A new class of integrable Lotka–Volterra systems

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Abstract

A parameter-dependent class of Hamiltonian (generalized) Lotka–Volterra systems is considered. We prove that this class contains Liouville integrable as well as superintegrable cases according to particular choices of the parameters. We determine sufficient conditions which result in integrable behavior, while we numerically explore the complementary cases, where these analytically derived conditions are not satisfied.

1 Introduction

The Lotka–Volterra system was introduced independently by Lotka [15] and Volterra [20] as a predator-prey model. Since then, many generalizations have been considered with applications to several scientific disciplines. These systems in general display rich dynamical behavior that varies according to the parameters that define each one of them. For example, there are Hamiltonian and non-Hamiltonian Lotka–Volterra systems, as well as integrable, non-integrable and chaotic ones. From the point of view of integrability, various kinds of generalized Lotka–Volterra systems have been extensively studied in the literature, e.g. [1, 4, 5, 6, 10, 11, 12, 16, 17, 19]. A numerical study of a 4–dimensional non-integrable Lotka–Volterra system can be found in [18].

In this paper, we study a parametric family of (generalized) Lotka–Volterra systems of the form

$$\dot{x}_i = x_i \left(\sum_{j>i} a_j x_j - \sum_{j<i} a_j x_j + r_i \right) , \quad a_i, r_i \in \mathbb{R}. \tag{1}$$

This family includes some particular interesting cases. The case of $r_i = 0$ and $a_i = 1$ came up in the study of a class of multi-sums of products in [13] which is related to integrals of periodic reductions of discrete integrable systems. It can be considered as a finite dimensional reduction of a Bogoyavlenskij lattice [2, 3] with fixed boundary conditions. The integrability of this case and its corresponding Kahan discretization has been studied in detail in [13]. In [14], the Liouville integrability and superintegrability of the more general cases, with $r_i = 0$ and arbitrary $a_i \in \mathbb{R}$, was proved and explicit solutions were given for the corresponding continuous and discrete systems. Motivated by these results, we aim to study the integrable and dynamical aspects of (1), with arbitrary parameters a_i and r_i in $\mathbb{R}$.

As is shown in Section 3, all the even-dimensional cases of (1) are Hamiltonian with respect to a log-canonical Poisson bracket and this also applies to odd dimensions under some extra conditions on the parameters r_i. A first approach to trace integrable cases is the following. We consider the integrals of the $r_i = 0$ case as they appear in [14], and we demand them to be in involution with the Hamiltonian function of (1). This restriction leads to a system for the parameters a_i and r_i. Solutions of this system provide necessary and sufficient conditions which ensure the pairwise involutivity of all the integrals (including the Hamiltonian). This procedure provides several Liouville integrable cases. By considering a permutation symmetry of the system more integrable cases appear as well as superintegrable cases according to particular choices of the parameters. These results appear in Sections 4–5.

In Section 6, we numerically explore the behavior of (1) with $n = 4$ for the cases where integrability is not proven by the analytical arguments of the previous sections. To this end we perform a series of numerical simulations for various different parameters which determine the system (1). Integrability or non-integrability is manifested by the Poincaré surfaces of section as well as the evolution of the largest Lyapunov exponent for various initial conditions at gradually increasing energies. We have strong indications that more integrable cases exist, however, we find non-integrable cases as well. Notable non-integrable examples are found for the 4-dimensional Lotka–Volterra system (1) with bounded trajectories in phase space, whose orbits demonstrate a particularly rich complexity.

2 A class of Lotka–Volterra systems

Generalized Lotka–Volterra or just *Lotka–Volterra systems* are systems of the form

$$\dot{x}_i = x_i \left(\sum_{j=1}^{n} A_{ij} x_j + r_i \right) , \qquad i = 1, \ldots, n , \tag{2}$$

where $A = (A_{ij})$ is any arbitrary $n \times n$ matrix, known as the *community matrix* and $\mathbf{r} = (r_1, \ldots, r_n)$ is a vector in $\mathbb{R}^n$.

In this paper, we are going to study a particular class of Lotka–Volterra systems, with community matrix

$$A = \begin{pmatrix} 0 & a_2 & a_3 & \dots & a_n \\ -a_1 & 0 & a_3 & \dots & a_n \\ -a_1 & -a_2 & 0 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \\ -a_1 & -a_2 & -a_3 & \dots & 0 \end{pmatrix} , \tag{3}$$

and parameters $a_1, \dots, a_n \in \mathbb{R}$. In this case, system (2) can be written as (1), or equivalently, as

$$\dot{x}_i = x_i \left(\sum_{j=1}^{n} P_{ij} a_j x_j + r_i \right) , \tag{4}$$

where P is the antisymmetric matrix

$$P_{ij} = \begin{cases} 1 - \delta_{ij} & \text{for } 1 \le i \le j \le n \ , \\ -1 & \text{for } 1 \le j < i \le n \ . \end{cases} \tag{5}$$

The special case of (1) with $\mathbf{r} = 0$ was extensively studied in [13, 14], where the Liouville and superintegrability of the corresponding systems were proved and explicit solutions were given. Here, our aim is to investigate the integrability of particular cases with $\mathbf{r} \neq 0$. In due course we mainly restrict our attention to the case that n is even.

3 Hamiltonian formalism

We consider the log-canonical Poisson structure

$$\{x_i, x_j\} = x_i x_j \ , \quad 1 \le i < j \le n. \tag{6}$$

The rank of this Poisson structure, for $x_1 \dots x_n \neq 0$, is n for even n, and $n-1$ for odd n. In the odd case, $C := \frac{x_1 x_3 \dots x_n}{x_2 x_4 \dots x_{n-1}}$ is a Casimir function.

Proposition 3.1. *For any even n, $a_i, r_i \in \mathbb{R}$ and $x_i > 0$, $i = 1, 2, \dots, n$, the Lotka–Volterra system* (1) *is Hamiltonian with respect to the Poisson structure* (6) *and the Hamiltonian function*

$$H(\mathbf{x}) = \sum_{i=1}^{n} (a_i x_i + k_i \log x_i),$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{k} = (k_1, \dots k_n)$ defined by $\mathbf{k} = P^{-1}\mathbf{r}$.

In terms of the parameters k_i , the system is written as

$$\dot{x}_i = x_i \left(\sum_{j=1}^{n} P_{ij} (a_j x_j + k_j) \right) . \tag{7}$$

For odd n, the matrix P is not invertible. Hence, the Hamiltonian structure of Prop. 3.1 does not include all the cases of (1) for arbitrary $\mathbf{r} \in \mathbb{R}^n$. However, for any n we can restrict our analysis to the Hamiltonian systems (7), i.e. systems (1) with $\mathbf{r} = P\mathbf{k}$.

By setting $u_i = \log x_i$, the Poisson bracket (6) becomes a constant one, that is $\{u_i, u_j\} = P_{ij}$, and the Hamiltonian function $H_u = \sum_{i=1}^{n}(a_i e^{u_i} + k_i u_i)$. In these coordinates our system is expressed as

$$\dot{u}_i = \sum_{j=1}^{n} P_{ij} a_j e^{u_j} + r_i = \sum_{j>i} a_j e^{u_j} - \sum_{j<i} a_j e^{u_j} + r_i, \text{ with } \mathbf{r} = P\mathbf{k}.$$

Remark 3.2. The parameters $a_i, \ldots, a_n$ of (1) can be rescaled to $a_1c_1, \ldots, a_nc_n$, by using the transformation $x_i \mapsto x_i/c_i$, for $c_i > 0$. This linear transformation preserves the Poisson bracket and gives rise to an equivalent Hamiltonian system with Hamiltonian $H_y = \sum_{i=1}^{n}(a_i c_i y_i + k_i \log c_i y_i)$, in the new variables $y_i = x_i/c_i$. For example, by setting $c_i = \frac{1}{|a_i|}$, all the nonzero a_i can be rescaled to 1 or -1. Hence, we can consider systems with parameters $a_i \in \{-1, 0, 1\}$ without any loss of generality.

