Manuscript prepared for Journal of Marine Science and Technology

Melnikov Integral Formula for Beam Sea Roll Motion Utilizing a Non-Hamiltonian Exact Heteroclinic Orbit (Part II)

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Keywords: Melnikov's method, Beam Seas, Roll Motion, Non-Hamiltonian Exact Heteroclinic Orbit

Abstract

In the research filed of nonlinear dynamical system theory it is well known that a homoclinic/heteroclinic point leads to unpredictable motions, such as chaos. Melnikov's method enables us to judge whether the system has a homoclinic/heteroclinic orbit. Therefore, in order to assess a vessels safety against capsizing, Melnikov's method has been applied for the investigations of chaos that appears in beam sea rolling. This is because chaos is closely related to capsizing incidents. In a previous paper [1)], the formula to predict the capsizing boundary by applying Melnikov's method to analytically obtain the non-Hamiltonian heteroclinic orbit, was proposed. However, in that paper, limited numerical investigation had been carried out. Therefore further comparative research between the analytical and numerical results is conducted, with the result being that the formula is validated.

1. Introduction

Currently, investigations into chaos of nonlinear vessel rolling in beam seas have been extensively investigated [e.g. 2)-11)], with Melnikov's method being effectively applied [12)]. Melnikov's method can establish the onset of the heteroclinic point which assures the existence of the horseshoe map via the Smale-Birkoff theorem [13)]. For instance, Kan & Taguchi [5)] implied that the threshold of fractal metamorphoses in the control plane obtained from the Melnikov analysis of the non-biased roll equation could be applicable for a vessel's stability criterion. On the other hand, Spyrou [11)] investigated the biased roll equation with an appropriate variable transformation and performed the Melnikov analysis using an analytically obtained homoclinic orbit. A nonlinear system, however, is not necessarily solvable, so in general it is difficult to analytically obtain the separatrix closed loop. Consequently Wu and McCue [14)] applied the extended Melnikov's method [15)] for a numerically obtained heteroclinic orbit based upon the Endo and Chua's work [16)].

In order to apply Melnikov's method, it is required to obtain the separatrix closed loop for the autonomous part of the full system. Thanks to recent advances in nonlinear science, several solitary solutions have already been found via methods using nonlinear equations [17)]. Maki et al. [1)] pointed out that the escape equation used by Kan & Taguchi [6)] is identical with FHN (FitzHugh-Nagumo) with the exception of some of the coefficients. They investigated analytically the heteroclinic orbit in the time domain by using the solution technique that is originally used for analysing nonlinear waves, and then extended Melnikov's method proposed by Salam [15)] was applied. The paper [1)], however, mainly addresses the analytical formulation, and limited numerical results were presented. In this paper, it is the objective to numerically validate the proposed formula, and carry out additional analysis of the escape equation.This paper is structured as follows: firstly, following the brief explanation of the formulation for the biased roll equation, the analytical results of the heteroclinic orbit are validated by using numerical bifurcation analysis. Then the results of the Melnikov integral are shown, and the obtained threshold of fractal metamorphoses that appeares in the control plane is compared with numerical simulation result.

2. Non-Hamiltonian heteroclinic obit

In order to apply Melnikov's method introduced by Salam [15], it is required to obtain the non-Hamiltonian heteroclinic orbit. Although it is difficult to estbalish the exact solution of a nonlinear equation, the solutions of the equation could be found by employing a solution technique used in nonlinear waves. Maki et al. [1] applied this technique for the escape equation, and then provided the heteroclinic orbit and its condition. In this paper, the same methodology is used but the treatment of the bias term that appears in the equation is slightly different from that presented in the previous paper[1]. Thus, it is shown that the following brief reformulation is suitable for the numerical validation.

Consider the following biased roll equation with linear damping and nonlinear cubic term in the restoring force as follows:

$$I\frac{d^2\Phi}{dt^2}+N\frac{d\Phi}{dt}+W\cdot GM\cdot\Phi\left(1-\Phi/\Phi_V\right)\left(1+a\cdot\Phi/\Phi_V\right)=M_r\cos\left(\omega t+\delta\right) \tag{2.1}$$,

Where a : the coefficient representing the bias of roll equation, GM :the metacentric height, I : the moment of inertia in roll, M_r : the amplitude of the 1st order wave-induced roll moment, N : the damping coefficient in roll, t : time, W : the ship mass, Φ : the roll angle and Φ_V : the angle of vanishing stability. The appropriate non-dimensionalization for (2.1) yields:

$$\ddot{\phi}+\beta\dot{\phi}+\mu\cdot\phi(1-\phi)(1+a\cdot\phi)=\gamma\cos(\omega t+\delta) \tag{2.2}$$,

where:

$$\begin{cases}\phi\equiv\Phi/\Phi_V\\ \beta\equiv N/I,\ \mu\equiv W\cdot GM/I,\ \gamma\equiv M_r/I\Phi_V\end{cases} \tag{2.3}$$.

