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Invariant Differential Positivity and Consensus on Lie Groups

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Abstract: Differential positivity of a dynamical system refers to the property that its linearization along trajectories is positive, that is, infinitesimally contracts a smooth cone field defined in the tangent bundle. The property can be thought of as a generalization of monotonicity, which is differential positivity in a linear space with respect to a constant cone field. Differential positivity induces a conal order which places significant constraints on the asymptotic behavior of solutions. This paper studies differentially positive systems defined on Lie groups, which constitute an important and basic class of manifolds with the structure of a homogeneous space. The geometry of a Lie group allows for the generation of invariant cone fields over the tangent bundle given a single cone in the Lie algebra. We outline the mathematical framework for studying differential positivity of a nonlinear flow on a Lie group with respect to an invariant cone field and motivate the use of this analysis framework in nonlinear control, and, in particular in nonlinear consensus theory.

Keywords: Differential Analysis, Positivity, Monotone Systems, Nonlinear Spaces, Manifolds, Lie Groups, Consensus, Synchronization

1. INTRODUCTION

In *differential analysis* of a dynamical system, we seek to derive statements about the global behavior of a nonlinear system by studying the linearization of the system, also known as the *variational* or *prolonged system*. The key idea is that the local behavior of a system can often capture important aspects of the global behavior. The prominent illustration of this phenomenon in nonlinear control is contraction theory, as proposed in the seminal work of Lohmiller and Slotine (1998). Contraction analysis theory exploits the property that a local or infinitesimal measure of contraction can be used to infer stability without the need to present an explicit construction of distance between converging trajectories. See also Sontag (2010); Forni and Sepulchre (2014b); van der Schaft (2013); Forni and Sepulchre (2013); Forni et al. (2013); Simpson-Porco and Bullo (2014) for recent developments of this framework.

One important motivation for a differential analysis is when the state space of the system is not a vector space, but a nonlinear manifold. In many applications, the manifold is not arbitrary but has a *homogeneous structure* in that it is a *quotient manifold* or a *homogeneous space* in the sense of Lie theory. A homogeneous space is defined as a manifold on which a Lie group acts in a transitive way, meaning that any point on the manifold can be mapped onto any other point by an element of a group of transformations that act on the space. Such a space can be endowed with the structure of a quotient manifold in a natural way. A fundamental significance of such spaces is that the *local window* at each point can be made the same everywhere. That is, local descriptions of laws can be made independent of the locality of the data window.

There is a strong incentive in such situations to make the differential analysis *invariant*, that is, to *a priori* include the symmetry of the state-space in the analysis. For instance, invariant contraction analysis would seek to study contraction with respect to an *invariant* Riemannian metric on a given homogeneous space. In this case, the homogeneity of the metric ensures that local measures of length are invariant over the manifold.

The present paper focuses on *differentially positive* dynamical systems on Lie groups, which constitute an important class of homogeneous spaces. The notion of differentially positive systems was introduced by Forni and Sepulchre (2015) as systems whose linearizations along trajectories are positive. Recall that positive systems are defined as linear behaviors that leave a cone invariant Bushell (1973). Such systems find many applications in control engineering, including to stabilisation Muratori and Rinaldi (1991); Farina and Rinaldi (2000), observer design Bonnabel et al. (2011), and distributed control Moreau (2004). An important feature of positivity is that it restricts the behavior of a system, as seen in Perron Frobenius theory. To illustrate these ideas, let the vector space $\mathcal{V}$ be the state space of the system and consider the linear dynamics $\dot{x} = Ax$ on $\mathcal{V}$. Such a system is said to be positive with respect to a pointed convex solid cone $\mathcal{K} \subseteq \mathcal{V}$ if $e^{At}\mathcal{K} \subseteq \mathcal{K}$, for all $t > 0$, where $e^{At}\mathcal{K} := \{e^{At}x : x \in \mathcal{K}\}$. Perron Frobenius theory demonstrates that if the system is strictly positive in the sense that the transition map e^{At} maps the boundary of the cone $\mathcal{K}$ into its interior, then any trajectory $e^{At}x$,

★ C. Mostajeran is supported by the Engineering and Physical Sciences Research Council (EPSRC) of the United Kingdom.

$x \in \mathcal{K}$, converges asymptotically to the subspace spanned by the unique dominant eigenvector of A.

The study of differential positivity of a system defined on a manifold necessitates the construction of a cone field which assigns to each point a cone that lies in the tangent space at that point. Naturally, we are interested in constructing cone fields that capture some form of invariance with respect to the underlying symmetries of the nonlinear spaces of interest.