In the present work we will restrict to the even-dimensional case; however, a similar approach can be considered for odd dimensions. Some additional comments on odd-dimensional cases as well as two examples, for $n = 3$ and $n = 5$, are given in the appendix.

4 Liouville integrability

Following [14], we introduce the functions

$$v_i := a_1x_1 + \cdots + a_ix_i, \quad i = 1, \ldots, n.$$

If $a_1a_2 \ldots a_n \neq 0$, the functions v_i define new coordinates on $\mathbb{R}^n$ but generally this is not true. Furthermore, for any even n we define the functions

$$J_m(\mathbf{x}) = \frac{x_1x_3 \ldots x_{2m-1}}{x_2x_4 \ldots x_{2m}}, \; I_m(\mathbf{x}) = \frac{x_{2m+2}x_{2m+4} \ldots x_n}{x_{2m+1}x_{2m+3} \ldots x_{n-1}}, \; F_m(\mathbf{x}) = v_{2m} I_m(\mathbf{x}),$$

for $m = 1, \ldots, \frac{n}{2}$. We also set

$$H_0 := F_{n/2} = v_n = \sum_{i=1}^{n} a_i x_i,$$

which corresponds to the Hamiltonian function in the case of $\mathbf{r} = \mathbf{k} = 0$. So, the generic Hamiltonian of (1) is written as

$$H = H_0 + \sum_{i=1}^{n} k_i \log x_i.$$

In [14], it is proved that for any $m, l \in \{1, \ldots, \frac{n}{2}\}$,

$$\{J_m, J_l\} = \{F_m, F_l\} = \{F_m, H_0\} = 0 \ , \tag{8}$$

as well as the following theorem which establishes the Liouville integrability of the system in the case of $\mathbf{r} = 0$.

Theorem 4.1. *Suppose that n is even. Let ℓ denote the smallest integer such that $a_{\ell+1} \neq 0$ and let $\lambda := \left[\frac{\ell}{2}\right]$. The $\frac{n}{2}$ functions $J_1, J_2, \ldots, J_\lambda, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1}, H_0$ are pairwise in involution and functionally independent.*

Here, our first goal is to determine the parameters a_i and $\mathbf{r}$, so that the more general system (1) inherits the same integrals as the $\mathbf{r} = 0$ case which ensure Liouville integrability. In the following, we always assume that n is even.

Lemma 4.2. *For any $m = 1, \ldots, \frac{n}{2} - 1$*

$$\{F_m, H\} = I_m \sum_{j=1}^{2m} k_j (v_j + v_{j-1} - v_{2m}) \ . \tag{9}$$

Proof. Since $\{F_m, H\} = \{F_m, H_0 + \sum_{j=1}^{n} k_j \log x_j\}$ and $\{F_m, H_0\} = 0$ (from (8)), we get

$$\{F_m, H\} = \sum_{j=1}^{n} k_j \{F_m, \log x_j\} = \sum_{j=1}^{n} k_j \{v_{2m} I_m, \log x_j\}. \tag{10}$$

Now $\{v_{2m}, \log x_j\} = \sum_{i=1}^{2m} \{x_i, x_j\} \frac{a_i}{x_j}$, so it follows that

$$\{v_{2m}, \log x_j\} = \begin{cases} v_{2m} \ , & \text{for } 2m < j \ , \\ v_j + v_{j-1} - v_{2m} \ , & \text{for } j \leq 2m \ . \end{cases}$$

Also, we have

$$\{I_m, x_j\} = \begin{cases} -I_m x_j \ , & \text{for } 2m < j \ , \\ 0 \ , & \text{for } j \leq 2m \end{cases}$$

(this identity was proved in [14]) and

$$\{I_m, \log x_j\} = \{I_m, x_j\} \frac{1}{x_j} = \begin{cases} -I_m \ , & \text{for } 2m < j \ , \\ 0 \ , & \text{for } j \leq 2m \ . \end{cases}$$

Therefore, we see that

$$\{v_{2m} I_m, \log x_j\} = \begin{cases} 0 \ , & \text{for } 2m < j \ , \\ (v_j + v_{j-1} - v_{2m}) I_m \ , & \text{for } j \leq 2m \ , \end{cases}$$

and by substituting in (10) we derive (9). □

We can recast the sum that appears in (9) to derive

$$\sum_{j=1}^{2m} k_j(v_j + v_{j-1} - v_{2m}) = \sum_{j=1}^{2m} x_j a_j(-\sum_{i=1}^{j-1} k_i + \sum_{i=j+1}^{2m} k_i) \ ;$$

hence, from Lemma 4.2, the next proposition follows.

Proposition 4.3. *Suppose that n is even. For every $m = 1, \dots, \frac{n}{2} - 1$, $\{F_m, H\} = 0$ if and only if $S_{jm} = 0$, for every $j = 1, \dots, 2m$, where*

$$S_{jm} = a_j(-\sum_{i=1}^{j-1} k_i + \sum_{i=j+1}^{2m} k_i). \tag{11}$$

Solutions of the system $S_{jm} = 0$, for $m = 1, \dots, \frac{n}{2} - 1$ and $j = 1, \dots, 2m$, provide conditions on the parameters a_i and k_i ensuring that the functions $F_1, F_2, \dots, F_{\frac{n}{2}-1}$ are first integrals of the system. Moreover, according to (8), these integrals are pairwise in involution. Therefore, in the case where $F_m \neq 0$, for all $m = 1, \dots, \frac{n}{2} - 1$, these conditions on the parameters provide Liouville integrability. For example, in the particular case where $a_j \neq 0$, for every $j = 1, \dots, n$, the corresponding system implies the unique solution $k_1 = k_2 = \cdots = k_{n-2} = 0$.

Corollary 4.4. *For $a_1 a_2 \dots a_n \neq 0$, $k_1 = k_2 = \cdots = k_{n-2} = 0$, $k_{n-1}, k_n \in \mathbb{R}$ and $\mathbf{r} = P\mathbf{k} = (k_{n-1} + k_n, k_{n-1} + k_n, \dots, k_n, -k_{n-1})$, the Hamiltonian system (1) is Liouville integrable with first integrals $H, F_1, F_2, \dots, F_{\frac{n}{2}-1}$*[1].

Now, let $a_1 = a_2 = \cdots = a_\ell = 0$, $a_{\ell+1} \neq 0$ and $\lambda := \left[\frac{\ell}{2}\right]$. In such a case, $F_1 = \cdots = F_\lambda = 0$. So, for any choice of parameters there are not enough F-type integrals to ensure the integrability of the system. However, Theorem 4.1 suggests that we could probably replace the first λ missing F-integrals by λ J-integrals. Hence next, we are going to determine the conditions on the parameters to ensure that $\{J_m, H\} = 0$, for $m = 1, \dots, \lambda$ [2].

Lemma 4.5. *Let $a_1 = a_2 = \cdots = a_\ell = 0$, $a_{\ell+1} \neq 0$ and $\lambda := \left[\frac{\ell}{2}\right]$. Then,*

$$\{J_m, H\} = J_m(k_1 + k_2 + \cdots + k_{2m}), \tag{12}$$

for $m = 1, \dots, \lambda$.