In order to obtain the heteroclinic orbit of the homogenious part of (2.2), an addition of the parameter for both sides of the equation are as follows.

$$\ddot{\phi}+\beta\dot{\phi}+\mu\cdot\phi(1-\phi)(1+a\cdot\phi)+\sigma=\sigma+\gamma\cos(\omega t+\delta) \tag{2.4}$$,

Here σ can be calculated by using the procedure described in the previous paper [1], and the results are shown as follows:

$$\tilde{\mu}\left(\frac{1}{2}-\tilde{a}\right)\pm\tilde{\beta}\sqrt{\frac{\tilde{\mu}}{2}}=0 \tag{2.5}$$,

where

$$\begin{cases} \tilde{a} \equiv (\phi_2 - \phi_1)/(\phi_3 - \phi_1) \\ \tilde{\beta} \equiv \beta,\ \tilde{\mu} \equiv a\mu(\phi_3 - \phi_1)^2 \end{cases} \tag{2.6}$$

and ϕ_1, ϕ_2, ϕ_3 are the solutions of the following equation.

$$\mu \cdot \phi(1-\phi)(1+a\cdot\phi)+\sigma=0 \tag{2.7}$$

This is a third order polynomial with respect to ϕ and can be factorised using Cardano's method. In (2.5), a positive or a negative sign corresponds to the trajectory on the upper and/or lower phase plane, respectively. When the condition (2.5) is satisfied, heteroclinic orbit is realized, and then it can be represented in the time domain as:

$$\phi_0(t)=\phi_1+\frac{\phi_3-\phi_1}{1+e^{\pm\sqrt{0.5\tilde{\mu}}t}} \tag{2.8}$$

Additionally, this is achieved by using the quadratic function in the phase plane, thus

$$\dot{\phi}_0(t)=\mp\sqrt{\frac{\tilde{\mu}}{2}}\frac{(\phi_0(t)-\phi_1)(\phi_0(t)-\phi_3)}{\phi_3-\phi_1} \tag{2.9}$$

Note the double sign in the same order with eq.2.5, in eq's.2.8 and 2.9.It is now possible to compare the results using the proposed method and numerical bifurcation analysis. In this paper, the numerical bifurcation analysis proposed by Kawakami et al. [18)] is employed for finding the critical parameter σ. Using this method, all the conditions necessary for realizing the heteroclinic bifurcation, i.e. the equilibrium of saddle points, their eigenvalues, their eigenvectors, and the connection of both trajectories at the intermediate point, are simultaneously solved with Newton's method. Using Kawakami's method an allowable numerical displacement vector norm of 1.0^{-6} error is applied when using the Newton method. To further retain the numerical accuracy, a 5th order Runge-Kutta integral scheme is also applied.

In fig.1 the comparison of the critical value σ for the non-biased escape equation, i.e. when $a=1$, obtained by using these two methods. Whereas Fig.2 is when the bias. $a=0.9$. Since only a small discrepancy can be observed in these figures, the proposed analytical method is considered to be satisfactory. Fig.3 and fig.4 illustrate the heteroclinic orbits of the in phase plane spanned by ϕ and $\dot{\phi}$ obtained using these two methods. Note that the analytically obtained heteroclinic orbit is a quadratic function (2.9). It can also be observed that the two orbits are completely identical. Although the uniqueness of a heteroclinic orbit for this system cannot be proved, mutual agreement indicates that (2.8) locally it is consistent and represents the heteroclinic orbit of Equation (2.2). Since this solution, for obtaining the heteroclinic bifurcation point is quite simple and robust, it can be used easily to calculate the parameter set required, thus realizing the heteroclinic bifurcation as shown in fig.5.

3. Critical forcing

If an heteroclinic orbit is obtained, Melnikov's method is analytically applicable. Next the biased case is examined. Wu and McCue [14)] used

$$\ddot{\phi}+\beta\dot{\phi}+\mu\left(\phi-\phi^{3}\right)=(1-a)\left(\phi^{2}-\phi^{3}\right)+\gamma\cos(\omega t+\delta) \tag{3.1}$$

based upon the assumption of small $1-a$, as an alternative to directly manipulating (2.4), and then Melnikov integral

$$M(t_0)=\int_{-\infty}^{\infty}\dot{\phi}_0(t)\left[\gamma\cos\omega(t+t_0)+\sigma+(1-a)\left(\phi_0^2(t)-\phi_0^3(t)\right)\right]e^{\tilde{\beta}t}dt \tag{3.2}$$

is carried out for the heteroclinic orbit of the left hand side of (3.1). Note that the left hand side of (3.1) is the non-biased roll equation. By using the analytically obtained heteroclinic orbit (2.9), the critical forcing is obtained as follows:

$$\gamma=\frac{(1-a)\left(A_0I_0+A_1I_1+A_2I_2+A_3I_3\right)+\sigma I_0}{\sqrt{I_r^2+I_i^2}} \tag{3.3}$$

In this equation, the values I_r, I_i, I_0, I_1, I_2, I_3 can be calculated analytically as follows:

$$\begin{aligned}I_r&\equiv\mathrm{Re}\left[\frac{1}{4}\int_{-\infty}^{\infty}\frac{\exp\left(\tilde{\beta}t+i\omega t\right)}{\cosh^2(\tilde{c}t/2)}dt\right]=\mathrm{Re}\left[\frac{\pi\left(\tilde{\beta}+i\omega\right)\csc\left[\pi\left(\tilde{\beta}+i\omega\right)/\tilde{c}\right]}{\tilde{c}^2}\right]\mathrm{sgn}\,\tilde{c}\\&=-\frac{2\pi\left[\tilde{\beta}\cosh(\omega\pi/\tilde{c})\sin\left(\tilde{\beta}\pi/\tilde{c}\right)+\omega\cos\left(\tilde{\beta}\pi/\tilde{c}\right)\sinh(\omega\pi/\tilde{c})\right]}{\tilde{c}^2\left[\cos\left(2\tilde{\beta}\pi/\tilde{c}\right)-\cosh(2\omega\pi/\tilde{c})\right]}\mathrm{sgn}\,\tilde{c}\end{aligned} \tag{3.4}$$