A key motivating force behind the development of invariant differential positivity is to develop a theory of consensus on nonlinear spaces Sepulchre et al. (2008). The theory of consensus in linear spaces is well understood and relies heavily on the ideas of order and positivity. Consensus algorithms on linear spaces have been developed and shown to be efficient and robust Moreau (2004); Olfati-Saber (2006); Olfati-Saber et al. (2007); Jadbabaie and Lin (2003). In many applications, the agents evolve on nonlinear manifolds that are Lie groups Sarlette et al. (2010); Tron et al. (2012) or homogeneous spaces Sarlette and Sepulchre (2009). For example, oscillators evolve on the circle $\mathbb{S}^1$, satellite attitudes on $SO(3)$, and vehicles move in $SE(3)$. Exploiting suitable notions of order and positivity on homogeneous spaces offers the possibility of adapting the existing successful approaches of consensus theory on linear spaces to consensus on homogeneous spaces.

2. POSITIVITY, MONOTONICITY, AND DIFFERENTIAL POSITIVITY

2.1 Monotonicity as Differential Positivity

We briefly review the concept of differential positivity, recently introduced in Forni and Sepulchre (2015), emphasizing its relationship to the classical notions of positivity and monotonicity.

A cone in a vector space $\mathcal{V}$ is a subset $\mathcal{K} \subset \mathcal{V}$ that satisfies (i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, (ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for all $\lambda \in \mathbb{R}_{\geq 0}$, and (iii) $\mathcal{K} \cap -\mathcal{K} = \{0\}$. That is, we assume that a cone is closed, convex, and pointed. Furthermore, we assume that we are dealing with solid cones that contain $n := \dim \mathcal{V}$ linearly independent vectors. We define a *conal manifold* as a smooth manifold $\mathcal{M}$ endowed with a *cone field* $\mathcal{K}_{\mathcal{M}}$, which smoothly assigns a cone $\mathcal{K}_{\mathcal{M}}(x) \subseteq T_x\mathcal{M}$ to each point $x \in \mathcal{M}$.

A cone field $\mathcal{K}_{\mathcal{M}}$ gives rise to a *conal order* $\prec$ and the manifold $\mathcal{M}$ is said to be an *infinitesimally partially ordered* manifold when endowed with $\mathcal{K}_{\mathcal{M}}$. A continuous piecewise smooth curve $\gamma : [t_0, t_1] \to \mathcal{M}$ is called a *conal curve* if $\gamma'(t) \in \mathcal{K}_{\mathcal{M}}(\gamma(t))$, whenever the derivative exists. For points $x, y \in \mathcal{M}$, we write $x \prec y$ if there exists a conal curve $\gamma : [t_0, t_1] \to \mathcal{M}$ with $\gamma(t_0) = x$ and $\gamma(t_1) = y$. If the conal order is also antisymmetric, then it is a partial order. It is clear that $\prec$ defines a global partial order when $\mathcal{M}$ is a vector space and the cone field $\mathcal{K}_{\mathcal{M}}(x) = \mathcal{K}_{\mathcal{M}}$ is constant. Specifically, $x \prec y$ if and only if $y - x \in \mathcal{K}_{\mathcal{M}}$, for $x,\ y \in \mathcal{M}$. In general, however, $\prec$ is not a global partial order, since antisymmetry may fail. A simple example of this is provided by any conal order defined on the circle $\mathbb{S}^1$, which clearly fails to be global.

Let $\mathcal{M}$ be a smooth n-dimensional Riemannian manifold whose metric tensor is given by the smoothly varying inner product $\langle \cdot, \cdot \rangle_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$. Assume that $(\mathcal{M}, d)$ is a complete metric space, where $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ is the distance function on $\mathcal{M}$ induced by the norm $|\delta x| := \sqrt{\langle \delta x, \delta x \rangle}$, for $\delta x \in T_x\mathcal{M}$. Given a smooth map $F : \mathcal{M}_1 \to \mathcal{M}_2$ between smooth manifolds $\mathcal{M}_1$, $\mathcal{M}_2$, denote the differential of F at x by $dF(x) : T_x\mathcal{M}_1 \to T_{F(x)}\mathcal{M}_2$. Now consider a dynamical system Σ on $\mathcal{M}$ given by $\dot{x} = f(x)$, where f is a smooth vector field that assigns a tangent vector $f(x) \in T_x\mathcal{M}$ to each point $x \in \mathcal{M}$. We assume that the system is *forward complete* in the sense that the domain of any solution $x(\cdot)$ is of the form $[t_0, \infty)$.

