Dynamic Programming with Recursive Preferences

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Declaration

Except where otherwise acknowledged, I certify that this thesis is my original work. The thesis is within the 100,000 word limit set by the Australian National University.

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I sincerely dedicate this thesis to my Mother Ying Jiang, Father Yamin Ren, my Wife Jian Gao and my beloved Son George Hongyu Ren.

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Abstract

There is now a considerable amount of research on the deficiencies of additively separable preferences for effective modeling of economically meaningful behavior. Through analysis of observational data and the design of suitable experiments, economists have constructed progressively more realistic representations of agents and their choices. For intertemporal decisions, this typically involves a departure from the additively separable benchmark. A familiar example is the recursive preference framework of Epstein and Zin (1989), which has become central to the quantitative asset pricing literature, while also finding widespread use in applications ranging from optimal taxation to fiscal policy and business cycles.

This thesis presents three essays which examine mathematical research questions within the context of recursive preferences and dynamic programming. The focus is particularly on showing the existence and uniqueness of recursive utility processes under stationary and non-stationary consumption growth specifications, and on solving the closely related problem of optimality of dynamic programs with recursive preferences.

On the one hand, the thesis has been motivated by the availability of new and unexploited techniques for studying the aforementioned questions. The techniques in question build upon an alternative version of the theory of monotone concave operators proposed by Du (1989, 1990). They are typically well suited to analysis of dynamic optimality with a variety of recursive preference specifications.

On the other hand, motivation also comes from the demand side: while many useful results for dynamic programming within the context of recursive preferences have been obtained by existing literature, suitable results are still lacking for some of the most popular specifications for applied work, such as common parameterizations of the Epstein–Zin specification, or preference specifications that incorporate loss aversion and narrow framing into the Epstein–Zin framework, or ambiguity sensitive preference specifications. Accordingly, the thesis has sought to provide a new approach to dynamic optimality suitable for recursive preference specifications commonly used in modern economic analysis.

The approach to examining the problems of dynamic programming exploits the theory of monotone convex operators, which, while less familiar than that of monotone concave operators, turns out to be well suited to dynamic maximization. The intuition is that convexity is preserved under maximization, while concavity is not. Meanwhile, concavity pairs well with

minimization problems, since minimization preserves concavity. By applying this idea, a parallel theory for these two cases is established and it provides sufficient conditions that are easy to verify in applications.

Contents

Common Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegative numbers
$\mathbb{R}_{++}$	the set of positive numbers
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$\mathbb{E}$	a generic Banach space, unless otherwise specified
$\mathbf{0}$	the zero element (vector) of $\mathbb{E}$
$\mathbb{B}$	a generic Banach lattice
$\mathbb{K}$	a generic cone of $\mathbb{E}$
$\mathscr{B}(\mathbb{X})$	the Borel subsets of $\mathbb{X}$
$(\mathbb{X}, \mathscr{B}(\mathbb{X}))$	the measurable space on $\mathbb{X}$
$\mathring{\mathbb{X}}$	the set of interior points of $\mathbb{X}$
$\mathbb{R}^{\mathbb{X}}$	the set of all functions mapping from $\mathbb{X}$ to $\mathbb{R}$
$\mathbb{R}^{\mathbb{X}}_+$	the set of all nonnegative functions in $\mathbb{R}^{\mathbb{X}}$
$b(\mathbb{X})$	the set of all bounded functions in $\mathbb{R}^{\mathbb{X}}$
$m(\mathbb{X})$	the set of all measurable real-valued functions on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$
$m(\mathbb{X})_+$	the set of all nonnegative functions in $m(\mathbb{X})$
$bm(\mathbb{X})$	the set of all bounded functions in $m(\mathbb{X})$
$c(\mathbb{X})$	the set of all continuous functions in $m(\mathbb{X})$
$bc(\mathbb{X})$	the set of all bounded functions in $c(\mathbb{X})$
$im(\mathbb{X})$	the collection of all increasing (i.e., nondecreasing) functions in $m(\mathbb{X})$
$\mathcal{C}im(\mathbb{X})$	the collection of all concave functions in $im(\mathbb{X})$
$b_\kappa m(\mathbb{X})$	the collection of all κ-bounded functions in $m(\mathbb{X})$
$\Vert f \Vert_\infty$	the (supremum) norm $\sup_{x \in \mathbb{X}} \lvert f(x) \rvert$ on $b(\mathbb{X})$
$d_\infty(f, g)$	the distance $\Vert f - g \Vert_\infty$ on $b(\mathbb{X})$
$\Vert f \Vert_\kappa$	the (κ)–weighted supremum norm $\sup_{x \in \mathbb{X}} \{\lvert f(x) \rvert / \kappa(x)\}$ on $b_\kappa m(\mathbb{X})$
$[f]_\kappa$	the infimum operator $\inf_{x \in \mathbb{X}} \{\lvert f(x) \rvert / \kappa(x)\}$ on $b_\kappa m(\mathbb{X})$
$\mathscr{L}_0(\mu)$	the set $\mathscr{L}_0(\mathbb{X}, \Sigma, \mu)$ of μ–equivalence classes of measurable functions from $\mathbb{X}$ to $\mathbb{R}$
$\mathscr{L}_1(\mu)$	the collection of μ–integrable functions in $\mathscr{L}_0(\mu)$
$\Vert f \Vert_1$	the L_1–norm $\mu(f) := \int f \, \mathrm{d}\mu$ on $\mathscr{L}_1(\mu)$

E_t	the conditional expectation with respect to the period t information
$\mathcal{M}$	a non-additive Markov operator
$\mathcal{R}$	the Kreps–Porteus certainty equivalent operator
$\boldsymbol{A}^+$	the majorant of an operator $\boldsymbol{A}$
$\boldsymbol{A}^n$	the n-th composition (iteration) of the operator $\boldsymbol{A}$ with itself
$r(\boldsymbol{K})$	the spectral radius of a linear operator $\boldsymbol{K}$
$\mathbb{1}_\mathsf{X}$	the characteristic (indicator) function of set X
$B(x, \varepsilon)$	the ε-ball centered on x
$\stackrel{\text{IID}}{\sim} \mathscr{P}$	independent and identically distributed according to distribution $\mathscr{P}$

Chapter 1

Introduction

1.1 Overview of the Thesis

Through a combination of theory and analysis of observational and experimental data, economists have constructed progressively more realistic representations of agents and their choices. In the context of intertemporal choice, these preferences have come to include such features as independent sensitivity to intertemporal substitution and intratemporal risk (see, e.g., Kreps and Porteus (1978) or Epstein and Zin (1989)), desire for robustness (Hansen and Sargent (2008)), the impact of narrow framing (Barberis et al. (2006), Barberis and Huang (2009)) and sensitivity to ambiguity (e.g., Gilboa and Schmeidler (1989), Epstein and Schneider (2008), Klibanoff et al. (2009), Hayashi and Miao (2011), Ju and Miao (2012), Strzalecki (2013)).

While some of these models have found significant popularity in applied research,[1] traditional additively separable preferences with linear certainty equivalents still form the backbone of applied and quantitative work. This is partly due to the fact that optimal choice in the traditional setting is much better understood (see, e.g., Bellman (1957) or Blackwell (1965)). Specifications involving nonlinear recursive preferences have proved harder to handle. While early attempts to treat nonlinear recursive preferences in a discrete time dynamic programming framework used the contraction mapping arguments that had proved successful for additively separable models (see, e.g., Lucas and Stokey (1984)), it was soon realized that the Bellman operators generated by the most common recursive preference specifications are not supremum norm contractions, implying that the classical theory typified by Blackwell (1965) cannot be employed.[2]

This realization drove a second wave of theoretical analysis on the existence and unique-

[1] For example, the recursive preference specification of Epstein and Zin (1989) forms a core component of the quantitative asset pricing literature, while also finding use in applications ranging from optimal taxation to fiscal policy and business cycles. See, for example, Bansal and Yaron (2004), Hansen and Sargent (2008) or Schorfheide et al. (2018).

[2] In addition to Lucas and Stokey (1984), related work can be found in Boyd (1990), Durán (2003), Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007). It was Marinacci and Montrucchio (2010) who emphasized that sup norm contractivity fails for many economically reasonable aggregators, such as Thompson aggregators.

ness of solutions to recursively defined utility specifications and on the closely related problem of dynamic programming in the context of nonlinear recursive preferences, built instead around monotonicity and concavity. The approach by exploiting monotonicity and concavity has achieved a lot of success in the economic literature, since the earlier works of Coleman (1991), Lacker and Schreft (1991) and Datta et al. (2002). More recently, Marinacci and Montrucchio (2010), Borovička and Stachurski (2017) and Becker and Rincón-Zapatero (2017) exploited monotonicity and concavity to obtain a range of deep results on existence and uniqueness of recursive utilities. On the other side, Le Van et al. (2008) adapted the theory of monotone concave operators, as pioneered by Krasnosel'skiĭ and Ladyžhenskiĭ (1954), to dynamic programming problems. Moreover, Balbus (2016), Marinacci and Montrucchio (2017) and Bloise and Vailakis (2018) further extended these ideas. Like contraction maps, under certain regularity conditions, monotone concave operators possess unique and globally attracting fixed points—a highly attractive property in the context of dynamic programming.

Following those aforementioned works, this thesis goes a step further and utilizes an alternative version of the theory of monotone concave operators proposed by Du (1989, 1990), so as to deal with mathematical research questions within the context of recursive preferences and dynamic programming.

The initial contribution of the thesis is a new and tractable approach to obtain results for existence, uniqueness and global stability of recursive utility processes of the type suggested by Epstein and Zin (1989)—that is, results showing that the typical specifications of recursive preferences are well defined, in the sense that fixed consumption paths or policies uniquely identify lifetime utility. This approach is primarily built upon the monotone concave operator theory and is relatively general, allowing for both unbounded stationary and non-stationary consumption growth specifications. It is hoped that this approach will therefore be helpful in further integrating theoretical and empirical research which is based on the Epstein and Zin framework. In particular, we additionally present how our methodology can also be essentially applied to risk-sensitive models and to the usual Epstein–Zin utility specifications that allow for the framing of risks.

While having a good understanding of the existence and uniqueness of recursive utilities, we realize that our understanding of the closely related problem of optimality of dynamic programs with recursive preferences may be less complete. Despite important contributions, foundations have been lacking for some of the most popular specifications for applied work, such as certain empirically relevant parameterizations of Epstein–Zin preferences, or the narrow framing or ambiguity sensitive preferences discussed above.

Therefore, the major objective of the remaining studies in the present thesis is to resolve several of these outstanding problems by developing a set of sufficient conditions for abstract dynamic programs—including both additively separable and recursive preference models—that provide global convergence of the Bellman operator to the value function and optimality of the associated policies. We show that these conditions are well applied to a range of recursive preference specifications popular in applied settings, including standard Epstein–Zin models with constant elasticity of substitution aggregators, risk-sensitive and robust control models, narrow framing models and some kinds of ambiguity sensitive preferences.

Our approach to addressing problems of dynamic programming still builds on the monotone concave approach but with one significant difference: the relevant operators are convex. Put differently, we use monotone *convex* operators to study the maximization problems associated with dynamic programming. The main benefit is that, unlike concavity, convexity is preserved under the taking of pointwise suprema. Hence convexity pairs naturally with maximization. Moreover, under suitable conditions, monotone convex operators enjoy all the stability properties possessed by monotone concave operators. In the meantime, we find that concavity pairs well with minimization problems, since minimization preserves concavity. By applying this idea, a parallel theory for these two cases is established and it provides sufficient conditions that are easy to verify in applications.

In addition, we find that simple continuous transformations can be used to transform inherently concave problems into convex problems and vice versa. Through these transformations, one can shift between convex maximization problems and concave minimization problems on a case by case basis. In particular, we show how various preference specifications that have been recognized as concave can be modified so that they exhibit convexity rather than concavity. Furthermore, we show how our methodology can also be adapted to accommodate unbounded rewards.

1.2 Structure of the Thesis

Following this introduction, the thesis is divided into five main chapters (Chapters 2–6). The first of these (Chapter 2) is an exposition of the mathematical preliminaries. The primary objective is to set out the key ideas and definitions, in order to avoid having to repeat them in each of the relevant theory chapters. The second objective is to present an abstract fixed-point theorem that will be used in Chapter 3. Using a monotone iterative technique and the properties of cones, our theorem proves the existence and uniqueness results of solutions for a class of nonlinear operator equations, as well as their convergence of successive approximations. It is not only mathematically interesting but also economically important to study the problems of the existence and uniqueness of the utility process in the generalized recursive utility model.

The preliminary chapter is followed by three essays on showing existence and uniqueness of recursive utilities and on solving the optimality of dynamic programs in the context of recursive preferences.

The first essay (Chapter 3) studies the problem of the utility process in the Epstein–Zin recursive preferences model with constant elasticity of intertemporal substitution and relative risk aversion. In particular, our study utilizes the results obtained in Chapter 2 to provide an alternative approach to obtain results for the existence, uniqueness and convergence of solutions to Epstein–Zin recursive utility models with stationary or non-stationary consumption processes. Besides that, we also consider a class of discrete time recursive preference specifications that incorporate narrow framing into the classical Epstein–Zin recursive utility model. Under a compact-state Markovian environment, we prove that the recursive utility process with narrow framing uniquely exists.

Chapter 4—perhaps the core contribution of the thesis—establishes a new approach to the theory of discrete time dynamic programming that is designed to accommodate the recursive preferences specifications commonly used in modern economic analysis, while still supporting traditional additively separable rewards. The approach exploits the theory of monotone convex operators, which turns out to be well suited to the dynamic maximization problem. In the sequel, we show how our theory can be applied to a variety of well-known models, including the classical Epstein–Zin models with constant elasticity of substitution aggregators, risk-sensitive and robust control models, ambiguity sensitive models, and models that incorporate narrow framing into the standard Epstein–Zin framework. In each case we show that value function iteration converges to the value function and that Bellman's principle of optimality is valid. As a by-product, policy iteration is also provided in this essay.

As an extension of the work in Chapter 4, to ensure sufficient generality for economic applications, in Chapter 5 we allow for unbounded reward functions in dynamic programs. The corresponding sufficient conditions for optimality are then embedded into a space of potentially unbounded functions endowed with a generic weighted supremum norm. Once again, we resolve many economic problems by developing a set of sufficient conditions for abstract dynamic programs— including both additively separable and recursive preferences models— that provide global convergence of the Bellman operator to the value function and optimality of the associated policies.

Finally, the third essay (Chapter 6) proposes an alternative technique to study a generic dynamic program with recursive preferences. The key idea of this technique is to twist the standard metrics by inserting one part of the risk specifications found in recursive preference models. This part can be simply taken to be a continuous monotonic transformation associate with the ratio of relative risk aversion and elasticity of intertemporal substitution. By exploiting a geometric property of this transformation, we show that the Bellman operator is a contraction mapping, and the usual results of dynamic programming apply. In addition, this technique also allows us to use a variety of algorithms popular in the dynamic programming literature, such as value iteration and policy iteration. Moreover, as another additional benefit of our method, under the appropriate assumptions with respect to the reward function and production technology, the true value function can then be shown to possess additional properties such as monotonicity, concavity, and differentiability. Furthermore, we derive a general Euler equation for such dynamic programs.

Apart from some simple arguments, all proofs in the present thesis are deferred to the appendixes.

1.3 Existing Literature

On the mathematical side, the greatest influence on this thesis has been the theory of monotone concave operators. The pioneering, general theorems on monotone concave operators defined on partially ordered spaces were established by Krasnosel'skiĭ and Ladyženskiĭ (1954), and further refined in the monograph of Krasnosel'skiĭ (1964). They introduce the notion of u_0-concave operators (cf., Definition 2.1.8 in Chapter 2) and prove that a monotone increasing and u_0-concave operator has at most one positive fixed point (cf. Theorem 6.11 of Krasnosel'skiĭ (1964)).

It is well-known that there are three classical fixed point theorems, which are of fundamental importance to the development of functional analysis: (1) Banach fixed point theorem (1922)—making essential use of strict contractive operators; (2) Schauder fixed point theorem (1930)—making essential use of compact operators; and (3) Bourbaki–Kneser fixed point theorem (1940)—making essential use of set–theoretic ordering principles. Nevertheless, with the advent of complex problems posed by the natural sciences, scientists and mathematicians encountered a number of typical difficulties in solving certain nonlinear differential and integral equations or other closely related applications, where it was not possible to apply the contraction mapping theorem or the theory of compact operators.

In this context, how to eschew contraction or compactness conditions of the underlying fixed point theorems has become an active area of research. Over the past several decades, there have appeared a series of seminal works concerning the theory of positive operators with respect to a cone. Making use of the theory of cones and monotonicity of operators, existence theorems can be obtained under relatively general conditions. Furthermore, geometric properties—i.e., concavity and convexity—of the operators have frequently been exploited in establishing uniqueness and the convergence of successive approximations of a fixed point.

Besides the aforementioned u_0-concavity proposed by Krasnosel'skiĭ (1964), there have been several variations of the notion of concave operators, such as order concavity (cf. Definition 2.1.10) and α-concavity (cf. Definition 2.1.9) pioneered by Amann (1976) and Potter (1977), respectively.

Amann (1976) proves that a strongly monotone increasing and completely continuous operator defined on an order interval of a real Banach space has a unique fixed point in this order interval, provided that this operator is order concave or order convex. Following the seminal contribution of Amann (1976), Du (1989, 1990) generalizes and improves Amann's result by completely removing the compactness and continuity conditions on the operator.

Potter (1977) shows that for $\alpha \in \mathbb{R}_+$, increasing α-concave and decreasing $(-\alpha)$-convex operators have contraction ratios less than or equal to α, and then obtains existence results of solutions to the nonlinear eigenvalue problem $\boldsymbol{A}x = \lambda x$. While the underlying method in Potter (1977) is based upon the Contraction Mapping Principle with respect to Hilbert's projective metric, Guo and Lakshmikantham (1988) propose an alternative, geometric method to study solutions of increasing α-concave and decreasing $(-\alpha)$-convex operators, by means of the properties of cones and an order-theoretic technique.

More recently, following the pioneering work of Krasnosel'skiǐ (1964), Liang et al. (2006) obtain a sufficient and necessary condition for the existence and uniqueness of the fixed point of a kind of monotone increasing and u_0-concave operator under some relatively restrictive assumptions. Further, Zhao and Du (2007) generalize the notions of u_0-concave and u_0-convex operators. Hereby, they derive a set of sufficient conditions for the existence, uniqueness and monotone iterative techniques of positive fixed points of such generalized u_0-concave/convex operators, without assuming those operators to be compact or continuous.

These fruitful approaches by exploiting monotonicity and concavity of the operators have proven to be an indispensable and powerful tool in the analysis of nonlinear differential and integral equations. As shown by Liang et al. (2006), under certain conditions, these approaches based on different notions of concavity on operators have a theoretically close connection and may alternate with each other. Empirically, each of them has different strengths in solving typical classes of operator equations arising in the natural science applications, and these approaches, to some extent, may complement each other well.

In the present thesis, our mathematical techniques mainly build upon the fixed point theorems proved by Du (1989, 1990). In addition, based on Du's general results, we confine ourselves to a specific class of monotone, order concave operators and then extend his results in multiple directions. These include developing sufficient conditions that assure existence of a concretely upper solution for the nonlinear operator possessing certain linear majorants, and providing uniqueness results and convergence of the iteration method over whole solid cones rather than just bounded order intervals.

It is noteworthy that the fixed point problems based on u_0-concave and α-concave operators are usually defined on cones or the interior of the cones, while those based on order concave operators are often defined on order intervals. In this sense, the upper and lower solutions for the order concave operators play a crucial role in constructing an appropriate domain—order interval—of the fixed point problems. However, there has been some criticism of the examination of conditions of the upper and lower solutions (see, e.g., Zhai et al. (2010)). Motivated by this issue, on the one hand, we note that the zero vector is a natural choice for a lower solution of the positive operator, and hence concentrate on establishing some sufficient conditions that assure existence of an upper solution of that operator. These conditions are straightforward, intuitive (as illustrated in Figure 2.3 in Chapter 2) and easy to verify in some applications that will be shown later. On the other hand, when the cone is solid, we prove that the unique and globally attracting properties of a fixed point on the order interval can be extended to the whole cone. This extension is non-trivial and new to the literature, and thus may be useful in theory and applications.

While the theory of nonlinear differential and integral equations has been well investigated by making use of the monotone concave operators, economists have started to apply similar methods to nonlinear operators which arise in connection with various economic applications. For example, an early work by Coleman (1991) studies equilibrium in a stochastic production economy with an income tax and shows the existence and uniqueness of the consumption function by constructing a monotone, strongly subhomogeneous map.[3] Another early work is due to Lacker and Schreft (1991), who study equilibrium in stochastic economies with multiple means of payment and prove the existence and uniqueness of the nominal interest rate function by constructing a monotone u_0-concave operator. In addition, Montrucchio (1998) examines smoothness properties of the value and policy functions in a general class of discrete-time, Ramsey-type models with capital accumulation, and gives a new complete proof of Santos' theorem by using monotonicity and convexity of the operator to derive a contraction with respect to Thompson's metric. Following the seminal contribution of Coleman (1991), Datta et al. (2002) and Morand and Reffett (2003) utilize a lattice based approach and exploit monotonicity and strong subhomogeneity of nonlinear operators to obtain existence and uniqueness results for solutions to forward-looking recursive equations with capital and elastic labor supply. More recently, a variety of monotone concave operator theorems have been extensively applied to studies of recursive utility functions and the optimality problem of dynamic programs with recursive preferences. See, for example, Le Van et al. (2008), Marinacci and Montrucchio (2010, 2017), Balbus (2016), Borovička and Stachurski (2017), Becker and Rincón-Zapatero (2017), Bloise and Vailakis (2018) and Pavoni et al. (2018).

On the economic side, the most important reference of the present thesis was without doubt the classical recursive preferences paper of Epstein and Zin (1989). In attempting to examine the problem of stochastic recursive utilities, the pioneering works of Marinacci and Montrucchio (2010) and Hansen and Scheinkman (2012) serve as an inspiration for the first essay of the thesis. When moving to the problem of dynamic programming with recursive utilities, our work primarily builds upon the seminal contributions of Le Van et al. (2008) and Bloise and Vailakis (2018). In particular, we draw on several key ideas of Bloise and Vailakis (2018) and extend their analysis in multiple directions. These include developing an optimality theory for monotone convex operators, adopting transformations that allow for the use of convex operator methods, and providing a treatment of several sophisticated models used in recent applied works.

Following the seminal work of Kreps and Porteus (1978), Epstein and Zin (1989, 1991) introduce a class of recursive preferences which allow for breaking the link between risk aversion

[3] The notion of "strong subhomogeneity" of an operator defined on partially ordered spaces is essentially identical to the notion of "u_0-concavity" in Krasnosel'skiĭ (1964), which is often referred to as "strong sub-linearity" in Amann (1976) and Guo and Lakshmikantham (1988), and also referred to as "pseudo-concavity" in Coleman (1991).

and the willingness to substitute consumption intertemporally. These preferences have proved very useful in applied work in asset pricing and portfolio selection, and are becoming more prevalent in macroeconomics. In an infinite-horizon setting, we consider a discrete-time specification of recursive preferences of the type suggested by Kreps and Porteus (1978) and Epstein and Zin (1989);[4] namely, the utility process $\{U_t\}$ of consumption path $\{C_t\}$ is defined recursively in the form of

$$U_t = W\left(C_t,\ \mathcal{R}_t\left(U_{t+1}\right)\right).$$

Here, W is an aggregator function that maps today's consumption C_t and a function $\mathcal{R}$ of tomorrow's random continuation value U_{t+1} into a value U_t today. The term $\mathcal{R}_t(\cdot)$ is a "certainty equivalent" function that maps a random variable U_{t+1} that is measurable with respect to the next period's information into a random variable that is measurable with respect to the current period's information:

$$\mathcal{R}_t(U_{t+1}) := \phi^{-1}\left(\mathrm{E}_t\phi\left(U_{t+1}\right)\right), \tag{1.1}$$

where ϕ is a function that describes attitudes toward atemporal risk:

$$\phi(z) := \begin{cases} z^{1-\gamma} & \text{if } 0 < \gamma \neq 1, \\ \ln(z) & \text{if } \gamma = 1, \end{cases} \tag{1.2}$$

and γ is the coefficient of relative risk aversion (RRA). Here E_t stands for the conditional expectation operator with respect to the period t information. Thus, the certainty equivalent operator $\mathcal{R}_t$ adjusts the continuation value U_{t+1} for risk.

Epstein and Zin (1989, 1991) use the constant elasticity substitution (CES) aggregator

$$W\left(C, \mathcal{R}\right) = \begin{cases} \left[(1-\beta)C^{1-\rho} + \beta\mathcal{R}^{1-\rho}\right]^{\frac{1}{1-\rho}} & \text{if } 0 < \rho \neq 1, \\ C^{1-\beta}\mathcal{R}^{\beta} & \text{if } \rho = 1, \end{cases} \tag{1.3}$$

where $0 < \beta < 1$ is a subjective time discount factor and $1/\rho$ is the elasticity of intertemporal substitution (EIS).[5]

Epstein and Zin (1989) obtain remarkable results for the existence of the recursive utility process across a broad set of RRA and EIS parameters. These results were further strengthened by Ozaki and Streufert (1996), who provide a comprehensive analysis of existence and uniqueness of recursive, stochastic utility processes by introducing the notion of biconvergence. There the biconvergence can be viewed as a dually ordinal concept, which essentially requires that

[4] Kreps and Porteus (1978) first propose a finite horizon recursive utility with an expected utility conditional certainty equivalent. Epstein and Zin (1989) and Weil (1990) generalize the Kreps and Porteus model to an infinite horizon. Thus, the following utility is often called the Kreps–Porteus utility, or the Epstein–Zin utility, or the Epstein–Zin–Weil utility in the literature.

[5] In fact, there are many choices for the certainty equivalent operator and CES aggregator, but the above one is most popular due to its tractability in deriving asset pricing results (see, for example, Epstein and Zin (1991)). Setting $\gamma = \rho$ yields the special case of additive power utility. When $\gamma > \rho$, this parametrization indicates that agents are more concerned about uncertainty than predictable variation in consumption. A number of empirical studies reveal that agents indeed prefer early resolution of risk (uncertainty) about future consumption paths.

returns of any feasible path are able to be sufficiently discounted from above (upper convergence) and sufficiently discounted from below (lower convergence).

Marinacci and Montrucchio (2010) propose a new class of Thompson aggregators and study a class of quasi-arithmetic certainty equivalent operators that generalize those of Kreps and Porteus (1978).[6] Based on specific properties of the aggregators and the quasi-arithmetic operators, they provide a thorough analysis of existence, uniqueness and global attractivity of continuation value processes.[7] In particular, they utilize the monotonicity and concavity induced by the Thompson aggregator, along with subhomogeneity induced by the quasi-arithmetic operator, to derive a contraction with respect to the Thompson metric, and thus prove the relevant results for the corresponding general nonlinear stochastic equation. All of these aforementioned, seminal works escape a Markovian specification and treat uncertainty in a very general setting.

Recently, Becker and Rincón-Zapatero (2017) reconsider the theory of Thompson aggregators proposed by Marinacci and Montrucchio (2010). On one hand, the authors exploit an order continuity property of the Koopmans operator and hence, establish the existence results of extremal solutions to the Koopmans equation by applying a constructive Tarski-Kantorovich fixed point theorem, instead of the classical Tarski theorem as employed in Marinacci and Montrucchio (2010).[8] In particular, their existence theory of Thompson aggregators provides verification that successive approximation yields extremal solutions to the Koopmans equation derived from Thompson aggregators. On the other hand, under additional auxiliary conditions, the authors obtain the unique results of the solutions to the Koopmans equation while a specific formation of the monotone u_0-concave operator techniques is applied.

Motivated by an insightful observation that some of the parametric Epstein–Zin models used in practice may involve certain unbounded and non-stationary growth specifications for consumption processes, Hansen and Scheinkman (2012) study the infinite-horizon, discrete-time Epstein–Zin utility models within the context of a Markov environment. They establish existence and uniqueness results of solutions to the Epstein–Zin utility process, by exploiting a connection between the solution to the functional equation of utility recursion and to the Perron-Frobenius eigenvalue equation that occurs in the study of large deviations (cf. Hansen and Scheinkman (2009)). Recently, Borovička and Stachurski (2017) also study the existence, uniqueness and stability of solutions to the Epstein–Zin recursive utility models with consumption specifications analogous to those in Hansen and Scheinkman (2012), except that the underlying state space is assumed to be compact. Nevertheless, they derive a set of remarkable conditions that are both necessary and sufficient for the existence and uniqueness of the Epstein–Zin utility processes, as well as a globally convergent method of computation of those

[6] One can verify that the aggregator W in CES formation (as defined in (1.3)) possesses properties (W-iii) and (W-iv) in p. 1783-1784 of Marinacci and Montrucchio (2010) iff $0 \leqslant p < 1$, in which case W is a so-called Thompson aggregator; W possesses property (W-v) iff $\rho \leqslant 0$, from which we obtain this W is a so-called Blackwell aggregator.

[7] More precisely, under certain sufficient conditions concerning bounded consumption growth rate and/or interiority corresponding to consumption processes, they prove the existence, uniqueness and convergence of approximations of the recursive utility process for both Blackwell and Thompson aggregators.

[8] The notion of a constructive procedure as defined in Becker and Rincón-Zapatero (2017) means the use of successive approximations (of the underlying solutions to the Koopmans equation) indexed on the natural numbers.

solutions. In particular, to my best knowledge, there is no existing literature in this direction that is capable of proving the necessity. At the same time, Guo and He (2018) consider an extension of the classical Epstein–Zin recursive utility model by adding a new component concerning utility measures for investment gains and losses. Under a specific and finite-state Markovian specification, they obtain sharp results for existence, uniqueness and convergence of solutions to the generalized Epstein–Zin utility process.[9]

Regarding the literature review for the second theme of the present thesis, one of the most closely related studies is Bloise and Vailakis (2018), who analyze dynamic programming problems with bounded recursive utility. By exploiting the theory of monotone concave operators, they prove a set of significant optimality results. Their paper is carefully constructed and, as mentioned before, we draw on several key ideas and extend their line of work in multiple directions, especially by adopting transformations that allow for the use of convex operator methods. This leads us to treatment of relatively sophisticated models used in recent quantitative work, including those with narrow framing or ambiguity aversion.

A recent, alternative and related reference for the general theory of dynamic programming with recursive preferences is Marinacci and Montrucchio (2017), which presents valuable new methods for determining when monotone maps have unique fixed points.

Also related is Guo and He (2018). As another part of their study, they obtain results for existence, uniqueness and successive approximations of the solution to the Bellman equation for a portfolio selection problem with gain–loss utility. They provide sharp results under the assumption that the state space is finite and the exogenous state process is irreducible.

Another related paper is Balbus (2016), who considers a class of nonnegative recursive utilities with certain types of nonlinear aggregators and certainty equivalent operators, and studies the corresponding dynamic programming problem. His results for existence, uniqueness and convergence of solutions to recursive utilities and to the Bellman equation rest upon the theory of monotone α-concave operators.

Ozaki and Streufert (1996) not only provide a comprehensive study of the recovery of recursive preferences, but also solve the corresponding dynamic programming problem under a non-Markovian environment. Their results, by means of the biconvergence approach, are useful for studying dynamic programming with non-additive stochastic objectives in a very general setting. Recently, in a related study based on the biconvergence technique, Bich et al. (2018) establish existence, uniqueness and computation of the solution to the Bellman equation for deterministic dynamic programming problems under certain types of continuity properties imposed on temporal aggregators. In another related study, Schwarz (2003) utilizes the biconvergence property to solve the dynamic programming problem with a tractable separation of risk and ambiguity attitudes in the bounded utility function.

[9] To be more precise, under a Markov environment, the basic set-up of this paper for consumption specifications is also analogous to those in Hansen and Scheinkman (2012), except that the state space is restricted to be finite.

Our study also has some subtle connection to the recent work by Pavoni et al. (2018), who introduce a recursive dual approach suitable for limited commitment problems or other incentive-constrained programming problems. The authors construct a dual formulation of the applications they consider, which is recursive and such that the dual Bellman operator is contractive under a Thompson-like metric. Their theory can handle problems where preferences are specified via a general time aggregator and stochastic aggregator. In the problems we consider in the present thesis, forward-looking constraints are absent and we can directly consider the primal optimization problem. This allows us to avoid certain assumptions used to tie the primal problem to the dual and obtain contractivity on the dual side.

Chapter 2

Monotone Concave Operators Theory

2.1 Overview of the Method

This section gives a broad and heuristic overview of the mathematical preliminaries and techniques used in the thesis and forms a basis for the remaining chapters. Some fixed point results begin in Section 2.2 and will be mainly used in the first essay.

2.1.1 Outline

In many problems arising from models of natural sciences, economics and other dynamic systems, we need to investigate the existence of positive solutions with certain desired qualitative properties. A natural instrument for the investigation of positive solutions is the method of functional analysis in ordered spaces. In fact, what we normally understand by positivity and nonnegativity is developed through arbitrary cones, that is, closed convex subsets of the space under consideration. These cones automatically define a (partial order) relation by means of which certain elements can be compared better than crude estimates in terms of a norm.

The underlying technique to solve the positive solutions of operators is firstly to utilize a cone to introduce an order relation on a certain Banach space. Then many of the concepts and results which we have for real numbers under the ordinary relation of "$\leqslant$" can be carried over to Banach spaces, including, for instance, those of monotone increasing/decreasing and convex/concave functions, nested interval techniques, and so on. In addition, we will see later that many intuitively obvious results for monotonic real-valued functions can be generalized directly to operators on Banach spaces. Furthermore, as the laws of nature may prescribe definite bounds for solutions of some real-world problems, this, in turn, may imply the existence of solutions in the corresponding order intervals. This finding is often referred to as the *principle of upper- and lower-solutions* that will be discussed later, but for now we may understand it as it sounds: *The existence of both a lower- and an upper-solution, along with special geometric properties of the cone, yields the existence of a solution under rather general conditions.*

Although uniqueness conclusions are, in general, more easily obtained than existence conclusions, it is also interesting and important to know what suitable auxiliary conditions are needed in order to assure that the model must have a unique and stable solution. To answer this question, we require appropriate selection of operators and cones. In this thesis, our results heavily rest upon the theory of monotone concave operators developed by Du (1989, 1990). Du's fixed point theorem enables us to have uniqueness and global attracting properties of solutions to the operator equation that we aim to solve. In addition, it is worth emphasizing that we do not demand any particular topological properties (such as being a contraction, or nonexpansive, or condensing, or the like) as well as compactness and continuity conditions on the operator, while still getting all the same results from the Contraction Mapping Principle.

2.1.2 Real Banach Spaces with a Cone

In the sequel, we shall introduce the concept of a cone in a real Banach space. A partial ordering relation is then defined in a real Banach space by means of a cone. By an *ordered* real Banach space we normally mean a real Banach space together with a cone. Several important classes of cones are introduced and their geometrical characteristics are also discussed.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = [0, \infty)$.