$$\begin{aligned}I_i&\equiv\mathrm{Im}\left[\frac{1}{4}\int_{-\infty}^{\infty}\frac{\exp\left(\tilde{\beta}t+i\omega t\right)}{\cosh^2(\tilde{c}t/2)}dt\right]=\mathrm{Im}\left[\frac{\pi\left(\tilde{\beta}+i\omega\right)\csc\left[\pi\left(\tilde{\beta}+i\omega\right)/\tilde{c}\right]}{\tilde{c}^2}\right]\mathrm{sgn}\,\tilde{c}\\&=\frac{2\pi\left[-\omega\cosh(\omega\pi/\tilde{c})\sin\left(\tilde{\beta}\pi/\tilde{c}\right)+\tilde{\beta}\cos\left(\tilde{\beta}\pi/\tilde{c}\right)\sinh(\omega\pi/\tilde{c})\right]}{\tilde{c}^2\left[\cos\left(2\tilde{\beta}\pi/\tilde{c}\right)-\cosh(2\omega\pi/\tilde{c})\right]}\mathrm{sgn}\,\tilde{c}\end{aligned} \tag{3.5}$$

$$I_0\equiv\int_{-\infty}^{\infty}\frac{\exp\left(\tilde{\beta}t+\tilde{c}t\right)}{(1+\exp(\tilde{c}t))^2}dt=\frac{\tilde{\beta}\pi}{\tilde{c}^2\sin\left(\tilde{\beta}\pi/\tilde{c}\right)}\mathrm{sgn}\,\tilde{c} \tag{3.6}$$

$$I_1 \equiv \int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}t + 2\tilde{c}t\right)}{\left(1+\exp(\tilde{c}t)\right)^3} dt = \frac{\tilde{\beta}\left(\tilde{\beta}+\tilde{c}\right)\pi}{2!\tilde{c}^3 \sin\left(\tilde{\beta}\pi/\tilde{c}\right)} \operatorname{sgn}\tilde{c} \tag{3.7}$$

$$I_2 \equiv \int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}t + 3\tilde{c}t\right)}{\left(1+\exp(\tilde{c}t)\right)^4} dt = \frac{\tilde{\beta}\left(\tilde{\beta}+\tilde{c}\right)\left(\tilde{\beta}+2\tilde{c}\right)\pi}{3!\tilde{c}^4 \sin\left(\tilde{\beta}\pi/\tilde{c}\right)} \operatorname{sgn}\tilde{c} \tag{3.8}$$

$$I_3 \equiv \int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}t + 4\tilde{c}t\right)}{\left(1+\exp(\tilde{c}t)\right)^5} dt = \frac{\tilde{\beta}\left(\tilde{\beta}+\tilde{c}\right)\left(\tilde{\beta}+2\tilde{c}\right)\left(\tilde{\beta}+3\tilde{c}\right)\pi}{4!\tilde{c}^5 \sin\left(\tilde{\beta}\pi/\tilde{c}\right)} \operatorname{sgn}\tilde{c} \tag{3.9}$$

$$A_0 = \phi_1^2 (1-\phi_1) \tag{3.10}$$

$$A_1 = \phi_2 (\phi_3 - \phi_1)(2 - 3\phi_2) \tag{3.11}$$

$$A_2 = (\phi_3 - \phi_1)^2 (1 - 3\phi_1) \tag{3.12}$$

$$A_3 = -(\phi_3 - \phi_1)^3 \tag{3.13}$$

where

$$\tilde{c} \equiv \pm\sqrt{\tilde{\mu}/2} \tag{3.14}$$,

In following figures, the results based on (3.3) are plotted as 'Formula of a non-biased heteroclinic orbit'

On the other hand, the Melnikov integral can be carried out without transposing the part of the restoring term into right side since the non-biased roll equation is solved analytically as shown in previous section. In this case, the critical forcing can be obtained as:

$$\gamma = \frac{\sigma I_0}{\sqrt{I_r^2 + I_i^2}} \tag{3.15}$$

I_0, I_r and I_i, are the same as those shown in eqs.(3.4) to (3.6). Note, for the calculation of I_0, I_r and I_i, an heteroclinic orbit with respect to the biased-roll equation should be employed. In the following figures, the results based on (3.15) are plotted as 'Proposed formula'.

Fig.6 shows the final results of the critical forcing γ for the non-biased case. In this figure, The results are obtained by using the formula [14)] given by:

$$\gamma = \left[\frac{2\beta}{3} - \frac{\sqrt{2}(1-a)}{3}\right] \frac{\sinh\left(\pi\omega/\sqrt{2}\right)}{\pi\omega} \tag{3.16}$$.

And are plotted as 'Formula from Hamiltonian heteroclinic orbit' for comparative purposes. Note that (3.16) with a of 1.0 is identical with the formula obtained by Kan and Taguchi [6)]. From figure 6, it can be seen that there is only a minor discrepancy between the two. Figure 7 indicates comparative results of the critical forcing obtained by using several methods and it shows that the results of critical forcing do not wholly depend upon an assumed heteroclinic orbit. The reason is considered as follows. In this study, the extended Melnikov method introduced by Salam [15] is employed. The significant difference between the original method and the extended method is whether or not the damping term of a heteroclinic orbit is taken into account. However, it is well known that roll dumping is generally small, and that the contribution of its difference also becomes small. The proposed calculation technique is relatively complicated compared to that proposed by Kan & Taguchi [6] and the method proposed and by Spyrou et al. [11], and these method are considerably validated by numerical simulation. Therefore, these two methods are more practical and thus recommended.