Definition 1. The dynamical system Σ is said to be *differentially positive* with respect to the cone field $\mathcal{K}$ if

$$d\psi_t(x)\mathcal{K}(x) \subseteq \mathcal{K}(\psi_t(x)) \quad \forall x \in \mathcal{X}, \forall t \geq 0 \tag{1}$$

where $\psi_t(x)$ denotes the flow at time t from initial condition x. The system is said to be *strictly differentially positive* if in addition the differential $d\psi_T(x)$ uniformly maps the boundary of $\mathcal{K}(x)$ into the interior of $\mathcal{K}(\psi_T(x))$ for some $T > 0$.

The general definition of differentially positive systems requires the definition of a smooth cone field. The special case of differential positivity on a linear space with respect to a constant cone field is important for this paper in that it highlights the fact that invariant differential positivity in a linear space is precisely the local characterization of monotonicity. Indeed, recall that a dynamical system Σ on a vector space $\mathcal{V}$ endowed with a partial order $\preceq$ induced by some cone $\mathcal{K} \subseteq \mathcal{V}$ is said to be monotone if for any $x_1, x_2 \in \mathcal{V}$ the trajectories ψ_{t_0} satisfy

$$x_1 \preceq_{\mathcal{K}} x_2 \quad \Rightarrow \quad \psi_{t_0}(t, x_1) \preceq_{\mathcal{K}} \psi_{t_0}(t, x_2), \tag{2}$$

for all $t \geq t_0$. Now endow the manifold $\mathcal{V}$ with the constant cone field $\mathcal{K}_{\mathcal{V}}(x) := \mathcal{K}_{\mathcal{V}}$ and note that the infinitesimal difference $\delta x(\cdot) := \hat{x}(\cdot) - x(\cdot)$ between two ordered neighbouring solutions $x(t) \preceq_{\mathcal{K}_{\mathcal{V}}} \hat{x}(t)$ satisfies $\delta x(t) \in \mathcal{K}_{\mathcal{V}}(x(t))$, $\forall t \geq t_0$. Since $(x(\cdot), \delta x(\cdot))$ is a trajectory of the prolonged or variational system $\delta\Sigma$, this shows that the system is monotone if and only if it is differentially positive. That is, the system is monotone if and only if for any initial time t_0 and all $t \geq t_0$,

$$\delta x(t_0) \in \mathcal{K} \quad \Rightarrow \quad \delta x(t) \in \mathcal{K}. \tag{3}$$

A linear space is of course a Lie group with the group operation given by linear translations. A constant cone field thus has the direct interpretation of a cone field first defined at identity and then translated at every point in an invariant manner, see Section 3 for details.

2.2 The Hilbert Metric

Given a cone $\mathcal{K}$ in $\mathbb{R}^N$, let $M_{\mathcal{K}}(v_1, v_2) := \inf\{\lambda \in \mathbb{R}_{\geq 0} : \lambda v_2 - v_1 \in \mathcal{K}\}$, $m_{\mathcal{K}}(v_1, v_2) := \sup\{\lambda \in \mathbb{R}_{\geq 0} : v_1 - \lambda v_2 \in \mathcal{K}\}$, and $M_{\mathcal{K}}(v_1, v_2) := \infty$ when $\{\lambda \in \mathbb{R}_{\geq 0} : \lambda v_2 - v_1 \in \mathcal{K}\} = \varnothing$. The Hilbert metric $d_{\mathcal{K}}$ induced by $\mathcal{K}$ is defined as

$$d_{\mathcal{K}}(v_1, v_2) = \log\left(\frac{M(v_1, v_2)}{m(v_1, v_2)}\right), \tag{4}$$

for all $v_1, v_2 \in \mathcal{K} \setminus \{0\}$. Note that $d_{\mathcal{K}}$ is a projective metric with the invariance property $d_{\mathcal{K}}(\lambda_1 x_1, \lambda_2 x_2) = d_{\mathcal{K}}(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{K}$ and $\lambda_1, \lambda_2 > 0$ Bushell (1973). The following important theorem due to Birkhoff establishes a link between positivity and contraction of the Hilbert metric Birkhoff (1957).

Theorem 2. Let F be a map defined on a cone $\mathcal{K}$ which satisfies: (i) F maps the cone $\mathcal{K}$ to its interior, (ii) F is homogeneous of degree p in $\operatorname{int}\mathcal{K}$, i.e. $F(\lambda x) = \lambda^p F(x)$, and (iii) F is monotone with respect to the order induced by $\mathcal{K}$: $x \preceq y \Rightarrow F(x) \preceq F(y)$. Then F is a contraction for the Hilbert metric $d_{\mathcal{K}}$. Furthermore, for each $x_1, x_2 \in \mathcal{K}$,

$$d_{\mathcal{K}}(F(x_1), F(x_2)) \leq \tanh\left(\frac{1}{4}\Delta(F)\right) d_{\mathcal{K}}(x_1, x_2), \qquad (5)$$

where $\Delta(F) := \sup_{x_1,x_2\in\mathcal{K}} d_{\mathcal{K}}(F(x_1), F(x_2))$, is known as the projective diameter of F.