Proof. We consider $m \in \{1, \dots, \lambda\}$. From Theorem 4.1, it follows that $\{J_m, H_0\} = 0$. So,

$$\{J_m, H\} = \sum_{j=1}^{n} k_j \{J_m, \log x_j\} = \sum_{j=1}^{2m} k_j \{J_m, \log x_j\} + \sum_{j=2m+1}^{n} k_j \{J_m, \log x_j\}. \tag{13}$$

[1]The proof of the functional independence of the integrals is given in Prop. 4.7.

[2]For $m > \lambda$, J_m cannot be an integral of the system i.e. $\{J_m, H\} \neq 0$. So, the total number of F and J integrals cannot exceed $\frac{n}{2} - 1$.

Also,

$$\{J_m, x_j\} = \sum_{i=1}^{n} \{x_i, x_j\} \frac{\partial J_m}{\partial x_i} = \sum_{i=1}^{j-1} x_i x_j \frac{\partial J_m}{\partial x_i} - \sum_{i=j+1}^{n} x_i x_j \frac{\partial J_m}{\partial x_i}$$

and

$$x_i \frac{\partial J_m}{\partial x_i} = \begin{cases} (-1)^{i+1} J_m \ , & \text{for } 1 \leq i \leq 2m \ , \\ 0 \ , & \text{for } 2m < i \leq n \ . \end{cases}$$

Consequently, after some calculations we obtain

$$\{J_m, x_j\} = \begin{cases} J_m x_j \ , & \text{for } j \leq 2m \ , \\ 0 \ , & \text{for } j > 2m \end{cases} \quad \text{and} \quad \{J_m, \log x_j\} = \begin{cases} J_m \ , & \text{for } j \leq 2m \ , \\ 0 \ , & \text{for } j > 2m \ . \end{cases}$$

Substituting this into (13), we derive (12).

□

Finally, if we combine Lemma 4.5 with Prop. 4.3 we come up with the following theorem.

Theorem 4.6. *Suppose that n is even. Let ℓ denote the smallest integer such that $a_{\ell+1} \neq 0$ and let $\lambda = \left[\frac{\ell}{2}\right]$. The $\frac{n}{2}$ functions $J_1, J_2, \ldots, J_\lambda, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1}, H$ are pairwise in involution if and only if $k_{2i} = -k_{2i-1}$, for $i = 1, \ldots, \lambda$, and $S_{jm} = 0$, for $m = \lambda + 1, \ldots, \frac{n}{2} - 1$, $j = \ell + 1, \ldots, 2m$.*

Proof. Let $a_1 = a_2 = \cdots = a_\ell = 0$, $a_{\ell+1} \neq 0$ and $\lambda = \left[\frac{\ell}{2}\right]$. From Lemma 4.5, we conclude that $\{J_1, H\} = \{J_2, H\} = \cdots = \{J_\lambda, H\} = 0$ if and only if $k_1 + k_2 + \cdots + k_{2i} = 0$, for all $i = 1, \ldots, \lambda$, which is equivalent to $k_{2i} = -k_{2i-1}$, for $i = 1, \ldots, \lambda$. Also, from Prop. 4.3, we derive that for $m = \lambda + 1, \ldots, \frac{n}{2} - 1$, $\{F_m, H\} = 0$ if and only if $S_{jm} = 0$, for every $j = \ell + 1, \ldots, 2m$ (for $j = 1, \ldots \ell$, $S_{jm} = 0$, since $a_1 = \cdots = a_l = 0$). Finally, Theorem 4.1 shows that all the other pairs of functions are in involution too. □

We will close this section by proving the functional independence of the integrals.

Proposition 4.7. *For every even n, the functions $J_1, J_2, \ldots, J_\lambda, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1}$, H, are functionally independent.*

Proof. For $\mathbf{k} = 0$, $J_1, \ldots, J_\lambda, F_{\lambda+1}, \ldots, F_{\frac{n}{2}-1}, H$ are functionally independent. This follows from Theorem 4.1, since in this case H coincides with H_0. Hence, by continuity the same functions remain functionally independent for parameters $\mathbf{k}$ in a sufficiently small open neighborhood U of $\mathbf{k} = 0$. Now, let us consider any $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{R}^n$. Then there is $\mu > 0$ and $\mathbf{k}' = (k'_1 \ldots, k'_n) \in U$, such that $\mathbf{k} = \mu \mathbf{k}'$. Also, in view of Remark 3.2, we can rescale the a_i parameters to μa_i, by setting $y_i = x_i/\mu$. The Hamiltonian function in the new y-coordinates then becomes

$$H_y(\mathbf{y}) = \sum_{i=1}^{n} (a_i \mu y_i + k_i \log \mu y_i) = \mu \sum_{i=1}^{n} (a_i y_i + k'_i \log \mu y_i).$$

So, $dH_y = \mu dH'$, where $H'(\mathbf{y}) = \sum_{i=1}^{n}(a_i y_i + k_i' \log y_i)$, i.e. the Hamiltonian of the corresponding system with parameters a_i and k_i'. Therefore, from the functional independence of $J_1, \ldots, J_\lambda, F_{\lambda+1}, \ldots, F_{\frac{n}{2}-1}, H'$ that we proved, the functional independence of $J_1, \ldots, J_\lambda, F_{\lambda+1}, \ldots, F_{\frac{n}{2}-1}, H_y$ follows and consequently the functional independence of $J_1, \ldots, J_\lambda, F_{\lambda+1}, \ldots, F_{\frac{n}{2}-1}, H$ for all parameters a_i and k_i.

$\square$

5 Symmetry and superintegrability

In [13, 14], a second set of first integrals in involution has been introduced for the case of $\mathbf{r} = 0$. By considering this set of integrals we can derive more integrable cases of our system. The main observation to accomplish this is that system (7) remains invariant under the transformation $x_i \mapsto x_{n+1-i}$ and the reparametrization $a_i \mapsto -a_{n+1-i}$, $k_i \mapsto -k_{n+1-i}$, for $i = 1, \ldots, n$. Let us now consider the involution $\iota(x_1, x_2, \ldots, x_n) \mapsto (x_n, x_{n-1}, \ldots, x_1)$ and the functions

$$\tilde{J}_m = J \circ \iota \ , \ \tilde{I}_m = I \circ \iota \ \ , \ \tilde{F}_m = \tilde{v}_{2m}\tilde{I}_m \ ,$$

where $\tilde{v}_i := a_{n+1-i}x_{n+1-i} + a_{n+2-i}x_{n+2-i} + \cdots + a_n x_n$, for $i = 1, \ldots, n$. Then, by Theorem 4.6 and the described symmetry of the system we derive the next theorem.