Finally, we confirm whether the obtained critical forcing actually represents the bound of chaos or fractals. Since the obtained values using the proposed method are: γ of 0.06344, a of 0.975, β of 0.1 and ω of 0.8, a numerical calculation is carried out for γ of 0.07. Here, slightly above the critical forcing is chosen for numerical calculation. This is because fractal metamorphoses of basin boundary cannot be clearly observed at γ close to 0.06344. Fig.8 shows the onset of a safe basin erosion near this value. The black shaded part of the plot represents the non-capsizing region while white non-shaded part is a capsize region. From this figure it can be concluded that the critical forcing obtained by using Melnikov's integral formula can approximately demonstrate the onset of chaos and fractals.

Although this analysis is carried out for the escape equation having only linear damping terms the same procedure is of course applicable for equations having higher order damping terms. As an example, in Appendix 1,the method employing the equation having linear and quadratic damping terms is described , and an extended analysis for a 1 DoF roll equation having 4th order polynomial restoring term and a quadratic polynomial damping term is described in Appendix 2. Furthermore, it is worth noting that saddle-node bifurcation appears in the escape equation and the relation to the Melnikov analysis is demonstrated based on Yagasaki's work [19], in a previous paper [20].

4. Concluding remarks

The main conclusions to be drawn from this work can be summarized as follows:

1. The proposed equation representing the heteroclinic orbit from previous work is verified by numerical results.
2. By using an analytically obtained heteroclinic orbit, the Melnikov integral can be analytically evaluated. As a result, it is concluded that for the equation having a small damping term, such as the escape equation, whether or not the damping term is taken into account the calculation of the separatrix does not strongly influence the final result.

Acknowledgement

This research was partly supported by a Grant-in-Aid for Scientific Research of the Japan Society for Promotion of Science (no. 24360355). The authors express the sincere gratitude to the above organization. The authors are grateful to Mr. John Kecsmar from Ad Hoc Marine Designs Ltd. for his comprehensive review of this paper as an expert in small craft technology and a native English speaker.

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Appendix 1

In this paper, the damping term in the state equation is assumed as linear, but the critical forcing should be formulated for a general case. Therefore, the formulation is shown for the case of linear, quadratic and cubic dumpling:

$$\ddot{\varphi}+\beta\dot{\varphi}+\beta_2\dot{\varphi}^2\operatorname{sgn}\dot{\varphi}+\beta_3\dot{\varphi}^3+\varphi(1-\varphi)(\varphi-\tilde{a})=\sigma+F\sin\omega t \qquad \text{(A1-1).}$$

Manipulation of this equation easily leads to following expression:

$$\frac{d}{dt}\begin{pmatrix}\varphi\\ \dot{\varphi}\end{pmatrix}=\begin{pmatrix}\dot{\varphi}\\ -\beta\dot{\varphi}-\varphi(1-\varphi)(\varphi-\tilde{a})\end{pmatrix}+\begin{pmatrix}0\\ -\beta_2\dot{\varphi}^2\operatorname{sgn}\dot{\varphi}-\beta_3\dot{\varphi}^3+\sigma+F\sin\omega t\end{pmatrix}\equiv\mathbf{F}(\mathbf{x})+\mathbf{G}(t) \qquad \text{(A1-2).}$$

Then the Melnikov integral becomes:

$$\begin{aligned}M(t_0)&=\int_{-\infty}^{\infty}\mathbf{F}\left(\varphi^0(t)\right)\wedge\mathbf{G}(t+t_0)\cdot\exp\left(-\int_0^t \mathrm{tr}D\mathbf{F}(\mathbf{x})\,ds\right)dt\\ &=-\operatorname{sgn}\tilde{c}\int_{-\infty}^{\infty}\beta_2\tilde{c}^3\left(\varphi^0(t)\right)^3\left(1-\varphi^0(t)\right)^3\cdot\exp\left(\int_0^t\beta ds\right)dt\\ &\quad-\int_{-\infty}^{\infty}\beta_3\tilde{c}^4\left(\varphi^0(t)\right)^4\left(1-\varphi^0(t)\right)^4\cdot\exp\left(\int_0^t\beta ds\right)dt\\ &\quad+\int_{-\infty}^{\infty}F\sin\omega(t+t_0)\tilde{c}\varphi^0(t)\left(1-\varphi^0(t)\right)\cdot\exp\left(\int_0^t\beta ds\right)dt\\ &\quad+\int_{-\infty}^{\infty}\sigma\tilde{c}\varphi^0(t)\left(1-\varphi^0(t)\right)\cdot\exp\left(\int_0^t\beta ds\right)dt\\ &=F\tilde{c}\sqrt{I_r^2+I_i^2}\sin\left(\omega t_0+\tan^{-1}(I_i/I_r)\right)+\sigma\tilde{c}I(0)-\beta_2\tilde{c}^3K_2\operatorname{sgn}\tilde{c}-\beta_3\tilde{c}^4K_3\end{aligned} \qquad \text{(A1-3).}$$