In particular, note that a linear map A that is positive monotone with respect to a cone $\mathcal{K}$ satisfies all three conditions of Theorem 2. The Banach contraction mapping theorem applied to Birkhoff's theorem yields a powerful generalization of the Perron-Frobenius theorem. In the following $\sim$ denotes the equivalence relation on $\operatorname{int}\mathcal{K}$ defined by $x_1 \sim x_2$ if and only if there exists $\lambda > 0$ such that $x_2 = \lambda x_1$.

Theorem 3. If $(\operatorname{int}\mathcal{K}/\sim, d_{\mathcal{K}})$ is a complete metric space and $\Delta(F) < \infty$, then there exists a unique eigenvector of F in $\mathcal{K}$.

2.3 Consensus Theory in Linear Spaces

Continuous time linear consensus algorithms result in time-varying systems of the form

$$\dot{x} = A(t)x, \qquad (6)$$

where $A(t)$ is a *Metzler* matrix. That is, its rows sum to zero and its off-diagonal elements are non-negative:

$$A(t)\mathbf{1} = 0, \quad \text{and} \quad A_{ij}(t) \geq 0 \quad \text{for} \quad i \neq j, \qquad (7)$$

where $\mathbf{1} = (1, \dots, 1)^T$. Protocols of the form (6) arise from N nodes exchanging information about a scalar quantity $x_i(t)$ along communication edges (i, j) weighted by non-negative scalars a_{ij}:

$$\dot{x}_k = \sum_{i\in N_k} a_{ki}(x_i - x_k), \qquad (8)$$

where N_k denotes the set of all neighboring nodes of node k in the communication graph $(\mathcal{V}, \mathcal{E})$.

Tsitsiklis (1984) observed that the Lyapunov function

$$V(x) = \max_{1\leq i\leq N} x_i - \min_{1\leq i\leq N} x_i \qquad (9)$$

is never increasing along solutions of (6). Under suitable connectedness assumptions, it decreases uniformly in time. The non-quadratic nature of the Lyapunov function (9) is a key feature in the analysis of consensus algorithms. This property is intimately connected to the Hilbert metric. It is easy to show that the linear system (6) is positive with respect to the positive orthant $\mathcal{K} := \mathbb{R}^N_+$ in $\mathbb{R}^N$ as $A(t)$ is Metzler. Since $A(t)\mathbf{1} = 0$, Birkhoff's theorem implies that

$$d_{\mathcal{K}}(x(t), \mathbf{1}) \leq d_{\mathcal{K}}(x(0), \mathbf{1}), \qquad (10)$$

for all t. Thus, Birkhoff's theorem yields the Lyapunov function

$$V_B(x) = d_{\mathcal{K}}(x, \mathbf{1}) = \log\frac{\max_i x_i}{\min_i x_i} = \max_i \log x_i - \min_i \log x_i, \qquad (11)$$

which is clearly the same as the Tsitsiklis Lyapunov function in log coordinates. Furthermore, the contraction of the Hilbert metric (10) is strict provided that the positivity of $\dot{x} = A(t)x$ is strict, which is the case if the off-diagonal elements $A_{ij}(t)$ are positive.

The seminal paper of Moreau (2005) highlights the underlying geometry of the consensus algorithm (6), which is that the convex hull of the states $\{x_1, x_2, \dots, x_n\}$ never expands under the consensus update. The Lyapunov function (9) is a measure of the diameter of the convex hull. This insight leads to a number of nonlinear generalizations of consensus theory. For instance, the linear update (6) can be replaced by an arbitrary monotone update, without altering the convergence analysis.

A key contribution of Forni and Sepulchre (2015) is the generalization of Perron-Frobenius theory to the differential framework, whereby the Perron-Frobenius eigenvector of linear positivity theory is replaced by a Perron-Frobenius vector field $w(x)$ whose integral curves shape the attractors of the system. The main result on closed differentially positive systems is that the asymptotic behavior is either captured by a Perron-Frobenius curve γ such that

$$\gamma'(s) = w(\gamma(s)) \qquad (12)$$

at every point on γ; or is the union of the limit points of a trajectory that is nowhere aligned with the Perron-Frobenius vector field, which is a highly non-generic situation.