Given a real Banach space $\mathbb{E}$, the zero element (vector) of $\mathbb{E}$ is denoted by $\mathbf{0}$.

Definition 2.1.1. A subset $\mathbb{K}$ of a Banach space $\mathbb{E}$ is called a *cone* if the following conditions are satisfied:[1]

(i) the set $\mathbb{K}$ is non-empty, norm-closed and $\mathbb{K} \neq \{\mathbf{0}\}$;

(ii) if $u, v \in \mathbb{K}$ then $\alpha u + \beta v \in \mathbb{K}$ for all $\alpha, \beta \in \mathbb{R}_+$;

(iii) of each pair of vectors (points) $(u, -u)$ at least one element does not belong to $\mathbb{K}$, provided $u \neq \mathbf{0}$ (i.e., $u \in \mathbb{K}$ and $-u \in \mathbb{K}$ imply $u = \mathbf{0}$).

The cone $\mathbb{K}$ is called *solid* if it contains interior points in its norm topology, i.e., $\mathring{\mathbb{K}} \neq \emptyset$, where $\mathring{\mathbb{K}}$ denotes the set of interior points of $\mathbb{K}$.

Given a cone $\mathbb{K} \subset \mathbb{E}$, we define a *partial ordering relation* $\leqslant_{\mathbb{K}}$ *with respect to* $\mathbb{K}$ on the vectors belonging to $\mathbb{E}$ in the following manner: write $u \leqslant v$ if and only if $v - u \in \mathbb{K}$. On this basis, we define a strict partial ordering relation $<$ on $\mathbb{E}$, write $u < v$ provided $v - u \in \mathbb{K} \setminus \{\mathbf{0}\}$ (i.e., $u \leqslant v$ and $u \neq v$). Analogously, the symbol $\ll$ stands for a strong partial ordering so that $u \ll v$ means $v - u \in \mathring{\mathbb{K}}$ if $\mathbb{K}$ has a non-empty interior. In all that follows, we shall denote $\leqslant_{\mathbb{K}}$ by $\leqslant$ for short if there is no confusion.

[1] In fact, condition (ii) is equivalent to stating that a cone $\mathbb{K}$ is a convex set and that together with any point $u \in \mathbb{K}$, $\mathbb{K}$ contains the ray through u. Here, the *ray* through a point $u \in \mathbb{E}$ ($u \neq \mathbf{0}$) is defined to be the set of points αu for every $\alpha \in \mathbb{R}_+$.

In fact, there is a one-to-one correspondence between the cones of $\mathbb{E}$ and the order relations on $\mathbb{E}$ which are compatible with the linear structure of $\mathbb{E}$.

Proposition 2.1.1. *Let $\mathbb{E}$ be a real Banach space and $\mathbb{K} \subset \mathbb{E}$ be a cone. A partial order relation $\leqslant_{\mathbb{K}}$ induced by $\mathbb{K}$ is compatible with the linear structure of $\mathbb{E}$, i.e., whenever $u_i, v_i \in \mathbb{E}$, $u_i \leqslant_{\mathbb{K}} v_i$, for $i = 1, 2$, and $\alpha \in \mathbb{R}_+$, we have*

$$u_1 + u_2 \leqslant_{\mathbb{K}} v_1 + v_2, \qquad \text{and} \qquad \alpha u_i \leqslant_{\mathbb{K}} \alpha v_i \quad (\forall\, i = 1, 2).$$

Conversely, if $\leqslant$ is a partial order relation on $\mathbb{E}$ compatible with the linear structure of $\mathbb{E}$, then the set

$$\mathbb{K}_+ = \{u \in \mathbb{E} : \mathbf{0} \leqslant u\}$$

is a cone (called the positive cone henceforth) and $\leqslant = \leqslant_{\mathbb{K}_+}$.

The proof of the above proposition is immediate and hence is omitted here.

Definition 2.1.2. A Banach space endowed with a cone (equivalently, with a partial order relation compatible with its linear structure) is called an *ordered Banach space*.

Definition 2.1.3. Let $\mathbb{E}$ be a real Banach space, $\mathbb{K} \subset \mathbb{E}$ a cone and $\leqslant$ the partial order relation defined by $\mathbb{K}$. Then, the following statements are true.

(a) The cone $\mathbb{K}$ is called *reproducing* if every element $w \in \mathbb{E}$ can be represented in the form $w = u - v$ for some $u, v \in \mathbb{K}$ (i.e., $\mathbb{E} = \operatorname{span}(\mathbb{K}) = \mathbb{K} - \mathbb{K} = \{u - v \colon u, v \in \mathbb{K}\}$).[2]

(b) $\mathbb{K}$ is *total* if $\operatorname{span}(\mathbb{K})$ is dense in $\mathbb{E}$ (i.e., $\mathbb{E} = \overline{\mathbb{K} - \mathbb{K}}$).

(c) $\mathbb{K}$ is called *normal* if and only if there exists a positive constant N such that, for all $u, v \in \mathbb{E}$, we have $\|u\| \leqslant N\|v\|$ whenever $\mathbf{0} \leqslant u \leqslant v$. Sometimes the smallest number N is called the *normality constant* of $\mathbb{K}$.

(d) $\mathbb{K}$ is called *regular* if each increasing sequence which has an upper bound *in order* has a limit, and *fully regular* if each increasing sequence which is bounded *in norm* has a limit.

(e) $\mathbb{K}$ is called *minihedral* if $\sup\{u, v\}$ exists for all $u, v \in \mathbb{E}$, and *strongly minihedral* if every set which is bounded from above has a supremum.[3]

(f) The norm on $\mathbb{E}$ is called *monotonic* if $\mathbf{0} \leqslant u \leqslant v$ implies $\|u\| \leqslant \|v\|$, and *semi-monotonic* if $\|u\| \leqslant \lambda\|v\|$ for some $\lambda > 0$ and for all $u, v \in \mathbb{E}$ such that $\mathbf{0} \leqslant u \leqslant v$.

Remark 2.1.1. In fact, every solid cone is reproducing and hence also total.[4]

---

[2] Note that the elements u and v are not defined uniquely in the representation of $w = u - v$.

[3] The notation $\sup\{u, v\}$ is understood as the supremum of a pair of elements $u, v \in \mathbb{E}$; that means, (1) $u \leqslant \sup\{u, v\}$ and $v \leqslant \sup\{u, v\}$, and (2) $u \leqslant w$ and $v \leqslant w$ imply $\sup\{u, v\} \leqslant w$.

[4] To see this, pick any $u \in \mathring{\mathbb{K}}$, we note that $u \pm \varepsilon v \in \mathbb{K}$ for each $v \in \mathbb{E}$ with $\varepsilon > 0$ sufficiently small. Thus, $\mathbb{E} = \mathbb{K} - \mathbb{K}$ because $v = [(u + \varepsilon v) - (u - \varepsilon v)]/2\varepsilon$.

Regarding claim (c), it is clear that the normality constant $N \geqslant 1$. Indeed, taking $u = v \neq \mathbf{0}$, we have $N \geqslant 1$.[5]

Regarding claim (d), it is equivalent to stating that $\mathbb{K}$ is regular if a sequence $\{u_n\} \subset \mathbb{E}$ and $v \in \mathbb{E}$ satisfy the condition $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v$, then there exists $u \in \mathbb{E}$ such that $\|u_n - u\| \to 0$ as $n \to \infty$.

The definition of full regularity in claim (d) is equivalent to stating that $\mathbb{K}$ is fully regular if a sequence $\{u_n\} \subset \mathbb{E}$ and $v \in \mathbb{E}$ satisfy the conditions $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant \cdots$, and $M = \sup_n \|u_n\| < \infty$, then there exists $u \in \mathbb{E}$ such that $\|u_n - u\| \to 0$ as $n \to \infty$.

Definition 2.1.4. Let $\mathbb{E}$ be an ordered real Banach space which is partially ordered by a cone $\mathbb{K}$. For any $u, v \in \mathbb{E}$ with $u < v$, we define an *order interval* in $\mathbb{E}$ by

$$\begin{aligned}[u, v] &:= \{w \in \mathbb{E} : u \leqslant w \leqslant v\} \\ &= (u + \mathbb{K}) \cap (v - \mathbb{K}).\end{aligned}$$

Evidently, any order interval $[u, v]$ in an ordered Banach space is a closed convex subset. If the cone $\mathbb{K}$ is normal, then every order interval $[u, v]$ is bounded.[6]

Furthermore, in an ordered Banach space, by virtue of the property of the cone of being closed, we must always pass to the limit in inequalities. For instance, $\mathbf{0} \leqslant u_n$ for all $n \in \mathbb{N}_0$ and $\|u_n - u\| \to 0$ as $n \to \infty$ together imply $\mathbf{0} \leqslant u$. As a consequence, we have a generalized result such that $u_n \leqslant v_n$ for all $n \in \mathbb{N}_0$, $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$ together imply $u \leqslant v$.[7]

2.1.3 Riesz Spaces

Much of our theory takes place in the setting of a Riesz space. In what follows, we shall introduce such a space and discuss some characteristics of the elements in that space.

In order to define a Riesz space, let $(\mathbb{E}, \leqslant)$ be a partially ordered set.[8] We say that $(\mathbb{E}, \leqslant)$ is an *ordered real vector space* if $\mathbb{E}$ is a real vector space with the ordering relation $\leqslant$ that is compatible with the algebraic structure of $\mathbb{E}$ in the sense that it satisfies the following two properties:

(i) $u \leqslant v$ implies $u + w \leqslant v + w$ for each $w \in \mathbb{E}$; and

(ii) $u \leqslant v$ implies $\alpha u \leqslant \alpha v$ for every scalar $\alpha \in \mathbb{E}_+$.

On the other hand, a partially ordered set $(\mathbb{E}, \leqslant)$ is a *lattice* if each pair of elements $u, v \in \mathbb{E}$ has a supremum and an infimum. We denote the supremum and infimum of two elements $u, v \in \mathbb{E}$ by $u \vee v$ and $u \wedge v$, respectively.

[5] For a Banach lattice (complete Reize space), the normality constant is always equal to 1.

[6] To see this, we note that if we pick any $w \in [u, v]$, it follows that $\mathbf{0} \leqslant w - u \leqslant v - u$, and hence by normality of $\mathbb{K}$, $\|w - u\| \leqslant N\|v - u\|$.

[7] To see that it is so, we note that $u_n \leqslant v_n \implies v_n - u_n \in \mathbb{K} \implies v - u \in \mathbb{K} \implies u \leqslant v$.

[8] A *partially ordered set* $(\mathbb{E}, \leqslant)$ is a set $\mathbb{E}$ equipped with a partial order $\leqslant$. That is, the partial ordering relation $\leqslant$ is a transitive, reflexive and antisymmetric relation.

In this connection, if an ordered real vector space $(\mathbb{E}, \leqslant)$ is also a lattice, then $\mathbb{E}$ is called a *real Riesz space* (or a *real vector lattice*). It is worth noting that each element u in a real Riesz space $\mathbb{E}$ can be decomposed into positive and negative parts:

$$u = u^+ - u^-, \qquad \text{where } u^+ := u \vee \mathbf{0} \text{ and } u^- := (-u) \vee \mathbf{0}.$$

The absolute value $|u|$ of u is defined as $|u| := u^+ + u^-$. See, for example, Zaanen (1997) or Aliprantis and Border (2006).

A *Banach lattice* $\mathbb{B}$ is a Riesz space $(\mathbb{B}, \leqslant)$ equipped with a complete lattice norm $\|\cdot\|$ on $\mathbb{B}$.[9] A *lattice norm* $\|\cdot\|$ has the property that $|u| \leqslant |v|$ in $\mathbb{B}$ implies $\|u\| \leqslant \|v\|$. In addition, for any Banach lattice $\mathbb{B}$, we note that $\|\,|u|\,\| = \|u\|$ for each $u \in \mathbb{B}$.

2.1.4 Operators

In the sequel, we shall introduce some relevant definitions for an operator defined in some partially ordered space. The notion of a fixed point of an operator is introduced. In addition, we shall briefly discuss the underlying idea of the upper–lower solution method, which is a fundamental basis of our technique that will be used in the subsequent analysis.

In what follows, we take $(\mathbb{E}, \|\cdot\|)$ to be a real Banach space which is partially ordered by a cone $\mathbb{K}$ of $\mathbb{E}$ (i.e., $u \leqslant v$ if $v - u \in \mathbb{K}$). Let $\boldsymbol{A}$ be an operator and $\mathcal{D}(\boldsymbol{A})$ denote the domain of the operator $\boldsymbol{A}$.

The operator $\boldsymbol{A} \colon \mathcal{D}(\boldsymbol{A}) \subset \mathbb{E} \to \mathbb{E}$ is called *monotone increasing* (with respect to the cone $\mathbb{K}$) if for all $u, v \in \mathcal{D}(\boldsymbol{A})$, $u \leqslant v$ implies that $\boldsymbol{A}u \leqslant \boldsymbol{A}v$. The operator $\boldsymbol{A}$ is called *positive* if it leaves the cone $\mathbb{K}$ invariant, i.e., $\boldsymbol{A}(\mathbb{K}) \subset \mathbb{K}$. The operator $\boldsymbol{A}$ is called *strongly positive* (with respect to the cone $\mathbb{K}$ with interior) if, for each frontier (non-zero) element $u \in \mathbb{K}$, there is a natural number $N = N(u)$ such that $\mathbf{0} \ll \boldsymbol{A}^N u$; that is, $\boldsymbol{A}^N u$ is an interior point of $\mathbb{K}$ for each $u \in \mathbb{K} \setminus \{\mathbf{0}\}$ and for some $N \in \mathbb{N}$.

If $\boldsymbol{L}$ is a *linear operator* from $\mathbb{E}$ to itself, then the spectral radius of $\boldsymbol{L}$ is denoted by $r(\boldsymbol{L})$ and defined as the supremum of $|\lambda|$ over all λ in the spectrum of $\boldsymbol{L}$.[10] In what follows, the operator norm of any bounded linear operator $\boldsymbol{L}$ on $\mathbb{E}$ always refers to the induced norm $\|\boldsymbol{L}\| := \sup_{\|f\| \leqslant 1} \|\boldsymbol{L}f\| = \min\{M \geqslant 0 \colon \|\boldsymbol{L}f\| \leqslant M\|f\| \text{ for all } f \in \mathbb{E}\}$.

Remark 2.1.2. Given a linear operator $\boldsymbol{A}$ acting in the space $\mathbb{E}$ with the cone $\mathbb{K}$ and leaving this cone invariant, it is worth noting that such a positive linear operator $\boldsymbol{A}$ possesses the property of monotonicity.[11] To see this, we assume that a linear operator $\boldsymbol{A}$ is positive and pick any

[9] A Riesz space equipped with a lattice norm is called a *normed Riesz space*. In this sense, a Banach lattice is also referred to as a *complete* normed Riesz space.

[10] For more details of spectral theory, please refer to Riesz and Szőkefalvi-Nagy (1955), Krasnosel'skiĭ et al. (1972) or Aliprantis and Border (2006).

[11] In fact, a linear operator is positive if and only if it is monotone.

$u, v \in \mathbb{E}$ with $u \leqslant v$. Then, by the positivity of $\boldsymbol{A}$, $\mathbf{0} \leqslant v - u$ implies $\mathbf{0} \leqslant \boldsymbol{A}(v-u)$. As $\boldsymbol{A}$ is linear, $\mathbf{0} \leqslant \boldsymbol{A}v - \boldsymbol{A}u$, and hence $\boldsymbol{A}u \leqslant \boldsymbol{A}v$, which means that $\boldsymbol{A}$ is monotone increasing.

We say that an operator $\boldsymbol{A}^+$ is a *majorant* of the operator $\boldsymbol{A}$ on the cone $\mathbb{K}$ if, the operator $\boldsymbol{A}^+$ is greater than (dominating) $\boldsymbol{A}$ pointwise on $\mathbb{K}$ and write $\boldsymbol{A} \leqslant \boldsymbol{A}^+$ provided $\boldsymbol{A}f \leqslant \boldsymbol{A}^+ f$ for all $f \in \mathbb{K}$.

In general, we consider an operator equation (induced by an operator $\boldsymbol{A}$)

$$v = \boldsymbol{A}v \tag{2.1}$$

together with the corresponding iterative method

$$v_{n+1} = \boldsymbol{A}v_n \qquad (n \in \mathbb{N}_0). \tag{2.2}$$

A point v satisfying (2.1) is called a *fixed point* of the operator $\boldsymbol{A}$. The point v is said to be an *upper solution*, a *strict* upper solution, or a *strong* upper solution of (2.1) if and only if $\boldsymbol{A}v \leqslant v$, $\boldsymbol{A}v < v$, or $\boldsymbol{A}v \ll v$, respectively. Likewise, the prefix "upper" is replaced by "*lower*" when the respective inequalities are reversed. It is easy to see that an upper solution or a lower solution of (2.1) is strict or strong if and only if it is not a solution to (2.1).

In addition, the operator $\boldsymbol{A}$ transforming some subset $\mathcal{D}(\boldsymbol{A}) \subset \mathbb{E}$ into itself is called *globally asymptotically stable* on $\mathcal{D}(\boldsymbol{A})$ if $\boldsymbol{A}$ has a unique fixed point v^* in $\mathcal{D}(\boldsymbol{A})$ and the trajectory $\{\boldsymbol{A}^n v\}_{n\in\mathbb{N}}$ converges to v^* from any $v \in \mathcal{D}(\boldsymbol{A})$.

When we wish to solve some nonlinear increasing operators defined on some ordered real Banach spaces, we are primarily interested in a non-trivial solution of (2.1) in the cone $\mathbb{K}$, which is a question of a positive (different from $\mathbf{0}$) solution in $\mathbb{K}$.[12] A solution v is said to be a *positive solution* if $v > \mathbf{0}$ (i.e., $v \in \mathbb{K} \setminus \{\mathbf{0}\}$), and is called a *strongly positive solution* if $v \gg \mathbf{0}$.

The upper–lower solution method is usually expected in proving the existence of solutions and obtaining a convergence of the iteration.[13] The intuition of this method is straightforward in the case of a nonlinear equation in a one-dimensional space. Consider a question of the existence of positive real numbers x^* such that $x^* = f(x^*)$, where $f(x)$ is a continuous and nonnegative function for $x \in \mathbb{R}_+$. Evidently, it is sufficient for the existence of a positive solution that either small numbers r be found such that $f(r) > r$ and large numbers R such that $f(R) < R$ or small values r such that $f(r) < r$ and large values R such that $f(R) > R$ (as illustrated in Figure 2.1). It turns out that this elementary idea carries over to positive nonlinear operators acting in real Banach spaces with a cone.

[12] By an ordered real Banach space we mean a real Banach space together with a cone. It is worth noting that an ordered Banach space is not necessarily a Banach lattice. For example, if $\mathbb{E}$ is a regular ordered Banach space with the Riesz separation property and satisfies a condition that for any $u, v \geqslant \mathbf{0}$ we have $\|u+v\| = \|u\| + \|v\|$, then this ordered Banach space $\mathbb{E}$ is a Banach lattice. For more details, interested readers may refer to Davies (1968) or Schaefer (1974).

[13] The upper–lower solution method is also known as the super- and lower solution method.

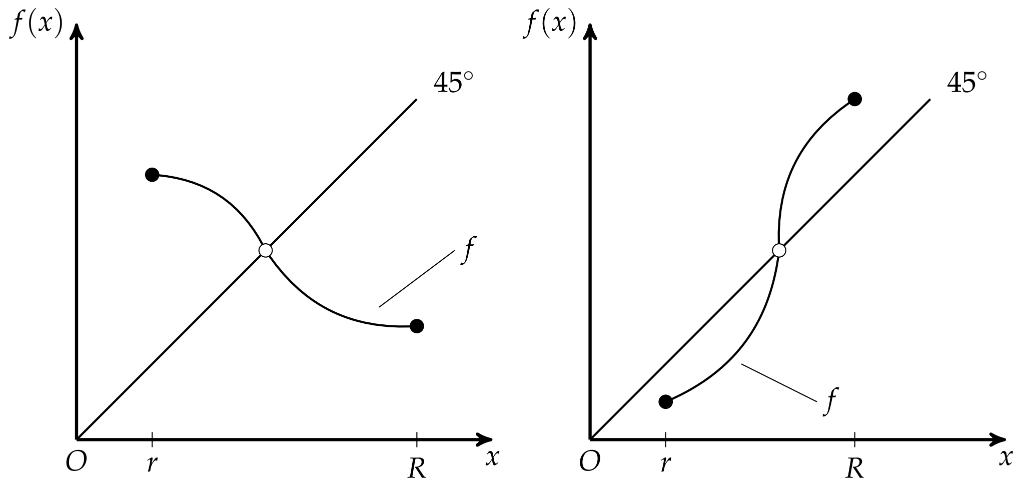

Figure 2.1

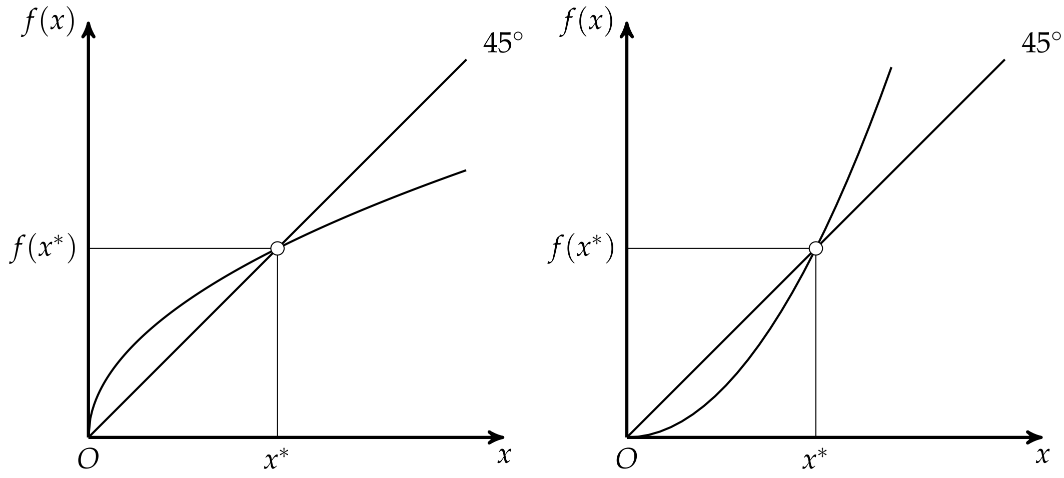

Figure 2.2

Furthermore, it is also interesting to know what conditions are needed in order that the upper–lower solution method works and solves the existence of a unique solution. In this situation, special criteria—the geometrical characteristics of a nonlinear operator— are found and studied for the uniqueness of a positive solution. For instance, let us consider a strictly increasing concave operator in the case of a one-dimensional space, which is a scalar function $f(x)$ defined on $\mathbb{R}_+$ (cf. Figure 2.2). It is clear to see that the graph of f has the property that on each segment $[0, x_0]$, this graph does not have points situated below the straight line passing through the origin and the point $(x_0, f(x_0)) \in \mathbb{R}^2_{++}$.[14] As a result, uniqueness of the positive fixed points is guaranteed by the strict concavity of f.

Therefore, in order to study the method for the proof of the existence of positive solutions, to find special criteria for the uniqueness of a positive solution, and to consider the question about the convergence of successive approximations to positive solutions, we require the suitable selection of new classes of operators.

[14] In other words, the function f satisfies $f(tx) > tf(x)$ for all $x > 0$ and $t \in (0, 1)$.

2.1.5 Monotone Concave Operators

In the following, we shall introduce some geometric definitions of an operator and then present fixed point theorems established by Du (1989, 1990), which are central to our analysis.

Definition 2.1.5. An operator $\boldsymbol{A}$ defined on a cone $\mathbb{K}$ of $\mathbb{E}$ is *super-additive* if, for any u and v in $\mathbb{K}$,

$$\boldsymbol{A}\,(u+v) \geqslant \boldsymbol{A}(u) + \boldsymbol{A}(v). \tag{2.3}$$

Definition 2.1.6. An operator $\boldsymbol{A}$ defined on a cone $\mathbb{K}$ of $\mathbb{E}$ is *subhomogeneous* if, for any v in $\mathbb{K}$ and any real number $t \in (0,1]$,[15]

$$\boldsymbol{A}\,(tv) \geqslant t\boldsymbol{A}\,(v)\,. \tag{2.4}$$

If the cone $\mathbb{K}$ is solid and

$$\boldsymbol{A}\,(tv) \gg t\boldsymbol{A}\,(v)\,,$$

for all v in $\mathbb{K} \setminus \{\mathbf{0}\}$ and any $t \in (0,1)$, then $\boldsymbol{A}$ is said to be *strongly subhomogeneous*.

Remark 2.1.3. It is worth noting that (2.4) is equivalent to

$$\boldsymbol{A}\,(sv) \leqslant s\boldsymbol{A}\,(v)\,, \tag{2.5}$$

for all v in $\mathbb{K}$ and any number $s \geqslant 1$.

To see this, pick any v in $\mathbb{K}$ and any $s \geqslant 1$, by virtue of (2.4), we have $\boldsymbol{A}(v) = \boldsymbol{A}(s^{-1}sv) \geqslant s^{-1}\boldsymbol{A}(sv)$; that is (2.5), as desired. Conversely, for any $t \in (0,1]$ and each v in $\mathbb{K}$, $\boldsymbol{A}(v) = \boldsymbol{A}(t^{-1}tv) \leqslant t^{-1}\boldsymbol{A}(tv)$, by (2.5); so (2.4) holds.

Definition 2.1.7. An operator $\boldsymbol{A}$ defined on a cone $\mathbb{K}$ of $\mathbb{E}$ is *positively homogeneous* (of the first degree) if, for any v in $\mathbb{K}$ and any real number $t \geqslant 0$,

$$\boldsymbol{A}\,(tv) = t\boldsymbol{A}\,(v)\,. \tag{2.6}$$

Definition 2.1.8. (see Krasnosel'skiĭ (1964), p. 187) Given an operator $\boldsymbol{A}\colon \mathbb{K} \to \mathbb{K}$, and $u_0 > \mathbf{0}$, $\boldsymbol{A}$ is called u_0*-concave*, if (i) for any $v > \mathbf{0}$, there exists $\lambda = \lambda(v) > 0$ and $\mu = \mu(v) > 0$ such that $\lambda u_0 \leqslant \boldsymbol{A}v \leqslant \mu u_0$, and (ii) for every $v \in \mathbb{K}$ satisfying $\lambda_1 u_0 \leqslant v \leqslant \mu_1 u_0$ (where $\lambda_1 = \lambda_1(v) > 0$, $\mu_1 = \mu_1(v) > 0$) and for every positive number $t \in (0,1)$, a number $\eta = \eta(v;t) > 0$ can be found such that $\boldsymbol{A}(tv) \geqslant (1+\eta)t\boldsymbol{A}v$.

Definition 2.1.9. (see Potter (1977), p. 94) Let $\mathbb{K}$ be solid. An operator $\boldsymbol{A}\colon \mathring{\mathbb{K}} \to \mathring{\mathbb{K}}$ is said to be α*-concave* if there exists a number $\alpha \in (0,1)$ such that $\boldsymbol{A}(tv) \geqslant t^{\alpha}\boldsymbol{A}v$ for all $v \in \mathring{\mathbb{K}}$ and $t \in (0,1)$.

Definition 2.1.10. (see Amann (1976), p. 690) Let $\mathcal{D}(\boldsymbol{A})$ be a convex set in $\mathbb{E}$ and $\boldsymbol{A}\colon \mathcal{D}(\boldsymbol{A}) \to \mathbb{E}$ be an operator. If for every $u, v \in \mathcal{D}(\boldsymbol{A})$ with $u \leqslant v$ and for every $t \in [0,1]$, we have that

$$\boldsymbol{A}\,(tu + (1-t)\,v) \geqslant t\boldsymbol{A}u + (1-t)\,\boldsymbol{A}v, \tag{2.7}$$

[15] Alternatively, subhomogeneous operators are often called "sublinear" by Amann (1976) and Guo and Lakshmikantham (1988).

then $\boldsymbol{A}$ is said to be an *order concave operator*.

Remark 2.1.4. If the operator $\boldsymbol{A}$ is order concave, then the operator $c\boldsymbol{A}$ is also order concave where c is an arbitrary positive number. If $\boldsymbol{A}$ is monotone increasing and order concave, then the operators $\boldsymbol{A}^2$, $\boldsymbol{A}^3$ and so on possess these properties. If there are two order concave operators $\boldsymbol{A}$ and $\boldsymbol{B}$ mapping from the same convex subset of $\mathbb{E}$ to $\mathbb{E}$, then the operator $\boldsymbol{A}+\boldsymbol{B}$ is order concave.

Lemma 2.1.1. *If an operator $\boldsymbol{A}$ defined on a cone $\mathbb{K}$ is positive and order concave, then it is subhomogeneous.*

Proof. Pick any $f \in \mathbb{K}$ and $\lambda \in [0,1]$. By the positivity and order concavity of $\boldsymbol{A}$, we have

$$\begin{aligned}\boldsymbol{A}(\lambda f) = \boldsymbol{A}(\lambda f + (1-\lambda)\mathbf{0}) &\geqslant \lambda \boldsymbol{A}(f) + (1-\lambda)\boldsymbol{A}(\mathbf{0}) \\ &\geqslant \lambda \boldsymbol{A}(f),\end{aligned}$$

which implies the subhomogeneity of $\boldsymbol{A}$, as desired. □

An order concave operator needs not automatically be a subhomogeneous operator. It is also worth noting that an order concave operator is not necessarily monotone and not necessarily continuous. In this connection, Du (1989, 1990) generalizes and improves Amann's result by removing the compactness and even continuity conditions on the operator. The main results of Du (1989, 1990) are stated below to keep our presentation reasonably self-contained.

Theorem 2.1.1. *(Du (1989), Theorem 2.1, p. 619) Let $(\mathbb{E}, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $\mathbb{K} \subset \mathbb{E}$. Suppose $\mathbb{K}$ is normal, $u, v \in \mathbb{E}$ with $u \leqslant v$, and $\boldsymbol{A}\colon [u,v] \to \mathbb{E}$ is a monotone increasing operator. Let $h := v - u$. If one of the following conditions holds:*

(i) $\boldsymbol{A}$ is concave and $\boldsymbol{A}u \geqslant u + \varepsilon h$, $\boldsymbol{A}v \leqslant v$,

(ii) $\boldsymbol{A}$ is convex and $\boldsymbol{A}u \geqslant u$, $\boldsymbol{A}v \leqslant v - \varepsilon h$,

where ε is a positive number with $0 < \varepsilon < 1$, then $\boldsymbol{A}$ has a unique fixed point v^* in $[u,v]$. In addition, for any $v_0 \in [u,v]$, if $v_{n+1} = \boldsymbol{A}v_n$ $(n \in \mathbb{N}_0)$, then we have

$$\|v_n - v^*\| \to 0 \quad (n \to \infty),$$

$$\|v_n - v^*\| \leqslant M(1-\varepsilon)^n \quad (n \in \mathbb{N}_0),$$

where M is a positive constant which is not dependent on v_0.

While the above result is established for an arbitrary Banach space with a normal cone, the following result is built upon a Banach space with a normal and solid cone. It is worth noting that the order interval chosen in both of these results does not need to lie in the cone $\mathbb{K}$.

Corollary 2.1.1. *(Du (1990), Theorem 3.1, p. 8) Suppose $\mathbb{K}$ is a normal solid cone of $\mathbb{E}$, $u, v \in \mathbb{E}$ with $u \leqslant v$, and $\boldsymbol{A}\colon [u, v] \to \mathbb{E}$ is a monotone increasing operator. If one of the following conditions holds:*

(i) $\boldsymbol{A}$ is concave and $\boldsymbol{A}u \gg u$, $\boldsymbol{A}v \leqslant v$,

(ii) $\boldsymbol{A}$ is convex and $\boldsymbol{A}u \geqslant u$, $\boldsymbol{A}v \ll v$,

then $\boldsymbol{A}$ has a unique fixed point v^* in $[u, v]$. In addition, for any $v_0 \in [u, v]$, if $v_{n+1} = \boldsymbol{A}v_n$ $(n \in \mathbb{N}_0)$, then we have $\|v_n - v^*\| \to 0$ as $n \to \infty$ and

$$\|v_n - v^*\| \leqslant Mr^n \quad (n \in \mathbb{N}_0),$$

where $0 < r < 1$ and $M > 0$ are constants.

This order-geometric approach is technically straightforward and intuitive. It stems from geometric ideas related to the study of fixed points for monotone increasing operators defined on partially ordered spaces. Another advantage of this approach is that it is constructive and can therefore be used to translate conditions in existence theorems into conditions which entail the applicability of various approximation methods, examination of convergence and even acquisition of error estimates. More importantly, no contraction or compactness properties are needed here, but like Banach fixed point theorem, under certain regularity conditions, Du's theorems yield that monotone order concave operators possess unique, globally attracting fixed points—a highly attractive and valuable result to the natural scientist, engineer and economist.

2.2 Fixed Points of Positive Operators in Ordered Banach Spaces

In this section, we shall study the existence and uniqueness of positive solutions to an abstract nonlinear operator equation of the form:

$$f = h + \boldsymbol{B}f \qquad (f \in \mathbb{K}), \tag{2.8}$$

where h is a fixed non-zero element of a certain cone $\mathbb{K}$ and $\boldsymbol{B}$ is a nonlinear operator which leaves the cone $\mathbb{K}$ invariant in a Banach space. To this end, certain linear operators are carefully chosen, serving as the corresponding majorants for the nonlinear operator $\boldsymbol{B}$. In addition, a set of sufficient conditions is provided and discussed to adapt to different characteristics of cones. In this context, an alternative technique is applied in the proofs, primarily applying Du's fixed point theorems, on a cone. Furthermore, based on Du's classical results, uniqueness and global attracting property of the solution to (2.8) over the whole cone are established.

In fact, the examination of conditions of the upper and lower solutions that are embedded in Du's fixed point theorems may have some difficulty in concrete problems. In other words, the direct implementation of Du's theorems in solving concrete problems may potentially depend on various requirements which arise from the specifics of the problem. Indeed, in applications, how to find admissible, lower and upper solutions of the nonlinear operator equation is a tricky task. In particular, seeking strong upper and/or strong lower solutions is one of the typical difficulties encountered with nonlinear problems.

In the sequel, we aim to establish some special criteria such that the existence of a positive upper solution is guaranteed. Those sufficient conditions have relatively weak requirements, making the underlying operator easy to deal with and thus making the corresponding, abstract theorems easy to implement in practice.