Here I, K_2 and K_3 are defined as follows:

$$I(\omega)\equiv\int_{-\infty}^{\infty}\varphi^0(t)\left(1-\varphi^0(t)\right)e^{\tilde{\beta}t}e^{i\omega t}dt \qquad \text{(A1-4),}$$

$$K_2 \equiv \int_{-\infty}^{\infty} (\varphi^0(t))^3 (1-\varphi^0(t))^3 \cdot \exp\left(\int_0^t \tilde{\beta} ds\right) dt \qquad \text{(A1-5)},$$

$$K_3 \equiv \int_{-\infty}^{\infty} (\varphi^0(t))^4 (1-\varphi^0(t))^4 \cdot \exp\left(\int_0^t \tilde{\beta} ds\right) dt \qquad \text{(A1-6)},$$

where we put $I_r = \mathrm{Re}[I]$ and $I_i = \mathrm{Im}[I]$. K_2 and K_3 can be calculated via Cauchy's integral theorem as follows:

$$K_2 \equiv \frac{\pi\beta(\beta^2 - 4\tilde{c}^2)(\beta^2 - \tilde{c}^2)}{5!\tilde{c}^6 \sin(\pi\beta / \tilde{c})} \operatorname{sgn} \tilde{c} \qquad \text{(A1-7)}$$

$$K_3 \equiv \frac{-\pi\beta(\beta^2 - 9\tilde{c}^2)(\beta^2 - 4\tilde{c}^2)(\beta^2 - \tilde{c}^2)}{7!\tilde{c}^8 \sin(\pi\beta / \tilde{c})} \operatorname{sgn} \tilde{c} \qquad \text{(A1-8)}$$

Note that a singular point of the equation (A1-5), i.e. $t = \pi i(2n+1)/\tilde{c}$, is a pole of order 6. Here n denotes the arbitrary integer. Therefore following condition must be held.

$$\frac{\sigma I(0) - \beta_2 \tilde{c}^2 K_2 \operatorname{sgn} \tilde{c} - \beta_3 \tilde{c}^3 K_3}{\sqrt{I_r^2 + I_i^2}} < F \qquad \text{(A1-9)}.$$

Appendix 2

In the main section, the 1 DoF roll equation with cubic, quadratic and linear restoring term is carried out. In this appendix, the study for the 1 DoF roll equation is shown, having 4th order polynomial restoring term and quadratic polynomial damping term.

$$\ddot{x} + \tilde{\beta}_1 \dot{x} + \tilde{\beta}_2 \dot{x}^2 + \tilde{\gamma} x(1-x)(x-k)(x-s_1) = 0 \qquad \text{(A2-1)}.$$

The reason why these terms are represented by higher order polynomial is to fit their original curves. Here making the following ansatz as a solution in time domain:

$$x = a_0 / (e^{-c_0 t} + b_0) \qquad \text{(A2-2)},$$

and substituting equation (A2-2) into (A2-1) yields:

$$\begin{aligned}
&a_0(-\tilde{\gamma} a_0^3 + \tilde{\gamma} a_0^2 b_0 + \tilde{\gamma} k a_0^2 b_0 + \tilde{\gamma} s_1 a_0^2 b_0 - \tilde{\gamma} k a_0 b_0^2 - \tilde{\gamma} s_1 a_0 b_0^2 - \tilde{\gamma} k s_1 a_0 b_0^2 + \tilde{\gamma} k s_1 b_0^3) \\
&+a_0 e^{-c_0 t}(\tilde{\gamma} a_0^2 + \tilde{\gamma} k a_0^2 + \tilde{\gamma} s_1 a_0^2 - 2\tilde{\gamma} k a_0 b_0 - 2\tilde{\gamma} s_1 a_0 b_0 - 2\tilde{\gamma} k s_1 a_0 b_0 + 3\tilde{\gamma} k s_1 a_0 b_0 + 3\tilde{\gamma} k s_1 b_0^2 + \tilde{\beta}_1 b_0^2 c_0 - b_0^2 c_0^2) \\
&+a_0 e^{-2c_0 t}(-\tilde{\gamma} k a_0 - \tilde{\gamma} s_1 a_0 - \tilde{\gamma} k s_1 a_0 + 3\tilde{\gamma} k s_1 a_0 b_0 + 2\tilde{\beta}_1 b_0 c_0 + \tilde{\beta}_2 a_0 c_0^2) \\
&+a_0 e^{-3c_0 t}(\tilde{\gamma} k s_1 + \tilde{\beta}_1 c_0 + c_0^2) = 0
\end{aligned} \qquad \text{(A2-3)}$$

Comparing both sides of the equation when considering:

$$a_0 = b_0 = 1 \qquad \text{(A2-4)},$$

we can obtain the following equations:

$$\begin{cases} \tilde{\gamma} - \tilde{\gamma}k - \tilde{\gamma}s_1 + \tilde{\gamma}ks_1 + \tilde{\beta}_1 c_0 - c_0^2 = 0 \\ -\tilde{\gamma}k - \tilde{\gamma}s_1 + 2\tilde{\gamma}ks_1 + 2\tilde{\beta}_1 c_0 + \tilde{\beta}_2 c_0^2 = 0 \\ \tilde{\gamma}ks_1 + \tilde{\beta}_1 c_0 + c_0^2 = 0 \end{cases} \quad \text{(A2-5a,b,c)}$$

Solving (A2-5c) with respect to $\tilde{\beta}_1$ yields:

$$\tilde{\beta}_1 = \frac{-\tilde{\gamma}ks_1 - c_0^2}{c_0} \quad \text{(A2-6).}$$

Substituting the above equation to (A2-5a) and (A2-5b), we can obtain:

$$\begin{cases} \tilde{\gamma} - \tilde{\gamma}k - \tilde{\gamma}s_1 - 2c_0^2 = 0 \\ -\tilde{\gamma}k - \tilde{\gamma}s_1 + \tilde{\beta}_2 c_0^2 - 2c_0^2 = 0 \end{cases} \quad \text{(A2-7a,b)}$$