It is insightful to revisit nonlinear consensus theory in linear spaces through invariant differential positivity. Consider the consensus model where the linear coupling implicit in (6) is replaced by a nonlinear protocol of the form

$$\dot{x}_k = \sum_{i\in N_k} f_{ki}(x_i - x_k), \qquad (13)$$

where N_k denotes the set of all neighboring nodes of node k in the communication graph $(\mathcal{V}, \mathcal{E})$, and the functions $f_{ki} : \mathbb{R} \to \mathbb{R}$ are odd, locally Lipschitz and strictly increasing for all edges $(i, k) \in \mathcal{E}$. Note that the variational or prolonged system associated with the nonlinear protocol (13) can be expressed as

$$\begin{cases} \dot{x} = \boldsymbol{F}(x)\mathbf{1}, \\ \dot{\delta x} = \boldsymbol{J}(x)\delta x \end{cases} \qquad (14)$$

where $x = (x_1, \dots, x_N)^T \in \mathbb{R}^N$, $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^N$, $F_{ki}(x) = f_{ki}(x_i - x_k)$, and

$$\begin{cases} J_{kk}(x) = -\sum_{i\in N_k} f'_{ki}(x_i - x_k), \\ J_{ki}(x) = f'_{ki}(x_i - x_k) \quad \text{if} \quad (k, i) \in \mathcal{E}, \\ J_{ki}(x) = 0 \quad \text{if} \quad (k, i) \notin \mathcal{E}. \end{cases} \qquad (15)$$

Note that the identity $\boldsymbol{J}(x)\mathbf{1} = 0$ captures the invariance of the consensus subspace $\{x \in \mathbb{R}^N : x_1 = \ldots = x_N\}$. Since the nonlinear coupling functions f_{ik} are assumed to be strictly increasing, the Jacobian matrix $\boldsymbol{J}(x)$ has non-negative off-diagonal elements. Indeed, it is easy to see that the system is differentially positive with respect to the invariant cone field generated by the positive orthant $\mathcal{K} = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_i \geq 0,\ i = 1, \dots, N\}$. This is made clear by observing that on each face of the cone given by $F_i := \{\delta x : \delta x_i = 0\}$, we have $\dot{\delta x} = \boldsymbol{J}(x)\delta x \geq 0$ if and only if $J_{ij}(x) \geq 0$ for all $i \neq j$. Moreover, in the case of a strongly connected graph, the differential positivity is strict. The attractor of the system is precisely the consensus subspace, which coincides with the Perron-Frobenius curve generated by the invariant vector field $w(x) = \mathbf{1}$.

3. INVARIANT DIFFERENTIAL POSITIVITY ON LIE GROUPS

3.1 Invariant Cone Fields on Lie Groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Recall that a cone field $\mathcal{K}_G$ smoothly assigns a cone $\mathcal{K}_G(g) \subset T_g G$ to each point $g \in G$. We say that the cone field $\mathcal{K}_G$ is *left-invariant* if

$$\mathcal{K}_G(g_1 g_2) = dL_{g_1}\big|_{g_2} \mathcal{K}_G(g_2), \quad \forall g_1, g_2 \in G, \tag{16}$$

where $L_g : G \to G$ denotes left translation, $L_g(a) := ga$, and $dL_{g_1}|_{g_2}$ is the push-forward or differential of L_{g_1} at g_2. Note that a left-invariant cone field is characterised by the cone in the tangent space at identity $T_e G = \mathfrak{g}$. That is, given a cone $\mathcal{K}$ in $\mathfrak{g}$, the corresponding left-invariant cone field is given by $\mathcal{K}_G^L(g) = dL_g|_e(\mathcal{K})$, for all $g \in G$. Similarly, one can define *right-invariant* cone fields on a Lie group. A right-invariant cone field is characterised by the cone at identity according to $\mathcal{K}_G^R(g) = dR_g|_e(\mathcal{K})$, for all $g \in G$, where R_g denotes right-translation.

Now recall that one method of describing cone fields $\mathcal{K}_G$ on G is using a suitable collection of smooth functions $k_i : TG \to \mathbb{R}$, $i \in \mathcal{I}$ as

$$\mathcal{K}_G(g) := \{\delta g \in T_g G : k_i(g, \delta g) \geq 0, \forall i \in \mathcal{I}\}. \tag{17}$$

Suppose that we are given a cone in $T_e G$ characterised by $\mathcal{K} = \{\delta g \in T_e G : k_i(\delta g) \geq 0, \forall i \in \mathcal{I}\}$. The corresponding left-invariant cone field on G is given by $\mathcal{K}_G^L(g) = \{\delta g \in T_g G : k_i\left(dL_{g^{-1}}|_g \delta g\right) \geq 0, \forall i \in \mathcal{I}\}$.