As before, let $(\mathbb{E}, \|\cdot\|)$ be a real Banach space, $\mathbb{K}$ be a cone of $\mathbb{E}$ and $\leqslant$ be the partial ordering defined by $\mathbb{K}$. We note that the solution of (2.8) can be constructed by a fixed point argument. Hereby we define an operator $\boldsymbol{A}\colon \mathbb{K} \to \mathbb{E}$ through

$$\boldsymbol{A}f := h + \boldsymbol{B}f. \tag{2.9}$$

Then, the solvability of (2.8) corresponds to the *fixed-point problem*

$$f = \boldsymbol{A}f.$$

Recall that $f^* \in \mathbb{K}$ is called a positive solution of (2.8) provided $f^* \in \mathbb{K}$ solves (2.8) and $f^* \neq \mathbf{0}$. In this sense, stating that $f^* \in \mathbb{K}$ is a positive solution of (2.8) is equivalent to stating that $f^* \in \mathbb{K} \setminus \{\mathbf{0}\}$ is a fixed point of the operator $\boldsymbol{A}$.

Remark 2.2.1. It is worth noticing that $\boldsymbol{A}$ is a well-defined positive operator on $\mathbb{K}$ (i.e., $\boldsymbol{A}(\mathbb{K}) \subset \mathbb{K}$). Indeed, since h is a fixed element in $\mathbb{K} \setminus \{\mathbf{0}\}$ and since the operator $\boldsymbol{B}$ acts in $\mathbb{K}$, by the convexity of the cone $\mathbb{K}$, we then conclude that $\boldsymbol{A}f = h + \boldsymbol{B}f$ lies in $\mathbb{K}$, and more precisely, $\boldsymbol{A}f$ lies in $\mathbb{K} \setminus \{\mathbf{0}\}$.

A central assumption for solving (2.8) is as follows:

Assumption 2.2.1. The nonlinear operator $\boldsymbol{B}\colon \mathbb{K} \to \mathbb{K}$ is order concave.

Lemma 2.2.1. *If Assumption 2.2.1 holds, then the operator $\boldsymbol{A}$ defined in (2.9) is monotone increasing.*

Lemma 2.2.1 follows directly from Lemma 3.4 of Du (1990) given that the operator $\boldsymbol{A}$ is positive and order concave (since $\boldsymbol{B}$ is positive and order concave by assumption, so is $\boldsymbol{A}$). This lemma is quite useful to establish the monotonicity of a positive operator, whenever the operator possesses order concavity.

Recall that the upper–lower solution method, in general, is utilized to construct a solution by iteration over an appropriate order interval that serves as an admissible subset of the domain of an operator. Since $\boldsymbol{A}$ is a positive operator, it is clear that the zero element $\mathbf{0}$ is a natural choice for a lower solution of (2.8). This observation implies that in order to adapt the upper–lower solution method to the fixed point theorems of Du (1989), the primary objective is to

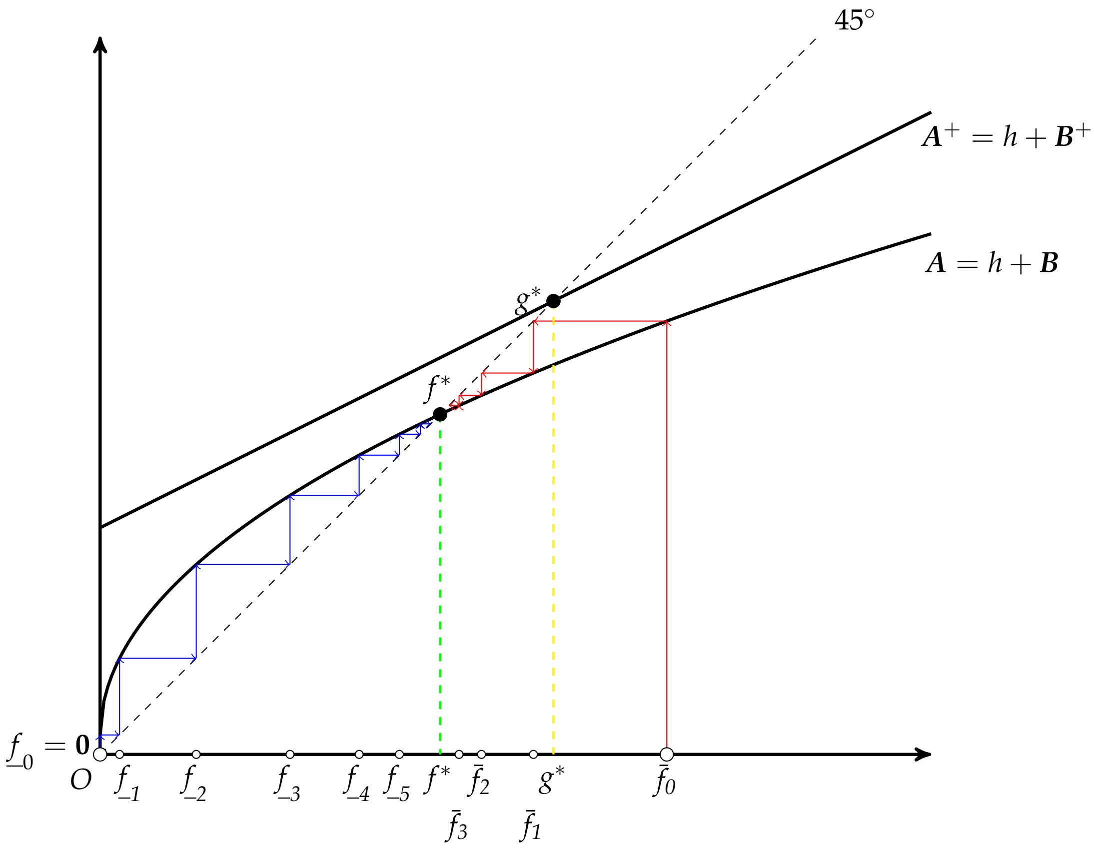

Figure 2.3

prove the existence of (or to construct) a positive upper solution of (2.8) on the cone $\mathbb{K}$. In the following, we shall show that the key to finding such an upper solution is making essential use of majorants of the operator being studied.

To this end, we make the following assumption.

Assumption 2.2.2. There exists a majorant of the operator $\boldsymbol{B}$ on $\mathbb{K}$, denoted by $\boldsymbol{B}^+$, such that $\boldsymbol{B}^+ \colon \mathbb{K} - \mathbb{K} \to \mathbb{K} - \mathbb{K}$ is a continuous linear operator with $\|\boldsymbol{B}^+\| < 1$.[16]

Lemma 2.2.2. *If the cone $\mathbb{K}$ is total and Assumption 2.2.2 holds, then (2.8) has an upper solution in $\mathbb{K}$, (i.e., the set $\{f \in \mathbb{K} : \boldsymbol{A}(f) \leqslant f\}$ is non-empty).*

In order to illustrate the underlying idea of our approach, we refer to Figure 2.3 in the special case of $\mathbb{E} = \mathbb{R}$. It turns out that monotone concave operators defined on partially ordered spaces present a behavior which is similar to the corresponding increasing concave scalar functions.

Proof of Lemma 2.2.2. Since the cone $\mathbb{K}$ of the real Banach space $\mathbb{E}$ is total, the set $\operatorname{span}(\mathbb{K}) = \mathbb{K} - \mathbb{K}$ is a dense linear subspace of $\mathbb{E}$. As the majorant $\boldsymbol{B}^+ \colon \mathbb{K} - \mathbb{K} \to \mathbb{K} - \mathbb{K}$ is a continuous linear operator, by the bounded linear transformation (BLT) extension theorem (see, e.g.,

[16] Recall that $\mathbb{K} - \mathbb{K} = \{u - v : u, v \in \mathbb{K}\}$.

Theorem 2.4.1 of Atkinson and Han (2010)), it can be uniquely extended to a continuous linear operator mapping from $\overline{\mathbb{K}-\mathbb{K}} = \mathbb{E}$ to $\mathbb{E}$, and the extended operator is denoted by $\overline{\boldsymbol{B}^+}$ henceforth. In addition, it follows from the BLT extension theorem that $\overline{\boldsymbol{B}^+}|_{\mathbb{K}-\mathbb{K}} = \boldsymbol{B}^+$ and $\|\overline{\boldsymbol{B}^+}\| = \|\boldsymbol{B}^+\|$.[17]

Let $\boldsymbol{I}$ denote the identity mapping on $\mathbb{E}$. As $\mathbb{E}$ is a real Banach space, by the Neumann series theorem (also known as the geometric series theorem), the condition $\|\overline{\boldsymbol{B}^+}\| = \|\boldsymbol{B}^+\| < 1$ implies that $\boldsymbol{I} - \overline{\boldsymbol{B}^+}$ is a bijection on $\mathbb{E}$, and hence its inverse $(\boldsymbol{I} - \overline{\boldsymbol{B}^+})^{-1}$ exists as a continuous linear operator on $\mathbb{E}$ with $(\boldsymbol{I} - \overline{\boldsymbol{B}^+})^{-1} = \sum_{n=0}^{\infty}(\overline{\boldsymbol{B}^+})^n$ (see, e.g., Theorem 2.3.1 of Atkinson and Han (2010)).

We now consider the following linear operator equation

$$g = h + \overline{\boldsymbol{B}^+}g, \qquad (g \in \mathbb{E}). \tag{2.10}$$

Observe that (2.10) can be expressed alternatively as $(\boldsymbol{I} - \overline{\boldsymbol{B}^+})g = h$. By the Neumann series theorem again, for any fixed h in $\mathbb{K} \setminus \{\boldsymbol{0}\}$, (2.10) has a unique solution $g^* = (\boldsymbol{I} - \overline{\boldsymbol{B}^+})^{-1}h \in \mathbb{K} \setminus \{\boldsymbol{0}\}$. In particular, the solution depends continuously on the choice of the function h. To see that the unique solution g^* lies in $\mathbb{K} \setminus \{\boldsymbol{0}\}$ whenever $h \in \mathbb{K} \setminus \{\boldsymbol{0}\}$, recall first that the nonlinear operator $\boldsymbol{B}$ leaves the cone $\mathbb{K}$ invariant. As $\mathbb{K}$ is a subset of $\mathbb{K} - \mathbb{K}$ and $\boldsymbol{B}$ is dominated by $\boldsymbol{B}^+$ on $\mathbb{K}$, for each $f \in \mathbb{K}$, we have $\boldsymbol{0} \leqslant \boldsymbol{B}f \leqslant \boldsymbol{B}^+f$, which in turn implies the positivity of the operator $\boldsymbol{B}^+$ (i.e., $\boldsymbol{B}^+(\mathbb{K}) \subset \mathbb{K}$). Further, by virtue of the positivity of $\boldsymbol{B}^+$, it follows from the fact $\overline{\boldsymbol{B}^+}|_{\mathbb{K}-\mathbb{K}} = \boldsymbol{B}^+$ (and thus $\overline{\boldsymbol{B}^+}(\mathbb{K}) = \boldsymbol{B}^+(\mathbb{K})$) that $\overline{\boldsymbol{B}^+}$ is also positive. Furthermore, this means that the operator $(\boldsymbol{I} - \overline{\boldsymbol{B}^+})^{-1} = \boldsymbol{I} + \overline{\boldsymbol{B}^+} + (\overline{\boldsymbol{B}^+})^2 + \cdots + (\overline{\boldsymbol{B}^+})^n + \cdots$ is also positive (monotone increasing) in $\mathbb{E}$.[18] Thus, $g^* = (\boldsymbol{I} - \overline{\boldsymbol{B}^+})^{-1}h = h + \overline{\boldsymbol{B}^+}h + (\overline{\boldsymbol{B}^+})^2h + \cdots + (\overline{\boldsymbol{B}^+})^nh + \cdots$ indeed lies in $\mathbb{K} \setminus \{\boldsymbol{0}\}$ given that $h \in \mathbb{K} \setminus \{\boldsymbol{0}\}$, as was to be shown.

Next, let us go back to operator Equation (2.8) and consider the fixed element g^* in $\mathbb{K} \setminus \{\boldsymbol{0}\}$. It follows that

$$\boldsymbol{A}g^* = h + \boldsymbol{B}g^* \leqslant h + \boldsymbol{B}^+g^* = h + \overline{\boldsymbol{B}^+}g^* = g^*.$$

Hence, g^* is an upper solution of (2.8), which completes the proof. □

Remark 2.2.2. In fact, our assumption imposed on a cone $\mathbb{K}$ is very weak, since most function spaces that are widely used in economic theory possess the reproducing property with respect to the positive cone, which in turn implies that those spaces naturally possess the total property as well (cf. Section 2.2.1).

In the sequel, we shall mainly focus on the cone $\mathbb{K}$ that is restricted to be reproducing. Meanwhile, instead of Assumption 2.2.2, we weaken the operator norm condition and assume the

[17] $\overline{\boldsymbol{B}^+}|_{\mathbb{K}-\mathbb{K}} = \boldsymbol{B}^+$ means that $\overline{\boldsymbol{B}^+}f = \boldsymbol{B}^+f$ for all $f \in \mathbb{K} - \mathbb{K}$. Please note that $\|\overline{\boldsymbol{B}^+}\| := \sup\{\|\overline{\boldsymbol{B}^+}f\|/\|f\| : f \in \mathbb{E} \setminus \{\boldsymbol{0}\}\}$ and $\|\boldsymbol{B}^+\| := \sup\{\|\boldsymbol{B}^+f\|/\|f\| : f \in \mathbb{E}_0 \setminus \{\boldsymbol{0}\},\ \mathbb{E}_0 := \mathbb{K} - \mathbb{K}\}$.

[18] Recalling Remark 2.1.2, for any linear operator, the concept of positivity is the same as that of monotone increasing.

following:

Assumption 2.2.3. There exists a majorant of the operator $\boldsymbol{B}$ on $\mathbb{K}$, denoted by $\boldsymbol{B}^+$, such that $\boldsymbol{B}^+ : \mathbb{E} \to \mathbb{E}$ is a continuous linear operator with $r(\boldsymbol{B}^+) < 1$.

Remark 2.2.3. Since $r(\boldsymbol{B}^+) \leqslant \|\boldsymbol{B}^+\|$, in order to obtain the spectral radius condition $r(\boldsymbol{B}^+) < 1$, it suffices that $\|\boldsymbol{B}^+\| < 1$. In fact, the precondition $\|\boldsymbol{B}^+\| < 1$ is convenient to implement in applications when the spectral radius condition is difficult to figure out.

Corollary 2.2.1. *If the cone $\mathbb{K}$ is reproducing and Assumption 2.2.3 holds, then operator Equation (2.8) has an upper solution in $\mathbb{K}$.*

Proof. To see this, we first note that $\mathbb{E} = \mathbb{K} - \mathbb{K}$ since $\mathbb{K}$ is reproducing. In addition, since $r(\boldsymbol{B}^+) < 1$, Gelfand's formula implies the existence of an $m \in \mathbb{N}$ such that $\|(\boldsymbol{B}^+)^m\| < 1$. As $\mathbb{E}$ is a real Banach space, this and the Neumann series theorem imply that $\boldsymbol{I} - \boldsymbol{B}^+$ is a bijection on $\mathbb{E}$ and hence its inverse exists as a continuous linear operator and equals $\sum_{i=0}^{\infty} (\boldsymbol{B}^+)^i$ (see, e.g., Corollary 2.3.3 of Atkinson and Han (2010)).

In particular, it follows from $r(\boldsymbol{B}^+) < 1$ that (2.10) has a unique solution $g^* = (\boldsymbol{I} - \boldsymbol{B}^+)^{-1} h$ and a stable iterative method on $\mathbb{E}$. That is, the sequence $\{g_n\}_{n \in \mathbb{N}}$ of successive approximations (generated through $g_n = h + \boldsymbol{B}^+ g_{n-1}$) converges, for every $h \in \mathbb{K} \setminus \{\boldsymbol{0}\}$ and for arbitrary initial element $g_0 \in \mathbb{E}$, to a unique solution g^* of (2.10).

The remaining part is identical to the proof in Lemma 2.2.2 given that $\overline{\boldsymbol{B}^+} \equiv \boldsymbol{B}^+$. □

In light of Lemma 2.2.1 and Corollary 2.2.1 and the preceding discussion, we are now ready to state our main result of the fixed point problem for the operator $\boldsymbol{A}$ defined in (2.9).

Theorem 2.2.1. *Let $\mathbb{E}$ be an ordered real Banach space with the positive cone $\mathbb{K}$ in $\mathbb{E}$.*

Suppose that

(i) the cone $\mathbb{K}$ is normal and solid;

(ii) Assumptions 2.2.1 and 2.2.3 hold;

(iii) the element h is an interior point of the cone $\mathbb{K}$, i.e., $h \in \mathring{\mathbb{K}}$.

Then we may conclude the following:

(1) **Existence and uniqueness**: *The operator Equation (2.8) has exactly one positive solution in $\mathring{\mathbb{K}}$, i.e., the corresponding operator $\boldsymbol{A}$ has exactly one positive fixed point f^* in $\mathring{\mathbb{K}}$. More precisely, this positive fixed point f^* is located in $\big(\boldsymbol{0}, (\boldsymbol{I} - \boldsymbol{B}^+)^{-1} h\big]$, where $\boldsymbol{I}$ is the identity operator on $\mathbb{K}$, and $\boldsymbol{B}^+ : \mathbb{E} \to \mathbb{E}$ is the majorant of $\boldsymbol{B}$.*[19]

[19] The order interval $\big(\boldsymbol{0}, (\boldsymbol{I} - \boldsymbol{B}^+)^{-1} h\big]$ is understood as the set $\{v \in \mathbb{E} : \boldsymbol{0} < v \leqslant (\boldsymbol{I} - \boldsymbol{B}^+)^{-1} h\}$.

(2) **Convergence of the iteration method**: *The sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of successive approximations converges to the solution,* f^**, for an arbitrary choice of initial point* f_0 *in* $\left[\mathbf{0},\ (\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h\right]$*; that is, if* $f_n = \boldsymbol{A}f_{n-1}$ *for every* $n \in \mathbb{N}$*, then we have that*

$$\lim_{n\to\infty} \|f_n - f^*\| = \lim_{n\to\infty} \|\boldsymbol{A}^n f_0 - f^*\| = 0,$$

whenever $f_0 \in \left[\mathbf{0},\ (\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h\right]$.

(3) **Error estimates**: *For each* $n \in \mathbb{N}_0$ *and any* $f_0 \in \left[\mathbf{0}, (\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h\right]$*, there exist a constant* $M > 0$ *and* $r \in (0,1)$ *such that, we have a priori error estimate*

$$\|f_n - f^*\| = \|\boldsymbol{A}^n f_0 - f^*\| \leqslant Mr^n \to 0, \qquad (\text{as } n \to \infty).$$

Here we take $M = N \cdot \|\boldsymbol{A}\mathbf{0}\| \cdot (1-r)^{-2}$*, where* N *is the normality constant of* $\mathbb{K}$.

(4) **Globally attractive property of the fixed point**: *For any initial condition* $f_0 \in \mathbb{K}$*, we have*

$$\lim_{n\to\infty} \|\boldsymbol{A}^n f_0 - f^*\| = 0.$$

The proof of Theorem 2.2.1 is given in Appendix A.1.

Remark 2.2.4. In the above results, Theorem 2.2.1 extends the original results of Theorem 3.1 of Du (1990) (cf. Corollary 2.1.1 in the present chapter). In particular, the uniqueness and global convergence results over an order interval in Theorem 3.1 of Du (1990) have been strengthened over the whole positive cone. Such extension is new to the literature and non-trivial.

In the following corollary, we relax the restrictive requirement of $\mathbb{K}$ to be reproducing and then replace condition (iii) in Theorem 2.2.1 with an interiority-like condition.

Corollary 2.2.2. *Let* $\mathbb{E}$ *be an ordered real Banach space with the positive cone* $\mathbb{K}$ *in* $\mathbb{E}$*. Suppose that*

(i) $\mathbb{K}$ *is normal and reproducing;*

(ii) *Assumptions 2.2.1 and 2.2.3 hold;*

(iii) *there exists a sufficiently small number* $\varepsilon \in (0,1)$ *such that* $\boldsymbol{A}\mathbf{0} \geqslant \varepsilon(\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h$.

Then we may conclude the following:

(1) **Existence and uniqueness**: *The operator Equation (2.8) has exactly one positive solution in* $\mathbb{K}$*, i.e.,* $\boldsymbol{A}$ *has exactly one positive fixed point* f^* *in* $\mathbb{K}$*. More precisely, this positive fixed point* f^* *is located in* $\left(\mathbf{0},\ (\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h\right]$*, where* $\boldsymbol{I}$ *is the identity operator on* $\mathbb{E}$ *and* $\boldsymbol{B}^+\colon \mathbb{E} \to \mathbb{E}$ *is the majorant of* $\boldsymbol{B}$.

(2) **Convergence of the iterative method**: *The sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of successive approximations converges to the solution,* f^**, for an arbitrary choice of initial point* f_0 *in* $\left[\mathbf{0},\ (\boldsymbol{I}-\boldsymbol{B}^+)^{-1}h\right]$*; that*

is, if $f_n = \boldsymbol{A} f_{n-1}$ $(n \in \mathbb{N})$, then we have that

$$\lim_{n\to\infty} \|f_n - f^*\| = \lim_{n\to\infty} \|\boldsymbol{A}^n f_0 - f^*\| = 0,$$

whenever $f_0 \in \big[\mathbf{0},\, (\boldsymbol{I} - \boldsymbol{B}^+)^{-1}h\big]$.

(3) **Error estimates**: *For all $n \in \mathbb{N}_0$, we have a priori error estimate*

$$\|f_n - f^*\| = \|\boldsymbol{A}^n f_0 - f^*\| \leqslant N \cdot \frac{r^n}{(1-r)^2} \cdot \|\boldsymbol{A}\mathbf{0}\|,$$

for all $f_0 \in [\mathbf{0}, (\boldsymbol{I} - \boldsymbol{B}^+)^{-1}h]$, where N is the normality constant of $\mathbb{K}$ and $r = 1 - \varepsilon$.

Proof of Corollary 2.2.2. Regarding claims (1)–(3), as $\mathbb{K}$ is reproducing, it then follows from Assumption 2.2.3 and hence Corollary 2.2.1 that there exists an upper solution of (2.8), which is $g^* = (\boldsymbol{I} - \boldsymbol{B}^+)^{-1}h$. Given that g^* is an upper solution, it immediately follows from assumption (iii) that the lower solution condition in Theorem 2.1.1 is satisfied.

Then, applying Theorem 3.1 of Du (1990) gives us the stated result. The almost identical proof of claims (1)–(3) has been given in Theorem 2.2.1, so we omit it here. □

Remark 2.2.5. It is noteworthy that Theorem 2.2.1 and Corollary 2.2.2 do not need to impose any continuity or compactness assumptions on the nonlinear operator. Moreover, the condition that the cone $\mathbb{K}$ is normal and solid (and hence reproducing) is automatically satisfied for many Banach spaces.

Since the nonlinear operator we consider transforms from the cone to itself, its admissible, underlying majorant usually maps from the same cone to itself as well. In this connection, we might need an extension of this majorant to satisfy an essential part of Assumption 2.2.3. The next lemma establishes a sufficient condition for the essential part of Assumption 2.2.3 when such underlying majorant is additive and bounded.

Lemma 2.2.3. *Let $\mathbb{E}$ be a real Archimedean Banach lattice which is partially ordered by the positive cone $\mathbb{K}$ of $\mathbb{E}$.*[20] *If $\boldsymbol{B}^+ : \mathbb{K} \to \mathbb{K}$ is a bounded additive mapping in the sense that*

(1) (Additivity) *for all f, g in $\mathbb{K}$, $\boldsymbol{B}^+(f+g) = \boldsymbol{B}^+ f + \boldsymbol{B}^+ g$; and*

(2) (Boundedness) *the operator norm $\|\boldsymbol{B}^+\| := \sup_{\{f\in\mathbb{K}:\|f\|\leqslant 1\}} \|\boldsymbol{B}^+ f\|$ is finite,*

then $\boldsymbol{B}^+$ extends uniquely to a positive bounded linear operator $\overline{\boldsymbol{B}^+}$ mapping from $\mathbb{E}$ to $\mathbb{E}$. In addition, the unique positive linear extension is given by the formula

$$\overline{\boldsymbol{B}^+}(f) = \overline{\boldsymbol{B}^+}(f^+) - \overline{\boldsymbol{B}^+}(f^-), \qquad (f \in \mathbb{E}). \tag{2.11}$$

[20] In fact, a condition that $\mathbb{E}$ is a real Archimedean normed Riesz space should suffice for this lemma. In other words, the completeness of the norm is not necessary.

2.2.1 Candidate Function Spaces

We shall close this chapter by giving some illustrative examples of those cones that are widely used in natural sciences and economic literature as well as in this thesis. In particular, some important properties of those cones are discussed, in order to keep the exposition of the thesis as self-contained as possible.

Example 2.2.1. Let $\mathbb{X}$ be a topological space. Both the vector space $c(\mathbb{X})$ of all continuous real-valued functions and the vector subspace $bc(\mathbb{X})$ of all bounded functions in $c(\mathbb{X})$ are Riesz spaces when the ordering is defined pointwise. Under the usual supremum norm $\|f\|_\infty := \sup\{|f(x)| : x \in \mathbb{X}\}$, $bc(\mathbb{X})$ is a Banach lattice. If $\mathbb{S}$ is a compact space, then the Riesz space $c(\mathbb{S})$ of all continuous real-valued functions on $\mathbb{S}$ under the supremum norm is also a Banach lattice.

By virtue of $f \leqslant g \iff f(x) \leqslant g(x)$ for all $x \in \mathbb{X}$, one can see that the supremum norm is monotonic, from which we know that both the positive cones $bc(\mathbb{X})_+$ of $bc(\mathbb{X})$ and $c(\mathbb{S})_+$ of $c(\mathbb{S})$ are normal and the normal constant $N = 1$. In addition, it is not hard to show that both $bc(\mathbb{X})_+$ and $c(\mathbb{S})_+$ are solid and hence reproducing.

Example 2.2.2. Let $\mathbb{X}$ be a non-empty set and assumed to be separable and metrizable. Let $m(\mathbb{X})$ denote the space of all (Borel) measurable real-valued functions defined on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$. Consider a function $\kappa \in m(\mathbb{X})$ with

$$\kappa(x) \geqslant 1, \qquad \forall x \in \mathbb{X}.$$

Denote by $b_\kappa m(\mathbb{X})$ the space of all (Borel) measurable real-valued functions f on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ such that $f(x)/\kappa(x)$ is bounded as x ranges over $\mathbb{X}$, and consider the *weighted supremum norm* (κ-norm) on this space $b_\kappa m(\mathbb{X})$:

$$\|f\|_\kappa := \sup_{x \in \mathbb{X}} \frac{|f(x)|}{\kappa(x)} = \left\| \frac{f}{\kappa} \right\|_\infty .$$

We refer to these functions with the finite κ-norm as the *κ-bounded* functions on $\mathbb{X}$. Clearly, the finite weighted supremum norm turns the set $b_\kappa m(\mathbb{X}) = \{f \in m(\mathbb{X}) : f \text{ is } \kappa\text{-bounded}\}$ into a real Banach space.[21]

On the other hand, it is clear that $b_\kappa m(\mathbb{X})$ is a Riesz space under the standard pointwise order, and for each element f in $b_\kappa m(\mathbb{X})$, its modulus $|f|$ is given by $|f|(x) = |f(x)|$ for every $x \in \mathbb{X}$. In addition, the κ-norm $\|\cdot\|_\kappa$ is a lattice norm on $b_\kappa m(\mathbb{X})$. To see this, pick any f, g in $b_\kappa m(\mathbb{X})$ with $|f| \leqslant |g|$; that is, $|f(x)| \leqslant |g(x)|$ for every $x \in \mathbb{X}$. Since the weight function κ is strictly positive, we have $|f(x)|/\kappa(x) \leqslant |g(x)|/\kappa(x)$ for each $x \in \mathbb{X}$, which implies that $\|f\|_\kappa \leqslant \|g\|_\kappa$. As a result, the finite weighted supremum norm turns the normed Riesz space $(b_\kappa m(\mathbb{X}), \|\cdot\|_\kappa)$ into a real Banach lattice.

[21] In fact, the finite weighted supremum norm $\|f\|_\kappa$ can be expressed alternatively as $\|f\|_\kappa = \inf\{M > 0 : |f(x)| \leqslant M\kappa(x), \forall x \in \mathbb{X}\} = \inf\{M > 0 : -Mw(x) \leqslant f(x) \leqslant M\kappa(x), \forall x \in \mathbb{X}\}$.

Remark 2.2.6. When the weight function κ is bounded (i.e., $\sup_{x\in\mathbb{X}}\kappa(x) \leqslant M$ for some number $M > 0$.), since $\|f\|_\kappa = \|f/\kappa\|_\infty$, the norms $\|\cdot\|_\kappa$ and $\|\cdot\|_\infty$ are equivalent, and hence $b_\kappa m(\mathbb{X}) = bm(\mathbb{X})$, where $bm(\mathbb{X})$ is the subspace of all bounded functions in $m(\mathbb{X})$. Therefore, the weighted supremum norms become relevant when κ is unbounded.

Let $b_\kappa m(\mathbb{X})_+ := \{f \in b_\kappa m(\mathbb{X}) : f \geqslant \mathbf{0}\}$. Evidently, the set $b_\kappa m(\mathbb{X})_+$ is the positive cone of $(b_\kappa m(\mathbb{X}), \|\cdot\|_\kappa)$.

In addition, since $f \leqslant g \iff f(x) \leqslant g(x)$ for all x in $\mathbb{X}$, it follows that the weighted supremum norm is monotonic, and hence $b_\kappa m(\mathbb{X})_+$ is normal with the normal constant $N = 1$.

Moreover, $b_\kappa m(\mathbb{X})_+$ is a solid cone in $b_\kappa m(\mathbb{X})$. The next lemma shows such a result. Given $f \in b_\kappa m(\mathbb{X})$, we define an infimum operator $[\cdot]$ on $b_\kappa m(\mathbb{X})$ through

$$[f]_\kappa = \inf_{x\in\mathbb{X}} \frac{f(x)}{\kappa(x)}.$$

Lemma 2.2.4. *The positive cone $b_\kappa m(\mathbb{X})_+$ is solid.*

The following lemma is a stronger version concerning the interior of $b_\kappa m(\mathbb{X})_+$.

Lemma 2.2.5.

$$b_\kappa \mathring{m}(\mathbb{X})_+ = \{f \in b_\kappa m(\mathbb{X})_+ : [f]_\kappa > 0\}.$$

Example 2.2.3. Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space. Then, $\mathscr{L}_0(\mu)$ is defined to be the set of equivalence classes of measurable functions from $\mathbb{X}$ to $\mathbb{R}$ that are equal μ-almost everywhere (a.e.). For $0 \leqslant p < \infty$, $\mathscr{L}_p(\mu)$ is defined to be the collection of all (μ-equivalence classes of) μ-measurable functions f for which

$$\|f\|_p := \left\{\int |f(x)|^p \mu(\mathrm{d}x)\right\}^{1/p} < \infty,$$

where the number $\|f\|_p$ is the L_p-*norm* of f. Similarly, the L_∞-norm (or the *essential supremum norm*) of a μ-measurable function $f \colon \mathbb{X} \to \mathbb{R}$ is defined by

$$\|f\|_\infty := \inf\{M > 0 : |f(x)| \leqslant M \text{ for } \mu\text{-almost all } x\}.$$

The collection of all equivalence classes of measurable functions f with $\|f\|_\infty < \infty$ is denoted by $\mathscr{L}_\infty(\mu)$.

In all cases of $0 \leqslant p \leqslant \infty$, the vector space $\mathscr{L}_p(\mu)$ is a Riesz space under the almost everywhere pointwise ordering.[22] Further, by the Riesz-Fischer theorem, for $1 \leqslant p \leqslant \infty$, the Riesz space $\mathscr{L}_p(\mu)$ equipped with the L_p-norm is a Banach lattice.

Since $f \leqslant g \iff f(x) \leqslant g(x)$ for almost all x, the L_p-norm is monotonic. Then for $1 \leqslant p \leqslant \infty$, the set $\mathscr{L}_p(\mu)_+ := \{f \in \mathscr{L}_p(\mu) : f(x) \geqslant 0,\ a.e.\ x \in \mathbb{X}\}$ is a normal positive cone, and its

[22] That is, $f \leqslant g$ in $\mathscr{L}_p(\mu)$ whenever $f(x) \leqslant g(x)$ for μ-almost every x.

normal constant $N = 1$. It is clear that $\mathscr{L}_p(\mu)_+$ is not a solid cone in $\mathscr{L}_p(\mu)$ for $1 \leqslant p < \infty$, while the positive cone $\mathscr{L}_\infty(\mu)_+$ is solid.[23] However, for $1 \leqslant p < \infty$, the positive cone $\mathscr{L}_p(\mu)_+$ is reproducing and fully regular (and so it is also regular). In addition, $\mathscr{L}_p(\mu)_+$ (with $1 \leqslant p < \infty$) is strongly minihedral.

[23] For more details of verifying a solid cone, interested readers may refer to Krasnosel'skiĭ (1964), Deimling (1985) or Guo et al. (2004).

Chapter 3

Solving Recursive Utilities in a Markovian Environment

3.1 Introduction

The first essay is devoted to the study of a specific class of recursive utilities proposed by Epstein and Zin (1989) and its extension associated with narrow framing proposed by Barberis et al. (2006) in the context of a Markov environment. It provides sufficient conditions for existence and uniqueness results, as well as global convergence of successive approximation, for the solutions to these recursive utility models. It is noteworthy that the recoverability of a recursive utility function does not rely on the Contraction Mapping Principle. Instead, an alternative approach through this essay is built upon the theory of monotone concave operators as introduced in Chapter 2.

Following the seminal contributions of Hansen and Scheinkman (2012), we go a step further and develop an analogous argument for the Epstein–Zin recursive utility models. In terms of results, the strengths of our approach are as follows: First, in the same general Markovian setting, while Hansen and Scheinkman (2012) show a somewhat nonconstructive proof of the existence of continuation value processes, we exploit the regularity of the positive cones of L_p spaces and then adapt an existence theorem on monotone operators to provide constructive proof for the existence of continuation value processes. Second, we obtain uniqueness of the solution for a broader range of parameterizations of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) that differ from unity, which is not fully considered in Hansen and Scheinkman (2012).[1] By exploiting the concavity of the relevant nonlinear operators, we adapt the fixed point theorem (stated in Chapter 2) on monotone concave operators to prove uniqueness and an iterative method of continuation value processes. It is clear that without uniqueness, Hansen and Scheinkman (2012) may only show that the continuation value process is locally attracting, in the sense that it can be computed merely by starting from

[1] In fact, they only obtain a uniqueness result for the case where $(1 - RRA)/(1 - 1/EIS) \geqslant 1$.

certain specific initial guesses. Hence, it is desirable to prove a constructive version of existence, uniqueness and global convergence of iteration results in order to provide systematic analysis for continuation value processes. Third, our approach is relatively general, allowing for unbounded, stationary and non-stationary consumption growth specifications.

