \Rightarrow \quad \psi_t(x_1) \sim \psi_t(x_2), \tag{8}$$

for all $t > 0$. The system is *strongly monotone* with respect to $\mathcal{C}$ if $x_1 \sim x_2$, $x_1 \neq x_2 \Rightarrow \psi_t(x_1) \approx \psi_t(x_2)$, for all $t > 0$.

Definition 3. Let $\mathcal{M}$ be a smooth manifold endowed with a smooth cone field $\mathcal{C}$ of rank k and consider a continuous-time system Σ given by $\dot{x} = f(x)$ on $\mathcal{M}$ with positive semiflow $\psi_t(x)$. We say that Σ is differentially positive with respect to $\mathcal{C}$ if

$$d\psi_t\big|_x \mathcal{C}(x) \subseteq \mathcal{C}(\psi_t(x)) \tag{9}$$

for all $x \in \mathcal{M}$ and $t \geq 0$. Furthermore, we say that Σ is strictly positive if $d\psi_t\big|_x (\mathcal{C}(x) \setminus \{0_x\}) \subset \operatorname{int} \mathcal{C}(\psi_t(x))$, for all $x \in \mathcal{M}$ and $t \geq 0$.

Now if Σ is a monotone system with respect to $\mathcal{C}$, it is easy to see that an infinitesimal difference δx between neighbouring ordered solutions at x satisfies $d\psi_t\big|_x \delta x \in \mathcal{C}(\psi_t(x))$, for all $t \geq 0$. That is, monotonicity with respect to a cone field $\mathcal{C}$ of rank k implies differential positivity with respect to $\mathcal{C}$.

3.3 Differential Perron-Frobenius theory for cones of rank $k \geq 2$

The following result from Fusco and Oliva (1991) is a generalization of the classical Perron-Frobenius theorem to higher rank cones.

Theorem 4. Let $\mathcal{C}$ be a cone of rank k in a vector space $\mathcal{V}$ of dimension n and suppose that $T \in \mathcal{L}(\mathcal{V})$ is a strictly positive linear map with respect to $\mathcal{C}$ so that $T(\mathcal{C} \setminus \{0\}) \subset \operatorname{int} \mathcal{C}$. Then there exist unique subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{V}$ such that $\dim \mathcal{W}_1 = k$, $\dim \mathcal{W}_2 = n - k$, $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$, which are T-invariant:

$$T(\mathcal{W}_i) \subseteq \mathcal{W}_i, \quad \text{for} \quad i = 1, 2, \tag{10}$$

and satisfy

$$\mathcal{W}_1 \subset \operatorname{int}\mathcal{C} \cup \{0\}, \quad \mathcal{W}_2 \cap \mathcal{C} = \{0\}. \tag{11}$$

Furthermore, denoting the spectrum of T restricted to $\mathcal{W}_i$ by $\sigma_i(T)$ for $i = 1, 2$, we have

$$|\lambda_1| > |\lambda_2|, \quad \forall \lambda_1 \in \sigma_1(T),\ \lambda_2 \in \sigma_2(T). \tag{12}$$

By replacing the classical Perron-Frobenius theorem with this generalization and the notion of differential positivity with respect to a pointed convex cone field with that of differential positivity with respect to a cone field of rank $k \geq 2$, we arrive at a generalization of differential Perron-Frobenius theory whereby the attractors of the system are shaped not by a Perron-Frobenius vector field, but by a smooth distribution of rank k.