3.2 Invariant Differential Positivity of Equations of the form $\dot{g} = g\,\Omega(g)$

Consider the dynamical system

$$\frac{d}{dt} g = g\,\Omega(g) \tag{18}$$

on a Lie group G where $\Omega : G \to \mathfrak{g}$. We identify $\mathfrak{g} = T_e G$ with $\mathbb{R}^n$ via the $^\vee$ map $\Omega \mapsto \Omega^\vee \in \mathbb{R}^n$, where $n = \dim G$. Thus, the system (18) is characterized by the map $\Omega^\vee : G \to \mathbb{R}^n$. The push-forward of this map takes the form

$$d\Omega^\vee\big|_g : T_g G \to \mathbb{R}^n \tag{19}$$

and maps tangent vectors $\delta g \in T_g G$ to vectors in $\mathbb{R}^n$. The linearization of (18) takes the form

$$\frac{d}{dt}\delta g = dL_g\big|_e \circ d\Omega^\vee\big|_g \delta g = g\, d\Omega^\vee\big|_g(\delta g). \tag{20}$$

Since we are working with a (left) invariant cone field, we can equivalently consider the linear map

$$A(g) := d\Omega^\vee\big|_g \circ dL_g\big|_e : \mathbb{R}^n \to \mathbb{R}^n \tag{21}$$

for each $g \in G$, where $\mathbb{R}^n$ is identified with $\mathfrak{g}$ through the $^\vee$ map. Differential positivity of the system (18) with respect to an invariant cone field generated by a cone $\mathcal{K} \subset \mathbb{R}^n$ reduces to the positivity of the linear map

$$\dot{x} = A(g)x, \quad x \in \mathbb{R}^n \tag{22}$$

with respect to $\mathcal{K}$ for all $g \in G$.

3.3 Example: Nonlinear Pendulum

Here we briefly review the differential positivity of the classical nonlinear planar pendulum equation which takes the form of a flow on the cylinder $G = \mathbb{S}^1 \times \mathbb{R}$ Forni and Sepulchre (2014a). Thinking of $\mathbb{S}^1$ as being embedded in the complex plane, we can represent elements of the Lie group G by $(e^{i\theta}, v)$. The pendulum equation can be written in the form $\dot{g} = dL_g|_e \Omega(g)$, where $\Omega : G \to \mathfrak{g}$ is specified by $\Omega^\vee = (\Omega_1, \Omega_2)^T$, where Ω_1 is a purely imaginary number and Ω_2 is real. We have:

$$\frac{d}{dt}\begin{pmatrix} e^{i\theta} \\ v \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \tag{23}$$

where

$$\begin{cases} \Omega_1 &= iv, \\ \Omega_2 &= -\sin\theta - kv + u, \end{cases} \tag{24}$$

$k \geq 0$ is the damping coefficient, and u is a constant torque input. Thus, the linearized dynamics at point $(e^{i\theta}, v)$ is governed by the linear map $A(g) : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A(g) = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -k \end{pmatrix} \tag{25}$$

It is easy to verify that the map $A(g) : \mathbb{R}^2 \to \mathbb{R}^2$ is strictly positive with respect to the cone

$$\mathcal{K} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_1 + x_2 \geq 0\}, \tag{26}$$

for $k \geq 2$ by showing that at any point on the boundary of the cone $\mathcal{K}$, the vector $A(g)(x_1, x_2)^T$ is oriented towards the interior of the cone for any $g \in G$.

4. MONOTONICITY AND CONSENSUS THEORY ON LIE GROUPS

4.1 Consensus in Nonlinear Spaces

An interpretation of the classical continuous time linear consensus algorithm on $\mathbb{R}^N$ with update

$$\dot{x}_k = \sum_{j=1}^{N} a_{jk}(t)(x_j(t) - x_k(t)). \tag{27}$$

is that the state of agent k evolves at each instant in time towards the arithmetic mean or average of its neighbors. Moreover, the linear *coupling* in (27) can be viewed as the gradient of a quadratic (i.e. Euclidean) distance. The linear coupling can be replaced by a nonlinear monotone coupling function, which can be thought of as the gradient of a convex distance function instead.

With this geometric interpretation, consensus algorithms can be defined on arbitrary Riemannian manifolds $\mathcal{M}$, where the average of neighboring points relies on the definition of the Riemannian metric and the notion of updating a point towards a new point is achieved by moving along a well-defined minimal geodesic connecting the two points.

4.2 Monotonicity on Lie Groups

The definition of monotonicity of a dynamical system relies on the existence of a well-defined partial order on the state space. Generally, unlike in $\mathbb{R}^n$, an invariant cone field on a

Lie group G does not necessarily yield a partial order on G, as is clear in the example of $\mathbb{S}^1$ where the conal order fails to satisfy antisymmetry. Nonetheless, a dynamical system can be seen to exhibit a well-defined notion of monotonic behavior on a Lie group if it is (i) differentially positive with respect to an invariant cone field, and (ii) is defined on an invariant subset $\mathcal{S} \subset G$ that is geodesically convex in the sense that for any two points $p_1, p_2 \in \mathcal{S}$, there exists a unique minimal geodesic contained in $\mathcal{S}$ which connects p_1, p_2.