Theorem 5.1. *Suppose that n is even. Let d denote the smallest integer such that $a_{n-d} \neq 0$ and let $\delta = \left[\frac{d}{2}\right]$. The $\frac{n}{2}$ functions $\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_\delta, \tilde{F}_{\delta+1}, \tilde{F}_{\delta+2}, \ldots, \tilde{F}_{\frac{n}{2}-1}, H$ are pairwise in involution if and only if $k_{n+1-2i} = -k_{n+2-2i}$, for $i = 1, \ldots, \delta$, and $\tilde{S}_{jm} = 0$, for $m = \delta+1, \ldots, \frac{n}{2}-1$, $j = d+1, \ldots, 2m$, where*

$$\tilde{S}_{jm} = a_{n+1-j}\Big(-\sum_{i=1}^{j-1} k_{n+1-i} + \sum_{i=j+1}^{2m} k_{n+1-i}\Big) \ .$$

Theorem 5.1, determines different values of the parameters of the system that lead to integrability. Furthermore, a combination of Theorems 4.6-5.1 provide some superintegrable cases. For any $\ell, d \in \{0, 1, \ldots, n-1\}$, we consider the following two sets of parameters:

$$\begin{aligned}
\Sigma_\ell = \{(\mathbf{a}, \mathbf{k}) \in \mathbb{R}^{2n} &: a_1 = a_2 = \ldots a_\ell = 0, a_{\ell+1} \neq 0, k_{2i} + k_{2i-1} = S_{jm} = 0, \\
&\text{for } i = 1, \ldots, \left[\frac{\ell}{2}\right], \ m = \left[\frac{\ell}{2}\right] + 1, \ldots, \frac{n}{2} - 1, \ j = \ell + 1, \ldots, 2m \ \}, \\
\tilde{\Sigma}_d = \{(\mathbf{a}, \mathbf{k}) \in \mathbb{R}^{2n} &: a_n = a_{n-1} = \cdots = a_{n-d+1} = 0, a_{n-d} \neq 0, \\
&k_{n+1-2i} + k_{n+2-2i} = \tilde{S}_{jm} = 0, \text{for } i = 1, \ldots, \left[\frac{d}{2}\right], \ m = \left[\frac{d}{2}\right] + 1, \ldots, \frac{n}{2} - 1, \\
&j = d + 1, ..., 2m \ \},
\end{aligned}$$

where $(\mathbf{a}, \mathbf{k}) := (a_1, \ldots, a_n, k_1, \ldots, k_n)$. Then we conclude with the following theorem.

Theorem 5.2. *If* $(\mathbf{a},\mathbf{k}) \in \Sigma_\ell \cup \tilde{\Sigma}_d$ *for some* $\ell, d \in \{0,1,\ldots,n-1\}$*, then for every even* n *system* (7) *with parameters* $\mathbf{a},\mathbf{k}$ *is Liouville integrable. If* $(\mathbf{a},\mathbf{k}) \in \Sigma_\ell \cap \tilde{\Sigma}_d$*, then the corresponding system* (7) *is superintegrable, i.e. it admits the following* $n-1$ *functionally independent integrals:*
$J_1, J_2, \ldots, J_\lambda, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{\frac{n}{2}-1}, \tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_\delta, \tilde{F}_{\delta+1}, \tilde{F}_{\delta+2}, \ldots, \tilde{F}_{\frac{n}{2}-1}, H.$

Example 5.3. The simplest interesting case is $n = 4$ (for $n = 2$ the system is always integrable since it is Hamiltonian). In this case we have,

$$\begin{aligned}
\Sigma_0 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_1 \neq 0, k_2 = a_2 k_1 = 0\},\\
\Sigma_1 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_1 = 0, a_2 \neq 0, k_1 = 0\},\\
\Sigma_2 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_1 = a_2 = 0, a_3 \neq 0, k_2 = -k_1\},\\
\Sigma_3 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_1 = a_2 = a_3 = 0, a_4 \neq 0, k_2 = -k_1\},\\
\tilde{\Sigma}_0 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_4 \neq 0, k_3 = a_3 k_4 = 0\},\\
\tilde{\Sigma}_1 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_4 = 0, a_3 \neq 0, k_4 = 0\},\\
\tilde{\Sigma}_2 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_4 = a_3 = 0, a_2 \neq 0, k_4 = -k_3\},\\
\tilde{\Sigma}_3 &= \{(\mathbf{a},\mathbf{k}) \in \mathbb{R}^8 : a_4 = a_3 = a_2 = 0, a_1 \neq 0, k_4 = -k_3\},
\end{aligned}$$

where $(\mathbf{a},\mathbf{k}) = (a_1,\ldots,a_4,k_1,\ldots,k_4)$. Now, using Theorem 5.2 we can detect different integrable and superintegrable cases. So for example, when $a_1 \ldots a_n \neq 0$, from $\Sigma_0 \cup \tilde{\Sigma}_0$ we come up with two integrable cases, for $\mathbf{k} = (0,0,k_3,k_4)$ and $\mathbf{k} = (k_1,k_2,0,0)$, while the only superintegrable case that is derived from $\Sigma_0 \cap \tilde{\Sigma}_0$ is when $\mathbf{k} = 0$. On the other hand, for $a_2 = 0$ and $a_1 a_3 a_4 \neq 0$, we derive the integrable cases with $\mathbf{k} = (k_1,0,k_3,k_4)$ and $\mathbf{k} = (k_1,k_2,0,0)$, and the superintegrable case for $\mathbf{k} = (k_1,0,0,0)$. Proceeding in this way, we can detect all the integrable and superintegrable cases given by Theorem 5.2.

6 Numerical results for $n = 4$

The purpose of this section is to explore numerically the behavior of 4-dimensional Lotka–Volterra systems of the form (1) and investigate their integrability in cases that are not described in the previous sections. In the rest of the paper we will restrict to the case of $a_1 = a_2 = a_3 = a_4 = 1$ and we vary only the k_i values. We perform a series of numerical calculations for the system

$$\dot{x}_i = x_i \left(\sum_{j=1}^{4} P_{ij}(x_j + k_j) \right), \quad i = 1,\ldots,4, \tag{14}$$

with different $k_1,\ldots,k_4$ values, which are complementary to the two integrable cases described by Theorem 5.2. We numerically integrate the system's equations of motion together with its variational equations to compute the value of the largest Lyapunov exponent λ. The variational equations of the system (14) are

$$\delta\dot{\mathbf{x}} = [J \cdot \nabla^2 H(\mathbf{x}(t))] \cdot \delta\mathbf{x} \quad , \tag{15}$$

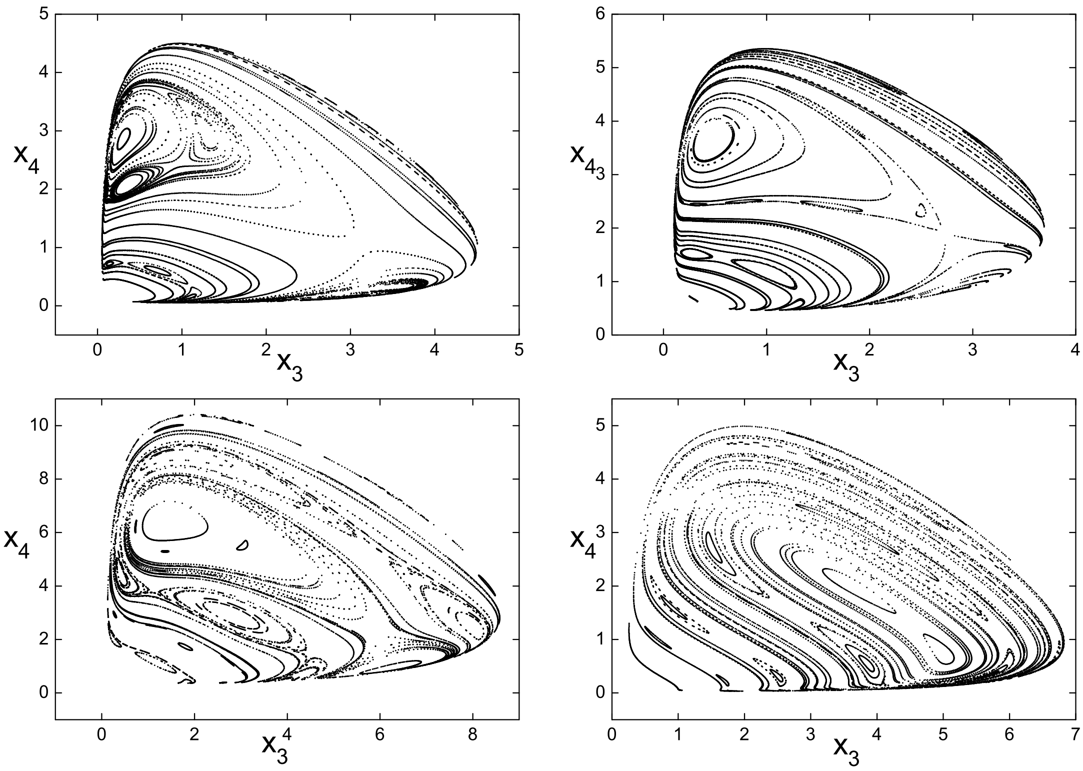

Figure 1: The Poincaré surface of section $x_2 = 1, x_1 > 1$ for the Lotka–Volterra system with $a_i = 1$ and $E = 6$ for various k_i, $i = 1, 2, 3, 4$ values: (a) $(k_1, k_2, k_3, k_4) = (-1, -2, -1, -1)$ (b) $(k_1, k_2, k_3, k_4) = (-1, -2, -1, -2)$, (c) $(k_1, k_2, k_3, k_4) = (-1, -4, -2, -3)$, (d) $(k_1, k_2, k_3, k_4) = (-1, -4, -2, -1)$.