Compared with the work of Borovička and Stachurski (2017), their analysis is mainly based on a fixed point argument closely related to the one we use, but is focused on the special case where the state space is restricted to be compact. In addition, on the one hand, they derive a set of remarkable conditions that are necessary as well as sufficient, both for existence and for uniqueness, while our approach can only provide sufficient conditions. On the other hand, under the same auxiliary conditions as in Borovička and Stachurski (2017), we additionally obtain results for existence, uniqueness and convergence of solutions to risk-sensitive models, while it is not considered in their work. Moreover, as an extension of this essay, we present how our approach can also tackle both the risk-sensitive model and the narrow framing model with consumption specifications analogous to those in Borovička and Stachurski (2017). Compared with the results of Guo and He (2018), our compactness assumption on the state space is relatively weaker than theirs and we provide a constructive and easy-to-check condition for the existence, uniqueness and globally attracting properties of solutions to these models.

The structure of the essay is as follows. Section 3.2 introduces some set-up and basic notations, and Section 3.2.3 discusses some underlying properties of certainty equivalent operators. Section 3.3 states the main results about solving Epstein–Zin utility processes with unbounded stationary and non-stationary consumption specifications. Section 3.4 provide an analysis of solving risk-sensitive recursion and narrow framing under the compactness restriction on state spaces.

3.2 Preliminaries and Basic Notations

In this section, we set basic notations and fix basic assumptions related to the state space and state process that will be in force for the remainder of this chapter.

3.2.1 Uncertainty and Information

Let $\mathbb{X}$ be a non-empty set, referred to hereafter as the *state space* and assumed to be Polish.[2] Let $\mathscr{B}(\mathbb{X})$ be the Borel sets of $\mathbb{X}$ and let Q be a stochastic kernel on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$, with $Q(x, B)$ understood as the transition probability that the state satisfies $X_{t+1} \in B$ given $X_t = x$.[3] In this chapter, we usually use the notation $\mathscr{B}$ for $\mathscr{B}(\mathbb{X})$ when no confusion is to be feared.

[2] That is, $\mathbb{X}$ is separable and completely metrizable. The Polish assumption is very weak, and used primarily to avoid measurability concerns.

[3] That is, Q is a function from $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ into $[0, 1]$ such that $x \mapsto Q(x, B)$ is $\mathscr{B}$-measurable for every $B \in \mathscr{B}(\mathbb{X})$ and $B \mapsto Q(x, B)$ is a probability measure on $\mathscr{B}(\mathbb{X})$ for every $x \in \mathbb{X}$. These definitions are standard. See, for example, Aliprantis and Border (2006) or Çinlar (2010).

In particular, the *state process* is a discrete time Markov process $\{X_t\}$ indexed over $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, which is induced by the transition (stochastic kernel) Q for a given initial probability measure P_0, and is defined on the canonical probability space $(\Omega, \mathscr{F}, \mathrm{P})$ and taking values in $\mathbb{X}$, where $\Omega = \mathbb{X}^\infty$ is the product space endowed with its σ-algebra $\mathscr{F} = \mathscr{B}^\infty$, and $\mathrm{P}\colon \mathscr{F} \to [0,1]$ is a countably additive probability measure. As usual, a filtration $\{\mathscr{F}_t\}_{t\in\mathbb{N}_0}$ is generated so that, at every time $t \in \mathbb{N}_0$, $\mathscr{F}_t$ is the least σ-algebra on Ω with respect to which the natural projection $x^t\colon \Omega \to \mathbb{X}^{t+1}$ is $\mathscr{B}^{t+1}$-measurable, where

$$x^t(\omega) := x^t(x_0, x_1, \ldots, x_t, x_{t+1}, \ldots) = (x_0, x_1, \ldots, x_t) \qquad (\omega \in \Omega)$$

Clearly, information is modeled through this increasing filtration $\{\mathscr{F}_t\}_{t\in\mathbb{N}_0}$ of σ-algebras contained in $\mathscr{F}$. This means that the conditional expectation

$$\mathrm{E}^{\mathrm{P}}[g(X_{t+1}) \mid \mathscr{F}_t] = \mathrm{E}_t\, g(X_{t+1}) = \int g(x')Q(X_t, \mathrm{d}x')$$

for every (essentially) bounded measurable $g\colon \mathbb{X} \to \mathbb{R}$ and every $t \in \mathbb{N}_0$.[4]

We say that probability measure π on $\mathscr{B}$ is *stationary* for Q if

$$\int Q(x, B)\pi(\mathrm{d}x) = \pi(B) \text{ for all } B \in \mathscr{B}.$$

3.2.2 Candidate Functions Spaces

Let $m(\mathbb{X})$ denote the space of all (Borel) measurable real-valued functions defined on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$. Given a function $\kappa \in m(\mathbb{X})$ with $\kappa(x) > 0$ for all $x \in \mathbb{X}$, we denote by $b_\kappa m(\mathbb{X})$ the space of all (Borel) measurable κ-bounded real-valued functions defined on $\mathbb{X}$. Furthermore, the positive cone $b_\kappa m(\mathbb{X})_+$ of $(b_\kappa m(\mathbb{X}), \|\cdot\|_\kappa)$ is defined naturally as

$$b_\kappa m(\mathbb{X})_+ := \{f \in b_\kappa m(\mathbb{X}) : f \geqslant \mathbf{0}\}.$$

Clearly, $b_\kappa m(\mathbb{X})$ is a real Banach lattice and more properties of this space can be referred to Example 2.2.2 in Chapter 2.

With π as the common marginal distribution of X_t, we take $\mathscr{L}_0(\pi)$ to be the vector space of equivalence classes of $\mathscr{B}$-measurable functions from $\mathbb{X}$ to $\mathbb{R}$ that are equal π-almost everywhere. Then we define

$$\mathscr{L}_p(\pi) := \text{ all } f \in \mathscr{L}_0(\pi) \text{ such that } \|f\|_p := \left\{\int |f(x)|^p \pi(\mathrm{d}x)\right\}^{1/p} < \infty.$$

Recalling Example 2.2.3 in Chapter 2, when $1 \leqslant p < \infty$, the pair $(\mathscr{L}_p(\pi), \|\cdot\|_p)$ is an ordered real Banach space contained in $\mathscr{L}_0(\pi)$. When endowed with the essential sup-norm $\|\cdot\|_\infty$, the

[4]In other words, the operator $\mathrm{E}_t := \mathrm{E}^{\mathrm{P}}[\ \cdot\ \mid \mathscr{F}_t\,]$ is expectation conditional on time t information with respect to $\mathscr{F}_t$, the σ-algebra generated by $X_0, \ldots, X_t$.

subspace $\mathscr{L}_\infty(\pi)$ of equivalence classes of essentially bounded measurable functions in $\mathscr{L}_0(\pi)$ is also an ordered real Banach space.

3.2.3 The Kreps–Porteus Certainty Equivalent Operator

Under a Markov environment setting, a class of Kreps–Porteus certainty equivalent operators as defined in (1.1) can be alternatively expressed as a *non-additive Markov operator* $\mathcal{M}\colon m(\mathbb{X})_+ \to m(\mathbb{X})_+$ in a form of

$$(\mathcal{M}f)(x) = \phi^{-1}\left(\int_{\mathbb{X}} \phi \circ f(x')Q(x, \mathrm{d}x')\right) \qquad (x \in \mathbb{X}),$$

where ϕ is a given strictly monotonic continuous real-valued function on some interval $I \subset \mathbb{R}_+$.

This non-additive Markov operator plays a central role in our analysis of recursive utility. There are two elementary properties of the non-additive Markov operator $\mathcal{M}$ defined above, which can be found in Marinacci and Montrucchio (2010), Balbus (2016) , Bloise and Vailakis (2018) and references cited therein.

(P1) For any nonnegative constant function d defined on $\mathbb{X}$, $\mathcal{M}(d) = d$.

(P2) $(\mathcal{M}f)(x) = (\mathcal{M}g)(x)$ whenever $\int \mathbb{1}_{\{y\in\mathbb{X}\colon f(y)=g(y)\}}Q(x, \mathrm{d}y) = 1$, for every nonnegative measurable function f and g on $\mathbb{X}$.[5]

In addition to basic properties (P1) and (P2), we aim to explore more properties of $\mathcal{M}$ by narrowing down a strictly monotone function ϕ to some more specific classes of functions. Our following results are established when we consider an abstract function ϕ in a power form of $\phi(s) := s^\theta$ with $0 \neq \theta < 1$.[6] That is, the non-additive Markov operator $\mathcal{M}\colon m(\mathbb{X})_+ \to m(\mathbb{X})_+$ corresponding to the exponent θ is defined through

$$(\mathcal{M}_\theta f)(x) := \left(\int_{\mathbb{X}} f(x')^\theta Q(x, \mathrm{d}x')\right)^{1/\theta}. \tag{3.1}$$

In order to eliminate some ambiguity of $\mathcal{M}_\theta$ arising when $\theta < 0$ and $f(x) = 0$ for some $x \in \mathbb{X}$, we adopt a convention that $0^\theta := \infty$ and $\infty^\theta := 0$. As a result, $\mathcal{M}_\theta$ is well defined for any nonnegative $\mathscr{B}$-measurable function f on $\mathbb{X}$. In addition, when $\theta < 0$ and for any nonnegative $\mathscr{B}$-measurable function f taking zero with positive probability, we have $\mathcal{M}_\theta(f) = \mathbf{0}$. In particular, this conventional extension leads naturally to a more consistent result of (P1) (i.e., $\mathcal{M}_\theta(\mathbf{0}) = \mathbf{0}$), regardless of the value of exponent θ.

With this convention, we are ready to state the following lemma.

Lemma 3.2.1. *If $\theta \in (-\infty, 0) \cup (0, 1)$, then the non-additive Markov operator $\mathcal{M}_\theta$ defined in (3.1) is super-additive.*

[5] That is, $f, g \in \mathscr{L}_0(\pi)_+$ given that the probability measure π on $\mathscr{B}$ which is stationary for Q exists.

[6] Here we temporarily exclude the case $\theta = 0$, since it will lead us to a logarithmic form of ϕ.

Lemma 3.2.1 follows directly from theorem **198** of Hardy et al. (1934), given that $Q(x, B)$ is a probability measure. In fact, this result can be also reviewed as the reversed Minkowski's Inequality.[7]

In addition, it is easy to see that the operator $\mathcal{M}_\theta$ defined in (3.1) is positively homogeneous in the sense of (2.6). Inspired by this fact, the following lemma provides a sufficient condition for an order concave operator.

Lemma 3.2.2. *If $\theta \in (-\infty, 0) \cup (0, 1)$, then $\mathcal{M}_\theta$ is order concave on $m(\mathbb{X})_+$.*

Proof. To see this, pick any f and g in $m(\mathbb{X})_+$ and any $\lambda \in (0, 1)$. Consider their convex combination $h := \lambda f + (1-\lambda)g$. Clearly, by the convexity of $m(\mathbb{X})_+$, h lies in $m(\mathbb{X})_+$, and we then have that

$$\begin{aligned}
\mathcal{M}_\theta(h) &= \mathcal{M}_\theta\left[\lambda f + (1-\lambda)g\right] \\
&\geqslant \mathcal{M}_\theta\left(\lambda f\right) + \mathcal{M}_\theta\left[(1-\lambda)g\right] \\
&= \lambda \mathcal{M}_\theta\left(f\right) + (1-\lambda)\mathcal{M}_\theta\left(g\right),
\end{aligned}$$

where the first inequality follows from the super-additivity of $\mathcal{M}_\theta$, as implied by Lemma 3.2.1, while the last equality from the positive homogeneity of $\mathcal{M}_\theta$. □

3.3 Applications

We now turn to applications of the recursive utility proposed in Epstein and Zin (1989). In the following, both stationary and non-stationary consumption processes will be considered. In each case, we shall study the existence, uniqueness and convergence of the iteration method for the Epstein–Zin recursive utility process. Throughout, the state process and notation are as specified in Section 3.2.1.

3.3.1 Unbounded Stationary Consumption

Beginning with the stationary case, let the state space $\mathbb{X}$ be Polish and let consumption be given by $C_t = c(X_t)$, where $c\colon \mathbb{X} \to \mathbb{R}$ is nonnegative and $\mathscr{B}$-measurable. Substituting (1.1) with (1.2) into (1.3) and using $C_t = c(X_t)$, we get

$$U_t^{1-\rho} = (1-\beta)c(X_t)^{1-\rho} + \beta\left(\mathrm{E}_t U_{t+1}^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}}. \tag{3.2}$$

Letting $\theta := (1-\gamma)/(1-\rho)$ and $\widehat{U}_t := U_t^{1-\rho}$, we can rewrite (3.2) as

$$\widehat{U}_t = (1-\beta)c(X_t)^{1-\rho} + \beta\left(\mathrm{E}_t \widehat{U}_{t+1}^{\theta}\right)^{1/\theta}. \tag{3.3}$$

[7] See, for example, DiBenedetto (2002).

We aim to seek a Markov solution $\widehat{U}_t = f(X_t)$ for some $f\colon \mathbb{X} \to \mathbb{R}_+$.

Before doing so, we make the following assumptions.

Assumption 3.3.1. There exist a continuous function $\kappa\colon \mathbb{X} \to (0, \infty)$ and nonnegative constants $d_\kappa \in (0,\, 1/\beta)$ such that

$$\int_{\mathbb{X}} \kappa(x') Q(x, \mathrm{d}x') \leqslant d_\kappa \kappa(x), \qquad \forall\, x \in \mathbb{X}. \tag{3.4}$$

Assumption 3.3.2. Given the function κ that is defined in Assumption 3.3.1, the consumption function $c\colon \mathbb{X} \to \mathbb{R}_+$ is $\mathscr{B}$-measurable, and there exist positive constants $\hat{d}_c < \infty$ and $\check{d}_c \in (0,\, \hat{d}_c)$ such that

$$\check{d}_c \kappa(x) \leqslant c(x)^{1-\rho} \leqslant \hat{d}_c \kappa(x), \qquad \forall\, x \in \mathbb{X}. \tag{3.5}$$

Remark 3.3.1. It is worth noting that stating $c(x)^{1-\rho} \geqslant \check{d}_c \kappa(x)$ for all $x \in \mathbb{X}$ and for some $0 < \check{d}_c < \hat{d}_c$ in (3.5) is equivalent to saying that $\inf_{x \in \mathbb{X}} c(x)^{1-\rho}/\kappa(x) \geqslant \check{d}_c > 0$. It can be reviewed as an interiority condition of consumption, and such an assumption has been commonly imposed in Marinacci and Montrucchio (2010) and Becker and Rincón-Zapatero (2017), etc.

Now, we are ready to solve the functional equation of the original problem (3.3)

$$f(x) = (1-\beta) c(x)^{1-\rho} + \beta \left(\int_{\mathbb{X}} f(x')^\theta Q(x, \mathrm{d}x') \right)^{1/\theta} \qquad (x \in \mathbb{X}), \tag{3.6}$$

along with the iteration method

$$f_{n+1}(x) = (1-\beta) c(x)^{1-\rho} + \beta \left(\int_{\mathbb{X}} f_n(x')^\theta Q(x, \mathrm{d}x') \right)^{1/\theta} \quad (x \in \mathbb{X};\, n = \mathbb{N}_0). \tag{3.7}$$

Proposition 3.3.1. *Let $0 \neq \theta < 1$. Suppose that Assumptions 3.3.1 and 3.3.2 are satisfied. Then, the following hold true:*

(i) Existence and uniqueness*: The functional Equation (3.6) has a unique solution $f^* \in b_\kappa m(\mathbb{X})_+$.*

(ii) Unique global attractivity of the fixed point*: For any initial seed $f_0 \in b_\kappa m(\mathbb{X})_+$, the iterative sequence $\{f_n\}_{n \in \mathbb{N}_0}$ constructed by (3.7) converges to f^* in $b_\kappa m(\mathbb{X})_+$.*

(iii) Convergence of the iterative method and error estimates*: For all $n \in \mathbb{N}_0$, we get a priori error estimate:*

$$\|f_n - f^*\|_\kappa \leqslant r^n (1-r)^{-2} \|f_1\|_\kappa.$$

for each $f_0 \in [\mathbf{0},\, g^*]$ and for some $r \in (0, 1)$, where g^* is an upper solution of (3.6).

In addition, if $\underline{f}_0$ is a lower solution and $\overline{f}_0$ is an upper solution of (3.6) with $\underline{f}_0 \leqslant \overline{f}_0$ on $b_\kappa m(\mathbb{X})_+$, then we have the error estimates $\underline{f}_n \leqslant f^* \leqslant \overline{f}_n$ on $b_\kappa m(\mathbb{X})_+$ for all $n \in \mathbb{N}_0$.

Remark 3.3.2. The preceding proposition utilizes the general fixed point results (built in Chapter 2) to solve the problem for the case in which $0 \neq \theta < 1$. The remaining case $\theta \geqslant 1$ has been extensively studied in existing literature by making use of the Contraction Mapping Principle (see, e.g., Epstein and Zin (1989), Marinacci and Montrucchio (2010) etc.).[8]

3.3.2 Non-stationary Consumption

Now let us turn to the case of non-stationary consumption. We still aim to seek a unique solution—a relevant function of the state—to the corresponding operator equation. Because the candidate function space is not solid, and the relevant operator is not necessarily a contraction, the usual fixed point theorems based on solid positive cones or contraction mappings do not apply. Instead, we will apply a theorem on monotone operators defined on the regular cones to prove the existence of at least one fixed point, and then adapt a theorem on concave operators to prove uniqueness.

Following Hansen and Scheinkman (2012), the normalized solution for recursive preferences satisfies

$$\left(\frac{U_t}{C_t}\right)^{1-\rho} = \zeta^{1-\rho} + \exp(-\delta)\left\{\mathcal{R}_t\left(\frac{U_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)\right\}^{1-\rho}, \tag{3.8}$$

where $\delta > 0$ and $\zeta > 0$ are preference parameters, and $\mathcal{R}_t$ is still the certainty equivalence operator defined as in (1.1) with (1.2). The parameters ρ and γ have the same definitions and parameter restrictions as given in Section 3.3.

As before we set $\theta := (1-\gamma)/(1-\rho)$ and wish to seek a Markovian solution $(U_t/C_t)^{1-\rho} := f(X_t)$ of (3.8) for some nonnegative $\mathscr{B}$-measurable function f on $\mathbb{X}$. In order to provide a close comparison, we follow Hansen and Scheinkman (2012) by invoking a change of measure so as to translate the recursive utility problem (3.8) into a fixed point problem associated with the functional equation

$$g(x) := \zeta^{1-\rho}e^{-1/\theta}(x) + \exp(-\xi)\left\{\tilde{\mathrm{E}}\left[g(X_{t+1})^{\theta} \mid X_t = x\right]\right\}^{1/\theta}, \tag{3.9}$$

(cf. Hansen and Scheinkman, 2012, p. 11968), where $\xi := \delta - \eta/\theta$. Moreover, $\exp(\eta)$ and $e(x) > 0$ are the Perron-Frobenius eigenvalue and the corresponding principal eigenfunction of the operator $\boldsymbol{P}$ defined by

$$(\boldsymbol{P}f)(x) := \mathrm{E}\left\{f(X_{t+1})\exp\left[(1-\gamma)\kappa(X_{t+1}, X_t, Y_{t+1})\right] \mid X_t = x\right\},$$

respectively. The change of measure from E to $\tilde{\mathrm{E}}$ is based on Perron-Frobenius theory and described in detail in Hansen and Scheinkman (2009, 2012). As a result, if g solves (3.9), then setting $f(x) := g(x)e(x)^{1/\theta}$ and $U_t = C_t f(X_t)^{1/(1-\rho)}$ yields a solution to (3.8). Hence the remaining study will concentrate on the solvability of (3.9).

To do so, we make a piece of notation first. Let $\tilde{Q}(x, \mathrm{d}x')$ be the stochastic kernel corresponding

[8] The key idea to solve this case is that the corresponding certainty equivalent operators possess subadditivity and positive homogeneity, which allow us to apply the weighted contraction mapping theorem.

$\tilde{\mathrm{E}}$ and let $\tilde{\pi}$ be a stationary distribution of $\tilde{Q}$. Then we write (3.9) into the nonlinear operator equation with functional notation as

$$g(x) = (\boldsymbol{T}g)(x) = h(x) + \exp(-\xi)\left\{\int g(x')^{\theta}\tilde{Q}(x, \mathrm{d}x')\right\}^{1/\theta} \quad (x \in \mathbb{X}), \tag{3.10}$$

where $h(x) := \zeta^{1-\rho} e^{-1/\theta}(x)$. If we further define a corresponding nonlinear operator $\tilde{\mathcal{M}}_{\theta}$ acting on all $\mathscr{B}$-measurable functions $g\colon \mathbb{X} \to \mathbb{R}_{+}$ via

$$\left(\tilde{\mathcal{M}}_{\theta} g\right)(x) := \left\{\int g(x')^{\theta}\tilde{Q}(x, \mathrm{d}x')\right\}^{1/\theta} \quad (x \in \mathbb{X}), \tag{3.11}$$

then in operator notation, a solution g of (3.10) translates to a fixed point of the operator $\boldsymbol{T} = h + \beta\tilde{\mathcal{M}}_{\theta}$, where $\beta := \exp(-\xi)$. It is clear that the form of the functional operator Equation (3.10) is now essentially identical to (2.8) with $\boldsymbol{B} := \beta\tilde{\mathcal{M}}_{\theta}$. Due to our candidate functions space in the present context may not possess a solid positive cone, so we shall aim to apply Corollary 2.2.2 to the operator $\boldsymbol{T}$ defined in (3.10).

Before doing so, we make the following assumptions.

Assumption 3.3.3. Let $\{(X_t, Y_t)\}_{t\in\mathbb{N}_0}$ be an underlying Markov process and satisfy the following conditions:

(a) The joint distribution of (X_{t+1}, Y_{t+1}) conditioned on (X_t, Y_t) depends only on X_t.

(b) Consumption dynamics evolve as $\log C_{t+1} - \log C_t = \tau(X_{t+1}, Y_{t+1}, X_t)$ for some real-valued function τ.

Assumption 3.3.4. Under the change of probability measure,

$$\lim_{t\to\infty} \tilde{\mathrm{E}}\left[\psi\left(X_t, Y_t\right) | X_0 = x\right] = \tilde{\mathrm{E}}\left[\psi\left(X_t, Y_t\right)\right]$$

for any bounded Borel-measurable function ψ. The expectation on the right-hand side uses a stationary distribution implied by the change in the transition distribution. We require that the convergence applies for almost all Markov states x under this stationary distribution.

Assumption 3.3.5. The parameter $\xi = \delta - \eta/\theta$ is strictly positive.

Assumption 3.3.6. $\tilde{\mathrm{E}}[e\left(x\right)^{-\frac{1}{\theta}}] < \infty$.

Assumption 3.3.7. $\tilde{\mathrm{E}}[e\left(x\right)^{-1}] < \infty$.

It is worth noticing that in order to be consistent with Hansen and Scheinkman's work, Assumptions 3.3.3 to 3.3.7 are the exactly same as Assumptions **1** and **5** in Hansen and Scheinkman (2012).

When we aim to seek a fixed point of $\boldsymbol{T}$ when $\theta \geqslant 1$, inspired by the work of Hansen and Scheinkman (2012) and Borovička and Stachurski (2017), it is technically convenient to introduce an auxiliary operator $\boldsymbol{S}$ defined on $\mathscr{L}_1(\tilde{\pi})_+$ given by

$$\boldsymbol{S}\ell(x) = \left\{ h(x) + \exp(-\xi) \left[\int \ell(x')\tilde{Q}(x, \mathrm{d}x') \right]^{1/\theta} \right\}^{\theta} \qquad (x \in \mathbb{X}) \tag{3.12}$$

with $h(x) = \zeta^{1-\rho} e^{-1/\theta}(x)$ defined as before. It is worth noting that $\boldsymbol{S}\ell \equiv \{\boldsymbol{T}(\ell^{1/\theta})\}^{\theta}$, from which we observe that there is a one-to-one correspondence between fixed points of $\boldsymbol{T}$ and fixed points of $\boldsymbol{S}$ for $\theta \geqslant 1$. In this case, invoking operator Equations (3.10) and (3.12) with $\ell(x) := g(x)^{1/\theta}$, we conclude that g is a fixed point of $\boldsymbol{T}$ in $\mathscr{L}_1(\tilde{\pi})_+$ if and only if ℓ is a fixed point of $\boldsymbol{S}$ in $\mathscr{L}_1(\tilde{\pi})_+$.

The following proposition reconstructs results for the existence of solutions to Epstein–Zin recursive utility process as studied in Hansen and Scheinkman (2012) without changing any assumptions therein.

Proposition 3.3.2. (Existence) *Suppose that*

(a) Assumption 3.3.3 holds,

(b) $e > \mathbf{0}$ is a solution to the Perron-Frobenius equation $\boldsymbol{P}e(x) = \exp(\eta)e(x)$ with $\exp(\eta)$ the associated eigenvalue, and satisfies Assumption 3.3.4,

(c) the subjective rate of discount satisfies Assumption 3.3.5.

Then for alternative ranges of the parameter θ, we have the following result:

(i) If $0 \neq \theta \leqslant 1$ and, additionally, Assumption 3.3.6 holds, then the operator $\boldsymbol{T}$ defined in (3.10) has a minimal fixed point g_* and a maximal fixed point g^* in $\mathscr{L}_1(\tilde{\pi})_+$.

(ii) If $\theta \geqslant 1$ and, additionally, Assumption 3.3.7 holds, then the operator $\boldsymbol{T}$ defined in (3.10) has a minimal fixed point g_* and a maximal fixed point g^* in $\mathscr{L}_1(\tilde{\pi})_+$.

Assumption 3.3.8. Denote by $\hat{g}_1(x) := \sum_{t=0}^{\infty} \exp(-t\xi)\tilde{\mathrm{E}}[e(X_t)^{-1/\theta} \,|X_0 = x]$ for all $x \in \mathbb{X}$. There exists a sufficiently small positive number ε such that

$$e^{-\frac{1}{\theta}} \geqslant \varepsilon \hat{g}_1, \qquad (\tilde{\pi}\text{-a.e.}).$$

Assumption 3.3.9. Denote by $\hat{g}_2(x) := \sum_{t=0}^{\infty} \exp(-t\xi)\tilde{\mathrm{E}}[e(X_t)^{-1} \,|X_0 = x]$ for all $x \in \mathbb{X}$. There exists a sufficiently small positive number ε such that

$$e^{-1} \geqslant \varepsilon \hat{g}_2, \qquad (\tilde{\pi}\text{-a.e.}).$$

Remark 3.3.3. The preceding two assumptions that we additionally posit can be reviewed as an interiority-like assumption with respect to the Perron-Frobenius eigenfunction e. In other words, these two assumptions imply that the scaled principle eigenfunction $e^{-1/\theta}$ (*resp.* e^{-1}) and the corresponding function $\hat{g}_1$ (*resp.* $\hat{g}_2$) are comparable elements in $\mathscr{L}_1(\tilde{\pi})$.

Proposition 3.3.3. (Uniqueness for $0 \neq \theta \leqslant 1$) *Suppose that Assumptions 3.3.3 through 3.3.5 hold. In addition, we assume that Assumptions 3.3.6 and 3.3.8 hold. Then we may conclude the following:*

(i) Existence and uniqueness*: The operator Equation* (3.10) *has a unique solution* $g^* \in \mathscr{L}_1(\tilde{\pi})_+$.

(ii) Convergence of the iterative method*: For any initial condition* $g_0 \in [\mathbf{0}, g^{+*}]$ *with* g^{+*} *being the fixed point of a majorant operator for* $\boldsymbol{T}$*, the iterative sequence* $\{g_n\}_{n\in\mathbb{N}_0}$ *constructed successively by* (3.10) *converges to* g^* *in* $[\mathbf{0}, g^{+*}]$.

(iii) Error estimate*: For all* $n \in \mathbb{N}_0$*, we get a priori error estimate:*

$$\|g_n - g^*\| \leqslant r^n(1-r)^{-2}\|g_1\|.$$

for each $g_0 \in [\mathbf{0}, g^{+*}]$ *and for some* $r \in (0,1)$*, where* g^{+*} *is the fixed point of a majorant operator for* $\boldsymbol{T}$.

Proposition 3.3.4. (Uniqueness for $\theta \geqslant 1$) *Suppose that Assumptions 3.3.3 through 3.3.5 hold. In addition, we assume that Assumptions 3.3.7 and 3.3.9 hold. Then we may conclude the following:*

(i) Existence and uniqueness*: The operator Equation* (3.12) *has a unique solution* $\ell^* \in \mathscr{L}_1(\tilde{\pi})_+$.

(ii) Convergence of the iterative method*: For any initial condition* $\ell_0 \in [\mathbf{0}, \ell^{+*}]$ *with* ℓ^{+*} *being the fixed point of a majorant operator for* $\boldsymbol{S}$*, the iterative sequence* $\{\ell_n\}_{n\in\mathbb{N}_0}$ *constructed successively by* (3.12) *converges to* ℓ^* *in* $[\mathbf{0}, \ell^{+*}]$.

(iii) Error estimate*: For all* $n \in \mathbb{N}_0$*, we get a priori error estimate:*

$$\|\ell_n - \ell^*\| \leqslant r^n(1-r)^{-2}\|\ell_1\|.$$

for each $\ell_0 \in [\mathbf{0}, \ell^{+*}]$ *and for some* $r \in (0,1)$*, where* ℓ^{+*} *is the fixed point of a majorant operator for* $\boldsymbol{S}$.

Remark 3.3.4. As can be seen from the proofs of Propositions 3.3.3 and 3.3.4 in Appendix A.2, such underlying majorant operators for $\boldsymbol{T}$ and $\boldsymbol{S}$ are able to be linear operators and both of them are easy to find and solve.

Remark 3.3.5. It is worth noting that Assumption 3.3.9 may not be necessary to the case for $\theta \geqslant 1$. In fact, as shown in some existing literature, when $\theta \geqslant 1$, Banach's Contraction Mapping Principle can apply to the operator $\boldsymbol{T}$ directly. In the present context, an analogous proof by using contraction mapping arguments is provided in Appendix A.2 (see Lemma A.2.1). Recently, in Proposition 6-(iii) of Guo and He (2018), the authors prove contraction mapping for a transformed operator $\boldsymbol{S}$ in the case of $\theta \in (0,1)$.

Remark 3.3.6. It is also worth noticing that Assumptions 3.3.8 and 3.3.9 hold automatically when we confine our analysis to the settings of Borovička and Stachurski (2017) and hence can be dispensed with under such an environment. As a consequence, we can obtain the essentially identical result as in Borovička and Stachurski (2017), except that the necessary conditions for existence and uniqueness are not available in the present chapter.[9]

3.4 Extensions

This section gives several examples of how the methodology proposed above can be essentially applied. The applications we consider are risk-sensitive models and the extension of Epstein–Zin recursive preferences models that incorporate loss aversion and narrow framing. To simplify the analysis, we confine ourselves to the compact state space and then follow the settings that are used in Borovička and Stachurski (2017). In this context, the existence, uniqueness and convergence of solutions to each of the aforementioned applications are studied.

3.4.1 Risk-Sensitive Model

So far we have abstracted the Epstein–Zin recursive utility specifications from the case $\rho = 1$, but now let us turn to the case in which $\rho = 1$ and $\gamma > 1$, and make the following assumptions.[10]

Assumption 3.4.1. The state process $\{X_t\}$ is time homogeneous and Markovian, taking values in some compact metric space $\mathbb{X}$. Henceforth, the stochastic kernel for $\{X_t\}$ is still denoted by Q.

The innovation process $\{Y_t\}$ is IID, independent of $\{X_t\}$, and takes values in some topological space $\mathbb{Y}$. Henceforth, the common distribution of each Y_t is a Borel probability measure on $\mathbb{Y}$ denoted by ν.

Assumption 3.4.2. Consumption dynamics evolve as

$$\ln C_{t+1} - \ln C_t = \tau\left(X_{t+1}, Y_{t+1}, X_t\right)$$

for some continuous real-valued function τ.

Invoking the specification of the aggregator in (1.3), we obtain the associated normalized solution for recursive preferences such that

$$\frac{U_t}{C_t} = \left\{\left[\mathrm{E}_t\left(\frac{U_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\right\}^{\beta},$$

[9] The advantage of our approach in this chapter is that we do not restrict the state space to be compact. Although the technical condition in Assumptions 3.3.8 and 3.3.9 may be nontrivial to test, our propositions still show an insightful result for unbounded theoretical models with non-compact states.

[10] Here we mainly focus on the application-oriented case—$\gamma > 1$, since in most financial and economic applied studies the coefficient of relative risk aversion γ is estimated to be strictly above 1. In addition, the case of $\gamma \in (0, 1]$ may raise some technical difficulties in finding an appropriate majorant linear operator for our targeted nonlinear operators.

and it follows from the consumption specification made in Assumption 3.4.2 that

$$\frac{U_t}{C_t} = \left\{ \left[\mathrm{E}_t \left(\frac{U_{t+1}}{C_{t+1}} \exp \left[\tau \left(X_{t+1}, Y_{t+1}, X_t \right) \right] \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right\}^{\beta}.$$

As before, given the Markov dynamics, we seek a solution $U_t / C_t =: f(X_t)$ for some function $f \colon \mathbb{X} \to \mathbb{R}_+$. In terms of the Markov solution $f(X_t)$, the above sequential equation can be expressed as

$$f(x) = \left\{ \left[\int f(x')^{1-\gamma} \int \exp \left[(1-\gamma) \tau(x', y', x) \right] \nu(\mathrm{d}y') Q(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} \right\}^{\beta}, \tag{3.13}$$

in a functional formation.

Inspired by the work of Hansen and Scheinkman (2012) and Borovička and Stachurski (2017), we consider an operator $\boldsymbol{K}$ defined on $c(\mathbb{X})$ by

$$\boldsymbol{K}\ell(x) = \int \ell(x') \int \exp \left[(1-\gamma) \tau(x', y', x) \right] \nu(\mathrm{d}y') Q(x, \mathrm{d}x'), \qquad (x \in \mathbb{X}). \tag{3.14}$$

Evidently, the operator $\boldsymbol{K}$ is linear and monotone increasing (and thus positive).

Assumption 3.4.3. The operator $\boldsymbol{K}$ defined in (3.14) is a compact and strongly positive operator from $c(\mathbb{X})$ to itself.

Remark 3.4.1. It is worth noting that Assumption 3.4.3 is fairly mild and holds in all applications we consider. We refer to Borovička and Stachurski (2017) for more comments about this assumption.

Since $\mathbb{X}$ is a compact metric space, $c(\mathbb{X})$ is a real ordered Banach space with the positive cone $c(\mathbb{X})_+$ having a non-empty interior. Hence, the Krein-Rutman Theorem applies to the operator $\boldsymbol{K}$ and gives us that $\boldsymbol{K}$ has exactly one eigenvector $e \in c(\mathbb{X})_+$ with $e \gg \mathbf{0}$ and $\|e\| = 1$, and the corresponding eigenvalue is $r(\boldsymbol{K}) > 0$ which is algebraically simple (see, e.g., Theorem 19.3 in Deimling (1985) or Theorem 7.C in Zeidler (1986)). In other words, as $\boldsymbol{K} \colon c(\mathbb{X}) \to c(\mathbb{X})$ is a linear, compact, strongly positive operator, there exists a unique eigenvector $e \in c(\mathring{\mathbb{X}})_+$ such that $\|e\| = 1$ and $\boldsymbol{K}e = r(\boldsymbol{K})e$.