In particular, in the case of consensus dynamics on a Lie group G, where the nonlinear coupling of the agents evolving on G is inherited from some possibly non-invariant Riemannian distance on G, the above notion of monotonicity on a Lie group can be used to analyse the convergence of the dynamics. This is illustrated in the next section using an example of consensus dynamics on $\mathbb{S}^1$, where we will make use of the following result from Forni (2015).

In the following theorem, we assume that

$$\left(\mathcal{K}(x) \cap \{\delta x \in T_x\mathcal{M} : |\delta x|_x = 1\}, d_{\mathcal{K}(x)}\right) \tag{28}$$

is a complete metric space for all $x \in \mathcal{C}$.

Theorem 4. Let Σ be a strictly differentially positive system with respect to a cone field $\mathcal{K}(x)$ in a compact and forward invariant region $\mathcal{C} \subseteq \mathcal{M}$. If there exists a complete vector field w satisfying $w(x) \in \operatorname{int}\mathcal{K}(x) \setminus \{0\}$ such that $\limsup_{t\to\infty} |\partial\psi_t(x)w(x)|_{\psi_t(x)} < \infty$, and for all $x \in \mathcal{C}$ and $t \geq 0$:

$$w(\psi_t(x)) = \frac{\partial\psi_t(x)w(x)}{|\partial\psi_t(x)w(x)|_{\psi_t(x)}}, \tag{29}$$

then there exists an integral curve of $w(x)$ whose image is an attractor for all the trajectories of Σ from $\mathcal{C}$.

4.3 Example: Kuramoto Flow on N-Torus $\mathbb{T}^N$

First we consider the consensus dynamics on $\mathbb{S}^1$

$$\dot{\theta}_k = \frac{1}{N}\sum_{i=1}^{N} \sin(\theta_i - \theta_k) \tag{30}$$

arising from the homogeneous Kuramoto flow generated by the interconnection of N agents $\theta_k \in \mathbb{S}^1$. The Kuramoto model can be thought of as a consensus model on the circle $\mathbb{S}^1$ (see e.g. Sepulchre et al. (2005); Jadbabaie and Lin (2003); Sarlette and Sepulchre (2011)).

The synchronization manifold can be expressed as

$$\mathcal{M}_{\text{sync}} = \{\vartheta \in \mathbb{S}^1 \times \dots \mathbb{S}^1 : \vartheta_1 = \dots = \vartheta_N\}, \tag{31}$$

where $\vartheta = (\vartheta_1, \dots, \vartheta_N)$. Note that the synchronization manifold $\mathcal{M}_{\text{sync}}$ corresponds to a single point on the quotient manifold $\mathbb{S}^1 \times \dots \times \mathbb{S}^1/\mathbb{S}^1$. Now the variational dynamics of (30) takes the form

$$\begin{cases} \dot{\vartheta} &= \dfrac{1}{N}F(\vartheta), \\ \dot{\delta\vartheta} &= \dfrac{1}{N}A(\vartheta)\delta\vartheta, \end{cases} \tag{32}$$

where $\mathbf{1} = (1, \dots, 1)^T$, and

$$\begin{cases} A_{kk}(\vartheta) &= -\displaystyle\sum_{i\neq k} \cos(\vartheta_i - \vartheta_k), \\ A_{ki}(\vartheta) &= \cos(\vartheta_i - \vartheta_k) \quad \text{for} \quad k \neq i. \end{cases} \tag{33}$$

The identity $A(\vartheta)\mathbf{1} = 0$ captures the invariance of the synchronization manifold $\mathcal{M}_{\text{sync}}$. Now consider the cone field $\mathcal{K}_{\mathbb{T}^N}$ on the N-torus $\mathbb{T}^N$ defined by

$$\mathcal{K}_{\mathbb{T}^N}(\vartheta, \delta\vartheta) := \{\delta\vartheta \in T_\vartheta\mathbb{T}^N : \mathbf{1}^T\delta\vartheta \geq 0,\ Q(\delta\vartheta) \geq 0\}, \tag{34}$$

where Q is the quadratic form

$$\begin{aligned} Q(\delta\vartheta) &= \delta\vartheta^T\mathbf{1}\mathbf{1}^T\delta\vartheta - \delta\vartheta^T\delta\vartheta \\ &= \left(\delta\vartheta_1 + \dots + \delta\vartheta_N\right)^2 - \left(\delta\vartheta_1^2 + \dots + \delta\vartheta_N^2\right). \end{aligned}$$