Recall that a smooth distribution $\mathcal{D}$ of rank k on a smooth manifold $\mathcal{M}$ is a rank-k smooth subbundle of $T\mathcal{M}$. A rank k distribution is often described by specifying a k-dimensional linear subspace $\mathcal{D}_x \subseteq T_x\mathcal{M}$ at each point $x \in \mathcal{M}$, and writing $\mathcal{D} = \cup_{x\in\mathcal{M}}\mathcal{D}_x$. It follows from the local frame criterion for subbundles that $\mathcal{D}$ is a smooth distribution if and only if each point $x \in \mathcal{M}$ has a neighborhood $\mathcal{U}$ on which there are smooth vector fields $X_1, ..., X_k$ such that $\{X_j|_{\tilde{x}} : j = 1, ..., k\}$ forms a basis for $\mathcal{D}_{\tilde{x}}$ at each point $\tilde{x} \in \mathcal{U}$. The distribution $\mathcal{D}$ is then said to be locally spanned by the vector fields X_j.

Given a smooth distribution $\mathcal{D} \subseteq T\mathcal{M}$, a nonempty immersed submanifold $\mathcal{N} \subseteq \mathcal{M}$ is said to be an *integral manifold* of $\mathcal{D}$ if

$$T_x\mathcal{N} = \mathcal{D}_x \quad \forall x \in \mathcal{N}. \tag{13}$$

The question of whether for a given distribution there exists an integral manifold is intimately connected to the notion of *involutivity* and characterized by the *Frobenius theorem*. A distribution $\mathcal{D}$ is said to be involutive if given any pair of smooth vector fields X, Y defined on an open subset of $\mathcal{M}$ such that $X_x, Y_x \in \mathcal{D}_x$ for each $x \in \mathcal{M}$, the Lie bracket $[X, Y]|_x$ also lies in $\mathcal{D}_x$. By the local frame criterion for involutivity, one can show that a distribution $\mathcal{D}$ is involutive if there exists a smooth local frame $\{X_j : j = 1, ..., k\}$ for $\mathcal{D}$ in a neighborhood of every point in $\mathcal{M}$ such that $[X_i, X_j]$ is a section of $\mathcal{D}$ for each $i, j = 1, ..., k$. The Frobenius theorem tells us that involutivity of a distribution is a necessary and sufficient condition for the existence of an integral manifold through every point Lee (2003).

4. INVARIANT DIFFERENTIAL POSITIVITY ON LIE GROUPS

Let G be a Lie group equipped with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle_g$ and consider a continuous-time dynamical system on G given by $\frac{d}{dt}g = f(g)$ for all $g \in G$, where f is a smooth vector field that assigns a vector $f(g) := X_g \in T_gG$ to each point $g \in G$. Let $\psi_t(g)$ denote the trajectory of Σ at time $t \in \mathbb{R}$ with initial point $g \in G$. Note that the flow $\psi_t : G \to G$ is a diffeomorphism with differential $d\psi_t|_g : T_gG \to T_{\psi_t(g)}G$. The linearization of Σ with respect to a left-invariant frame on G takes the form

$$\frac{d}{dt}\delta g = \lim_{t\to 0} \frac{dL_{g\psi_t(g)^{-1}}|_{\psi_t(g)} \circ d\psi_t|_g\, \delta g - \delta g}{t}. \tag{14}$$

To see this, note that the tangent vector $\delta g \in T_gG$ evolves under the flow to $d\psi_t|_g \delta g \in T_{\psi_t(g)}G$, which is then pulled back to T_gG using $dL_{g\psi_t(g)^{-1}}|_{\psi_t(g)}$ by left-invariance, in order to compute the derivative $\dot{\delta g}$ relative to a left-invariant frame.

The system on G can be rewritten as $\dot{g} = f(g) = g\,\Omega(g)$ for some smooth $\Omega : G \to \mathfrak{g}$. We identify $\mathfrak{g} = T_eG$ with $\mathbb{R}^n$ via the vectorization map $^\vee : \Omega \mapsto \Omega^\vee \in \mathbb{R}^n$. Thus, the system is characterized by the map $\Omega^\vee : G \to \mathbb{R}^n$. The push-forward of this map takes the form $d\Omega^\vee|_g : T_gG \to \mathbb{R}^n$ and maps tangent vectors $\delta g \in T_gG$ to vectors in $\mathbb{R}^n$. The linearization takes the form

$$\frac{d}{dt}\delta g = dL_g\big|_e \circ d\Omega^\vee\big|_g \delta g = g\, d\Omega^\vee\big|_g(\delta g). \tag{15}$$