In this connection, we use this Krein-Rutman eigenfunction to change the probability measure. Associated with such eigenfunction e and its corresponding eigenvalue $r(\boldsymbol{K})$, we consider and observe that a new kernel $\tilde{Q}$ defined by

$$\tilde{Q}(x, \mathrm{d}x') = \frac{e(x') \int \exp \left[(1-\gamma) \tau(x', y', x) \right] \nu(\mathrm{d}y') Q(x, \mathrm{d}x')}{r(\boldsymbol{K}) e(x)} \tag{3.15}$$

is a transition probability kernel such that $\int \tilde{Q}(x, \mathrm{d}x') = 1$.

Applying the change in probability measure $\tilde{Q}$ defined in (3.15) associated with the Krein-Rutman eigenfunction e, (3.13) translates to

$$f(x) = \left\{ \left[\int f(x')^{1-\gamma} r(\boldsymbol{K}) \frac{e(x)}{e(x')} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} \right\}^{\beta}$$
$$= [r(\boldsymbol{K}) \cdot e(x)]^{\frac{\beta}{1-\gamma}} \left\{ \left[\int \left(\frac{f(x')}{e(x')^{\frac{1}{1-\gamma}}} \right)^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} \right\}^{\beta}.$$

To use a multiplicative scaling of functions

$$g(x) := f(x) e(x)^{-\frac{1}{1-\gamma}},$$

the transformed counterpart to the above equation is

$$g(x) = r(\boldsymbol{K})^{\frac{\beta}{1-\gamma}} \cdot e(x)^{\frac{\beta-1}{1-\gamma}} \left[\left(\int g(x')^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right)^{\frac{1}{1-\gamma}} \right]^{\beta}. \tag{3.16}$$

In order to generate a simpler decomposition, we note that solving (3.16) for $g(x)$ is equivalent to solving the following equation

$$\tilde{g}(x) = \frac{\beta}{1-\gamma} \ln r(\boldsymbol{K}) + \frac{\beta - 1}{1-\gamma} \ln e(x) + \frac{\beta}{1-\gamma} \ln \left(\int \exp\left[(1-\gamma) \tilde{g}(x') \right] \tilde{Q}(x, \mathrm{d}x') \right) \tag{3.17}$$

for $\tilde{g}(x) := \ln g(x)$ for all $x \in \mathbb{X}$. It is worth noting that the recursion in a form of (3.17) is a special case of the so-called "risk-sensitive recursion".

Proposition 3.4.1. *Suppose that Assumptions 3.4.1 to 3.4.3 hold. Then, the following hold true:*

(i) Existence and uniqueness*: The functional Equation (3.17) has a unique solution* $\tilde{g}^* \in c(\mathbb{X})_+$.

(ii) Unique global attractivity of the fixed point*: For any initial condition* $\tilde{g}_0 \in c(\mathbb{X})_+$*, the iterative sequence* $\{\tilde{g}_n\}_{n \in \mathbb{N}_0}$ *constructed by (3.17) converges to* $\tilde{g}^*$ *in* $c(\mathbb{X})_+$.

(iii) Convergence of the iterative method and error estimates*: For all* $n \in \mathbb{N}_0$*, we get a priori error estimate:*

$$\|\tilde{g}_n - \tilde{g}*\| \leqslant r^n (1-r)^{-2} \|\tilde{g}_1\|.$$

for each $\tilde{g}_0 \in [\varepsilon \mathbb{1}_{\mathbb{X}}, \tilde{g}^{+*}]$ *and for some* $r \in (0,1)$*, where* $\varepsilon \mathbb{1}_{\mathbb{X}}$ *is a lower solution (with* $\varepsilon \in \mathbb{R}$*) and* $\tilde{g}^{+*}$ *is an upper solution of (3.17).*

In addition, if $\underline{f}_0$ *is a lower solution and* $\overline{f}_0$ *is an upper solution of (3.17) with* $\underline{f}_0 \leqslant \overline{f}_0$ *on* $c(\mathbb{X})_+$*, then we have the error estimates* $\underline{f}_n \leqslant \tilde{g}^* \leqslant \overline{f}_n$ *on* $c(\mathbb{X})_+$ *for all* $n \in \mathbb{N}_0$.

3.4.2 Recursive Utilities with Narrow Framing

In the sequel, we shall consider the Epstein–Zin recursive utility processes that incorporate loss aversion and narrow framing, as in, say, Barberis et al. (2006). That is,

$$U_t = W(C_t, \mathcal{R}_t(U_{t+1}) + B_t),$$

where W and $\mathcal{R}$ are defined as in (1.3) and (1.1) in the introduction Section 1.3, respectively. Relative to the usual recursive specification, B_t is a new term that captures loss aversion and narrow framing. By adding this new term to the second argument of the CES aggregator $W(\cdot,\cdot)$, the model allows for more flexibility, in the sense that the agent gets utility directly from the outcome of a specific gamble, rather than just indirectly via its contribution to the next period's wealth.

This framework has seen some success in addressing and explaining the equity premium puzzle and the stock market participation puzzle. Nevertheless, the closely related problem of existence and uniqueness of solutions to recursive utility processes with such preference specifications has not been completely investigated. Next, we shall study this interesting problem under Assumptions 3.4.1 to 3.4.3.

Following the approach of Hansen and Scheinkman (2012) again and by the homogeneity of the CES aggregator W, the normalized solution for recursive preferences satisfies

$$\frac{U_t}{C_t} = \left\{(1-\beta) + \beta \left\{\mathcal{R}_t\left(\frac{U_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right) + \frac{B_t}{C_t}\right\}^{1-\rho}\right\}^{\frac{1}{1-\rho}}, \tag{3.18}$$

where $\beta \in (0,1)$, $0 < \rho \neq 1$ and the certainty equivalent operator $\mathcal{R}$ are as stated in Section 3.3.2.

Invoking the consumption specification made in Assumption 3.4.2, it follows that

$$\frac{U_t}{C_t} = \left\{(1-\beta) + \beta \left\{\left[\mathrm{E}_t\left(\frac{U_{t+1}}{C_{t+1}}\exp\left[\tau(X_{t+1}, Y_{t+1}, X_t)\right]\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} + \frac{B_t}{C_t}\right\}^{1-\rho}\right\}^{\frac{1}{1-\rho}}.$$

To ensure that the utility recursion (3.18) is always well defined, in what follows, we make an assumption on investment utility.

Assumption 3.4.4. $B_t/C_t = b(X_t)$ for all $t \in \mathbb{N}_0$ and for some nonnegative real-valued continuous function b.

We now aim to seek a Markovian solution $(U_t/C_t) := f(X_t)$ of (3.18) for some nonnegative $\mathscr{B}$-measurable function f on $\mathbb{X}$. By virtue of Assumption 3.4.3, it follows from the results of the

Krein-Rutman eigenfunctions in Section 3.4.1 that

$$f(x) = \left\{ (1-\beta) + \beta \left\{ \left[\int f(x')^{1-\gamma} r(\boldsymbol{K}) \frac{e(x)}{e(x')} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} + b(x) \right\}^{1-\rho} \right\}^{\frac{1}{1-\rho}},$$

for all $x \in \mathbb{X}$ and so, after rearrangement,

$$\frac{f(x)}{e(x)^{\frac{1}{1-\gamma}}} = \left\{ \frac{1-\beta}{e(x)^{\frac{1}{\theta}}} + \beta r(\boldsymbol{K})^{\frac{1}{\theta}} \left\{ \left[\int \left(\frac{f(x')}{e(x)^{\frac{1}{1-\gamma}}} \right)^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} + \frac{b(x)}{[r(\boldsymbol{K}) e(x)]^{\frac{1}{1-\gamma}}} \right\}^{1-\rho} \right\}^{\frac{1}{1-\rho}}.$$

As before, we utilize a multiplicative scaling of functions

$$g(x) := \frac{f(x)}{e(x)^{\frac{1}{1-\gamma}}} \qquad \text{for all } x \in \mathbb{X},$$

from which we obtain the transformed counterpart as follows

$$g(x) = \left\{ h(x) + \beta r(\boldsymbol{K})^{\frac{1}{\theta}} \left\{ \left[\int g(x')^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} + j(x) \right\}^{1-\rho} \right\}^{\frac{1}{1-\rho}} \tag{3.19}$$

with

$$h(x) := \frac{1-\beta}{e(x)^{\frac{1}{\theta}}} \quad \text{and} \quad j(x) := \frac{b(x)}{[r(\boldsymbol{K}) e(x)]^{\frac{1}{1-\gamma}}}. \tag{3.20}$$

According to (3.19), we first define an operator $\boldsymbol{T}$ on $c(\mathbb{X})$ by

$$\boldsymbol{T} g(x) = \left\{ h(x) + \beta r(\boldsymbol{K})^{\frac{1}{\theta}} \left\{ \left[\int g(x')^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} + j(x) \right\}^{1-\rho} \right\}^{\frac{1}{1-\rho}} \tag{3.21}$$

where the functions h and j are as defined in (3.20).

For the sake of exposition, it is convenient to define an auxiliary operator $\boldsymbol{N}$ through

$$\boldsymbol{N} g(x) = \left[\int g(x')^{1-\gamma} \tilde{Q}(x, \mathrm{d}x') \right]^{\frac{1}{1-\gamma}} + j(x) \quad \text{for all } x \in \mathbb{X}. \tag{3.22}$$

Lemma 3.4.1. *Under Assumptions 3.4.1 to 3.4.4, if $0 < \gamma \neq 1$, then the operator $\boldsymbol{N}$ defined in (3.22) is monotone increasing and order concave on $c(\mathbb{X})_{++}$.*

Moreover, $\boldsymbol{N} g$ is strictly positive and continuous on $\mathbb{X}$ whenever $g \in c(\mathbb{X})_{++}$.

In addition, it is also convenient to define another auxiliary operator $\tilde{\boldsymbol{T}}$ by

$$\tilde{\boldsymbol{T}}g(x) = \left\{h(x) + \beta r(\boldsymbol{K})^{\frac{1}{\theta}} g(x)^{1-\rho}\right\}^{\frac{1}{1-\rho}} \quad \text{for all } g \in c(\mathbb{X})_{++},\ x \in \mathbb{X}. \tag{3.23}$$

Lemma 3.4.2. *Under Assumptions 3.4.1 to 3.4.4 with $\beta r(\boldsymbol{K})^{1/\theta} < 1$, if $0 < \rho \neq 1$, then the operator $\tilde{\boldsymbol{T}}$ defined in (3.23) is monotone increasing and order concave on $c(\mathbb{X})_{++}$.*

Moreover, $\tilde{\boldsymbol{T}}g$ is strictly positive and continuous on $\mathbb{X}$ whenever $g \in c(\mathbb{X})_{++}$.

In this connection, the operator $\boldsymbol{T}$ defined in (3.21) can be simply expressed as

$$\boldsymbol{T}g(x) = \tilde{\boldsymbol{T}} \circ \boldsymbol{N}g(x) \quad \text{for all } x \in \mathbb{X}.$$

Lemma 3.4.3. *Under Assumptions 3.4.1 to 3.4.4 with $\beta r(\boldsymbol{K})^{1/\theta} < 1$, if $0 < \rho \neq 1$ and $0 < \gamma \neq 1$, then the operator $\boldsymbol{T}$ defined in (3.21) is monotone increasing and order concave on $c(\mathbb{X})_{++}$.*

Moreover, $\boldsymbol{T}g$ is strictly positive and continuous on $\mathbb{X}$ whenever $g \in c(\mathbb{X})_{++}$.

In order to apply Du's theorem to solve (3.19), it only remains to show that there exist a strong lower solution and an upper solution of (3.19); that is, there exist w_1 and w_2 in $c(\mathbb{X})_{++}$ with $w_1 \leqslant w_2$ satisfying that $w_1 \ll \boldsymbol{T}w_1$ and $\boldsymbol{T}w_2 \leqslant w_2$.

Recalling the definition of the functions h and j in (3.20), it is obvious that both h and j are continuous on the compact metric space $\mathbb{X}$, from which we obtain the minimum and the maximum of h and j on $\mathbb{X}$, respectively, and henceforth denote by $\check{h} := \min h$, $\hat{h} := \max h$, $\check{j} := \min j$, $\hat{j} := \max j$. In particular, we have $\check{h} > 0$ and $\check{j} \geqslant 0$.

Lemma 3.4.4. *Under Assumptions 3.4.1 to 3.4.4 with $\beta r(\boldsymbol{K})^{1/\theta} < 1$, if $0 < \rho \neq 1$, then (3.19) has a strong lower solution w_1 and an upper solution w_2 in $c(\mathbb{X})_{++}$.*

It is worth noting that since $\mathbb{X}$ is compact, the space $c(\mathbb{X})_{++}$ is identical to the interior $c(\mathring{\mathbb{X}})_+$ of the positive cone $c(\mathbb{X})_+$.

Proposition 3.4.2. *Let $0 < \rho \neq 1$ and $0 < \gamma \neq 1$. In addition to Assumptions 3.4.1 to 3.4.3, if Assumption 3.4.4 holds and $\beta r(\boldsymbol{K})^{1/\theta} < 1$, then the following statements hold true:*

(i) Existence and uniqueness*: The functional Equation (3.19) has a unique solution $g^* \in c(\mathbb{X})_+$, more precisely, g^* lies in the interior of $c(\mathbb{X})_+$.*

(ii) Unique global attractivity of the fixed point*: For any initial condition $g_0 \in c(\mathbb{X})_{++}$, the iterative sequence $\{g_n\}_{n \in \mathbb{N}_0}$ constructed by (3.19) converges to g^* in $c(\mathbb{X})_+$.*

(iii) Convergence of the iterative method and error estimates*: For all $n \in \mathbb{N}_0$, we get a priori error estimate:*

$$\|g_n - g^*\| \leqslant r^n (1-r)^{-2} \|\boldsymbol{T}g_0\|.$$

for each $g_0 \in [w_1, w_2]$ and for some $r \in (0,1)$, where w_1 is a (strict) lower solution and w_2 is an upper solution of (3.19).

In addition, we have the error estimates $\boldsymbol{T}^n w_1 \leqslant g^* \leqslant \boldsymbol{T}^n w_2$ on $c(\mathbb{X})_+$ for all $n \in \mathbb{N}_0$.

3.4.3 Narrow Framing that Allows Gain and Loss Utility to be Negative

Sometimes, the gain and loss utility is not necessarily nonnegative. Thus, in what follows, Assumption 3.4.4 will be replaced by

Assumption 3.4.5. $B_t/C_t = b(X_t)$ for all $t \in \mathbb{N}_0$ and for some real-valued continuous function b.

Assumption 3.4.6. The following conditions are satisfied as follows:

$$\left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} > \frac{-\check{j}}{1-\delta^{1/\rho}} \quad \text{whenever } 0<\rho<1, \text{and}$$
$$\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} > \frac{-\check{j}}{1-\delta^{1/\rho}} \quad \text{whenever } \rho>1.$$

Remark 3.4.2. It is obvious that if $\check{j} \geqslant 0$, Assumption 3.4.6 is automatically satisfied. This means that the following results are more general and perfectly cover the results of Proposition 3.4.2.[11]

Remark 3.4.3. It is worth emphasizing that when $0 < 1-\rho < 1$, we have the following relations

$$\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} \geqslant \left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} > \frac{-\check{j}}{1-\delta^{1/\rho}} \geqslant \frac{-\hat{j}}{1-\delta^{1/\rho}}, \tag{3.24}$$

and when $1-\rho<0$, we have

$$\left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} \geqslant \left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} > \frac{-\check{j}}{1-\delta^{1/\rho}} \geqslant \frac{-\hat{j}}{1-\delta^{1/\rho}}. \tag{3.25}$$

Proposition 3.4.3. *Let $0 < \rho \neq 1$ and $0 < \gamma \neq 1$. In addition to Assumptions 3.4.1 to 3.4.3, if Assumptions 3.4.5 and 3.4.6 hold and $\beta r(\boldsymbol{K})^{1/\theta} < 1$, then the following statements hold true:*

(i) Existence and uniqueness: *The functional Equation (3.19) has a unique solution $g^* \in c(\mathbb{X})_+$, more precisely, g^* lies in the interior of $c(\mathbb{X})_+$.*

(ii) Unique global attractivity of the fixed point: *For any initial condition $g_0 \in c(\mathbb{X})_{++}$, the iterative sequence $\{g_n\}_{n\in\mathbb{N}_0}$ constructed by (3.19) converges to g^* in $c(\mathbb{X})_+$.*

[11] It is worth mentioning that Assumption 3.4.6 in the present essay is quite similar to Assumption 3 in Guo and He (2018).

(iii) Convergence of the iterative method and error estimates*: For all $n \in \mathbb{N}_0$, we get a priori error estimate:*

$$\|g_n - g^*\| \leqslant r^n (1-r)^{-2} \|\boldsymbol{T} g_0\|.$$

for each $g_0 \in [w_1, w_2]$ and for some $r \in (0,1)$, where w_1 is a lower solution and w_2 is an upper solution of (3.19).

In addition, we have the error estimates $\boldsymbol{T}^n w_1 \leqslant g^* \leqslant \boldsymbol{T}^n w_2$ on $c(\mathbb{X})_+$ for all $n \in \mathbb{N}_0$.

Chapter 4

Dynamic Programming with Recursive Preferences: Optimality and Applications

4.1 Introduction

In this essay we develop a set of sufficient conditions for abstract dynamic programs—including both additively separable and recursive preference models—that provide global convergence of the Bellman operator to the value function and optimality of the associated policies. These conditions are shown to apply to a range of recursive preference specifications popular in applied settings, including standard Epstein–Zin models with constant elasticity of substitution (CES) aggregators, risk-sensitive and robust control models, and ambiguity sensitive preferences such as those proposed by Ju and Miao (2012) as well as the narrow framing models proposed by Barberis et al. (2006). In each case we show that value function iteration converges uniformly to the value function and that Bellman's principle of optimality is valid. As a by-product, we also obtain the results of the policy iteration algorithm.

By analyzing observational data and the outcome of experiments, choice theorists have steadily constructed more realistic representations of economic agents and their preferences. For intertemporal decisions, this has led to departure from the simple, additively separable benchmark. A familiar example is the recursive preference framework of Epstein and Zin (1989), which has become central to the quantitative asset pricing literature, while also finding widespread use in applications ranging from optimal taxation to fiscal policy and business cycles. Another example is related to the narrow framing model of Barberis et al. (2006) and the recursive smooth ambiguity model of Ju and Miao (2012), which both successfully generate features of empirical asset pricing data that fail to arise under more traditional preferences.[1]

[1]Other important contributions to the literature on narrow framing include Barberis and Huang (2009), and the literature on recursive smooth ambiguity preferences includes Klibanoff et al. (2009) and Hayashi and Miao (2011).

While the kinds of recursive preference specifications discussed above have obtained clear empirical support, *optimal* choice in the presence of such specifications remains far less well understood than the classic, additively separable framework studied in, say, Bellman (1957), Blackwell (1965) or Stokey et al. (1989). For example, while early attempts to treat recursive preferences in a dynamic programming framework continued to use the contraction mapping arguments that had been successful for additively separable models (see, e.g., Lucas and Stokey (1984)), it was soon realized that the Bellman operators generated by the most common recursive preference specifications are not supremum norm contractions.[2]

This realization drove a second wave of theoretical analysis built around certain types of concavity exhibited by many intertemporal preferences, often using the theory of monotone concave operators pioneered by Krasnosel'skiĭ and Ladyžhenskiĭ (1954). Like contraction maps, under certain regularity conditions, monotone concave operators have unique, globally attracting fixed points—a highly attractive property in the context of dynamic programming.[3] Initial work along these lines can be found in Le Van et al. (2008). Some different but related approaches are pursued in Marinacci and Montrucchio (2017) and Bloise and Vailakis (2018).

On the one hand, monotone concave operator theory has been successful in providing results about the existence and uniqueness of recursive utilities—that is, in showing that the preference specifications are well defined, in the sense that fixed consumption paths or policies uniquely identify lifetime utility (see, for example, Marinacci and Montrucchio (2010, 2017)). On the other hand, it has been only modestly successful in providing results for dynamic programming itself, such as optimality of policies and global stability of the Bellman operator. By this we mean that, while a range of results have been obtained through skillful application of monotone concave operator theory, those results fail to accommodate some of the most popular specifications for applied work, such as common parameterizations of the Epstein–Zin specification as mentioned previously, or narrow framing models, or some kinds of ambiguity sensitive preferences.

In our view, the most central cause of these difficulties is a mismatch between monotone concave operator theory and the maximization step associated with the Bellman operator. This mismatch is caused by the fact that concavity is not preserved under maximization. In particular, the pointwise supremum of a family of concave functions is not, in general, concave. Thus, while concavity may be ideal for a task such as proving existence of recursive utilities (under a fixed policy or consumption process), that same concavity often breaks down when maximization is brought into the picture.

For this reason, we take an approach that, while inspired by monotone concave operator methods, has one significant difference: the relevant operators are *convex*. Put differently, we use

[2]In addition to Lucas and Stokey (1984), related work can be found in Boyd (1990), Durán (2003), Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007). It was Marinacci and Montrucchio (2010) who emphasized that sup norm contractivity fails for many economically reasonable aggregators, such as Thompson aggregators.

[3]This property was used the show existence of Markov equilibria in the presence of distortions in Datta et al. (2002), Morand and Reffett (2003) and several related papers.

monotone *convex* operators for maximization problems. Unlike concavity, convexity *is* preserved under the taking of pointwise suprema. Hence, convexity pairs naturally with maximization. Moreover, under suitable conditions, monotone convex operators enjoy all the properties possessed by monotone concave operators.

At the same time, we argue that the theory of monotone concave operators is ideal for *minimization* problems. This is because concavity is preserved by minimization, in the sense that the infimum of a family of concave functions is concave. Thus, any concavity inherent in the dynamic program flows naturally into the Bellman operator.

As the last piece of this puzzle, we note that simple continuous transformations can be used to transform inherently concave problems into convex problems and vice versa. Through these transformations, one can shift between convex maximization problems and concave minimization problems on a case by case basis. In particular, we show how preference specifications that have been recognized as concave can be modified so that they exhibit convexity rather than concavity.

Our theoretical framework departs from the separate specification of aggregator and certainty equivalent that has been popular in the economic literature since Kreps and Porteus (1978). Instead we adopt the abstract dynamic programming framework developed and collated by Bertsekas (2013). In abstract dynamic programming, the most cohesive sufficient conditions are still driven by contractions or semi-contractive properties (see, e.g., Bertsekas (2013), Chapters 2–3). The monotone-convex and monotone-concave results set out below offer an alternative branch of cohesive and broadly applicable methods.

The remainder of the essay is structured as follows: Section 4.2 contains our main results. Section 4.3 presents applications. Section 4.4 concludes.

4.2 General Results

Let X and A be separable metric spaces, called the *state* and *action space* respectively. Let $\mathbb{R}^{\mathsf{X}}$ represent all functions from X to $\mathbb{R}$ and let $\|\cdot\|$ denote the supremum norm on the bounded functions in $\mathbb{R}^{\mathsf{X}}$. For f and g in $\mathbb{R}^{\mathsf{X}}$, the statement $f \leqslant g$ means $f(x) \leqslant g(x)$ for all $x \in \mathsf{X}$. Let Γ be a non-empty correspondence from X to A, referred to below as the *feasible correspondence*. We understand $\Gamma(x)$ as representing all actions available to the controller in state x. The correspondence Γ in turn defines the set of *feasible state-action pairs*

$$\mathbb{G} := \{(x, a) \in \mathsf{X} \times \mathsf{A} : a \in \Gamma(x)\}.$$

Let

- w_1 and w_2 be bounded continuous functions in $\mathbb{R}^{\mathsf{X}}$ satisfying $w_1 \leqslant w_2$,
- $\mathscr{V}$ be all Borel-measurable functions v in $\mathbb{R}^{\mathsf{X}}$ satisfying $w_1 \leqslant v \leqslant w_2$, and

- $\mathscr{C}$ be the continuous functions in $\mathscr{V}$.

Both $\mathscr{V}$ and $\mathscr{C}$ are understood as classes of candidate value functions. The functions w_1 and w_2 serve as lower and upper bounds for lifetime value respectively. Their role will be clarified below.

Current and future payoffs are subsumed into a *state-action aggregator* Q, which maps a feasible state-action pair (x, a) and function v in $\mathscr{V}$ into a real value $Q(x, a, v)$. The interpretation of $Q(x, a, v)$ is total lifetime rewards, contingent on current action a, current state x and the use of v to evaluate future states. In other words, $Q(x, a, v)$ corresponds to the right-hand side of the Bellman equation when v represents the value function.

The central role of convexity and concavity was discussed in the introduction. To implement the corresponding restrictions, we call Q *value-convex* if

$$Q(x, a, \lambda v + (1 - \lambda) w) \leqslant \lambda Q(x, a, v) + (1 - \lambda) Q(x, a, w)$$

for each $(x, a) \in \mathbb{G}$, $\lambda \in [0, 1]$ and v, w in $\mathscr{V}$. Similarly, Q will be called *value-concave* when the reverse inequality holds (i.e., when $-Q$ is value-convex). One of these restrictions will be imposed on each problem we consider.

We also impose some basic properties that will be assumed in every case:

Assumption 4.2.1. The following conditions hold:

(a) The feasible correspondence Γ is compact valued and continuous.

(b) The map $(x, a) \mapsto Q(x, a, v)$ is Borel-measurable on $\mathbb{G}$ whenever $v \in \mathscr{V}$ and continuous on $\mathbb{G}$ whenever $v \in \mathscr{C}$.

(c) The state-action aggregator satisfies

$$v \leqslant v' \implies Q(x, a, v) \leqslant Q(x, a, v') \quad \text{for all } (x, a) \in \mathbb{G}. \tag{4.1}$$

(d) The functions w_1 and w_2 satisfy

$$w_1(x) \leqslant Q(x, a, w_1) \quad \text{and} \quad Q(x, a, w_2) \leqslant w_2(x) \tag{4.2}$$

for all (x, a) in $\mathbb{G}$.

The primary role of conditions (a) and (b) is to obtain the existence of solutions. If the state and action space are discrete (finite or countably infinite) then we adopt the discrete topology, in which case the continuity requirements in (a) and (b) are satisfied automatically, while the compactness requirement on Γ is satisfied if $\Gamma(x)$ is finite for each x.

Condition (c) imposes the natural requirement that higher continuation values increase lifetime values, while condition (d) is a consistency requirement that allows w_1 and w_2 to act as lower and upper bounds for lifetime value. The conditions in Assumption 4.2.1 are held to be true throughout the remainder of the paper.

Let Σ be a family of maps from X to A, referred to below as the set of all *feasible policies*, such that each $\sigma \in \Sigma$ is Borel-measurable and satisfies $\sigma(x) \in \Gamma(x)$ for all $x \in \mathsf{X}$.

Lemma 4.2.1. *The map $w(x) := Q(x, \sigma(x), v)$ is an element of $\mathscr{V}$ for all $v \in \mathscr{V}$.*

Proof. Borel measurability of $(x, a) \mapsto Q(x, a, v)$ and σ imply that w is Borel-measurable on X. Moreover, since $w_1 \leqslant v$, the inequalities in (4.1) and (4.2) imply $w_1(x) \leqslant Q(x, \sigma(x), w_1) \leqslant Q(x, \sigma(x), v)$ for all x. In particular, $w_1 \leqslant w$. A similar argument gives $w \leqslant w_2$, so $w \in \mathscr{V}$. □

Given $\sigma \in \Sigma$, a function $v_\sigma \in \mathscr{V}$ that satisfies

$$v_\sigma(x) = Q(x, \sigma(x), v_\sigma) \quad \text{for all } x \in \mathsf{X} \tag{4.3}$$

is called a *σ-value function*. The value $v_\sigma(x)$ can be interpreted as the lifetime value of following policy σ. Its existence and uniqueness are discussed below.

4.2.1 Maximization

We begin by studying maximization of value. Our key assumption is that the state-action aggregator satisfies value-convexity and possesses a strong upper solution:

Assumption 4.2.2. (Convex Program) The following conditions are satisfied:

(a) Q is value-convex.

(b) There exists an $\varepsilon > 0$ such that $Q(x, a, w_2) \leqslant w_2(x) - \varepsilon$ for all $(x, a) \in \mathbb{G}$.

Note that part (b) is a strengthening of one of the conditions in (4.2).

Proposition 4.2.1. *If Assumption 4.2.2 holds, then, for each σ in Σ, the set $\mathscr{V}$ contains exactly one σ-value function v_σ.*

Proposition 4.2.1 assures us that the value v_σ of a given policy σ is well defined. From this foundation we can introduce optimality concerning a maximization decision problem. In particular, in the present setting, a policy $\sigma^* \in \Sigma$ is called *optimal* if

$$v_{\sigma^*}(x) \geqslant v_\sigma(x) \quad \text{for all } \sigma \in \Sigma \text{ and all } x \in \mathsf{X}.$$

The *maximum value function* associated with this planning problem is the map v^* defined at $x \in \mathsf{X}$ by

$$v^*(x) = \sup_{\sigma \in \Sigma} v_\sigma(x). \tag{4.4}$$

One can show from conditions (c) and (d) of Assumption 4.2.1 that v^* is well defined as a real valued function on X and satisfies $w_1 \leqslant v^* \leqslant w_2$.

A function $v \in \mathscr{V}$ is said to satisfy the *Bellman equation* if

$$v(x) = \max_{a \in \Gamma(x)} Q(x, a, v) \text{ for all } x \in \mathsf{X}. \tag{4.5}$$

The *Bellman operator* T associated with our abstract dynamic program is a map sending v in $\mathscr{C}$ into

$$Tv(x) = \max_{a \in \Gamma(x)} Q(x, a, v). \tag{4.6}$$

Since v is in $\mathscr{C}$, existence of the maximum is guaranteed by Assumption 4.2.1. It follows from Berge's theorem of the maximum that Tv is an element of $\mathscr{C}$.[4] Evidently solutions to the Bellman equation in $\mathscr{C}$ exactly coincide with fixed points of T.

The convex program conditions lead to the following central result:

Theorem 4.2.1. *If Assumption 4.2.2 holds, then*

(a) The Bellman equation has exactly one solution in $\mathscr{C}$ and that solution is v^*.

(b) If v is in $\mathscr{C}$, then $T^n v \to v^*$ uniformly on X as $n \to \infty$.

(c) A policy σ in Σ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} Q(x, a, v^*) \text{ for all } x \in \mathsf{X}.$$

(d) At least one optimal policy exists.

The fixed point and convergence results for T in Theorem 4.2.1 rely on a fixed point theorem for monotone convex operators due to Du (1989), reprinted in Section 2.1 of Zhang (2012). In those references, convergence is shown to be uniformly geometric, in the sense that there exist constants $\lambda \in (0, 1)$ and $K \in \mathbb{R}$ such that

$$\|T^n v - v^*\| \leqslant \lambda^n K \text{ for all } n \in \mathbb{N} \text{ and } v \in \mathscr{C}.$$

Policy Function Iteration

In the sequel, we discuss a computational method for solving dynamic programs, which is known as the *policy function iteration* or *Howard's (policy) improvement algorithm*.

This typical policy iteration (PI for short) consists of the following steps:

[4] For Berge's theorem, interested readers may refer to pages 115–116 of Berge (1963), or Theorem 17.31 in Aliprantis and Border (2006), or Theorem B.1.3 in page 340 of Stachurski (2009).

1. (Policy evaluation) Pick an (initial) feasible policy $\sigma^0 \in \Sigma$, and compute the σ^k-value function v_{σ^k} associated with operating forever with that policy σ^k:

$$v_{\sigma^k}(x) = Q(x, \sigma^k(x), v_{\sigma^k}) \quad \text{for all } x \in \mathsf{X},$$

with $k = 0, 1, 2, \ldots$.

2. (Policy improvement) Generate a new policy $\sigma^{k+1} \in \Sigma$ that solves the intertemporal (two-period) problem (implied by v_{σ^k}), i.e.,

$$\sigma^{k+1}(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} Q(x, a, v_{\sigma^k}) \quad \text{for all } x \in \mathsf{X}.$$

3. Iterate over k to convergence on steps 1 and 2.

It turns out that the sequence of policy-value functions v_{σ^k} generated by the above PI algorithm converges to v^* and, with a sensible stopping rule, the resulting policy is approximately optimal. The following proposition proves a basic value improvement property (i.e., global convergence of the PI algorithm), and shows that the exact optimal policy is obtained in finite time when the set of policies is finite.

Proposition 4.2.2. (Convergence of PI) *Let Assumption 4.2.2 hold true, and let $\{\sigma^k\}_{k \geqslant 0}$ be a sequence generated by the PI algorithm. Then for all integers $k \geqslant 0$, we have $v_{\sigma^{k+1}} \geqslant v_{\sigma^k}$, with equality if and only if $v_{\sigma^k} = v^*$. Moreover,*

$$\lim_{k \to \infty} \|v_{\sigma^k} - v^*\| = 0,$$

and if the set of policies is finite, we have $v_{\sigma^k} = v^*$ for all large k.

4.2.2 Minimization

Next we treat minimization. In this setting, the convexity and strong upper solution in Assumption 4.2.2 are replaced by concavity and a strict lower solution.

In order to maintain consistency with other sources, we admit some overloading of terminology relative to Section 4.2.1 on maximization. For example, the optimal policy will now reference a minimizing policy rather than a maximizing one, and the Bellman equation will shift from maximization to minimization. The relevant definition will be clear from the context.

The next assumption is analogous to Assumption 4.2.2, which was used for maximization.

Assumption 4.2.3. (Concave Program) The following conditions are satisfied:

(a) Q is value-concave.

(b) There exists an $\varepsilon > 0$ such that $Q(x, a, w_1) \geqslant w_1(x) + \varepsilon$ for all $(x, a) \in \mathbb{G}$.

Note that part (b) is a strengthening of one of the conditions in (4.2).

Proposition 4.2.3. *If Assumption 4.2.3 holds, then, for each σ in Σ, the set $\mathscr{V}$ contains exactly one σ-value function v_σ.*

Proposition 4.2.3 mimics Proposition 4.2.1, assuring us that, in the present context, the cost v_σ of a given policy σ is well defined. A policy $\sigma^* \in \Sigma$ is then called *optimal* if

$$v_{\sigma^*}(x) \leqslant v_\sigma(x) \quad \text{for all } \sigma \in \Sigma \text{ and all } x \in \mathsf{X}.$$

The *minimum cost function* associated with this planning problem is the function v^* defined at $x \in \mathsf{X}$ by

$$v^*(x) = \inf_{\sigma \in \Sigma} v_\sigma(x). \tag{4.7}$$

A function $v \in \mathscr{V}$ is said to satisfy the *Bellman equation* if

$$v(x) = \min_{a \in \Gamma(x)} Q(x, a, v) \qquad \text{for all } x \in \mathsf{X}. \tag{4.8}$$

The *Bellman operator* S associated with our abstract dynamic program is a map sending v in $\mathscr{C}$ into

$$Sv(x) = \min_{a \in \Gamma(x)} Q(x, a, v). \tag{4.9}$$

Analogous to Theorem 4.2.1, we have

Theorem 4.2.2. *If Assumption 4.2.3 holds, then*

(a) The Bellman equation (4.8) has exactly one solution in $\mathscr{C}$ and that solution is the minimum cost function v^*.