(A2-7b) can be rewritten as:

$$c_0^2 = \frac{\tilde{\gamma}k + \tilde{\gamma}s_1}{-2 + \tilde{\beta}_2} \quad \text{(A2-8),}$$

so that substituting the above expression into (A2-7a), following relationship is obtained.

$$-\frac{\tilde{\gamma}\left[2 + \tilde{\beta}_2(-1 + k + s_1)\right]}{-2 + \tilde{\beta}_2} = 0 \quad \text{(A2-9)}$$

Solving the above equation with respect to $\tilde{\beta}_2$ and assuming $\tilde{\beta}_2 \neq -2$, we can obtain

$$\tilde{\beta}_2 = -\frac{2}{-1 + k + s_1} \quad \text{(A2-10).}$$

If this relationship is satisfied, the solution with regard to polynomial approximated equation is determined as follows:

$$x = 1 \Big/ \left\{ 1 + \exp\left[-\frac{\sqrt{-\tilde{\gamma}(-1 + k + s_1)}}{\sqrt{2}} t \right] \right\} \quad \text{(A2-11).}$$

Here c_0 is assumed as positive value as:

$$c_0 = \sqrt{\frac{-\tilde{\gamma}(-1 + k + s_1)}{2}} \quad \text{(A2-12),}$$

so that, substitution of above equation into (A6) yields:

$$\tilde{\beta}_1 = \frac{\tilde{\gamma}(-1 + k + s_1 - 2ks_1)}{\sqrt{-2\tilde{\gamma}(-1 + k + s_1)}} \quad \text{(A2-13).}$$

This is the condition of heteroclinic bifurcation. Obviously, the condition $\tilde{\gamma} < 0$ and $s_1 > 0$, or , $\tilde{\gamma} > 0$ and $s_1 < 0$ is required. Here we briefly consider the heteroclinic orbit. Eliminating the time *t* from equation (A2-11) yields the trajectory in the phase plane consists of x and $\dot{x}$ as quadratic equations. However, if we substitute the trajectory having a quadratic form into equation (A2-1), this equation is not satisfied. This is because the equation (A2-13) is only defined between the two saddle points. Note that equation (A2-10) and equation (A2-13) should be simultaneously satisfied, and it implies that the solution surface is formed in 4 dimensional parameter plane consisting of $\tilde{\beta}_1, \tilde{\beta}_2, k, s_1 k$. Note, therefore, that the obtained heteroclinic trajectory cannot represent the all the trajectories of equation (A2-1).

Using the quadratic form of the trajectory in the time domain, chaos that appears in equation (A2-1) can be studied. Considering the following relationships:

$$\mathbf{F}(q^0(t)) \wedge \mathbf{G}(t+t_0) = \dot{\varphi}(b_0 + b\sin\omega t) \tag{A2-14}$$

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \tilde{\gamma}\left[4x^3 - 3(k+s_1+1)x^2 + 2(ks_1+k+s_1)x - ks_1\right] & -\tilde{\beta}_1 - 2\tilde{\beta}_2\dot{x} \end{pmatrix} \tag{A2-15}$$

$$\mathrm{tr}D\mathbf{F}(\mathbf{x}) = -\tilde{\beta}_1 - 2\tilde{\beta}_2\dot{x} \tag{A2-16}$$

then the Melnikov integral becomes:

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} \mathbf{F}(\varphi^0(t)) \wedge \mathbf{G}(t+t_0) \cdot \exp\left(-\int_0^t \mathrm{tr}D\mathbf{F}(\mathbf{x})\,ds\right)dt \\ &= \int_{-\infty}^{\infty} b\sin\omega(t+t_0)\tilde{c}\varphi^0(t)(1-\varphi^0(t)) \cdot \exp\left(\int_0^t (\tilde{\beta}_1 - 2\tilde{\beta}_2\dot{x})\,ds\right)dt \\ &\quad + \int_{-\infty}^{\infty} b_0\tilde{c}\varphi^0(t)(1-\varphi^0(t)) \cdot \exp\left(\int_0^t (\tilde{\beta}_1 - 2\tilde{\beta}_2\dot{x})\,ds\right)dt \\ &= b\tilde{c}(I_i'\cos\omega t_0 + I_r'\sin\omega t_0) + b_0\tilde{c}I'(0) \\ &= b\tilde{c}\sqrt{I_r'^2 + I_i'^2}\,\sin\left(\omega t_0 + \tan^{-1}(I_i'/I_r')\right) + b_0\tilde{c}I'(0) \end{aligned} \tag{A2-17}$$

where $I(\omega)$ is defined with the following Fourier transformation:

$$I(\omega) \equiv \int_{-\infty}^{\infty} \varphi^0(t)(1-\varphi^0(t))e^{\tilde{\beta}_1 t - 2\tilde{\beta}_2\varphi^0(t)}e^{i\omega t}\,dt \tag{A2-18}$$

This equation has a form shown as follows:

$$I(\omega) = \frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}_1 t - \dfrac{2\tilde{\beta}_2}{1+\exp(-\tilde{c}t)} + i\omega t\right)}{\cosh^2(\tilde{c}t/2)}dt = \frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp\left(\tilde{\beta}_1 t - \tilde{\beta}_2 - \tilde{\beta}_2\tanh(\tilde{c}t/2) + i\omega t\right)}{\cosh^2(\tilde{c}t/2)}dt \tag{A2-19}$$

However, it is considered to be that the further analytical manipulation is difficult.

Figures legends

Figure 1 Comparison of parameter σ obtained by using analytical method and numerical method with a of 1.0.