Since we are working with a left-invariant cone field, we can equivalently consider the linear map

$$A(g) := d\Omega^\vee\big|_g \circ dL_g\big|_e : \mathbb{R}^n \to \mathbb{R}^n \tag{16}$$

for each $g \in G$, where $\mathbb{R}^n$ is identified with $\mathfrak{g}$ through the $^\vee$ map. Differential positivity with respect to an invariant cone field of rank k generated by a cone $\mathcal{C} \subset \mathbb{R}^n$ reduces to the positivity of the linear map $\dot{x} = A(g)x$ with respect to $\mathcal{C}$ for all $g \in G$.

The following theorem provides an explicit generalization of differential Perron-Frobenius theory to systems that are invariantly differentially positive with respect to higher rank cone fields on Lie groups. In the interest of brevity, we have elected to include the proof in a future journal publication which expands upon this work. Instead, we shall focus on the application of the theorem in this manuscript.

Theorem 5. Let Σ be a uniformly strictly differentially positive system with respect to a left-invariant cone field $\mathcal{C}$ of rank k in a bounded, connected, and forward invariant region $S \subseteq G$ of a Lie group G equipped with a bi-invariant Riemannian metric. If the distribution of rank k of dominant eigenspaces of linearizations of Σ in S is involutive and satisfies

$$\limsup_{t\to\infty} \|d\psi_t|_g w(g)\|_{\psi_t(g)} < \infty, \tag{17}$$

for every $g \in S$, $t \geq 0$, and $w(g) \in \mathcal{D}_g$, then there exists an integral manifold $\mathcal{N}$ of $\mathcal{D}$ that is an attractor for all the trajectories of Σ from S.

5. CONSENSUS ON $SO(3)$

Let G be a compact Lie group with a bi-invariant Riemannian metric. Consider a network of N agents g_k represented by an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ evolving on G. For a given element $g_k \in G$, the Reimannnian exponential and logarithm maps are denoted by $\exp_{g_k} : T_{g_k}G \to G$ and $\log_{g_k} : U_{g_k} \to T_{g_k}G$, respectively, where $U_{g_k} \subset G$ is the maximal set containing g_k for which $\exp_{g_k}$ is a diffeomorphism. For any commnication edge $(k, i) \in \mathcal{E}$, define

$$\theta_{ki} = d(g_k, g_i) \quad \text{and} \quad u_{ki} = \frac{\log_{g_k} g_i}{\|\log_{g_k} g_i\|}, \tag{18}$$

where d denotes the Riemannian distance on G. Let $\operatorname{inj}(G)$ denote the injectivity radius of G. A class of consensus dynamics can be defined on G by

$$\dot{g}_k = \sum_{i:(k,i)\in\mathcal{E}} f(\theta_{ki})\, u_{ki}, \tag{19}$$

where $f : [0, \mathrm{inj}(G)] \to \mathbb{R}$ is a suitable reshaping function that is differentiable on $(0, \mathrm{inj}(G))$ and satisfies $f(0) = 0$.

Here we consider a network of N agents evolving on the space of rotations $SO(3)$. Associate to each agent a state $g_k \in SO(3) = \{R \in \mathbb{R}^{3\times 3} : R^T R = I,\ \det R = 1\}$, where $e = I$ denotes the identity element and matrix in $SO(3)$. The Lie algebra of $SO(3)$ is the set of 3×3 skew symmetric matrices, and is denoted by $\mathfrak{so}(3)$. For any tangent vector $\delta g_k \in T_{g_k}G$, there exist $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^3$ such that

$$\delta g_k = g_k \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = g_k \Omega, \tag{20}$$

where $\Omega \in \mathfrak{so}(3)$. We can thus identify any $\delta g_k \in T_{g_k}G$ with a vector $\Omega^\vee = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ via the $^\vee$ map.