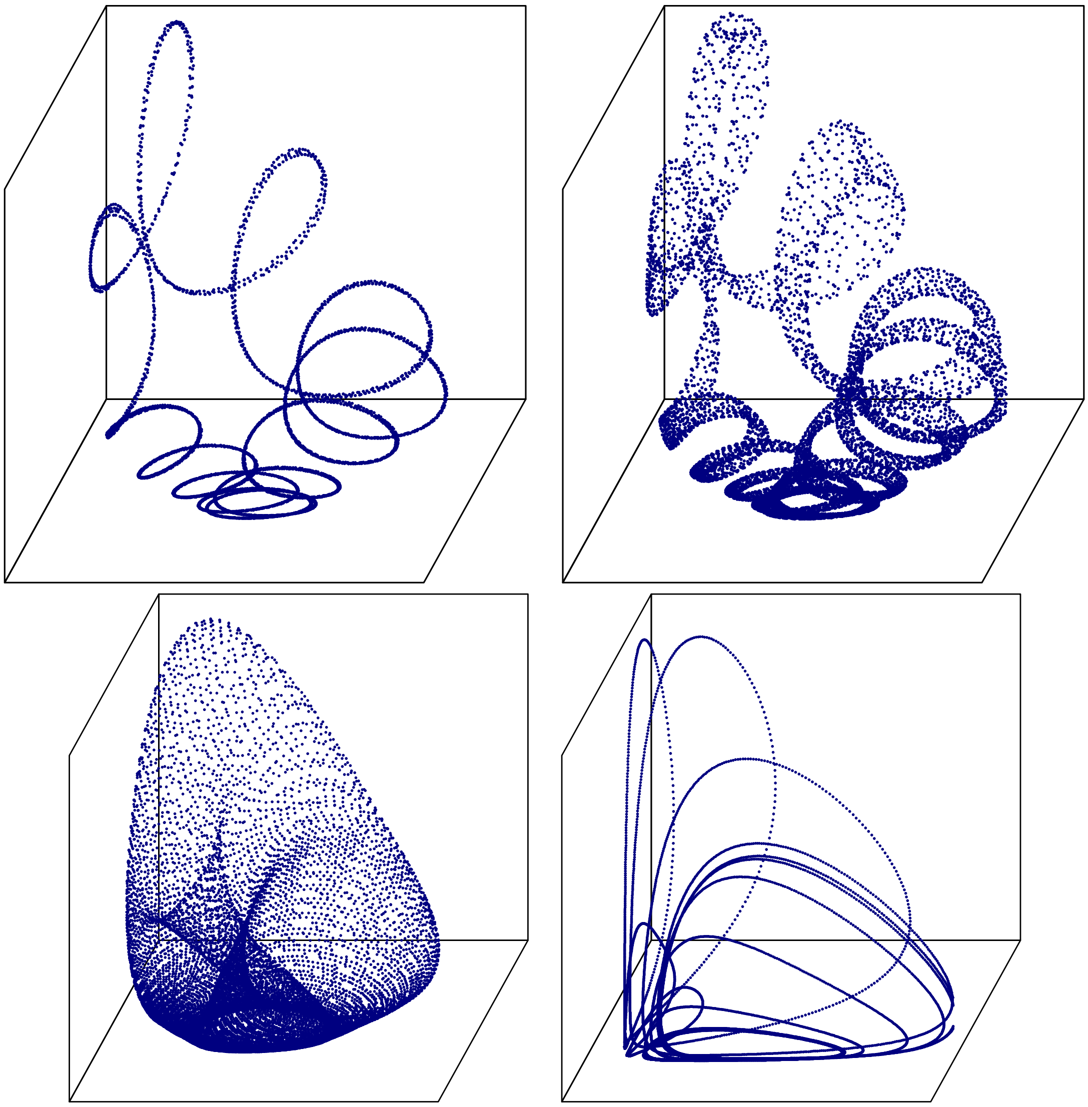

Figure 2: 3D projections on the x_2, x_3, x_4 plane for the system with $(k_1, k_2, k_3, k_4) = (-1, -4, -2, -1)$ and for initial conditions: (a) close to a fixed point of Fig.1(d) ($E = 6$), (b) on an ellipse around the fixed point ($E = 6$), (c) randomly chosen from Fig.1(d) ($E = 6$) and (d) randomly chosen at a higher total energy ($E = 20$) exhibiting chaotic behavior.

where $\delta\mathbf{x} = (\delta x_1, \delta x_2, \delta x_3, \delta x_4)$ is a vector which evolves on the tangent space of the system (14) and $\nabla^2 H$ denotes the Hessian matrix of the Hamiltonian function H calculated along the reference orbit $\mathbf{x}(t)$ of the system (14). In particular, we used the classical Runge–Kutta forth-order scheme with time-step $\tau = 10^{-4}$ for the numerical integration of the systems (14) and (15), which conserved the energy $E = H(\mathbf{x})$ of the system (14) with accuracy of more than 8 significant figures during integration times of the order of a few thousand. The indicator which controls of the relative energy error is

$$RE = \log_{10} \left| \frac{E(t) - E_0}{E_0} \right| \quad ,$$

where E_0 is the initial energy of the system and $E(t)$ the actual energy during the numerical integration.

For $k_i < 0$, $i = 1, \ldots 4$, the point $\mathbf{x}_0 = (-k_1, -k_2, -k_3, -k_4)$ is an elliptic fixed point of the system. Furthermore, in this case $H(\mathbf{x})$ admits a global minimum at $\mathbf{x}_0$ and all the orbits of the system are bounded.

We start our numerical study with examples of bounded motion, which correspond to negative values for all k_i. In Fig.1 some Poincaré surfaces of section $x_2 = 1$, $x_1 > 1$ are shown for different $k_i < 0$ values at $E = 6$. However, at this energy level all of them exhibit regular behavior. These Poincaré surfaces of section are constructed for a grid of initial conditions on the x_3, x_4 plane, with $x_2 = 1$ and x_1 found numerically by Newton's method requiring that $H(\mathbf{x}) = E$. We find a rich morphology consisting of periodic and quasiperiodic trajectories, island chains as well as separatrices. Each fixed point on the Poincaré surface represents a periodic orbit, while the ellipse-like curves correspond to quasiperiodic trajectories lying on tori. Fig.2 presents different trajectories projected on the x_2, x_3, x_4 plane for the system with $(k_1, k_2, k_3, k_4) = (-1, -4, -2, -1)$ which corresponds to Fig.1(d). The first three panels of Fig.2 correspond to $E = 6$ and the last one to $E = 20$.