(b) If v is in $\mathscr{C}$, then $S^n v \to v^*$ uniformly on X as $n \to \infty$.

(c) A policy σ in Σ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmin}_{a \in \Gamma(x)} Q(x, a, v^*) \;\; \textit{for all } x \in \mathsf{X}.$$

(d) At least one optimal policy exists.

Analogous to the result in maximization, the policy iteration algorithm in minimization consists of the following steps:

1. (Policy evaluation) Pick an (initial) feasible policy $\sigma^0 \in \Sigma$, and compute the σ^k-value function v_{σ^k} associated with operating forever with that policy σ^k:

   $$v_{\sigma^k}(x) = Q(x, \sigma^k(x), v_{\sigma^k}) \quad \text{for all } x \in \mathsf{X},$$

   with $k = 0, 1, 2, \ldots$.

2. (Policy improvement) Generate a new policy $\sigma^{k+1} \in \Sigma$ that solves the intertemporal (two-period) problem (implied by v_{σ^k}), i.e.,

$$\sigma^{k+1}(x) \in \operatorname*{argmin}_{a \in \Gamma(x)} Q(x, a, v_{\sigma^k}) \quad \text{for all } x \in \mathsf{X}.$$

3. Iterate over k to convergence on steps 1 and 2.

Analogous to Proposition 4.2.2, the following proposition establishes a globally convergent property of the PI algorithm for a minimization problem, as well as finite convergence for the case where the set of policies is finite.

Proposition 4.2.4. (Convergence of PI) *Let Assumption 4.2.3 hold true, and let $\{\sigma^k\}$ be a sequence generated by the PI algorithm. Then for all k, we have $v_{\sigma^{k+1}} \leqslant v_{\sigma^k}$, with equality if and only if $v_{\sigma^k} = v^*$. Moreover,*

$$\lim_{k \to \infty} \|v_{\sigma^k} - v^*\| = 0,$$

and if the set of policies is finite, we have $v_{\sigma^k} = v^*$ for all large k.

4.3 Applications

In this section we study a collection of applications, showing how the general results in Section 4.2 can be used to solve the dynamic programming problems discussed in the introduction.

4.3.1 An Additively Separable Decision Process

Before treating more sophisticated preference specifications, it is worth noting that the results stated above can be applied in the standard additive separable case, alongside the more traditional Bellman–Blackwell contraction mapping approach to dynamic programming. To see this, consider the generic additively separable dynamic programming model of Stokey et al. (1989) with the Bellman equation

$$v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ F(s,y,z) + \beta \int v(y,z') P(z, \mathrm{d}z') \right\} \tag{4.10}$$

over $(s,z) \in \mathsf{S} \times \mathsf{Z}$. Here S and Z are compact metric spaces containing possible values for the endogenous and exogenous state variables, respectively.[5] Let the transition function P on Z have the Feller property, let the feasible correspondence $\Gamma \colon \mathsf{S} \times \mathsf{Z} \to \mathsf{S}$ be compact valued and continuous, let $F \colon \mathbb{G} \to \mathbb{R}$ be continuous, and let β lie in $(0,1)$.

We translate this model to our environment by taking $x := (s,z)$ to be the state, $\mathsf{X} := \mathsf{S} \times \mathsf{Z}$ to

[5] Please note that a compact metric space is separable.

be the state space, $a = y \in \mathsf{S}$ to be the action, and setting

$$Q((s,z),y,v) = F(s,y,z) + \beta \int v(y,z')P(z,\mathrm{d}z').$$

Since F is continuous on a compact set, there exists a finite constant M with $|F| \leqslant M$.[6] For the bracketing functions w_1 and w_2 we fix $\varepsilon > 0$ and adopt the constant functions

$$w_1 \equiv -\frac{M}{1-\beta} \quad \text{and} \quad w_2 \equiv \frac{M+\varepsilon}{1-\beta}.$$

The conditions of Assumption 4.2.1 are all satisfied. Conditions (a) and (b) are true by assumption and condition (c) is trivial to verify. To see that condition (d) of Assumption 4.2.1 holds, we note that w_1 and w_2 lie in $bc\mathsf{X}$. In addition, for any given $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_1) = F(s,y,z) - \beta\frac{M}{1-\beta} \geqslant -M - \beta\frac{M}{1-\beta} = w_1(s,z).$$

Similarly,

$$Q((s,z),y,w_2) = F(s,y,z) + \beta\frac{M+\varepsilon}{1-\beta} \leqslant M + \beta\frac{M+\varepsilon}{1-\beta} = w_2(s,z) - \varepsilon.$$

The last inequality gives not only $Q(x,a,w_2) \leqslant w_2(x)$, as required for part (d) of the assumption, but also the stronger condition in part (b) of Assumption 4.2.2. Thus, to verify the requirements of Theorem 4.2.1, we need only check the convexity condition in part (a) of Assumption 4.2.2. But this is immediate from the linearity of expectations. Hence Theorem 4.2.1 applies.

4.3.2 Epstein–Zin Preferences

Kreps and Porteus (1978) and Epstein and Zin (1989) propose an alternative specification of lifetime value that separates and independently parameterizes intertemporal elasticity of substitution and risk aversion. To be more precise, Epstein and Zin (1989) propose the following preferences that are defined recursively by the CES aggregator

$$U_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left\{\mathcal{R}_t(U_{t+1})\right\}^{1-\rho}\right]^{\frac{1}{1-\rho}} \quad (0 < \rho \neq 1),$$

where $\{C_t\}$ is a consumption path, U_t is the utility value of the path onward from time t, and $\mathcal{R}_t$ is the Kreps–Porteus certainty equivalent operator

$$\mathcal{R}_t(U_{t+1}) = \left(\mathrm{E}_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \quad (0 < \gamma \neq 1).$$

Here, E_t stands for the conditional expectation with respect to the period t information. The

[6]The domain $\mathbb{G}$ of F is compact in the product topology by Tychonoff's theorem.

value $1/\rho$ represents elasticity of intertemporal substitution (EIS) between the composite good and the certainty equivalent, while γ governs the level of relative risk aversion (RRA) with respect to atemporal gambles. The most empirically relevant case is $\rho < \gamma$, implying that the agent prefers early resolution of uncertainty. We focus on this case in the followings.[7]

Under Epstein–Zin preferences, the generic additively separable Bellman equation in (4.10) becomes

$$v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left[\int v(y,z')^{1-\gamma} P(z, \mathrm{d}z') \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} \tag{4.11}$$

for each $(s,z) \in \mathsf{S} \times \mathsf{Z}$, where, here and below,

$$r(s,y,z) := (1-\beta) F(s,y,z)^{1-\rho}.$$

We impose the same conditions on the primitives discussed in Section 4.3.1. In particular, $\beta \in (0,1)$, F is continuous, P is Feller, Γ is continuous and compact valued and both S and Z are compact. To ensure $F(s,y,z)^{1-\rho}$ is always well defined, we also assume that F is everywhere positive.

The Case $\rho < \gamma < 1$

As in Hansen and Scheinkman (2012), we begin with the continuous strictly increasing transformation $\hat{v} = v^{1-\gamma}$, which allows us to rewrite (4.11) as

$$\hat{v}(s,z) = \max_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left[\int \hat{v}(y,z') P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \tag{4.12}$$

where

$$\theta := \frac{1-\gamma}{1-\rho}.$$

Since this transformation is bijective, there is a one-to-one correspondence between v and $\hat{v}$, in the sense that v solves (4.11) if and only if $\hat{v}$ solves (4.12). Note that in the current setting we have $\theta \in (0,1)$.

The state-action aggregator Q corresponding to (4.12) is

$$Q((s,z),y,v) = \left\{ r(s,y,z) + \beta \left[\int v(y,z') P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^{\theta}. \tag{4.13}$$

[7] We mainly concentrate on the case of $\rho < \gamma$ here, primarily because this case is application-oriented and the other case $\rho \geqslant \gamma$ may cause some technical difficulties in showing the corresponding Bellman operators to be convex or concave.

For the bracketing functions w_1 and w_2, we fix $\delta > 0$ and take the constant functions

$$w_1 := \left(\frac{m}{1-\beta}\right)^\theta \quad \text{and} \quad w_2 := \left(\frac{M+\delta}{1-\beta}\right)^\theta,$$

where

$$m := \min_{((s,z),y)\in\mathbb{G}} r(s,y,z) \text{ and } M := \max_{((s,z),y)\in\mathbb{G}} r(s,y,z). \tag{4.14}$$

These values are finite and positive, since F is continuous and positive on a compact domain.[8] Being constant, w_1 and w_2 are continuous.

We now show that the conditions of Assumptions 4.2.1 and 4.2.2 are all satisfied. Regarding Assumption 4.2.1, condition (a) is true by assumption, while condition (b) follows immediately from the continuity imposed on F and the Feller property of P. Condition (c) is easy to verify, since, for any $b > 0$, the scalar map

$$\psi(t) := (b + \beta t^{1/\theta})^\theta \qquad (t \geqslant 0) \tag{4.15}$$

is monotone increasing. To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_1) = \left\{r(s,y,z) + \beta\frac{m}{1-\beta}\right\}^\theta \geqslant \left\{m + \beta\frac{m}{1-\beta}\right\}^\theta = w_1(s,z).$$

Similarly,

$$Q((s,z),y,w_2) = \left\{r(s,y,z) + \beta\frac{M+\delta}{1-\beta}\right\}^\theta \leqslant \left\{M + \beta\frac{M+\delta}{1-\beta}\right\}^\theta,$$

or, with some rearranging,

$$Q((s,z),y,w_2) \leqslant \left\{\frac{M+\delta}{1-\beta} - \delta\right\}^\theta < w_2(s,z). \tag{4.16}$$

Hence condition (d) of Assumption 4.2.1 holds. In fact, (4.16) implies that our choice of w_2 also satisfies the uniformly strict inequality in (b) of Assumption 4.2.2.[9]

It only remains to check value-convexity of Q. But this is implied by the convexity of ψ defined in (4.15), which holds whenever $0 < \theta \leqslant 1$, along with linearity of the integral. The conclusions of Theorem 4.2.1 now follow.

[8] In this case, positivity of F can be weakened to nonnegativity.
[9] To be precise, condition (b) holds when $\varepsilon := [(M+\delta)/(1-\beta)]^\theta - [(M+\delta)/(1-\beta) - \delta]^\theta$.

The Case $\rho < 1 < \gamma$

To treat this case we again apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation (4.11). But now $1 - \gamma$ is negative, leading to the *minimization* problem

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left[\int \hat{v}(y,z') P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \tag{4.17}$$

for each $(s,z) \in \mathsf{X}$. The state-action aggregator Q corresponding to (4.17) is still as defined in (4.13). Note that in the current setting we have $\theta < 0$.

As (4.17) is a minimization problem, we aim to apply Theorem 4.2.2. For the bracketing functions w_1 and w_2, we take the constant functions

$$w_1 := \left(\frac{M + \delta}{1 - \beta} \right)^{\theta} \quad \text{and} \quad w_2 := \left(\frac{m}{1 - \beta} \right)^{\theta},$$

where δ is a positive constant and m and M are as defined in (4.14).

The conditions of Assumptions 4.2.1 and 4.2.3 are all satisfied. Regarding Assumption 4.2.1, the arguments verifying conditions (a) to (c) are identical to those in Section 4.3.2. To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_1) = \left\{ r(s,y,z) + \beta \frac{M+\delta}{1-\beta} \right\}^{\theta} \geqslant \left\{ M + \beta \frac{M+\delta}{1-\beta} \right\}^{\theta},$$

or, with some rearranging,

$$Q((s,z),y,w_1) \geqslant \left\{ \frac{M+\delta}{1-\beta} - \delta \right\}^{\theta} > w_1(s,z). \tag{4.18}$$

Similarly, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_2) = \left\{ r(s,y,z) + \beta \frac{m}{1-\beta} \right\}^{\theta} \leqslant \left\{ m + \beta \frac{m}{1-\beta} \right\}^{\theta},$$

and the last term is equal to $w_2(s,z)$. Hence, condition (d) of Assumption 4.2.1 is verified. Furthermore, it is immediately clear from (4.18) that our choice of w_1 also satisfies the uniformly strict inequality in (b) of Assumption 4.2.3.

It only remains to check the value-concavity of Q. But this follows directly from the concavity of the function ψ defined in (4.15), as implied by $\theta < 0$, along with linearity of the integral. We have now checked all conditions of Theorem 4.2.2.

The Case $1 < \rho < \gamma$

We now turn to the model in the case where the coefficient of relative risk aversion is still strictly greater than 1 but the intertemporal elasticity of substitution is less than 1, as is commonly found in the literature.[10] As before, we apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation (4.11) and, since $1 - \gamma < 0$, the transformed counterpart leads us to the minimization problem as defined in (4.17). Note that $\theta > 1$ in the current setting.

As (4.17) is a minimization problem, we aim to apply Theorem 4.2.2. For the bracketing functions w_1 and w_2, we take the constant functions

$$w_1 := \left(\frac{m-\delta}{1-\beta}\right)^{\theta} \quad \text{and} \quad w_2 := \left(\frac{M}{1-\beta}\right)^{\theta},$$

for some positive $\delta < m$, where m and M are as defined in (4.14).

Assumptions 4.2.1 and 4.2.3 are again satisfied. Regarding Assumption 4.2.1, the arguments of verifying conditions (a) to (c) are identical to those in 4.3.2. To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_1) = \left\{r(s,y,z) + \beta\frac{m-\delta}{1-\beta}\right\}^{\theta} \geqslant \left\{m + \beta\frac{m-\delta}{1-\beta}\right\}^{\theta},$$

or, with some rearranging

$$Q((s,z),y,w_1) \geqslant \left\{\frac{m-\delta}{1-\beta} + \delta\right\}^{\theta} > w_1(s,z). \tag{4.19}$$

Similarly, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$Q((s,z),y,w_2) = \left\{r(s,y,z) + \beta\frac{M}{1-\beta}\right\}^{\theta} \leqslant \left\{M + \beta\frac{M}{1-\beta}\right\}^{\theta} = w_2(s,z).$$

Hence condition (d) of Assumption 4.2.1 holds. In fact (4.19) implies that our choice of w_1 also satisfies the uniformly strict inequality in (b) of Assumption 4.2.3.

Value-concavity of Q is a direct consequence of the concavity of ψ, which holds again when $\theta > 1$, along with linearity of the integral. The conclusions of Theorem 4.2.2 now follow.[11]

4.3.3 Risk-Sensitive Preferences

In this section, we consider an economy with a representative agent having risk-sensitive preferences, as in, say, Hansen and Sargent (2008), Gottardi et al. (2015), or Bäuerle and Jaśkiewicz

[10] See, for example, Hall (1988), Farhi and Werning (2008) and Basu and Bundick (2017).

[11] Recently, Guo and He (2018) get the uniqueness of the solution to the dynamic programming (DP) equation by using a different method that might also be applicable here. In their paper, the authors do not assume any concavity or convexity for the operator of the DP equation, but they show that the solution to the DP equation therein must be the optimal value of the portfolio selection problem and thus is automatically unique.

(2018). The generic Bellman equation associated with risk-sensitive preferences is

$$v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ r(s,y,z) - \frac{\beta}{\theta} \ln \left[\int \exp\left(-\theta\, v(y,z')\right) P(z, \mathrm{d}z') \right] \right\} \tag{4.20}$$

for each $(s,z) \in \mathsf{S} \times \mathsf{Z}$. Here, $r \colon \mathbb{G} \to \mathbb{R}$ is a continuous one-period reward function. The parameter $\theta > 0$ captures the risk sensitivity, while other primitives are as discussed in Section 4.3.1. In particular, $\beta \in [0,1)$, P is Feller, Γ is continuous and compact valued and both S and Z are compact.

Applying the continuous bijective transformation $\hat{v} \equiv \exp(-\theta v)$ to v in the Bellman equation (4.20) leads to the minimization problem

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \exp \left\{ -\theta \left\{ r(s,y,z) - \frac{\beta}{\theta} \ln \left[\int \hat{v}(y,z') P(z, \mathrm{d}z') \right] \right\} \right\}. \tag{4.21}$$

We translate (4.21) to our environment by taking $\mathsf{X} := \mathsf{S} \times \mathsf{Z}$ to be the state space, $a = y \in \mathsf{S}$ to be the action, and setting

$$Q((s,z),y,v) = \exp \left\{ -\theta \left\{ r(s,y,z) - \frac{\beta}{\theta} \ln \left[\int v(y,z') P(z, \mathrm{d}z') \right] \right\} \right\}. \tag{4.22}$$

Since r is continuous, there exists a finite constant M with $|r| \leqslant M$. For the bracketing functions, we fix $\delta > 0$ and take the constant functions

$$w_1 := \exp \left[-\theta \left(\frac{M}{1-\beta} + \delta \right) \right] \quad \text{and} \quad w_2 := \exp \left[-\theta \left(\frac{-M}{1-\beta} \right) \right].$$

Assumptions 4.2.1 and 4.2.3 are all satisfied. Regarding Assumption 4.2.1, the steps verifying conditions (a) and (b) are identical to those in Section 4.3.2. Condition (c) clearly holds, since, for any $b \in \mathbb{R}$, the scalar map

$$\phi(t) := \exp \left[-\theta \left(b - \frac{\beta}{\theta} \ln t \right) \right] = \exp(-\theta b) t^{\beta} \qquad (t > 0) \tag{4.23}$$

is monotone increasing. To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_1) &= \exp \left(-\theta \left\{ r(s,y,z) + \beta \left(\frac{M}{1-\beta} + \delta \right) \right\} \right) \\ &\geqslant \exp \left(-\theta \left\{ M + \beta \left(\frac{M}{1-\beta} + \delta \right) \right\} \right) \end{aligned}$$

or, with some rearranging,

$$Q((s,z),y,w_1) \geqslant \exp \left(-\theta \left\{ \frac{M}{1-\beta} + \beta\delta \right\} \right) > w_1(s,z). \tag{4.24}$$

Similarly, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$
\begin{aligned}
Q((s,z),y,w_2) &= \exp\left(-\theta\left\{r(s,y,z) + \beta\left(\frac{-M}{1-\beta}\right)\right\}\right) \\
&\leqslant \exp\left(-\theta\left\{-M - \beta\frac{M}{1-\beta}\right\}\right),
\end{aligned}
$$

and the last term is equal to $w_2(s,z)$. Hence condition (d) of Assumption 4.2.1 holds. In addition, it is obvious from (4.24) that our choice of w_1 also satisfies the uniformly strict inequality in part (b) of Assumption 4.2.3.[12]

Finally, condition (a) of Assumption 4.2.3, which is value-concavity of Q, follows directly from the concavity of the function ϕ defined in (4.23), along with linearity of the integral. The conclusions of Theorem 4.2.2 now follow.

4.3.4 Ambiguity

Extending earlier work by Epstein and Zin (1989) and Klibanoff et al. (2009), Ju and Miao (2012) propose and study a recursive smooth ambiguity model where lifetime value satisfies

$$
V_t(C) = \left[(1-\beta)C^{1-\rho} + \beta\{\mathcal{R}_t(V_{t+1}(C))\}^{1-\rho}\right]^{1/(1-\rho)} \tag{4.25}
$$

with

$$
\mathcal{R}_t(V_{t+1}(C)) = \left\{\mathrm{E}_{\mu_t}\left(\mathrm{E}_{\pi_{\theta,t}}\left[V_{t+1}^{1-\gamma}(C)\right]\right)^{(1-\eta)/(1-\gamma)}\right\}^{1/(1-\eta)}. \tag{4.26}
$$

As before, $\beta \in (0,1)$, ρ is the reciprocal of the EIS and γ governs risk aversion, while η satisfies $0 < \eta \neq 1$ and captures ambiguity aversion. If $\eta = \gamma$, the decision maker is ambiguity neutral and (4.25)–(4.26) reduces to the classical recursive utility model of Epstein and Zin (1989). The decision maker displays ambiguity aversion if and only if $\gamma < \eta$. We focus primarily on the case $0 < \rho < 1 < \gamma < \eta$, which is the most empirically relevant.[13]

As a generic formulation of the preferences of Ju and Miao (2012), we consider the Bellman equation

$$
v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left\{ \int \left[\int v(y,z')^{1-\gamma} \pi_\theta(z, \mathrm{d}z') \right]^{\frac{1-\eta}{1-\gamma}} \mu(z, \mathrm{d}\theta) \right\}^{\frac{1-\rho}{1-\eta}} \right\}^{\frac{1}{1-\rho}} \tag{4.27}
$$

where $(s,z) \in \mathsf{S} \times \mathsf{Z}$. We assume both S and Z to be compact, Γ to be continuous and compact valued, F to be continuous and everywhere positive. The set Θ is a finite parameter space, each element of which is a vector of parameters in the specification of the exogenous state process.

[12] To be precise, condition (b) of Assumption 4.2.3 holds when we set $\varepsilon := \exp\{-\theta[M/(1-\beta)+\beta\delta]\} - \exp\{-\theta[M/(1-\beta)+\delta]\}$.

[13] The calibration used in Ju and Miao (2012) is $(\rho,\gamma,\eta) = (0.66, 2.0, 8.86)$. See p. 574.

Given any $\theta \in \Theta$, the transition function π_θ on Z is assumed to have the Feller property. Given any $z \in \mathsf{Z}$, the distribution $\mu(z, \cdot)$ maps subsets of Θ to $[0, 1]$ and evolves as a function of the exogenous state process. We suppose that μ is continuous in z for each $\theta \in \Theta$.

The Case $\rho \neq 1$

Applying the continuous bijective transformation $\hat{v} \equiv v^{1-\eta}$ to v in the Bellman equation (4.27) leads to the minimization problem

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left\{ \int \left[\int \hat{v}(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \right]^{\frac{1}{\xi_1}} \mu(z, \mathrm{d}\theta) \right\}^{\frac{1}{\xi_2}} \right\}^{\xi_2} \tag{4.28}$$

for all $(s,z) \in \mathsf{S} \times \mathsf{Z}$, where, here and below,

$$\xi_1 := \frac{1-\gamma}{1-\eta} \quad \text{and} \quad \xi_2 := \frac{1-\eta}{1-\rho}.$$

Since this transformation is bijective, there is a one-to-one correspondence between v and $\hat{v}$, in the sense that v solves (4.27) if and only if $\hat{v}$ solves (4.28). Note that in the current setting we have $\xi_1 \in (0,1)$ and $\xi_2 < 0$.

We translate this model to our environment by taking $\mathsf{X} := \mathsf{S} \times \mathsf{Z}$ to be the state space, $a = y$ to be the action taking values in S, and setting the state-action aggregator Q to

$$Q((s,z), y, \hat{v}) = \left\{ r(s,y,z) + \beta \left\{ \int \left[\int \hat{v}(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \right]^{\frac{1}{\xi_1}} \mu(z, \mathrm{d}\theta) \right\}^{\frac{1}{\xi_2}} \right\}^{\xi_2}. \tag{4.29}$$

As (4.28) is a minimization problem, we aim to apply Theorem 4.2.2. For the bracketing functions w_1 and w_2, we fix $\delta > 0$ and take the constant functions

$$w_1 := \left(\frac{M + \delta}{1 - \beta} \right)^{\xi_2} \quad \text{and} \quad w_2 := \left(\frac{m}{1-\beta} \right)^{\xi_2},$$

where the real numbers m and M are as defined in Section 4.3.2.

Assumptions 4.2.1 and 4.2.3 are satisfied. Regarding Assumption 4.2.1, condition (a) is true by assumption. Conditions (b) and (c) are proved in Lemma A.3.10 in the appendix. To verify condition (d), for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z), y, w_1) &= \left\{ r(s,y,z) + \beta \left\{ \int \left(\frac{M+\delta}{1-\beta} \right)^{\xi_2} \mu(z, \mathrm{d}\theta) \right\}^{1/\xi_2} \right\}^{\xi_2} \\ &= \left\{ r(s,y,z) + \beta \frac{M+\delta}{1-\beta} \right\}^{\xi_2} \geqslant \left\{ M + \beta \frac{M+\delta}{1-\beta} \right\}^{\xi_2}, \end{aligned}$$

where the first equality follows directly from the definition of Q and the fact that for any non-negative constant function d, $[\int d(z')^{\xi_1} \pi_\theta(z, \mathrm{d}z')]^{1/\xi_1} = d$. Furthermore, with some rearranging, we obtain

$$Q((s,z), y, w_1) \geqslant \left\{ \frac{M+\delta}{1-\beta} - \delta \right\}^{\xi_2} > w_1(s,z). \tag{4.30}$$

Similarly, for fixed $((s,z), y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z), y, w_2) &= \left\{ r(s,y,z) + \beta \left\{ \int \left(\frac{m}{1-\beta} \right)^{\xi_2} \mu(z, \mathrm{d}\theta) \right\}^{1/\xi_2} \right\}^{\xi_2} \\ &= \left\{ r(s,y,z) + \beta \frac{m}{1-\beta} \right\}^{\xi_2} \leqslant \left\{ m + \beta \frac{m}{1-\beta} \right\}^{\xi_2} = w_2(s,z). \end{aligned}$$

Hence condition (d) of Assumption 4.2.1 indeed holds true. Moreover, it is clear from (4.30) that our choice of w_1 also satisfies the uniformly strict inequality in part (b) of Assumption 4.2.3.

Condition (a) of Assumption 4.2.3 (i.e., value-concavity of Q) is also satisfied, as shown in Lemma A.3.10 of the appendix. The conclusions of Theorem 4.2.2 now follow.

The Case $\rho = 1$

In the limiting case with $\rho = 1$, the generic ambiguity recursion (4.26) becomes

$$\begin{aligned} U_t(C) =& (1-\beta) \ln C_t \\ &+ \frac{\beta}{1-\eta} \ln \left\{ \mathrm{E}_{\mu_t} \exp \left(\frac{1-\eta}{1-\gamma} \ln \left(\mathrm{E}_{\pi_{\theta,t}} \exp \left((1-\gamma) U_{t+1} \right) \right) \right) \right\}, \end{aligned} \tag{4.31}$$

where $U_t = \ln V_t$.[14] The generic Bellman equation in (4.27) becomes

$$\begin{aligned} v(s,z) = \max_{y \in \Gamma(s,z)} & \left\{ r(s,y,z) + \frac{\beta}{1-\eta} \times \right. \\ \times \ln & \left[\int \exp \left(\frac{1}{\xi_1} \ln \left(\int \exp \left((1-\gamma) v(y,z') \right) \pi_\theta(z, \mathrm{d}z') \right) \right) \mu(z, \mathrm{d}\theta) \right] \left. \vphantom{\int} \right\}, \end{aligned} \tag{4.32}$$

for each $(s,z) \in \mathsf{S} \times \mathsf{Z}$. The one-period return function r is still assumed to be continuous but is no longer restricted to being positive, while other primitives are as discussed in Section 4.3.4.

Applying the transformation $\hat{v} \equiv \exp[(1-\eta)v]$ to v in the Bellman equation (4.32) leads us to

[14] This specification connects with risk-sensitive control and robustness, as studied by Hansen and Sargent (2008). In particular, there are two risk-sensitivity adjustments in (4.31).

the minimization problem

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \exp\Bigg((1-\eta)\Bigg\{ r(s,y,z) + \frac{\beta}{1-\eta} \times \\ \times \ln\left[\int \exp\left(\frac{1}{\xi_1} \ln\left(\int \exp\left(\xi_1 \ln \hat{v}(y,z')\right) \pi_\theta(z, \mathrm{d}z') \right) \right) \mu(z, \mathrm{d}\theta) \right] \Bigg\} \Bigg). \tag{4.33}$$

With some rearranging, (4.33) can be written as

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \exp\Bigg((1-\eta)\Bigg\{ r(s,y,z) + \frac{\beta}{1-\eta} \times \\ \times \ln\left[\int \left(\int \hat{v}(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \right)^{1/\xi_1} \mu(z, \mathrm{d}\theta) \right] \Bigg\} \Bigg). \tag{4.34}$$

Note that we still have $\xi_1 \in (0,1)$ and $\eta > 1$ in the current setting with ambiguity aversion. The state-action aggregator Q corresponding to (4.34) is

$$Q((s,z),y,\hat{v}) = \exp\Bigg((1-\eta)\Bigg\{ r(s,y,z) + \frac{\beta}{1-\eta} \times \\ \times \ln\left[\int \left(\int \hat{v}(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \right)^{1/\xi_1} \mu(z, \mathrm{d}\theta) \right] \Bigg\} \Bigg). \tag{4.35}$$

Since the return function r is continuous on a compact set, there exists a finite constant M such that $|r| \leqslant M$. Hence for the bracketing function w_1 and w_2, we fix $\delta > 0$ and take the constant functions

$$w_1 := \exp\left((1-\eta) \left(\frac{M}{1-\beta} + \delta \right) \right) \quad \text{and} \quad w_2 := \exp\left((1-\eta) \left(\frac{-M}{1-\beta} \right) \right).$$

As (4.34) is the minimization problem, we aim to apply Theorem 4.2.2. Again, Assumptions 4.2.1 and 4.2.3 are all satisfied.

Regarding Assumption 4.2.1, condition (a) is trivial. Conditions (b) and (c) follow from Lemma A.3.12 in the appendix. To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_1) &= \exp\left((1-\eta) \left\{ r(s,y,z) + \beta \left(\frac{M}{1-\beta} + \delta \right) \right\} \right) \\ &\geqslant \exp\left((1-\eta) \left\{ M + \beta \left(\frac{M}{1-\beta} + \delta \right) \right\} \right), \end{aligned}$$

or, with some rearranging,

$$Q((s,z),y,w_1) \geqslant \exp\left((1-\eta) \left\{ \frac{M}{1-\beta} + \beta\delta \right\} \right) > w_1(s,z). \tag{4.36}$$

Similarly, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_2) &= \exp\left((1-\eta)\left\{r(s,y,z) + \beta\left(\frac{-M}{1-\beta}\right)\right\}\right) \\ &\leqslant \exp\left((1-\eta)\left\{-M - \beta\frac{M}{1-\beta}\right\}\right) = w_2(s,z). \end{aligned}$$

Hence condition (d) of Assumption 4.2.1 holds.

In fact, it is immediately clear from (4.36) that our choice of w_1 also satisfies the uniformly strict inequality in part (b) of Assumption 4.2.3. Regarding part (a) of Assumption 4.2.3, value-concavity of Q is immediate from Lemma A.3.12. We have now checked all conditions of Theorem 4.2.2, and the conclusions of that theorem now follow.

4.3.5 Narrow framing

In this section we study the recursive preferences that incorporate both first-order risk aversion and narrow framing, as in, say, Barberis et al. (2006) or Barberis and Huang (2009), which can be expressed as

$$U_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left\{\left(\mathrm{E}_t\, U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} + b_0 \mathrm{E}_t\left(\sum_{i=1} \bar{u}(\tilde{G}_{i,t+1})\right)\right\}^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $b_0 \geqslant 0$ is a parameter controlling the degree of narrow framing, while $\tilde{G}_{i,t+1}$ represents the specific gamble the agent is taking by investing in asset i whose uncertainty will be resolved between period t and $t+1$.[15] First-order risk aversion is introduced through the piecewise linearity of $\bar{u}(\cdot)$.[16] Relative to the recursive specification in Section 4.3.2, the new term prefixed by b_0 shows that the agent obtains utility directly from the outcomes of gambles $\{\tilde{G}_{i,t+1}\}_i$ over and above what those outcomes mean for total wealth risk, rather than just indirectly via its contribution to the next period's wealth. Other primitives are as discussed in Section 4.3.2. For the parameters ρ and γ, we assume that either $1 < \rho < \gamma$ or $\rho < 1 < \gamma$.

Under the preceding preference specification, the generic Bellman equation becomes

$$v(s,z) = \max_{y\in\Gamma(s,z)} \left\{r(s,y,z) + \beta\left[\left(\int v(y,z')^{1-\gamma} P(z,\mathrm{d}z')\right)^{\frac{1}{1-\gamma}} + B(s,y,z)\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}}. \tag{4.37}$$

As before, $\beta \in (0,1)$ and we suppose that the one-period return function $r(s,y,z)$ is positive and continuous on $\mathbb{G}$, while the aggregate gambling utility function $B(s,y,z)$ is assumed to be nonnegative and continuous on $\mathbb{G}$.

[15] A zero value of the parameter b_0 means no narrow framing at all, while a large value of b_0 indicates that $G_{i,t+1}$ is evaluated almost completely in isolation from other risks.

[16] The piecewise-linear specification of $\bar{u}(\cdot)$ in Barberis et al. (2006) is defined by $\bar{u}(x) = x\mathbb{1}\{x \geqslant 0\} + \lambda x \mathbb{1}\{x < 0\}$ with $\lambda > 1$.

We again apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation (4.37). As $1-\gamma$ is negative, the transformed counterpart leads us to the minimization problem

$$\hat{v}(s,z) = \min_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left[\left(\int \hat{v}(y,z') P(z, \mathrm{d}z') \right)^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta}, \tag{4.38}$$

where $\theta := (1-\gamma)/(1-\rho)$. The state-action aggregator is

$$Q((s,z),y,\hat{v}) = \left\{ r(s,y,z) + \beta \left[\left(\int \hat{v}(y,z') P(z, \mathrm{d}z') \right)^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta}. \tag{4.39}$$

Lemma 4.3.1. *Let Q be as defined in (4.39). If either $\rho < 1 < \gamma$ or $1 < \rho < \gamma$, then there exist continuous strictly positive functions w_1, w_2 on $\mathsf{S} \times \mathsf{Z}$, $w_1 < w_2$, such that*

(SL) there exists an $\varepsilon > 0$ such that $Q((s,z),y,w_1) \geqslant w_1(s,z) + \varepsilon$ for all $((s,z),y) \in \mathbb{G}$; and

(U) $Q((s,z),y,w_2) \leqslant w_2(s,z)$ for all $((s,z),y) \in \mathbb{G}$.

The proof is deferred to the appendix.

The conditions of Assumptions 4.2.1 and 4.2.3 are all satisfied. Regarding Assumption 4.2.1, condition (a) is true by assumption, while condition (b) follows immediately from the continuity imposed on r and B and the Feller property of P. Condition (c) is easy to verify, since, for any fixed constants $c > 0$ and $b \geqslant 0$, the scalar map

$$\psi(t) := \left\{ c + \beta \left[t^{\frac{1}{1-\gamma}} + b \right]^{1-\rho} \right\}^{\theta} \qquad (t > 0) \tag{4.40}$$

is monotone increasing. Condition (d) and part (b) of Assumption 4.2.3 have been verified by Lemma 4.3.1.