Assume that $SO(3)$ is equipped with the standard bi-invariant metric characterized by $\langle \Omega_1, \Omega_2 \rangle_{\mathfrak{so}(3)} = (\Omega_1^\vee)^T \Omega_2^\vee$, for $\Omega_1, \Omega_2 \in \mathfrak{so}(3)$, and let ∇ denote its Levi-Civita connection. Consider a consensus protocol of the form $\dot{g}_k = \sum_{i:(k,i)\in\mathcal{E}} f(\theta_{ki}) u_{ki}$, where the reshaping function $f : [0, \pi] \to \mathbb{R}$ is differentiable on $(0, \pi)$, satisfies $f(0) = 0$, $f(\theta) > 0$ and $f'(\theta) > 0$ for $\theta \in (0, \pi)$.

We consider the linearization of (19) in the form

$$\nabla_{\delta g_k} \left(\sum_{i:(k,i)\in\mathcal{E}} f(\theta_{ki}) \, \frac{\log_{g_k} g_i}{\|\log_{g_k} g_i\|} \right), \tag{21}$$

which is a measure of the change in $\sum_{i:(k,i)\in\mathcal{E}} f(\theta_{ki}) u_{ki}$ when g_k changes infinitesimally in the direction of $\delta g_k \in T_{g_k}G$. The term corresponding to the communication edge $(k, i) \in \mathcal{E}$ in this linearization can be expressed using a symmetric operator acting on two tangent vectors where

$$A_{ki}(\xi_{g_i}, \eta_{g_i}) := \langle \nabla_{\xi_{g_i}} \left(f(\theta_{ki}) u_{ki} \right), \eta_{g_i} \rangle_{g_i}, \tag{22}$$

for $\xi_{g_i}, \eta_{g_i} \in T_{g_i}G$. In the subsequent analysis, we make use of the following result from Ferreira et al. (2013).

Lemma 6. Let $B(g_k)$ be a sufficiently large geodesic ball centred at g_k and $\gamma : [0, \theta_{ki}] \to B(g_k)$ denote the unit speed geodesic from g_k to $g_i \in B(p)$. Then

$$\begin{aligned} A_{ki}(\xi_{g_i}, \eta_{g_i}) &= f'(\theta_{ki}) \langle \xi_{g_i}^{\parallel}, \eta_{g_i} \rangle_{g_i} \\ &\quad + \frac{f(\theta_{ki})}{\theta_{ki}} \sum_{j=1}^{3} c_{\lambda_j}(\theta_{ki}) \langle \xi_{g_i}^{\perp}, E_{g_i}^j \rangle \langle \eta_{g_i}, E_{g_i}^j \rangle, \end{aligned} \tag{23}$$

where $\{E_{g_i}^j\}$ is an orthonormal basis which diagonalizes the linear operator $\mathcal{R} : T_{g_i}G \to T_{g_i}G$, $\mathcal{R}(\xi_{g_i}) = R(\xi_{g_i}, \dot{\gamma}_{g_i})\dot{\gamma}_{g_i}$ with eigenvalues λ_j: $\mathcal{R}(E_{g_i}^j) = \lambda_j E_{g_i}^j$, where R denotes the curvature endomorphism on the Riemannian manifold G. The coefficients c_{λ_j} in (23) are given by

$$c_\lambda(t) = \begin{cases} \sqrt{\lambda} t / \tan(\sqrt{\lambda} t) & \lambda > 0 \\ 1 & \lambda = 0 \\ \sqrt{-\lambda} t / \tanh(\sqrt{-\lambda} t) & \lambda < 0, \end{cases} \tag{24}$$

and the $\parallel$ and $\perp$ signs denote parallel and perpendicular components of the vector with respect to the velocity vector of γ, respectively.