We find qualitatively similar behavior to the examples of Fig.1 for $k_1 = \cdots = k_4 = -1$, as Fig.3 indicates. In the Poincaré surface of section $x_2 = 1$, $x_1 > 1$ of Fig.3(a), which corresponds to the energy $E = 4.2$, there is no evidence of chaotic behavior. We verify this result in Fig.4(a) by computing the largest Lyapunov exponent λ, which approximately decays as $1/t$ for randomly chosen initial conditions. Similarly with the well-known Hénon–Heiles model [9], chaotic dynamics in the Lotka–Volterra system (14) for $k_i < 0$ (or $k_i = -1$) emerges for larger values of the energy. In the rest of the panels of Fig.3, where the total energy E is gradually increased, we observe a gradual transformation of fixed points and ellipses–like curves, while at energies of the order of $E = 30$ (Fig.3(d)) the chaotic motion is not only evident but also prevails over the ordered motion. The largest Lyapunov exponent at this energy, which is plotted in Fig.4(b), converges to a positive value $\lambda \simeq 0.01$.

As we have seen in example 5.3, the only integrable cases for $n = 4$, $\mathbf{a} = (1, 1, 1, 1)$ predicted by Theorem 5.2 are for $\mathbf{k} = (0, 0, k_3, k_4)$, $k_3, k_4 \in \mathbb{R}$ or $\mathbf{k} = (k_1, k_2, 0, 0)$, $k_1, k_2 \in \mathbb{R}$. We choose $(k_1, k_2, k_3, k_4) = (0, 0, -1, -1)$, for which the quantity $(x_1 + x_2)x_4/x_3$ is preserved besides the Hamiltonian. Fig.5(a) displays the evolution of the four variables $\log x_i$ in time for a random choice of initial conditions. It turns out that x_2 decays asymptotically to zero, approximately like $e^{-0.63t}$, while the rest variables x_1, x_3, x_4 asymptotically approach

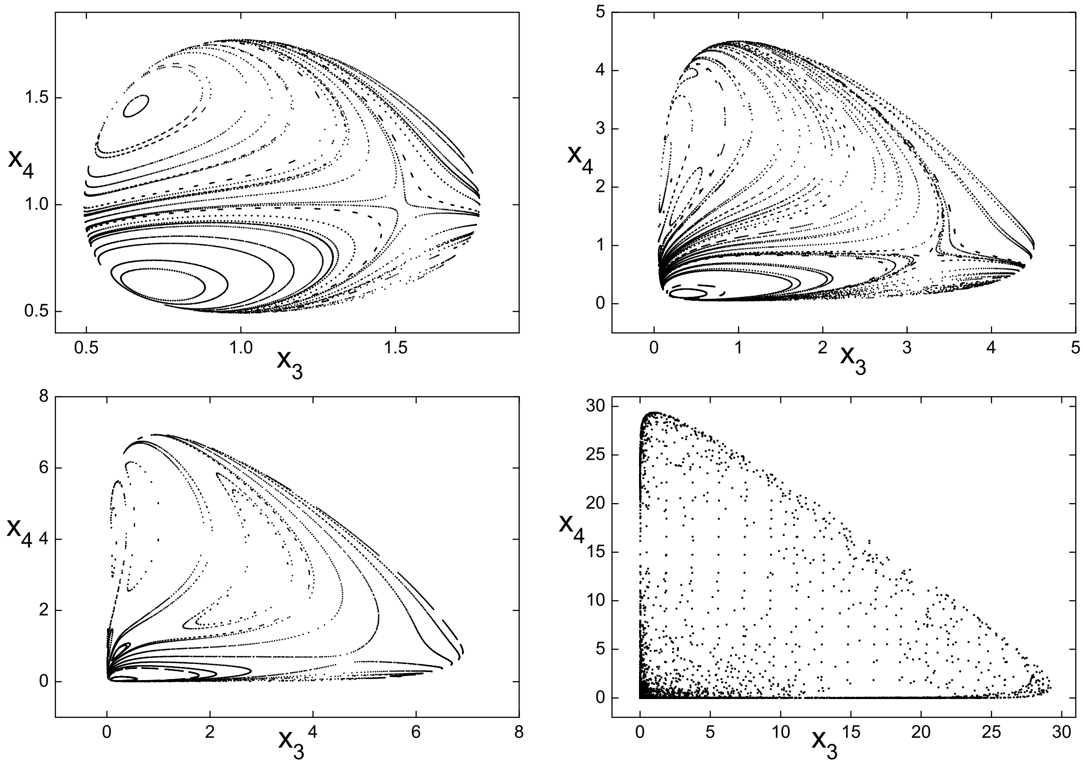

Figure 3: The Poincaré surface of section $x_2 = 1, x_1 > 1$ for the Lotka–Volterra system with $a_i = 1$ and $k_i = -1$, $i = 1, 2, 3, 4$ for the energies: (a) $E = 4.2$, (b) $E = 6$, (c) $E = 8$, (d) $E = 29$.

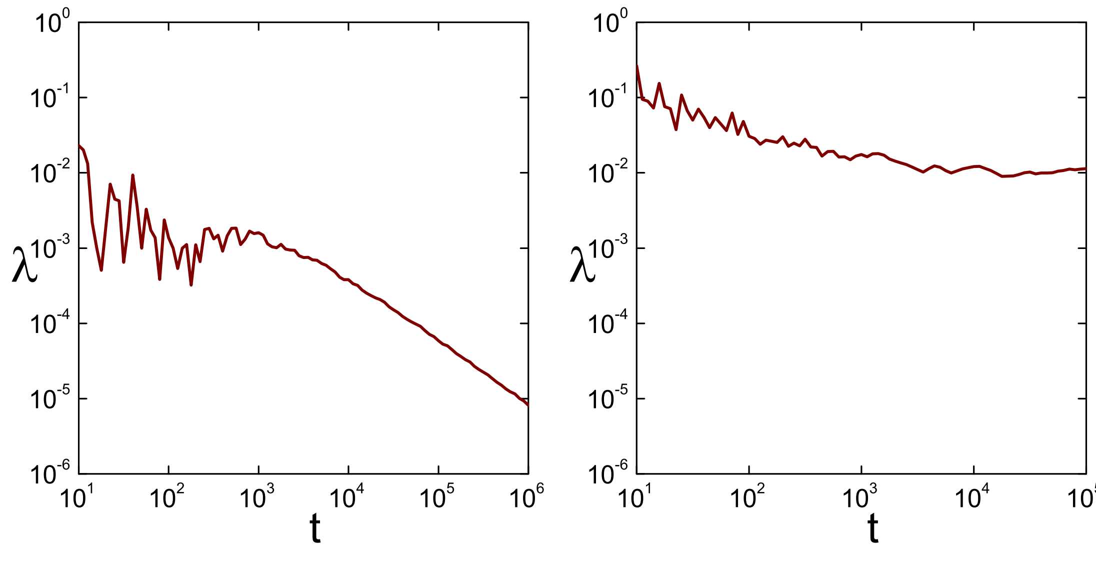

Figure 4: The largest Lyapunov exponent λ for the Lotka–Volterra system with $a_i = 1$ and $k_i = -1$, $i = 1, 2, 3, 4$ for the energies: (a) $E = 4.2$ and (b) $E = 29$.