Figure 2 Comparison of parameter σ obtained by using analytical method and numerical method with a of 0.9

Figure 3 Comparison of phase trajectories obtained by using analytical method and numerical method with a of 1.0 and β of 0.1

Figure 4 Comparison of phase trajectories obtained by using analytical method and numerical method with a of 0.9 and β of 0.1.

Figure 5 Parameter set of the heteroclinic bifurcation points.

Figure 6 Comparison of critical forcing for non-biased roll, i.e. a of 1.0.

Figure 7 Comparison of critical forcing with a of 0.975

Figure 8 An Example of fractal metamorphoses of basin boundary with a of 0.975, β of 0.1, ω of 0.8 and F of 0.07, which is slightly above the critical forcing.

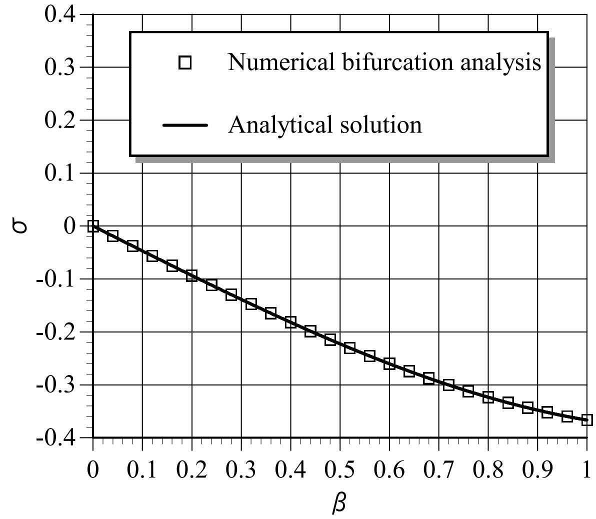

Figure 1 Comparison of parameter σ obtained by using analytical method and numerical method with a of 1.0.

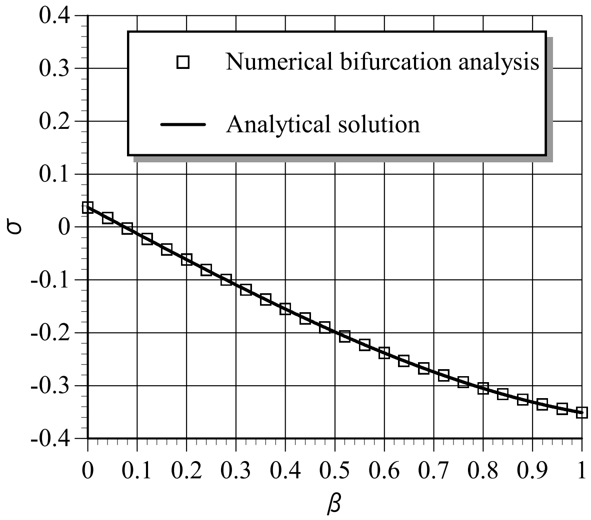

Figure 2 Comparison of parameter σ obtained by using analytical method and numerical method with a of 0.9.

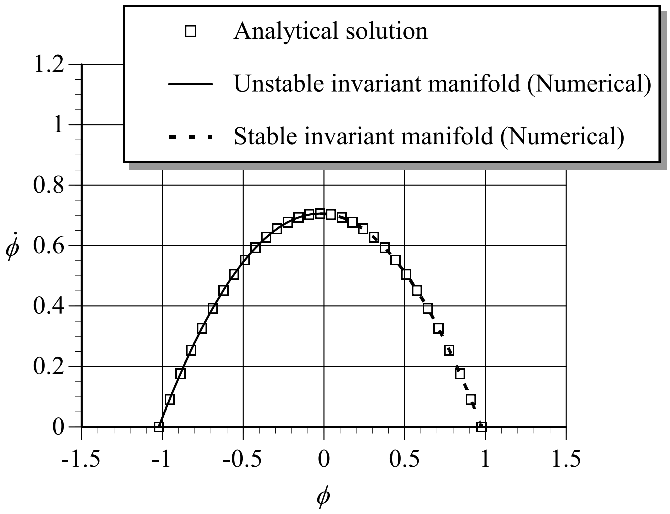

Figure 3 Comparison of phase trajectories obtained by using analytical method and numerical method with a of 1.0 and β of 0.1.

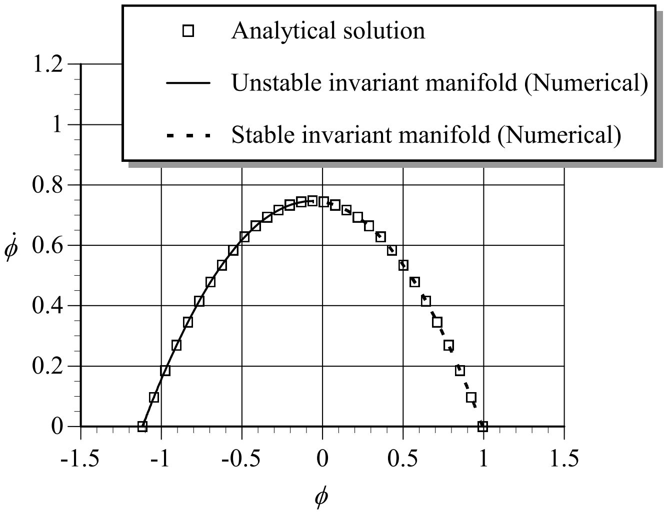

Figure 4 Comparison of phase trajectories obtained by using analytical method and numerical method with a of 0.9 and β of 0.1.