Note that (34) is clearly an invariant cone field since the defining inequalities in each tangent space $T_\vartheta\mathbb{T}^N$ are independent of $\vartheta \in \mathbb{T}^N$. On the boundary of the cone $\mathcal{K}(\vartheta, \delta\vartheta)$, we have $Q(\delta\vartheta) := \delta\vartheta^T\mathbf{1}\mathbf{1}^T\delta\vartheta - \delta\vartheta^T\delta\vartheta = 0$ and using $A(\vartheta)\mathbf{1} = 0$, we see that the derivative of $Q(\delta\vartheta)$ along the trajectories of the prolonged dynamics satisfies

$$\frac{d}{dt}Q(\delta\vartheta) = -\frac{1}{N}\delta\vartheta^T(A(\vartheta)^T + A(\vartheta))\delta\vartheta > 0 \tag{35}$$

for

$$\vartheta \in \mathbb{T}^N_{\pi/2} := \{\vartheta \in \mathbb{T}^N : |\vartheta_k - \vartheta_i| < \frac{\pi}{2},\ i, k = 1, \dots, N\}. \tag{36}$$

That is, for $\vartheta \in \mathbb{T}^N_{\pi/2}$, the system is strictly differentially positive with respect to the invariant cone field $\mathcal{K}_{\mathbb{T}^N}$. Note that the invariant vector field

$$w(\vartheta) = \frac{1}{\sqrt{N}}\mathbf{1} \in \operatorname{int}\mathcal{K}_{\mathbb{T}^N}(\vartheta, \delta\vartheta) \tag{37}$$

clearly satisfies the conditions in Theorem 4. Therefore, all trajectories from a compact and forward invariant region $\mathcal{C} \subseteq \mathbb{T}^N_{\pi/2}$ asymptotically converge to the synchronization manifold $\mathcal{M}_{\text{sync}}$.

Moreau observed that his convergence analysis applies to the nonlinear model (30) provided that all initial conditions lie in the same semi-circle. The dynamics can then be mapped to a monotone system on the real line through a nonlinear change of coordinates. It is not difficult to see that (36) implies that assumption not just for the complete graph in (30) but also if the graph is strongly connected. The argument can be extended to an arbitrary subset of the circle by modifying the coupling function; that is, by changing the underlying metric. This trick has been used in Sarlette and Sepulchre (2011) and Tron et al. (2012). In the limit of the entire open circle $(-\pi, \pi)$, one obtains a monotone discontinuous coupling, which connects the Kuramoto model to pulse-coupled models of integrate-and-fire oscillators Mauroy and Sepulchre (2012). The differential viewpoint adopted in this paper provides a geometric interpretation of those results and suggests that they extend to arbitrary Lie groups.

Modify the synchronization model (30) to

$$\dot{\vartheta}_k = \frac{1}{N}\sum_{i=1}^{N} f(\vartheta_i - \vartheta_k), \tag{38}$$

where the coupling function $f : [-\pi, \pi] \to \mathbb{R}$ is a twice differentiable function on $(-\pi, \pi)$ which satisfies $f(0) = 0$, and $f'(\vartheta) > 0$ for all $\vartheta \in (-\pi, \pi)$. The variational dynamics is modified to (32) with

$$\begin{cases} A_{kk}(\vartheta) &= -\displaystyle\sum_{i\neq k} f'(\vartheta_i - \vartheta_k), \\ A_{ki}(\vartheta) &= f'(\vartheta_i - \vartheta_k) \quad \text{for} \quad k \neq i. \end{cases} \tag{39}$$

In this case, the system is strictly differentially positive for all

$$\vartheta \in \mathbb{T}^N_\pi := \{\vartheta \in \mathbb{T}^N : |\vartheta_k - \vartheta_i| < \pi,\ i,k = 1,\ldots,N\}. \quad (40)$$

It follows that the modified consensus dynamics (38), where the 2π-periodic coupling function f is strictly increasing on $(-\pi,\pi)$ and discontinuous at π, is almost globally asymptotically convergent.

5. CONCLUSION

The study of consensus algorithms in linear spaces has historically received considerable attention due to numerous applications in distributed computation and control science. In recent years, consensus problems defined on state spaces that are not linear but instead are highly symmetric nonlinear spaces, such as Lie groups, have attracted increasing interest due to the ubiquity of such spaces in applications. In this work, we have formulated a framework for invariant differential positivity on Lie groups, discussed its relation to monotonicity, and outlined how invariant differential positivity can be used to study synchronization on Lie groups, including an extended example involving synchronization on $\mathbb{S}^1$. Future work will seek to apply the theory of invariant differential positivity to study consensus on more complicated Lie groups such as $SO(3)$.

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