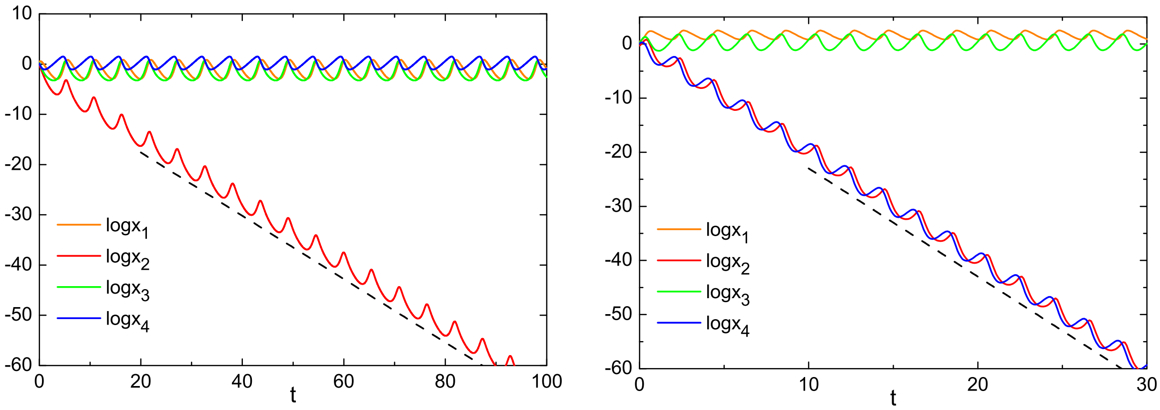

Figure 5: The evolution in time of the phase space variables for the integrable cases: (a) $(k_1, k_2, k_3, k_4) = (0, 0, -1, -1)$ and (b) $(k_1, k_2, k_3, k_4) = (-2, -2, -2, 2)$.

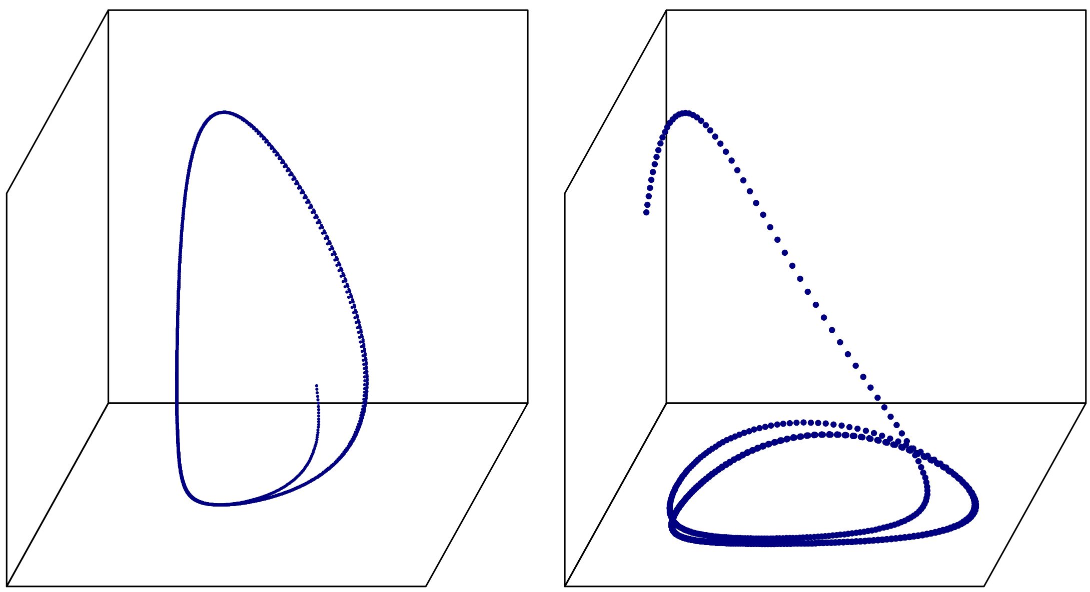

Figure 6: The trajectories projected on the 3D plane x_1, x_3, x_4 plane for the integrable systems: (a) $(k_1, k_2, k_3, k_4) = (0, 0, -1, -1)$ and (b) $(k_1, k_2, k_3, k_4) = (-2, -2, -2, 2)$.

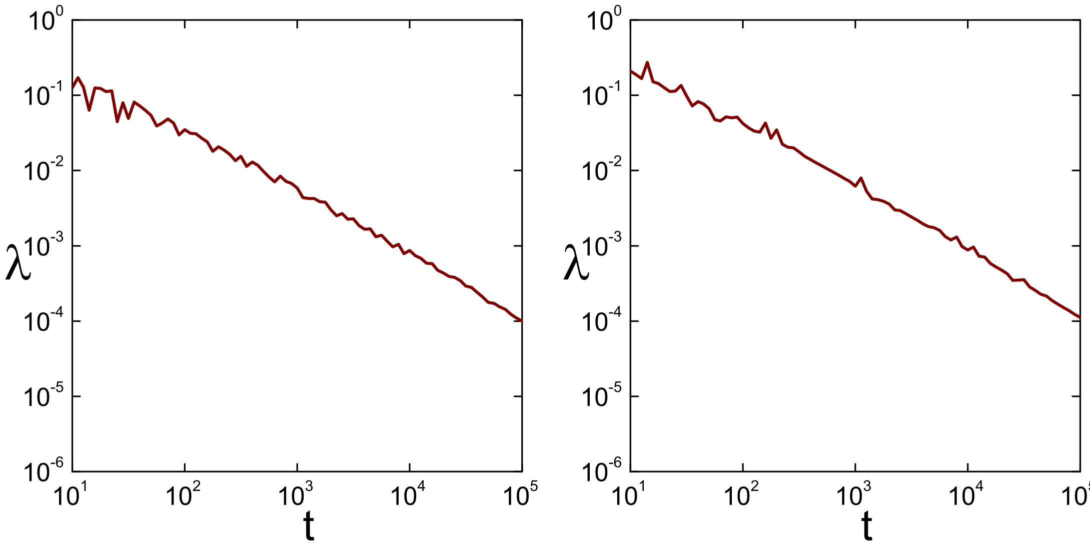

Figure 7: The largest Lyapunov exponent λ for the system with $(k_1, k_2, k_3, k_4) = (-2, -2, -2, 2)$ at (b) $E = 10$ and (c) $E = 72$.

a periodic orbit, as is illustrated in Fig.6(a). However, a similar behavior appears in other cases, not described as integrable by Theorem 5.2. Such an example is given in Fig.5(b) and corresponds to $(k_1, k_2, k_3, k_4) = (-2, -2, -2, 2)$. It turns our that the variables x_2 and x_4 tend asymptotically to zero as e^{-2t}, while x_1 and x_3 asymptotically converge to the periodic orbit shown in Fig.6(b). Furthermore, we carefully examine the largest Lyapunov exponent λ in Fig.7 for constantly increasing energies and we find that $\lambda \propto 1/t$, even when $E = 72$, which strongly indicates that the system is integrable in this case too.

Similarly to the case $(k_1, k_2, k_3, k_4) = (-2, -2, -2, 2)$ we find other cases which display integrable behavior, manifested by asymptotically vanishing Lyapunov exponents. Few of the cases that we checked are listed in the following table

k_1	k_2	k_3	k_4
1	1	-1	-1
1	-1	1	-1
1	-1	-1	1
-1	-1	1	-1
-1	1	-1	-1
1	-1	-1	-1

Finally, based on our numerical findings and observations, we conjecture that chaotic motion for the $n = 4$ system (7) emerges when $a_i > 0$, $k_i < 0$ and $a_i < 0$, $k_i > 0$.

7 Conclusions

We presented a new class of Hamiltonian parametric Lotka–Volterra systems with non-zero linear terms and we proved that, for particular choices of parameters, Liouville integrability and superintegrability is established. Different choices of parameters when $n = 4$, not described by the theory, were studied numerically, showing that both chaotic and new integrable cases appear. Concerning these new cases with integrable behavior, we aim to study them in detail in order to detect additional integrals and complete our investigation by including all the odd dimensional cases too.