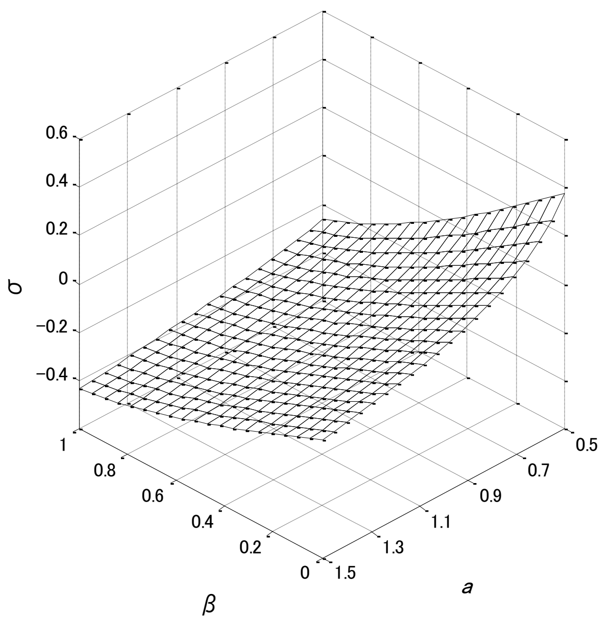

Figure 5　Parameter set of the heteroclinic bifurcation points.

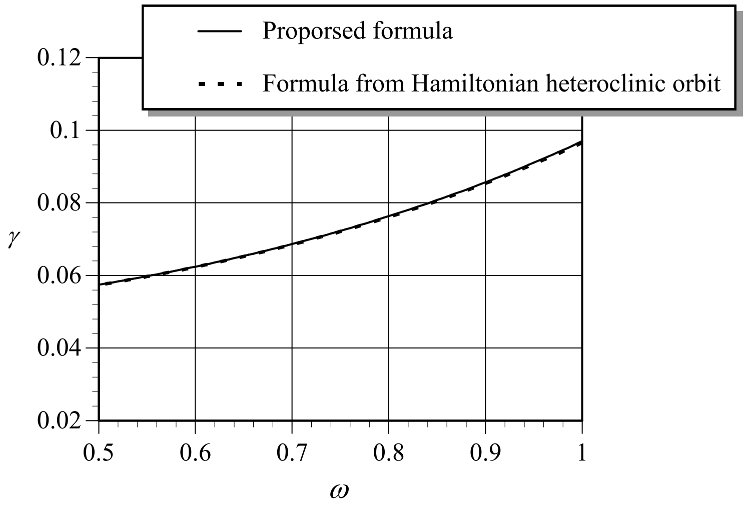

Figure 6 Comparison of critical forcing for non-biased roll, i.e. *a* of 1.0 and β of 0.1.

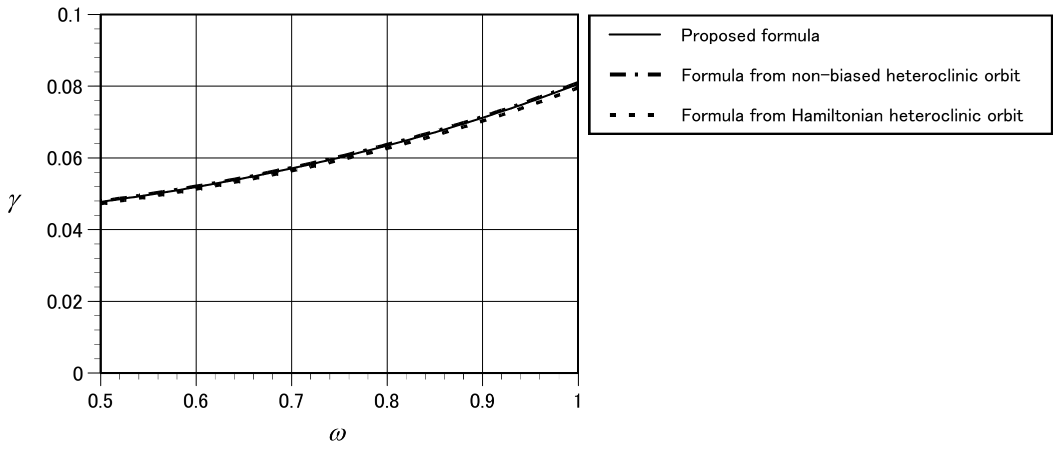

Figure 7 Comparison of critical forcing with a of 0.975 and β of 0.1.

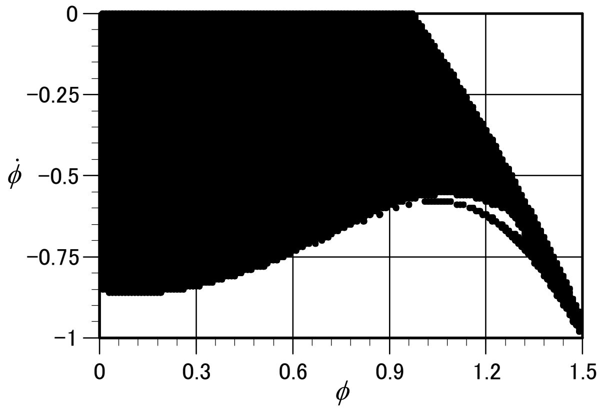

Figure 8 An Example of fractal metamorphoses of basin boundary with a of 0.975, β of 0.1, ω of 0.8 and γ of 0.07, which is slightly above the critical forcing.