It only remains to check value-concavity of Q. But this follows directly from the concavity of ψ defined in (4.40), as implied by either $\rho < 1 < \gamma$ or $1 < \rho < \gamma$, along with linearity of the integral.[17] Hence all conditions of Theorem 4.2.2 are verified and the conclusions of that theorem follow.

4.4 Conclusion

Recursive preference models have allowed economists to successfully replicate important empirical phenomena in a range of different settings. To date, most attempts to provide a theory of dynamic programming for recursive preferences models commensurate with those available for traditional additivity separable preferences have focused on exploiting concavity available

[17] For more details of the proof regarding the concavity of ψ, please refer to Lemma A.3.13 in the appendix.

in some classes of preferences. Here we instead used convexity, which pairs well with maximization, allowing us to provide conditions for optimality that are relatively simple, general enough to include many classes of preferences and strong in their conclusions.

Chapter 5

Extension I: Dynamic Programming with Recursive Preferences and Unbounded Rewards

5.1 Introduction

As one extension of our ideas in the preceding chapter (Chapter 4), to ensure sufficient generality for economic applications, we now consider an Epstein–Zin recursive preference model with possibly unbounded shocks and with rewards that are allowed to be unbounded from above. Both features are conceptually important and empirically useful.

It is well known that for Markov decision problems with unbounded rewards, there is no general theory for dynamic programming to work. One difficulty is that there is no general fixed point theorem to guarantee the existence of a solution to the Bellman equation. The Contraction Mapping Theorem cannot be simply applied to the set of unbounded and continuous functions because the sup norm on this set is not well-defined. One may apply other fixed point theorems such as the Schauder Fixed-Point Theorem or the Tarski Fixed-Point Theorem to find a fixed point. However, there is still no general theorem to guarantee its uniqueness and global convergence. Indeed, unlike the time-additive expected utility model, most recursive utility models are nonlinear, which induces subtle technical difficulties about the solvability of the Bellman equation. Important innovations tailored to economic applications can be found in Durán (2003), Le Van and Vailakis (2005), Rincón-Zapatero and Rodríguez-Palmero (2007), Martins-da Rocha and Vailakis (2010) and Bäuerle and Jaśkiewicz (2018).

Perhaps the most common approach to treating unbounded rewards in the broader field of dynamic programming has been one involving contraction mapping arguments in a setting of weighted supremum norms (see, e.g., Bertsekas (2013)).[1] However, as many researchers

[1] This idea appears in the mathematics literature (e.g., Wessels (1977)) and is extended by Boyd (1990) and further developed by Durán (2000, 2003) in economics.

have pointed out, the dynamic programming equation that we consider may not be solved by using contraction mapping principle. See, for example, Ozaki and Streufert (1996), Le Van and Vailakis (2005), Balbus (2016), Marinacci and Montrucchio (2017), Bich et al. (2018), and Bloise and Vailakis (2018), etc.. In connection with this, we show that similar ideas as in Chapter 4 can be applied when contractivity fails or is difficult to obtain. In particular, we show how the results from the preceding chapter can be generalized to accommodate unbounded rewards.

A natural idea to apply the theory of monotone concave operators to unbounded and continuous functions is to restrict to a subset of these functions by utilizing the weighted supremum norm. Each function in this subset is bounded by some positive function and thus one can define a supremum norm weighted by this function. Under this weighted supremum norm, the subset is a complete metric space and even a Banach lattice (cf. Example 2.2.2 in Chapter 2). As a result, the relevant cone theory can be applied to this Banach lattice and the techniques of monotone concave/convex operators can therefore be built upon the Banach lattice.

Section 5.2 states the general results of optimization problems. Section 5.3 gives applications.

5.2 General Results

Unless otherwise specified, the notation of Chapter 4 will continue to be used throughout this chapter.

Let X, A, Γ, $\mathbb{G}$, Q and Σ be defined as in Section 4.2 of Chapter 4. Given a real-valued continuous function ℓ defined on a separable metric space X with $\inf_{x \in \mathsf{X}} \ell(x) > 0$, let

- w_1 and w_2 be ℓ-bounded continuous functions in $\mathbb{R}^{\mathsf{X}}$ satisfying $w_1 \leqslant w_2$,[2]
- $\mathscr{V}$ be all Borel-measurable functions v in $\mathbb{R}^{\mathsf{X}}$ satisfying $w_1 \leqslant v \leqslant w_2$, and
- $\mathscr{C}$ be the continuous functions in $\mathscr{V}$.

As before, both $\mathscr{V}$ and $\mathscr{C}$ represent classes of candidate value functions, and the functions w_1 and w_2 serve as lower and upper bounds for lifetime value, respectively.

In the following, we impose some basic properties that will be assumed in each case:

Assumption 5.2.1. The following conditions hold:

(a) The feasible correspondence Γ is compact valued and continuous.

(b) The map $(x, a) \mapsto Q(x, a, v)$ is Borel-measurable on $\mathbb{G}$ whenever $v \in \mathscr{V}$ and continuous on $\mathbb{G}$ whenever $v \in \mathscr{C}$.

(c) The state-action aggregator satisfies

$$v \leqslant v' \implies Q(x, a, v) \leqslant Q(x, a, v') \quad \text{for all } (x, a) \in \mathbb{G}. \tag{5.1}$$

[2] Recall that a function $w \colon \mathsf{X} \to \mathbb{R}$ is *ℓ-bounded* if $w(x)/\ell(x)$ is bounded as x ranges over X.

(d) The functions w_1 and w_2 satisfy

$$w_1(x) \leqslant Q(x, a, w_1) \quad \text{and} \quad Q(x, a, w_2) \leqslant w_2(x) \tag{5.2}$$

for all (x, a) in $\mathbb{G}$.

Remark 5.2.1. The above assumption is exactly the same as Assumption 4.2.1 of Chapter 4. Hence, the detailed exposition of the role for each condition is omitted here.

Given $\sigma \in \Sigma$, a function v_σ in $\mathscr{V}$ that satisfies

$$v_\sigma(x) = Q(x, \sigma(x), v_\sigma) \quad \text{for all } x \in \mathsf{X} \tag{5.3}$$

is called a *σ-value function*. The value $v_\sigma(x)$ can be understood as the lifetime value of following policy σ now and forever, starting from current state x. Its existence and uniqueness is discussed below.

5.2.1 Maximization

Analogous to the bounded case, our key assumption for a maximization problem is that the state-action aggregator Q satisfies value-convexity and possesses a strong upper solution in an appropriate functions space:

Assumption 5.2.2. (Convex Program) The following conditions are satisfied:

(a) Q is value-convex.

(b) There exists an $\varepsilon > 0$ such that $Q(x, a, w_2) \leqslant w_2(x) - \varepsilon \ell(x)$ for all $(x, a) \in \mathbb{G}$.

Note that part (b) is a strengthening of one of the conditions in (5.2).

Proposition 5.2.1. *If Assumptions 5.2.1 and 5.2.2 hold, then for each σ in Σ, the set $\mathscr{V}$ contains exactly one σ-value function v_σ.*

Proposition 5.2.1 assures us that the value v_σ of a given policy σ is well defined. From this foundation we can introduce optimality concerning a maximization decision problem. In particular, in the present setting, a policy $\sigma^* \in \Sigma$ is called *optimal* if

$$v_{\sigma^*}(x) \geqslant v_\sigma(x) \quad \text{for all } \sigma \in \Sigma \text{ and all } x \in \mathsf{X}.$$

The *maximum value function* associated with this planning problem is the map v^* defined at $x \in \mathsf{X}$ by

$$v^*(x) = \sup_{\sigma \in \Sigma} v_\sigma(x). \tag{5.4}$$

One can show from conditions (c) and (d) of Assumption 5.2.1 that v^* is well defined as a real valued function on X and satisfies $w_1 \leqslant v^* \leqslant w_2$.

A function $v \in \mathscr{V}$ is said to satisfy the *Bellman equation* if

$$v(x) = \max_{a \in \Gamma(x)} Q(x, a, v) \text{ for all } x \in \mathsf{X}. \tag{5.5}$$

The *Bellman operator* T associated with our abstract dynamic program is a map sending v in $\mathscr{C}$ into

$$Tv(x) = \max_{a \in \Gamma(x)} Q(x, a, v). \tag{5.6}$$

Since v is in $\mathscr{C}$, the existence of the maximum is guaranteed by Assumption 5.2.1. It follows from Berge's theorem of the maximum that Tv is an element of $\mathscr{C}$. Evidently solutions to the Bellman equation in $\mathscr{C}$ exactly coincide with fixed points of T.

The convex program conditions lead to the following central result and the resulting policy iteration algorithm:

Theorem 5.2.1. *If Assumptions 5.2.1 and 5.2.2 hold, then*

(a) The Bellman equation has exactly one solution in $\mathscr{C}$ and that solution is v^*.

(b) If v is in $\mathscr{C}$, then $T^n v \to v^*$ uniformly on X as $n \to \infty$.

(c) A policy σ in Σ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} Q(x, a, v^*) \text{ for all } x \in \mathsf{X}.$$

(d) At least one optimal policy exists.

One can verify that part (b) of Assumption 5.2.2 implies the state-action aggregator Q has a strong upper solution within the context of the Banach space endowed with weighted supremum norms. Hence, the proof of Theorem 5.2.1 is analogous to that of Theorem 4.2.1 of Chapter 4, and thus we will omit proof here.

Proposition 5.2.2. (Convergence of PI) *Let Assumptions 5.2.1 and 5.2.2 hold true, and let $\{\sigma^k\}_{k \geqslant 0}$ be a sequence generated by the PI algorithm. Then for all integers $k \geqslant 0$, we have $v_{\sigma^{k+1}} \geqslant v_{\sigma^k}$, with equality if and only if $v_{\sigma^k} = v^*$. Moreover,*

$$\lim_{k \to \infty} \|v_{\sigma^k} - v^*\| = 0,$$

and if the set of policies is finite, we have $v_{\sigma^k} = v^*$ for all large k.

5.2.2 Minimization

Analogous to Assumption 5.2.2 which was used for maximization, in this setting, we shall replace the value-convexity and the condition regarding a strong upper solution with value-concavity and the condition associated with a strong lower solution as follows

Assumption 5.2.3. (Concave Program) The following conditions are satisfied:

(a) Q is value-concave.

(b) There exists an $\varepsilon > 0$ such that $Q(x, a, w_1) \geqslant w_1(x) + \varepsilon\ell(x)$ for all $(x, a) \in \mathbb{G}$.

Note that part (b) is a strengthening of one of the conditions in (5.2).

Proposition 5.2.3. *If Assumptions 5.2.1 and 5.2.3 hold, then for each σ in Σ, the set $\mathscr{V}$ contains exactly one σ-value function v_σ.*

Proposition 5.2.3 mimics Proposition 5.2.1, assuring us that, in the present context, the cost v_σ of a given policy σ is well defined. A policy $\sigma^* \in \Sigma$ is then called *optimal* if

$$v_{\sigma^*}(x) \leqslant v_\sigma(x) \quad \text{for all } \sigma \in \Sigma \text{ and all } x \in \mathsf{X}.$$

The *minimum cost function* associated with this planning problem is the function v^* defined at $x \in \mathsf{X}$ by

$$v^*(x) = \inf_{\sigma \in \Sigma} v_\sigma(x). \tag{5.7}$$

A function $v \in \mathscr{V}$ is said to satisfy the *Bellman equation* if

$$v(x) = \min_{a \in \Gamma(x)} Q(x, a, v) \qquad \text{for all } x \in \mathsf{X}. \tag{5.8}$$

The *Bellman operator* S associated with our abstract dynamic program is a map sending v in $\mathscr{C}$ into

$$Sv(x) = \min_{a \in \Gamma(x)} Q(x, a, v). \tag{5.9}$$

The concave program conditions lead to the following central result with its corresponding policy iteration method:

Theorem 5.2.2. *If Assumptions 5.2.1 and 5.2.3 hold, then*

(a) The Bellman equation has exactly one solution in $\mathscr{C}$ and that solution is v^*.

(b) If v is in $\mathscr{C}$, then $S^n v \to v^*$ uniformly on X as $n \to \infty$.

(c) A policy σ in Σ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmin}_{a \in \Gamma(x)} Q(x, a, v^*) \ \textit{ for all } x \in \mathsf{X}.$$

(d) At least one optimal policy exists.

Proposition 5.2.4. (Convergence of PI) *Let Assumptions 5.2.1 and 5.2.3 hold true, and let* $\{\sigma^k\}$ *be a sequence generated by the PI algorithm. Then for all* k*, we have* $v_{\sigma^{k+1}} \leqslant v_{\sigma^k}$*, with equality if and only if* $v_{\sigma^k} = v^*$*. Moreover,*

$$\lim_{k \to \infty} \|v_{\sigma^k} - v^*\| = 0,$$

and if the set of policies is finite, we have $v_{\sigma^k} = v^*$ *for all large* k*.*

5.3 Applications

Now we move to the study of a collection of applications, showing how the general results in Section 5.2 can be used to solve the dynamic programming problems discussed in the introduction.

5.3.1 An additively Separable Decision Process

In this subsection, we present how the results stated above can be applied in the standard additive separable case, alongside the more traditional Bellman–Blackwell contraction mapping approach to dynamic programming, as found in, say, Boyd (1990). To see this, consider the generic additively separable discounted dynamic programming model of Stokey et al. (1989) with the Bellman equation

$$v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ F(s,y,z) + \beta \int v(y,z') P(z, \mathrm{d}z') \right\} \tag{5.10}$$

over $(s,z) \in \mathsf{S} \times \mathsf{Z}$. Here S and Z are separable metric spaces containing possible values for the endogenous and exogenous state variables, respectively. Let the transition function P on Z have the Feller property, let the feasible correspondence $\Gamma \colon \mathsf{S} \times \mathsf{Z} \to \mathsf{S}$ be compact valued and continuous, let the one-period return function $F \colon \mathbb{G} \to \mathbb{R}$ be continuous, and let the subjective discount factor β lie in $(0,1)$.

In order to accommodate the case where F is allowed to be unbounded, for our last assumption we replace the boundedness of F by

Assumption 5.3.1. There exist a continuous function $\kappa \colon \mathsf{S} \times \mathsf{Z} \to [1, \infty)$ and constants $M \in \mathbb{R}_+$ and $c \in (0, 1/\beta)$ satisfying the conditions

$$\sup_{y \in \Gamma(s,z)} |F(s,y,z)| \leqslant M\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.11}$$

$$\sup_{y \in \Gamma(s,z)} \int \kappa(y,z') P(z, \mathrm{d}z') \leqslant c\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}. \tag{5.12}$$

In addition the map $(y,z) \mapsto \int \kappa(y,z') P(z, \mathrm{d}z')$ is continuous on $\mathsf{S} \times \mathsf{Z}$.

Remark 5.3.1. The above assumption is mild and regular (see, e.g., Boyd (1990), Durán (2003), Matkowski and Nowak (2011) and Stachurski (2009), etc.).

We translate this model to our environment by taking $x := (s, z)$ to be the sate, $\mathsf{X} := \mathsf{S} \times \mathsf{Z}$ to be the state space, $a = y \in \mathsf{S}$ to be the action, $\ell = \kappa$ to be the weight function, and setting

$$Q((s,z), y, v) = F(s, y, z) + \beta \int v(y, z') P(z, \mathrm{d}z').$$

For the bracketing functions w_1 and w_2, we fix $\delta > 0$ and adopt the functions

$$w_1 := -\frac{M}{1-\beta c}\kappa \quad \text{and} \quad w_2 := \frac{M+\delta}{1-\beta c}\kappa.$$

Since κ is continuous on $\mathsf{S} \times \mathsf{Z}$, we note that such w_1 and w_2 are κ-bounded continuous functions on $\mathsf{S} \times \mathsf{Z}$ and satisfy $w_1 \leqslant w_2$.

In the sequel, we show that all conditions of Theorem 5.2.1 are satisfied. Regarding Assumption 5.2.1, conditions (a) and (c) are trivial to verify. While the measurability of the map $((s,z), y) \mapsto Q((s,z), y, v)$ in condition (b) is true by assumption, the continuity of that map is derived by the last condition in Assumption 5.3.1 along with the Feller property of P and the continuity imposed on F.[3]

To verify condition (d), observe that, for any given $((s,z), y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z), y, w_1) &= F(s, y, z) + \beta \int -\frac{M}{1-\beta c}\kappa(y, z') P(z, \mathrm{d}z') \\ &\geqslant -M\kappa(s,z) + \beta \int -\frac{M}{1-\beta c}\kappa(y, z') P(z, \mathrm{d}z') \\ &\geqslant -M\kappa(s,z) - \beta \frac{M}{1-\beta c} c\kappa(s,z) = w_1(s,z), \end{aligned}$$

where the first inequality follows from (5.11), while the second follows from (5.12).

Similarly, making use of the conditions in Assumption 5.3.1 again, we have

$$\begin{aligned} Q((s,z), y, w_2) &= F(s, y, z) + \beta \int \frac{M+\delta}{1-\beta c}\kappa(y, z') P(z, \mathrm{d}z') \\ &\leqslant M\kappa(s,z) + \beta \int \frac{M+\delta}{1-\beta c}\kappa(y, z') P(z, \mathrm{d}z') \\ &\leqslant M\kappa(s,z) + \beta \frac{M+\delta}{1-\beta c} c\kappa(s,z) = w_2(s,z) - \delta\kappa(s,z). \end{aligned}$$

It is worth noting that the last inequality gives not only $Q((s,z), y, w_2) \leqslant w_2$, as required for part (d) of Assumption 5.2.1, but also the stronger condition in part (b) of Assumption 5.2.2.[4] Thus, it only remains to check the convexity condition in part (a) of Assumption 5.2.2. But it

[3] For detailed proof of this statement, readers may refer to Lemma 1 of Durán (2003) or Lemma 12.2.20 in Stachurski (2009).

[4] Taking $\varepsilon \equiv \delta$, the condition in part (b) holds automatically.

is immediate from the linearity of expectation. Hence, all requirements of Theorem 5.2.1 have been verified and that theorem applies and yields the stated results.

5.3.2 Models with Epstein–Zin Recursive Preferences

In the following subsections, we aim to treat more sophisticated preference specifications proposed by Kreps and Porteus (1978) and Epstein and Zin (1989). To this end, recalling the preference specifications that we considered in Section 4.3.2 in Chapter 4, we have the generic Bellman equation under Epstein–Zin preferences as follows

$$v(s,z) = \max_{y \in \Gamma(s,z)} \left\{ r(s,y,z) + \beta \left[\int v(y,z')^{1-\gamma} P(z, \mathrm{d}z') \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}$$

for each $(s,z) \in \mathsf{S} \times \mathsf{Z}$, where, here and below,

$$r(s,y,z) := (1-\beta) F(s,y,z)^{1-\rho}.$$

In parameterization, the value $1/\rho$ indicates the level of EIS while γ governs the level of RRA. Henceforth, let a parameter θ be defined by

$$\theta := \frac{1-\gamma}{1-\rho}.$$

In what follows, we impose the same conditions on the primitives discussed in Section 5.3.1. In particular, F is continuous, P is Feller, Γ is continuous and compact valued and both S and Z are arbitrary separable metric spaces. To ensure $F(s,y,z)^{1-\rho}$ is always well defined, we also assume that F is everywhere positive.

The Case $\rho \leqslant \gamma < 1$

In this current setting, we have $0 < \theta \leqslant 1$. In addition to the basic assumptions, we make the following assumptions.

Assumption 5.3.2. There exist a continuous function $\kappa \colon \mathsf{S} \times \mathsf{Z} \to [1, \infty)$ and constants $M \in \mathbb{R}_+$ and $c \in (0, 1/\beta^\theta)$ satisfying the conditions

$$\sup_{y \in \Gamma(s,z)} r(s,y,z) \leqslant M\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.13}$$

$$\sup_{y \in \Gamma(s,z)} \int \kappa(y,z') P(z, \mathrm{d}z') \leqslant c\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}. \tag{5.14}$$

In addition the map $(y,z) \mapsto \int \kappa(y,z') P(z, \mathrm{d}z')$ is continuous on $\mathsf{S} \times \mathsf{Z}$.

As in Section 4.3.2 in Chapter 4, we apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation, and thus the transformed counterpart leads us to a maximization problem with a state-action aggregator that is exactly the same as in (4.13):

$$Q((s,z),y,\hat{v}) = \left\{ r(s,y,z) + \beta \left[\int \hat{v}(y,z')P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta}.$$

Moreover, we can transform this current problem into our framework by setting the weight function $\ell \equiv \kappa$ to accommodate unbounded return functions.

As this is a maximization problem, we aim to apply Theorem 5.2.1. For the bracketing functions w_1 and w_2, we fix a constant $\delta > 0$ and then adopt the functions

$$w_1 := \left(\frac{L}{1-\beta} \right)^{\theta} \quad \text{and} \quad w_2 := \left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right)^{\theta} \cdot \kappa,$$

where L is the infimum of $r(s,y,z)$ over $\mathbb{G}$. We note that in the current setting, this L can be zero. In addition, it is clear that such w_1 and w_2 defined above are κ-bounded, continuous and satisfying $w_1 \leqslant w_2$.

The conditions of Assumption 5.2.1 are all satisfied. Condition (a) is true by assumption, while the proof of verifying condition (c) is identical to that in Section 4.3.2 of Chapter 4. To see that condition (b) holds true, since measurability of Q is obvious, we shall verify the continuity of Q and pick any $\hat{v} \in \mathscr{C}$. By virtue of Assumption 5.3.2 with the Feller property of P, Lemma 12.2.20 in Stachurski (2009) applies and gives us that the map $(y,z) \mapsto \int \hat{v}(y,z')P(z,\mathrm{d}z')$ is continuous on $\mathsf{S} \times \mathsf{Z}$. It then follows from the continuity of r that the map $((s,z),y) \mapsto Q((s,z),y,\hat{v})$ is continuous on $\mathbb{G}$, as desired.

To see that condition (d) holds, observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned}
Q((s,z),y,w_2) &= \left\{ r(s,y,z) + \beta \left[\int \left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right)^{\theta} \cdot \kappa(y,z')P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \\
&\leqslant \left\{ M\kappa(s,z) + \beta \left[\int \left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right)^{\theta} \cdot \kappa(y,z')P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \\
&\leqslant \left\{ M\kappa(s,z) + \beta \left[\left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right)^{\theta} \cdot c\kappa(s,z) \right]^{1/\theta} \right\}^{\theta} \\
&= \left\{ M\kappa(s,z) + \beta \left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right) c^{1/\theta} \cdot \kappa(s,z)^{1/\theta} \right\}^{\theta} \\
&\leqslant \left\{ \left[M + \beta \left(\frac{M+\delta}{1-\beta c^{1/\theta}} \right) c^{1/\theta} \right] \kappa(s,z)^{1/\theta} \right\}^{\theta} \\
&= \left\{ \frac{M+\beta\delta c^{1/\theta}}{1-\beta c^{1/\theta}} \right\}^{\theta} \kappa(s,z) = \left\{ \frac{M+\delta}{1-\beta c^{1/\theta}} - \delta \right\}^{\theta} \kappa(s,z),
\end{aligned}$$

where the first equality follows from the definition of w_2. The first and second inequalities follow directly from (5.13) and (5.14) in Assumption 5.3.2, respectively, while the last follows from the fact that $\kappa \leqslant \kappa^{1/\theta}$.[5] It then follows that

$$Q((s,z),y,w_2) \leqslant \left\{ \frac{M+\delta}{1-\beta c^{1/\theta}} - \delta \right\}^{\theta} \kappa(s,z) < \left\{ \frac{M+\delta}{1-\beta c^{1/\theta}} \right\}^{\theta} \kappa(s,z), \tag{5.15}$$

and the last term is equal to $w_2(s,z)$.

On the other hand, we have

$$Q((s,z),y,w_1) = \left\{ r(s,y,z) + \beta \frac{L}{1-\beta} \right\}^{\theta} \geqslant \left\{ L + \beta \frac{L}{1-\beta} \right\}^{\theta} = w_1(s,z).$$

Hence condition (d) of Assumption 5.2.1 holds.

In fact, the uniformly strict inequality (5.15) implies that our choice of w_2 also satisfies the stronger condition in part (b) of Assumption 5.2.2.[6]

Thus, to verify the requirements of Theorem 5.2.1, we need only check the convexity condition in part (a) of Assumption 5.2.2. But this has been already proved in Section 4.3.2 of Chapter 4, and thus is omitted here. Therefore, Theorem 5.2.1 applies now and the conclusions of that theorem follow.

The Case $\rho < 1 < \gamma$

We note that this parameterization setting implies $\theta < 0$. In addition to the basic assumptions, we make the following assumptions.

Assumption 5.3.3. There exist a continuous function $\kappa \colon \mathsf{S} \times \mathsf{Z} \to (0, \infty)$, positive constants M, L with $L \leqslant M$, and c, $d \in (1/\beta^{\theta}, \infty)$ with $d \leqslant c$ satisfying the conditions

$$\sup_{y \in \Gamma(s,z)} r(s,y,z) \leqslant M\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.16}$$

$$\inf_{y \in \Gamma(s,z)} r(s,y,z) \geqslant L\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.17}$$

$$\sup_{y \in \Gamma(s,z)} \int \kappa(y,z')^{\theta} P(z, \mathrm{d}z') \leqslant c\kappa(s,z)^{\theta} \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.18}$$

$$\inf_{y \in \Gamma(s,z)} \int \kappa(y,z')^{\theta} P(z, \mathrm{d}z') \geqslant d\kappa(s,z)^{\theta} \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}. \tag{5.19}$$

In addition the map $(y,z) \mapsto \int \kappa(y,z')^{\theta} P(z, \mathrm{d}z')$ is continuous on $\mathsf{S} \times \mathsf{Z}$.

[5] Since $\kappa \geqslant 1$ and since $1/\theta \geqslant 1$, it is clear that $\kappa \leqslant \kappa^{1/\theta}$.

[6] To be precise, condition (b) holds when $\varepsilon := [(M+\delta)/(1-\beta c^{1/\theta})]^{\theta} - [(M+\delta)/(1-\beta c^{1/\theta}) - \delta]^{\theta}$, along with $\ell \equiv \kappa$.

As in Section 4.3.2 of Chapter 4, we apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation. Since $1-\gamma < 0$, the transformed counterpart leads us to a minimization problem with state-action aggregator

$$Q((s,z),y,\hat{v}) = \left\{ r(s,y,z) + \beta \left[\int \hat{v}(y,z')P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta}.$$

Furthermore, we can write this problem into our framework by setting the weight function $\ell \equiv \kappa^{\theta}$ to accommodate unbounded return functions.

As it is a minimization problem, we now aim to apply Theorem 5.2.2. For the bracketing functions w_1 and w_2, we fix a constant $\delta > 0$, and then adopt the functions

$$w_1 := \left(\frac{M+\delta}{1-\beta d^{1/\theta}} \right)^{\theta} \cdot \kappa^{\theta} \quad \text{and} \quad w_2 := L^{\theta} \cdot \kappa^{\theta}.$$

Evidently, the functions w_1 and w_2 are κ^{θ}-bounded, continuous and satisfying $w_1 \leqslant w_2$.

In the sequel, we show that all conditions of Theorem 5.2.2 are satisfied. Regarding Assumption 5.2.1, the arguments of verifying conditions (a) and (b) are essentially identical to those in Section 5.3.2. Condition (c) and value-concavity (in Assumption 5.2.3) have been already shown in Section 4.3.2 of Chapter 4.

It remains to be shown that condition (d) holds. To see that this is so, observe that, for any $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_1) &= \left\{ r(s,y,z) + \beta \left[\int \left(\frac{M+\delta}{1-\beta d^{1/\theta}} \right)^{\theta} \cdot \kappa(y,z')^{\theta} P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \\ &\geqslant \left\{ M\kappa(s,z) + \beta \left[\int \left(\frac{M+\delta}{1-\beta d^{1/\theta}} \right)^{\theta} \cdot \kappa(y,z')^{\theta} P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^{\theta} \\ &\geqslant \left\{ M\kappa(s,z) + \beta \left[\left(\frac{M+\delta}{1-\beta d^{1/\theta}} \right) d^{1/\theta} \cdot \kappa(s,z) \right] \right\}^{\theta} \\ &= \left\{ \frac{M+\beta\delta d^{1/\theta}}{1-\beta d^{1/\theta}} \right\}^{\theta} \kappa(s,z)^{\theta} = \left\{ \frac{M+\delta}{1-\beta d^{1/\theta}} - \delta \right\}^{\theta} \kappa(s,z)^{\theta} \end{aligned}$$

where the first equality follows from the definition of w_1, while the first and second inequalities follow from (5.16) and (5.19), respectively. Further, it follows that

$$Q((s,z),y,w_1) \geqslant \left\{ \frac{M+\delta}{1-\beta d^{1/\theta}} - \delta \right\}^{\theta} \kappa(s,z)^{\theta} > \left\{ \frac{M+\delta}{1-\beta d^{1/\theta}} \right\}^{\theta} \kappa(s,z)^{\theta}, \tag{5.20}$$

and the last term is equal to $w_1(s,z)$. Similarly, we have

$$
\begin{aligned}
Q((s,z),y,w_2) &= \left\{ r(s,y,z) + \beta \left[\int L^\theta \cdot \kappa(y,z')^\theta P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^\theta \\
&\leqslant \left\{ L\kappa(s,z) + \beta \left[\int L^\theta \cdot \kappa(y,z')^\theta P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^\theta \\
&\leqslant \left\{ L\kappa(s,z) + \beta \left[L \cdot c^{1/\theta} \kappa(s,z) \right] \right\}^\theta \\
&= \left\{ L(1+\beta c^{1/\theta}) \right\}^\theta \kappa(s,z)^\theta \leqslant L^\theta \kappa(s,z)^\theta = w_2(s,z).
\end{aligned}
$$

Hence condition (d) is satisfied. Furthermore, the uniformly strict inequality (5.20) implies that our choice of w_1 also satisfies the stronger condition in part (b) of Assumption 5.2.3.[7]

Hence, all conditions of Theorem 5.2.2 have now been verified and the conclusions of that theorem follow.

The Case $1 < \rho < \gamma$

In this current setting, we note that $\theta > 1$. In addition to the basic assumptions, we make the following assumptions.

Assumption 5.3.4. There exist a continuous function $\kappa \colon \mathsf{S} \times \mathsf{Z} \to [1,\infty)$, positive constants M, L with $L \leqslant M$, and $c \in (0, 1/\beta^\theta)$ and $d \in [0, 1/\beta^\theta)$ satisfying the conditions

$$\sup_{y \in \Gamma(s,z)} r(s,y,z) \leqslant M\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.21}$$

$$\inf_{y \in \Gamma(s,z)} r(s,y,z) \geqslant L\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.22}$$

$$\sup_{y \in \Gamma(s,z)} \int \kappa(y,z')^\theta P(z,\mathrm{d}z') \leqslant c\kappa(s,z)^\theta \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}, \tag{5.23}$$

$$\inf_{y \in \Gamma(s,z)} \int \kappa(y,z') P(z,\mathrm{d}z') \geqslant d\kappa(s,z) \quad \text{for all } (s,z) \in \mathsf{S} \times \mathsf{Z}. \tag{5.24}$$

In addition the map $(y,z) \mapsto \int \kappa(y,z')^\theta P(z,\mathrm{d}z')$ is continuous on $\mathsf{S} \times \mathsf{Z}$.

As in Section 5.3.2, we apply the continuous transformation $\hat{v} \equiv v^{1-\gamma}$ to the Bellman equation and, since $1-\gamma < 0$, the transformed counterpart leads us to the minimization problem with state-action aggregator

$$Q((s,z),y,\hat{v}) = \left\{ r(s,y,z) + \beta \left[\int \hat{v}(y,z') P(z,\mathrm{d}z') \right]^{1/\theta} \right\}^\theta .$$

[7] To be precise, condition (b) holds when $\varepsilon := [(M+\delta)/(1-\beta d^{1/\theta}) - \delta]^\theta - [(M+\delta)/(1-\beta d^{1/\theta})]^\theta$, along with $\ell \equiv \kappa^\theta$.

Furthermore, we write this problem into our framework by setting the weight function $\ell \equiv \kappa^\theta$ to accommodate unbounded return functions.

In this way, we aim to apply Theorem 5.2.2. For the bracketing functions w_1 and w_2, we fix a constant δ such that $0 < \delta < L$, and then adopt the functions

$$w_1 := (L-\delta)^\theta \cdot \kappa \quad \text{and} \quad w_2 := \left(\frac{M}{1-\beta c^{1/\theta}}\right)^\theta \cdot \kappa^\theta.$$

Note that $\kappa \leqslant \kappa^\theta$, since $\theta > 1$ and $\kappa \geqslant 1$. It then follows that w_1 and w_2 are both κ^θ-bounded. In addition, the positivity and the continuity of κ directly imply the positivity and continuity of w_1 and w_2. Hence, such w_1 and w_2 are positive κ^θ-bounded continuous functions in $\mathbb{R}^\mathsf{X}$ with $w_1 \leqslant w_2$.

Regarding Assumption 5.2.1, the arguments of verifying conditions (a) and (b) are identical to those in Section 5.3.2, while condition (c) and value-concavity of Q (in Assumption 5.2.3) have been proved in Section 4.3.2 of Chapter 4.

To check condition (d), observe that, for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_1) &= \left\{ r(s,y,z) + \beta \left[\int (L-\delta)^\theta \cdot \kappa(y,z') P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^\theta \\ &\geqslant \left\{ L\kappa(s,z) + \beta \left[\int (L-\delta)^\theta \cdot \kappa(y,z') P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^\theta \\ &\geqslant \left\{ L\kappa(s,z) + \beta (L-\delta) \left[d\kappa(s,z) \right]^{1/\theta} \right\}^\theta \\ &\geqslant \left\{ \left[L + \beta(L-\delta) d^{1/\theta} \right] \cdot \kappa(s,z)^{1/\theta} \right\}^\theta, \end{aligned} \tag{5.25}$$

where the first and second inequalities immediately follow from (5.22) and (5.24) in Assumption 5.3.4, respectively, while the last follows from the fact that $\kappa^{1/\theta} \leqslant \kappa$. Further, with some rearranging, we obtain

$$Q((s,z),y,w_1) \geqslant \left[L - \delta + \beta L d^{1/\theta} + \delta(1-\beta d^{1/\theta}) \right]^\theta \kappa(s,z) > (L-\delta)^\theta \kappa(s,z),$$

and the last term is equal to $w_1(s,z)$. Similarly, we have

$$\begin{aligned} Q((s,z),y,w_2) &= \left\{ r(s,y,z) + \beta \left(\frac{M}{1-\beta c^{1/\theta}} \right) \left[\int \kappa(y,z')^\theta P(z, \mathrm{d}z') \right]^{1/\theta} \right\}^\theta \\ &\leqslant \left\{ M\kappa(s,z) + \beta \left(\frac{M}{1-\beta c^{1/\theta}} \right) \left[c\kappa(s,z)^\theta \right]^{1/\theta} \right\}^\theta \\ &= \left[\frac{M}{1-\beta c^{1/\theta}} \right]^\theta \kappa(s,z)^\theta = w_2(s,z) \end{aligned}$$

where the inequality follows from (5.21) and (5.23) in Assumption 5.3.4. Hence condition (d) is verified.