In the present work we restricted our analysis to the even-dimensional case; however, a similar approach can be considered for odd dimensions. Finally, we believe that a similar approach can be considered for integrable Lotka–Volterra systems with different community matrices, or integrable deformations of them such as the systems presented in [6, 7, 8], by inserting parametric linear terms in the corresponding vector fields.

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A Comments and examples on the odd dimensional cases

As it is stated in Section 3, in the odd dimensional cases the described Hamiltonian formalism, i.e. the log-canonical Poisson structure (6) along with the Hamiltonian $H(\mathbf{x}) = \sum_{i=1}^{n}(a_i x_i + k_i \log x_i)$, is not sufficient to include all the cases of vector fields (1) for arbitrary r_i, since matrix (5) is not invertible. Therefore, in this setting we can only restrict to the cases with $\mathbf{r} = P\mathbf{k}$, that is systems of the form (7). For $n = 3$, the integrability of (7) follows directly from its Hamiltonian formalism and the existence of the Casimir function $\frac{x_1 x_3}{x_2}$. More interesting integrable cases emerge for odd $n > 3$, by considering the corresponding integrals of the $\mathbf{k} = 0$ case as they appear in [14] and the corresponding permutation symmetry of the system. We will illustrate this in the following example for $n = 5$.

Let us consider the system

$$\dot{x}_i = x_i \left(\sum_{j=1}^{5} P_{ij}(a_j x_j + k_j) \right) , \quad i = 1, \dots, 5, \tag{16}$$

with parameters $\mathbf{a} = (a_1, \dots, a_5), \mathbf{k} = (k_1, \dots, k_5) \in \mathbb{R}^5$. According to [14], for $\mathbf{k} = \mathbf{0}$ this system admits the first integral

$$F = \frac{x_5}{x_4}(a_1 x_1 + a_2 x_2 + a_3 x_3).$$

We compute its Poisson bracket with the Hamiltonian $H = \sum_{i=1}^{n}(a_i x_i + k_i \log x_i)$ of (16) to get

$$\{F, H\} = \frac{x_5}{x_4}\left(a_1(k_2 + k_3)x_1 + a_2(k_3 - k_1)x_2 - a_3(k_1 + k_2)\right).$$

Hence, F is a first integral of (16) if and only if

$$a_1(k_2 + k_3) = a_2(k_3 - k_1) = a_3(k_1 + k_2) = 0. \tag{17}$$

If the parameters $\mathbf{a}, \mathbf{k}$ satisfy (17), then the integral F in addition to the Casimir function $C = \frac{x_1 x_3 x_5}{x_2 x_4}$ ensures the complete integrability of the system. Furthermore, the invariance of (16) under the transformation $x_i \mapsto x_{6-i}$, $a_i \mapsto -a_{6-i}$, $k_i \mapsto -k_{6-i}$, implies that

$$\tilde{F} = \frac{x_1}{x_2}(a_5 x_5 + a_4 x_4 + a_3 x_3).$$

is a first integral of (16) if and only if

$$a_5(k_4 + k_3) = a_4(k_3 - k_5) = a_3(k_5 + k_4) = 0. \tag{18}$$

So we conclude that system (16) is integrable if the parameters $\mathbf{a}, \mathbf{k}$ satisfy (17) or (18).

For example, in the case of $\mathbf{a} \neq 0$, system (16) is integrable if $k_3 = -k_2 = k_1$ or $k_3 = -k_4 = k_5$, while the case of $k_5 = -k_4 = k_3 = -k_2 = k_1$ which leads to superintegrability is equivalent to the $\mathbf{k} = 0$ case.

References

[1] Á. Ballesteros, A. Blasco and F. Musso, Integrable deformations of Lotka–Volterra systems, *Phys. Lett. A*, **375**(38) (2011), 3370–3374.

[2] O. I. Bogoyavlenskij, Some constructions of integrable dynamical systems, *Izv. Akad. Nauk SSSR Ser. Mat.*, **51**(4) (1987), 737–766.

[3] O. I. Bogoyavlenskij, Integrable Lotka-Volterra systems, *Regul. Chaotic Dyn.*, **13**(6) (2008), 543–556.

[4] T. Bountis and P. Vanhaecke, Lotka–Volterra systems satisfying a strong Painlevé property, *Phys. Lett. A.*, **380**(47) (2016), 3977–3982.

[5] S. A. Charalambides, P. A. Damianou, and C. A. Evripidou, On generalized Volterra systems, *J. Geom. Phys.*, **87** (2015), 86–105.

[6] P. A. Damianou, C. A. Evripidou, P. Kassotakis and P. Vanhaecke, Integrable Reductions of the Bogoyavlenskij–Itoh Lotka-Volterra Systems, *J. Math. Phys.*, **58** (2017), 032704.

[7] C. A. Evripidou, P. Kassotakis and P. Vanhaecke, Integrable deformations of the Bogoyavlenskij–Itoh Lotka–Volterra systems, *Regul. Chaot. Dyn.*, **22** (2017), 721–739.

[8] C. A. Evripidou, P. Kassotakis and P. Vanhaecke, Integrable reductions of the dressing chain, *arXiv:1903.02876*.

[9] M. Hénon and C. Heiles, The applicability of the third integral of motion: Some numerical experiments, *Astron. J.*, **69**(1) (1964), 73–79.

[10] B. Hernández–Bermejo and V. Fairén, Hamiltonian structure and Darboux theorem for families of generalized Lotka–Volterra systems, *J. Math. Phys.*, **39**(11) (1998), 6162–6174.

[11] Y. Itoh, Integrals of a Lotka–Volterra system of odd number of variables, *Progr. Theoret. Phys.*, **78**(3) (1987), 507–510.

[12] Y. Itoh, A combinatorial method for the vanishing of the Poisson brackets of an integrable Lotka–Volterra system, *J. Phys. A* **42**(2) (2009), 025201.

[13] P. H. van der Kamp, T. E. Kouloukas, G. R. W. Quispel, D. T. Tran and P. Vanhaecke, Integrable and superintegrable systems associated with multi-sums of products, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 470 (2014), 20140481.

[14] T. E. Kouloukas, G. R. W. Quispel and P. Vanhaecke, Liouville integrability and superintegrability of a generalized Lotka–Volterra system and its Kahan discretization, *J. Phys. A: Math. Theor.* **49** (2016), 225201.

[15] A. J. Lotka. Analytical theory of biological populations. The Plenum Series on Demographic Methods and Population Analysis. Plenum Press, New York, 1998. Translated from the 1939 French edition and with an introduction by David P. Smith and Hélène Rossert.

[16] O. Ragnisco and M. Scalia, The Volterra Integrable case, *arXiv:1903.03595*.

[17] Y. B. Suris and O. Ragnisco, What is the relativistic Volterra lattice?, *Comm. Math. Phys.*, **200**(2) (1999), 445–485.

[18] J. A. Vano, J. C. Wildenberg, M. B. Anderson, J. K. Noel and J. C. Sprott, Chaos in low-dimensional Lotka–Volterra models of competition, *Nonlinearity* **19** (2006), 2391–2404.

[19] A. P. Veselov and A. V. Penskoï, On algebro–geometric Poisson brackets for the Volterra lattice, *Regul. Chaotic Dyn.* **3**(2) (1998), 3–9.

[20] V. Volterra, Leçons sur la théorie mathématique de la lutte pour la vie.Les Grands Classiques Gauthier–Villars, Paris, 1931. Reprint 1990.