The formula in (23) simplifies considerably when the space G has constant sectional curvature as is the case for $SO(3)$. For such spaces, the eigenvalues of the operator $\mathcal{R}$ are constant and equal to the sectional curvature, except for one, which is 0 and corresponds to the eigenvector $\dot{\gamma}$. This result is used to obtain the matrix representation of the linearization of the consensus dynamics that follows.

Writing $\boldsymbol{g} = (g_1, \cdots, g_N)$, the dynamical system can be expressed as $\frac{d}{dt}\boldsymbol{g} = F(\boldsymbol{g})$, where F is a smooth vector field on $SO(3)^N$. The linearization of the system can be expressed in the form

$$\frac{d}{dt}\boldsymbol{x} = \mathcal{A}(\boldsymbol{g})\, \boldsymbol{x}, \tag{25}$$

where $\boldsymbol{x} \in \mathbb{R}^{3N}$ is the vector representation of

$$(\delta g_1, \cdots, \delta g_N) \in T_{\boldsymbol{g}} SO(3)^N \tag{26}$$

with respect to a left-invariant frame $\{E_k^l|_{\boldsymbol{g}}\}_{l=1,2,3;k=1,\cdots,N}$ of $SO(3)^N$ formed as the Cartesian product of N copies of a left-invariant orthonormal frame of $SO(3)$. For each $\boldsymbol{g}$, the linear map $\mathcal{A}(\boldsymbol{g})$ has the $3N \times 3N$ matrix representation of the form

$$\begin{cases} \mathcal{A}_{kk}(\boldsymbol{g}) &= -\sum_{i:(k,i)\in\mathcal{E}} A(\theta_{ki}), \\ \mathcal{A}_{ki}(\boldsymbol{g}) &= A(\theta_{ki}) \quad \text{if} \quad (k,i) \in \mathcal{E}, \\ \mathcal{A}_{ki}(\boldsymbol{g}) &= 0 \quad \text{if} \quad (k,i) \notin \mathcal{E}. \end{cases} \tag{27}$$

with respect to the orthonormal basis $\{E_k^l|_{\boldsymbol{g}}\}$ of $\mathbb{R}^{3N}$, where $A(\theta_{ki})$ is a 3×3 block matrix whose spectrum is given by

$$\left\{ f'(\theta_{ki}),\ \frac{1}{2} f(\theta_{ki}) \cot\left(\frac{\theta_{ki}}{2}\right),\ \frac{1}{2} f(\theta_{ki}) \cot\left(\frac{\theta_{ki}}{2}\right) \right\}. \tag{28}$$

Thus, for $\theta_{ki} \in (0, \pi)$, $A(\theta_{ki})$ is positive definite.

Let $\mathbf{1}_1$ denote the vector consisting of N copies of $\boldsymbol{e}_1 = (1, 0, 0)^T$ with respect to the orthonormal basis $\{E_k^l\}$ of $T_{\boldsymbol{g}} SO(3)^N \cong \mathbb{R}^{3N}$. Similarly, define $\mathbf{1}_2$ and $\mathbf{1}_3$ using N copies of $\boldsymbol{e}_2 = (0, 1, 0)^T$ and $\boldsymbol{e}_3 = (0, 0, 1)^T$, respectively. We define the invariant cone field $\mathcal{C}_{SO(3)^N}(\boldsymbol{g}, \delta\boldsymbol{g})$ of rank 3 by

$$\begin{aligned} Q(\boldsymbol{x}) := \boldsymbol{x}^T \mathbf{1}_1 {\mathbf{1}_1}^T \boldsymbol{x} + \boldsymbol{x}^T \mathbf{1}_2 {\mathbf{1}_2}^T \boldsymbol{x} + \boldsymbol{x}^T \mathbf{1}_3 {\mathbf{1}_3}^T \boldsymbol{x} \\ -\mu\, \boldsymbol{x}^T \boldsymbol{x} \geq 0, \end{aligned} \tag{29}$$

where $\mu \in (0, 3N)$ is a parameter and $\boldsymbol{x}$ is the vector representation of $\delta\boldsymbol{g} \in T_{\boldsymbol{g}} SO(3)^N$ with respect to the orthonormal basis $\{E_k^l\}$ of $T_{\boldsymbol{g}} SO(3)^N \cong \mathbb{R}^{3N}$. Observe that $\mathcal{D} = \mathrm{span}\{\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3\} \subset \mathrm{int}\, \mathcal{C}$, at every point $\boldsymbol{g} \in SO(3)^N$. Furthermore, it is not difficult to see that $\mathcal{D}$ is an involutive left-invariant distribution of rank 3 with an integral manifold isomorphic to a single copy of $SO(3)$ diagonally embedded in $SO(3)^N$. In particular, the consensus manifold $\mathcal{M}_{\text{sync}} = \{\boldsymbol{g} \in SO(3)^N : g_1 = \cdots = g_N\}$ is the integral manifold through the identity element $\boldsymbol{e} \in SO(3)^N$.

Noting that $\mathcal{A}(\boldsymbol{g})\mathbf{1}_j = 0$, for $j = 1, 2, 3$, we find that the time derivative of Q along trajectories of the variational dynamics takes the form

$$\frac{d}{dt} Q(\boldsymbol{x}) = -2\mu\, \boldsymbol{x}^T \mathcal{A}(\boldsymbol{g}) \boldsymbol{x} \tag{30}$$

$$= 2\mu \sum_{(k,i)\in\mathcal{E}} (\boldsymbol{x}_k - \boldsymbol{x}_i)^T A(\theta_{ki}) (\boldsymbol{x}_k - \boldsymbol{x}_i) \geq 0, \tag{31}$$

where $\boldsymbol{x}_k \in \mathbb{R}^3$ consists of the elements of $\boldsymbol{x} \in \mathbb{R}^{3N}$ at the entries $3k-2$, $3k-1$, $3k$. It is clear that for a connected graph $\dot{Q} > 0$, unless $\boldsymbol{x}_i = \boldsymbol{x}_k$ for all i, k. This demonstrates strict differential positivity of the consensus dynamics with respect to the cone field $\mathcal{C}$ for the monotone coupling function f, whenever $\theta_{ki} < \pi$. Thus, by Theorem 5, for any bounded, connected, and forward invariant region $S \subseteq \{\boldsymbol{g} \in SO(3)^N : d(g_i, g_k) < \pi, \ \forall (i,k) \in \mathcal{E}\}$, there exists a unique integral manifold of $\mathcal{D}$ that is an attractor for all of the trajectories from S. In particular, if $d(g_i, g_k) < \pi/2$ for all $g_i, g_k \in SO(3)$, then the attractor is clearly the three-dimensional synchronization manifold $\mathcal{M}_{\text{sync}} \cong SO(3)$.

6. CONCLUSION

A generalization of linear positivity theory is obtained when one replaces the notion of a dominant eigenvector with that of a dominant eigenspace of dimension $k \geq 2$. For such systems, it is natural to characterize positivity by the contraction of a generalized cone of rank k in place of a convex cone as in classical positivity theory. One can naturally extend this generalization to nonlinear systems by introducing the notion of differential positivity with respect to cone fields of rank $k \geq 2$. The resulting theory is a generalization of differential Perron-Frobenius theory whereby a distribution of rank k consisting of dominant eigenspaces of linearizations of the system shapes the attractors of the system. As illustrated with an example of consensus on $SO(3)$, this framework can be used to study systems whose attractors arise as integral submanifolds of the distribution.

In future work, we will seek to exploit the framework of invariant differential positivity with respect to polyhedral higher rank cone fields to study consensus on Lie groups for networks with directed and time-varying graphs, or coupling functions f_{ik} that are not symmetric in the indices i, k. These networks give rise to consensus dynamics that cannot be formulated as gradient dynamics and hence cannot be tackled via quadratic Lyapnunov theory. It is for these systems that the differential positivity approach may prove to be particularly powerful.

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