So far we have only shown w_1 as a lower solution of Q as required for part (d) of Assumption 5.2.1. Thus, to satisfy all the requirements of Theorem 5.2.2, it only remains to prove that w_1 is a strong lower solution of Q, as required for part (b) of Assumption 5.2.3. Observe from (5.25) that to show the condition in part (b), it is sufficient to show that there exists an $\varepsilon > 0$ such that

$$\left\{L\kappa(s,z) + \beta(L-\delta)\left[d\kappa(s,z)\right]^{1/\theta}\right\}^{\theta} \geqslant w_1(s,z) + \varepsilon\kappa(s,z)^{\theta} \tag{5.26}$$

for all $(s,z) \in \mathsf{S} \times \mathsf{Z}$. To this end, for fixed $(s,z) \in \mathsf{S} \times \mathsf{Z}$, consider

$$\begin{aligned}
&\frac{\left\{L\kappa(s,z) + \beta(L-\delta)\left[d\kappa(s,z)\right]^{1/\theta}\right\}^{\theta} - w_1(s,z)}{\kappa(s,z)^{\theta}} \\
&= \left\{L + \beta(L-\delta)d^{1/\theta}\cdot\kappa(s,z)^{1/\theta-1}\right\}^{\theta} - (L-\delta)^{\theta}\kappa(s,z)^{1-\theta} \\
&\geqslant L^{\theta} - (L-\delta)^{\theta}\kappa(s,z)^{1-\theta} \geqslant L^{\theta} - (L-\delta)^{\theta} > 0
\end{aligned}$$

where the first and second inequalities follow from the facts that $\kappa^{1/\theta-1} \geqslant 0$ and that $\kappa^{1-\theta} \leqslant 1$, respectively. Hence, condition (5.26) holds when we take $\varepsilon := L^{\theta} - (L-\delta)^{\theta}$, which is what we needed to show for condition (b) in Assumption 5.2.3.

Therefore, we have now checked all conditions of Theorem 5.2.2 and the conclusions of that theorem follow.

Chapter 6

Risk-Adjusted Metrics for Recursive Preference Theory

6.1 Introduction

In principle, additive separability can easily be bypassed by specifying dynamic preferences recursively. At the same time, the uptake of recursive preference methods in applied work has been slow. With notable exceptions, the overwhelming majority of studies use the familiar additively separable framework. We conjecture that a large part of this reluctance is due to technical difficulties, either real or perceived. This is perhaps due to the fact that the Bellman operators associated with the models typically fail to be contractive in any standard metric (see, e.g., Rincón-Zapatero and Rodríguez-Palmero (2007), Marinacci and Montrucchio (2010, 2017), Balbus (2016) or Bloise and Vailakis (2018)).

In this essay, we show that some of the most popular recursive preference models in terms of applications are in fact contractive, *once a modification is made to the standard metric that accounts for risk preferences.* When this risk-adjusted metric is adopted, the Bellman operator is a contraction mapping and the usual results apply: iteration with the Bellman operator from a natural set of initial conditions generates a sequence that converges to the value function, and a policy is optimal if and only if it maximizes the right-hand side of the Bellman equation at each state.[1] Moreover, we can also use a variety of algorithms popular in the dynamic programming literature, such as policy iteration and optimistic policy iteration.

To be more precise, the basic idea of our work is to twist the standard metrics traditionally used for dynamic programming by inserting one part of the risk preferences found in recursive

[1] It is worth noting that the transformation technique used in Hansen and Scheinkman (2012) and Guo and He (2018) is about transforming the original problem to a new modified problem and trying to solve this modified problem by establishing the contraction argument with the standard metrics, and then utilizing the one-to-one correspondence of this transformation to recover the solution to the original problem from the solution to the modified problem. In contrast, the risk-adjusted metric method is about twisting the standard metrics in order to ensure that some original problems which can not be solved by the contraction mapping theorem with the standard metrics, could be directly solved by the contraction mapping theorem under the twisted metrics.

utility models. This part can be seen as a continuous monotonic transformation associated with the ratio of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS). With additional important characterizations of the continuous monotonic transformation, we show that the twisted metric associated with this specific transformation is complete. Then, we exploit a geometric property of this transformation function, which leads us to obtain contraction mapping results. This geometric property plays a crucial role in establishing the contraction mapping results of the policy value operator and the Bellman operator.

The framework we propose is relatively general, and particularly well suited to handling a specific class of Epstein–Zin preferences, which is essential given their popularity within the literature on recursive preferences.

One of the attractive features of our framework is that the classical theory of stationary discrete time dynamic programming (see, e.g., Bellman (1957), Blackwell (1965) or Stokey et al. (1989)) can be obtained as a special case. In particular, if preferences are additively separable, then all of the standard results for Markov decision processes become simple corollaries of our main findings.

On the negative side, this means that our approach shares the weaknesses of the standard theory, such as difficulty in handling certain economic models where rewards are unbounded. While our theory is based on top of weighted supremum norms, which allow for some unboundedness and can perhaps be understood as the limit of the standard theory (see the insightful discussion in Chapter 2 of Bertsekas (2013)), this does not solve all issues with unboundedness, particularly when rewards are unbounded from below.

Another purpose of this essay is to provide easy-to-check conditions under which stochastic recursive models are represented by a well defined recursive optimization problem, and in turn, characterized by its associated Bellman equation.

We follow the conventional approach to recursive dynamic programming by showing first that the dynamic programming equation indeed has a solution. To this end, a Bellman (maximization) operator is defined whose fixed points are solutions to the dynamic programming (Bellman) equation. We then seek solutions to the Bellman equation by proving that the Bellman operator is a contraction on our twisted metric space. The domain of the Bellman operator is this twisted metric space being complete, which can be seen as the class of admissible functions: candidates to solve the Bellman equation and, ultimately, to be the value function. To accommodate reward functions that are not necessarily bounded from above, the admissible functions are considered in a space of potentially unbounded functions equipped with the twisted metric induced by the weighted supremum norm. In particular, unbounded rewards are permitted, provided they do not cause true value functions to diverge. As a result, making use of the Banach fixed point theorem yields the existence, uniqueness and global attractivity of the fixed point of the Bellman operator.

In order to show the principle of optimality, an alternative operator is often needed to be considered and studied. That is, an intertemporal recursion operator which is defined corresponding to each stationary (Markov) policy. Hereby, the fixed point of this operator is the corresponding policy value function. Then, applying our general fixed point results established in Section 6.2 leads us to obtain the existence and uniqueness results of the policy value function. In particular, as a consequence of Banach fixed point theorem, the global convergences of the iterative method for the fixed point of the Bellman operator and for the policy value function are obtained. In the sequel, we prove that the value function is the unique solution to the Bellman equation, and optimal policy exists.

As a benefit of our method, under the appropriate assumptions with respect to the reward function and production technology, the true value function can then be shown to possess other important properties such as monotonicity, concavity, and differentiability. In addition, as a by-product, we derive a general Euler equation for the dynamic programs that we consider.

Finally, we reconsider a special case where reward function is bounded and prove that the convergence in twisted metric topology implies the original convergence in supremum-norm topology. Meanwhile, we also derive the convergence of value iterations in twisted metric topology.

This essay is organized as follows. In Section 6.2, we establish an abstract fixed point result, which will be utilized to show the existence, uniqueness and globally attracting results of the (Markov) policy value function. Following that, we provide some illustrative examples associated with those abstract settings of the fixed point result in Section 6.3. Section 6.4 formulates a dynamic programming problem, that is, we describe the stochastic set-up and derive the objective function in the recursive formulation. Section 6.5 provides main results; that is, showing the existence, uniqueness and global stability of the solution to the Bellman equation and the optimality result. In Section 6.6, we exploit some potentially useful properties of the value function. Under some regular assumptions, it turns out that the value function is strictly increasing, concave and differentiable. In addition, we derive a general Euler equation corresponding to the dynamic programming problem. Finally, in Section 6.7, we present some applications of our theory, which include a stochastic growth model and an income fluctuation model.

6.2 General Fixed Point Results

We begin with a general fixed point result that will later be applied. In all of what follows, we take $(\mathbb{E}, \leqslant)$ to be a real Riesz space and let $\|\cdot\|$ be a map from $\mathbb{E}$ to $[0, +\infty]$ where, when restricted to

$$\mathbb{B} := \{u \in \mathbb{E} \colon \|u\| < +\infty\},$$

$\|\cdot\|$ becomes a complete lattice norm on $\mathbb{B}$.[2] In particular, $(\mathbb{B}, \|\cdot\|)$ is a Banach lattice contained in $\mathbb{E}$.

[2] Obviously, $\mathbb{B}$ is a linear subspace of $\mathbb{E}$.

In the sequel, we introduce a map Φ from the positive cone $\mathbb{E}_+$ of $\mathbb{E}$ to itself. This map Φ is called an *attitude-adjusted transformation (operator)* if

(i) $\Phi\colon \mathbb{E}_+ \to \mathbb{E}_+$ is a bijection on $\mathbb{E}_+$; and

(ii) for any u, v, w in $\mathbb{E}_+$, we have

$$|\Phi(u+w) - \Phi(v+w)| \leqslant |\Phi u - \Phi v| \,. \tag{6.1}$$

The reasoning behind our terminology will be clarified in the following examples and applications.

The *preimage* of the positive cone $\mathbb{B}_+ \subset \mathbb{E}_+$ of the Banach lattice $\mathbb{B}$ under the map Φ is the subset of $\mathbb{E}_+$ defined by

$$\Phi^{-1}(\mathbb{B}_+) = \{u \in \mathbb{E}_+ \colon \Phi u \in \mathbb{B}_+\} \subset \mathbb{E}_+.$$

For the sake of simplicity, we typically denote by $\mathbb{F}$ the preimage $\Phi^{-1}(\mathbb{B}_+)$ of the positive cone $\mathbb{B}_+$ under Φ throughout this chapter. Below, the set $\mathbb{F}$ will be our candidate function space for fixed points.

6.2.1 An Attitude-Adjusted Metric

Before progressing further, we need to induce a metric on $\mathbb{F}$ that twists the metric on $\mathbb{B}$ using the attitude-adjusted operator Φ. To this end, let a map $d_\Phi\colon \mathbb{F} \times \mathbb{F} \to \mathbb{R}$ be defined by

$$d_\Phi(u, v) := \|\Phi u - \Phi v\| \qquad (\forall u, v \in \mathbb{F}), \tag{6.2}$$

where $\|\cdot\|$ is the complete lattice norm on $\mathbb{B}$.

The following lemma shows that by our construction the map d_Φ defined in (6.2) is not only a well-defined metric on $\mathbb{F}$ but also a complete metric on $\mathbb{F}$.

Lemma 6.2.1. *The map d_Φ is a complete metric on $\mathbb{F}$.*

6.2.2 The Fixed Point Result

Recall that $(\mathbb{E}, \leqslant)$ is a fixed real Riesz space and $(\mathbb{B}, \|\cdot\|)$ is a Banach lattice contained in $\mathbb{E}$. Let $\Phi\colon \mathbb{E}_+ \to \mathbb{E}_+$ be a fixed attitude-adjusted operator, and $\mathbb{F}$ be the preimage of $\mathbb{B}_+ \subset \mathbb{E}_+$ under Φ.

We now consider an operator $\boldsymbol{A}$ from $\mathbb{F}$ to $\mathbb{E}_+$, defined by

$$\boldsymbol{A} f = h + \Phi^{-1} \boldsymbol{K} \Phi f \qquad (f \in \mathbb{F}), \tag{6.3}$$

where

(i) h is a fixed element of $\mathbb{F}$; and

(ii) $\boldsymbol{K}$ is a linear monotone increasing operator from $\mathbb{B}$ to itself.

Definition 6.2.1. A subset $\mathcal{D}$ of a real vector space $\mathbb{E}$ is a *semi-module* if, for any elements u and v in $\mathcal{D}$, the element $u+v$ of $\mathbb{E}$ is also in $\mathcal{D}$.

Definition 6.2.2. An operator $\boldsymbol{A}$ defined on a semi-module $\mathcal{D}$ is *sub-additive* if, for any u and v in $\mathcal{D}$, we have $\boldsymbol{A}(u+v) \leqslant \boldsymbol{A}u + \boldsymbol{A}v$.

We note that any convex cone is a semi-module. Evidently, the positive cones $\mathbb{E}_+$ and $\mathbb{B}_+$ are pointed convex cones and hence are semi-modules.[3]

To see that $\boldsymbol{A}$ does in fact map elements of $\mathbb{F}$ to $\mathbb{E}_+$, pick any $f \in \mathbb{F}$ and observe that $\Phi f \in \mathbb{B}_+$ by the definition of $\mathbb{F}$. Since $\boldsymbol{K}$ is linear and monotone increasing, and hence positive, we then have $\boldsymbol{K}\Phi f \in \mathbb{B}_+$. It then follows that $\Phi^{-1}(\boldsymbol{K}\Phi f)$ lies in $\mathbb{F} \subset \mathbb{E}_+$. Given that h also belongs to $\mathbb{F} \subset \mathbb{E}_+$, we conclude that $\boldsymbol{A}f = h + \Phi^{-1}\boldsymbol{K}\Phi f$ lies in $\mathbb{E}_+$.

Assumption 6.2.1. The attitude-adjusted operator Φ is sub-additive on $\mathbb{E}_+$.

Lemma 6.2.2. *If Assumption 6.2.1 holds, then the operator $\boldsymbol{A}$ defined in (6.3) maps $\mathbb{F}$ to itself.*

Proof. As shown above, given any $f \in \mathbb{F}$, the image $\boldsymbol{A}f$ can be expressed as the addition of two elements of $\mathbb{F}$, i.e., $\boldsymbol{A}f = h + g$ with $h \in \mathbb{F}$ and $g := \Phi^{-1}\boldsymbol{K}\Phi f \in \mathbb{F}$.

Since $\boldsymbol{A}f \in \mathbb{E}_+$ and Φ is sub-additive on $\mathbb{E}_+$, we have

$$\Phi(\boldsymbol{A}f) = \Phi(h+g) \leqslant \Phi h + \Phi g.$$

By virtue of the properties of the lattice norm on $\mathbb{B}$, we have $\|\Phi(\boldsymbol{A}f)\| \leqslant \|\Phi h + \Phi g\| \leqslant \|\Phi h\| + \|\Phi g\|$. It then follows from $\Phi h, \Phi g \in \mathbb{B}_+$ that $\|\Phi(\boldsymbol{A}f)\| < +\infty$. Hence, together with $\Phi(\boldsymbol{A}f) \in \mathbb{E}_+$, the finiteness of $\|\Phi(\boldsymbol{A}f)\|$ implies $\Phi(\boldsymbol{A}f) \in \mathbb{B}_+$; that is, $\boldsymbol{A}f \in \mathbb{F}$ as desired. □

We can now state the main fixed point results, as two theorems with one corollary.

Theorem 6.2.1. *If Assumption 6.2.1 holds and $\|\boldsymbol{K}\| < 1$, then we may conclude the following:*

(a) Existence and uniqueness*: The operator $\boldsymbol{A}$ defined in (6.3) has exactly one fixed point f^* on $\mathbb{F}$;*

(b) Convergence of the iteration*: The sequence $\{f_n\}$ of successive approximations ($f_{n+1} = \boldsymbol{A}f_n$, $\forall n \in \mathbb{N}_0$) converges to the fixed point f^*, for an arbitrary choice of initial point f_0 in $\mathbb{F}$;*

[3] Recall that a subset $\mathbb{C}$ of a vector space $\mathbb{E}$ is a *pointed convex cone* if (1) $a\mathbb{C} \subset \mathbb{C}$ for each $a \in \mathbb{R}_+$; (2) $\mathbb{C}$ is convex, i.e., $u, v \in \mathbb{C}$ implies $au + bv \in \mathbb{C}$ for all $a, b \in \mathbb{R}_+$; and (3) $\mathbb{C}$ is pointed, i.e., $\mathbb{C} \cap -(\mathbb{C}) = \{\mathbf{0}\}$. In particular, conditions (1) and (2) together amount to $\mathbb{C} + \mathbb{C} \subset \mathbb{C}$.

(c) Error estimates*: For all $n \in \mathbb{N}_0$, we have a priori error estimate*

$$d_\Phi(f_n, f^*) \leqslant \|\boldsymbol{K}\|^n (1 - \|\boldsymbol{K}\|)^{-1} d_\Phi(f_0, f_1),$$

and a posteriori error estimate

$$d_\Phi(f_{n+1}, f^*) \leqslant \|\boldsymbol{K}\| (1 - \|\boldsymbol{K}\|)^{-1} d_\Phi(f_n, f_{n+1});$$

(d) Rate of convergence*: For all $n \in \mathbb{N}_0$, we have*

$$d_\Phi(f_{n+1}, f^*) \leqslant \|\boldsymbol{K}\| d_\Phi(f_n, f^*).$$

Theorem 6.2.2. *(Sharpening of Theorem 6.2.1) If Assumption 6.2.1 holds and $r(\boldsymbol{K}) < 1$, then there exists an $N \in \mathbb{N}$ such that $\boldsymbol{A}^N$ is a strict contraction map on $(\mathbb{F}, d_\Phi)$.*

Remark 6.2.1. Since $r(\boldsymbol{K}) \leqslant \|\boldsymbol{K}\|$, to obtain the spectral radius condition $r(\boldsymbol{K}) < 1$, it suffices that $\|\boldsymbol{K}\| < 1$. In fact, as will be clear from the proof, if $\|\boldsymbol{K}\| < 1$, then $\boldsymbol{A}$ itself is a strict contraction mapping, with contraction coefficient $\|\boldsymbol{K}\|$.

Corollary 6.2.1. *Under the conditions of Theorem 6.2.2, the following statements hold:*

(a) *$\boldsymbol{A}$ has exactly one fixed point f^* in $\mathbb{F}$; and*

(b) *$d_\Phi(\boldsymbol{A}^k f, f^*) \to 0$ as $k \to \infty$ for any $f \in \mathbb{F}$.*

Proof. This follows from the fact that $(\mathbb{F}, d_\Phi)$ is a complete metric space and $\boldsymbol{A}^N$ is a strict contraction for some $N \in \mathbb{N}$. See, for example, the corollary in p. 272 of Wagner (1982). □

6.3 Illustrative Examples

In the following, we shall give some concrete examples to illustrate the abstract notions used in the preceding section.

6.3.1 Candidate Function Spaces

Let $\mathbb{S}$ be a non-empty set and assumed to be Polish.[4] Let $\mathscr{B}(\mathbb{S})$ be the Borel sets of $\mathbb{S}$. Consider the space $\mathbb{R}^{\mathbb{S}}$ of all functions from $\mathbb{S}$ to $\mathbb{R}$. Some common subspaces form Riesz spaces under the standard notions of pointwise addition, scalar multiplication, and pointwise order. Examples include

- the set $m(\mathbb{S})$ of all Borel-measurable functions from $\mathbb{S}$ to $\mathbb{R}$; and

[4]That is, $\mathbb{S}$ is separable and completely metrizable. The Polish assumption is very weak, and used primarily to avoid measurability concerns.

- the set $c(\mathbb{S})$ of all continuous functions from $\mathbb{S}$ to $\mathbb{R}$.

Certain subsets of these spaces become Banach lattices when paired with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{S}} |f(x)|$. These include

- the set $bm(\mathbb{S})$ of all bounded functions in $m(\mathbb{S})$; and
- the set $bc(\mathbb{S})$ of all bounded functions in $c(\mathbb{S})$.

Furthermore, given a continuous function $\kappa\colon \mathbb{S} \to (0,\infty)$, when equipped with the weighted supremum norm $\|\cdot\|_\kappa$, the Riesz spaces $b_\kappa m(\mathbb{S})$ of all κ-bounded functions in $m(\mathbb{S})$ and $b_\kappa c(\mathbb{S})$ of all κ-bounded functions in $c(\mathbb{S})$ are all Banach lattices.[5]

6.3.2 Attitude-Adjusted Transformations

Recalling the attitude-adjusted transformations on positive cones of a Riesz space that were defined in Section 6.2, we now consider some illustrative examples for such transformations, all of which relate to compositions of payoffs with some monotone increasing concave scalar functions.

Before doing so, we present an important lemma, which plays a crucial role in our further analysis.

Lemma 6.3.1. *If a function $\phi\colon \mathbb{R} \to \mathbb{R}$ is increasing and concave or decreasing and convex, then for every $a, b \in \mathbb{R}$ and any $c \in \mathbb{R}_+$, we have*

$$|\phi(a+c) - \phi(b+c)| \leqslant |\phi(a) - \phi(b)| \,. \tag{6.4}$$

Example 6.3.1. Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a bijection on $\mathbb{R}_+$ and be strictly increasing and concave. If the map $\Phi\colon m(\mathbb{S})_+ \to m(\mathbb{S})_+$ is defined by $\Phi u = \phi \circ u$, then Φ is an attitude-adjusted transformation on $m(\mathbb{S})_+$. Evidently, Φ is onto. Moreover, it is one-to-one because $\phi \circ u = \phi \circ v$ implies $u = v$ for any in $u, v \in m(\mathbb{S})_+$ by strict monotonicity of ϕ. Inequality (6.1) corresponding to Φ follows directly from the fact that ϕ is increasing and concave on $\mathbb{R}_+$. Evidently, invoking Lemma 6.3.1 and applying (6.4) pointwise for nonnegative functions u, v and w in $m(\mathbb{S})$ give us (6.1).

In particular, a function $\phi\colon \mathbb{R}_+ \ni t \mapsto t^\theta \in \mathbb{R}_+$ with $\theta \in (0,1]$ possesses all the aforementioned properties.

Example 6.3.2. Let $(\Phi f)(x) = \phi \circ f(x)$ where the setting of ϕ is the same as in Example 6.3.1. Note that the strict monotonicity and bijectivity of ϕ implies the continuity of ϕ on $\mathbb{R}_+$. It then follows that Φ maps $c(\mathbb{S})_+$ onto itself, since the composition of continuous functions is still continuous. Φ is also one-to-one on $c(\mathbb{S})_+$ by strict monotonicity of ϕ on $\mathbb{R}_+$. Being increasing

[5] For more details, please refer to Example 2.2.2 in Section 2.2.1 of Chapter 2.

and concave, the function ϕ also satisfies (6.4), which in turn implies that (6.1) holds. Hence Φ is an attitude-adjusted transformation on $c(\mathbb{S})_+$.

The following example shows what kinds of attitude-adjusted transformations are sub-additive.

Example 6.3.3. Take again the setting of ϕ in Example 6.3.1. Observe that the function ϕ leaves nonnegative real numbers $\mathbb{R}_+$ invariant (i.e., $\phi(\mathbb{R}_+) \subset \mathbb{R}_+$). It then follows that Φ is sub-additive on $m(\mathbb{S})_+$ due to subadditivity of ϕ on the real line (as follows from the stated properties on ϕ—see, e.g., Theorem 1.4.3 in Rosenbaum (1950)), which gives rise to the pointwise inequality

$$\phi \circ [u(x) + v(x)] \leqslant \phi[u(x)] + \phi[v(x)] \qquad (x \in \mathbb{S}), \tag{6.5}$$

for any $u, v \in m(\mathbb{S})_+$.

6.4 Dynamic Programming with Generic Recursive Preferences

6.4.1 The Model Framework

In the sequel, we consider an abstract infinite-horizon stochastic dynamic programming problem, that is defined by a tuple $(\mathbb{S}, \mathbb{A}, \Gamma, r, \mathbb{Z}, F, \beta, \phi)$ of objects satisfying the following conditions:

A1: $\mathbb{S}$ is a Borel space, referred to hereafter as a *state* space.[6]

A2: $\mathbb{A}$ is a Borel space of actions of the agent, referred to hereafter as an *action* space.

A3: Γ is a non-empty correspondence mapping $x \in \mathbb{S}$ into $\mathbb{A}$. For each $x \in \mathbb{S}$ the set $\Gamma(x)$ is interpreted as the collection of all feasible actions for the agent when the current state is x.

A4: $r\colon \operatorname{gr}\Gamma \to \mathbb{R}_+$ is a Borel-measurable instantaneous *return* function, where $\operatorname{gr}\Gamma := \{(x,a) \in \mathbb{S} \times \mathbb{A} \colon a \in \Gamma(x)\}$ denotes a graph of Γ, which is also called the set of feasible state/action pairs.

A5: $\mathbb{Z}$ is a Borel space, referred to hereafter as a *shock* space. Let $\{W_t\}_{t\in\mathbb{N}}^{\infty}$ be a sequence of random shocks that is assumed to be an independent and identically distributed (IID) random process defined on a canonical probability space $(\Omega, \mathscr{F}, \mathcal{P})$, where $\Omega = \mathbb{Z}^\infty$ is the product space endowed with its σ-algebra $\mathscr{F}$, and $\mathcal{P}\colon \mathscr{F} \to [0,1]$ is a countably additive probability measure. The marginal distribution is denoted by $\mathscr{P} \in \Delta(\mathbb{Z})$.[7]

A6: $F\colon \operatorname{gr}\Gamma \times \mathbb{Z} \ni (x,a,z) \mapsto F(x,a,z) \in \mathbb{S}$ is a measurable *transition* function, which captures the dynamics.

A7: $\beta \in (0,1)$ is a time discount factor.

[6] By a Borel space we mean a non-empty Borel subset of a complete separable metric space.

[7] The notation $\Delta(\mathbb{Z})$ denotes the space of probability measures on the measurable space $(\mathbb{Z}, \mathscr{B}(\mathbb{Z}))$.

A8: $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is some continuous monotonic transformation measuring the relation between risk aversion and EIS, which will be stated in detail below.

Before we proceed, we make the following assumptions.

Assumption 6.4.1. The correspondence $\Gamma\colon \mathbb{S} \to \mathbb{A}$ is continuous and compact valued.

Assumption 6.4.2. The transition function F is a Carathéodory function in the sense that: (1) for each $z \in \mathbb{Z}$, the function $F(\cdot,\cdot,z)\colon \operatorname{gr}\Gamma \ni (x,a) \mapsto F(x,a,z) \in \mathbb{S}$ is continuous; and (2) for each $(x,a) \in \mathbb{S} \times \mathbb{A}$, the function $F(x,a,\cdot)\colon \mathbb{Z} \to \mathbb{S}$ is Borel-measurable.

Assumption 6.4.3. The reward function $r\colon \operatorname{gr}\Gamma \to \mathbb{R}_+$ is continuous.

Assumption 6.4.4. The map $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing, concave, C^1-diffeomorphism and satisfies[8]

$$\phi(0) = 0, \qquad \lim_{s\to+\infty} \phi(s) = +\infty, \quad \text{and} \quad \phi(ts) = \phi(t)\cdot\phi(s) \qquad (\forall\, t,s \in \mathbb{R}_+).$$

Remark 6.4.1. One can verify that Assumption 6.4.4 is equivalent to assuming the function ϕ has the form $\phi(t) = t^\theta$ for some $\theta \in (0,1]$. In the context of non-additive Markov operators, the function ϕ may have the form $\phi(t) = Mt^\theta$ for some constants $M > 0$ and $\theta \in (0,1]$, since the effect of the constant M can be canceled out.

Assumption 6.4.5. There exists a continuous increasing function $\kappa\colon \mathbb{S} \to [1,\infty)$ and constants $R \in \mathbb{R}_+$ and $N_\kappa \in [0, 1/\phi(\beta))$ satisfying the conditions:

$$\sup_{a\in\Gamma(x)} |\phi \circ r(x,a)| \leqslant R\kappa(x) \qquad (x \in \mathbb{S}), \tag{6.6}$$

$$\sup_{a\in\Gamma(x)} \int \kappa\,[F(x,a,z)]\,\mathscr{P}(\mathrm{d}z) \leqslant N_\kappa \kappa(x) \qquad (x \in \mathbb{S}). \tag{6.7}$$

In addition, the map $(x,a) \mapsto \int \kappa\,[F(x,a,z)]\,\mathscr{P}(\mathrm{d}z)$ is continuous on $\operatorname{gr}\Gamma$.

Remark 6.4.2. Except for the assumption of the transformation function ϕ, the remaining assumptions used here are quite regular and analogous to those imposed in Chapter 12 of Stachurski (2009) and in Bäuerle and Jaśkiewicz (2018).

In what follows, we denote by $P(x,a;\mathrm{d}y)$ a distribution over $\mathbb{S}$ for each feasible state/action pair $(x,a) \in \operatorname{gr}\Gamma$. In other words, $P(x,a;\mathrm{B})$ is interpreted as the conditional probability that the next period state will be $X_{t+1} \in \mathrm{B}$ when the current state is $X_t = x$ and the current action $A_t = a$. For instance, according to the next period state is determined according to the transition function F such as

$$X_{t+1} = F\left(X_t,\, A_t,\, W_{t+1}\right),$$

[8] The notation C^1 stands for the collection of all differentiable functions whose derivative is continuous. Given two manifolds $\mathbb{X}$ and $\mathbb{Y}$, a differentiable map $\phi\colon \mathbb{X} \to \mathbb{Y}$ is called a *diffeomorphism* if it is a bijection and its inverse $\phi\colon \mathbb{Y} \to \mathbb{X}$ is differentiable as well. If in addition, these functions are continuously differentiable, then ϕ is called a C^1-diffeomorphism.

then

$$P(x,a;\mathrm{B}) = \int \mathbb{1}_{\mathrm{B}}\left[F\left(x,\, a,\, z\right)\right] \mathscr{P}(\mathrm{d}z). \tag{6.8}$$

If there is no ambiguity, we typically write the distribution $P(x,a;\mathrm{d}y)$ as $P_a(x,\mathrm{d}y)$ for simplicity.

In order to construct a sensible optimization problem, we confine the agent to policies in the set of *stationary Markov policies*.[9] Under a stationary Markov policy, the agent's behavior is described by a Borel-measurable function σ mapping each possible $x \in \mathbb{S}$ into a feasible action $a \in \Gamma(x)$. Denote by Σ the set of all Borel-measurable mappings $\sigma\colon \mathbb{S} \to \mathbb{A}$ such that $\sigma(x) \in \Gamma(x)$ for each $x \in \mathbb{S}$. We often refer to Σ simply as the set of feasible policies.

In this way, for each $\sigma \in \Sigma$, we obtain a stochastic recursive sequence

$$X_{t+1} = F\left(X_t, \sigma\left(X_t\right), W_{t+1}\right), \qquad \{W_t\}_{t\in\mathbb{N}} \overset{\text{IID}}{\sim} \mathscr{P}$$

for the state $\{X_t\}_{t\in\mathbb{N}_0}$, and hence a stochastic kernel $P_\sigma(x,\mathrm{d}y)$ on $\mathbb{S}$ given by[10]

$$P_\sigma(x,\mathrm{B}) := \int \mathbb{1}_{\mathrm{B}}\left[F\left(x,\sigma(x),z\right)\right] \mathscr{P}(\mathrm{d}z) \qquad (x \in \mathbb{S},\ \mathrm{B} \in \mathscr{B}(\mathbb{S})) \tag{6.9}$$

where $\mathbb{1}_{\mathrm{B}}$ is the characteristic (indicator) function of the set B.

To set up the problem, we consider a generic recursive preference of the form

$$U_t = u(C_t) + \beta\,\phi^{-1}\left(\mathrm{E}_t\phi \circ U_{t+1}\right), \tag{6.10}$$

and then define an *intertemporal recursion operator* $\boldsymbol{T}_\sigma$ corresponding to a stationary Markov policy $\sigma \in \Sigma$ as follows:

$$\boldsymbol{T}_\sigma w(x) := r\left(x,\sigma(x)\right) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,\sigma(x),z)]\mathscr{P}(\mathrm{d}z)\right) \tag{6.11}$$

for any $w \in m(\mathbb{S})_+$ and each $x \in \mathbb{S}$. A function w is *recursive* when $w = \boldsymbol{T}_\sigma w$.[11]

For the sake of simplicity, it is convenient to define the function

$$r_\sigma\colon S \ni x \mapsto r\left(x,\sigma(x)\right) \in \mathbb{R}_+$$

so that $r_\sigma(x)$ is the reward at x when the agent follows policy σ.

Invoking (6.9), it is also convenient to define a non-additive Markov operator $\boldsymbol{M}_\sigma$ correspond-

[9] For such a policy, the agent makes the same decision after observing $X_t = x$ as after observing $X_{t+n} = x$ at some later date $t+n$ with $n \geqslant 1$.

[10] Here, $\mathscr{B}(\mathbb{S})$ denotes the σ-algebra of Borel subsets of $\mathbb{S}$.

[11] The equation $w = \boldsymbol{T}_\sigma w$ is sometimes refereed to as Koopmans equation.

ing to the stochastic kernel P_σ as follows

$$\begin{aligned} \boldsymbol{M}_\sigma w(x) &:= \phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,\sigma(x),z)]\mathscr{P}(\mathrm{d}z)\right) \\ &= \phi^{-1}\left(\int \phi \circ w\,(y)\, P_\sigma(x,\mathrm{d}y)\right) \qquad (w \in m(\mathbb{S})_+,\ \ x \in \mathbb{S}). \end{aligned}$$

Hence, in operator notation, (6.11) translates to $\boldsymbol{T}_\sigma w = r_\sigma + \beta \boldsymbol{M}_\sigma w$ for every $w \in bm(\mathbb{S})_+$.

Now we follow the approach of Hansen and Sargent (1995) and adopt the recursion specification in the form of (6.10). For an initial state $x \in \mathbb{S}$ and a stationary Markov policy $\sigma \in \Sigma$, we define a *t-stage total discounted utility* as follows

$$J_t(x,\sigma) := \underbrace{(\boldsymbol{T}_\sigma \circ \cdots \circ \boldsymbol{T}_\sigma)}_{t+1 \text{ times}} \mathbf{0}(x) = \boldsymbol{T}_\sigma^{t+1}\mathbf{0}(x), \qquad (t \in \mathbb{N}_0) \tag{6.12}$$

where $\mathbf{0}$ is a constant function such that $\mathbf{0}(x) \equiv 0$ for every $x \in \mathbb{S}$. $\boldsymbol{T}_\sigma^{t+1}$ stands for the $(t+1)$-th composition (iterate) of the operator $\boldsymbol{T}_\sigma$ with itself, which is acting on the function $\mathbf{0}$.

Intuitively, $\boldsymbol{T}_\sigma^{t+1}\mathbf{0}$ is equal to the policy value function for the t-horizon problem with the terminal reward $r_\sigma = 0$. For instance, if $t = 1$, the definition of (6.12) is read as follows

$$\begin{aligned} J_1(x,\sigma) &= \boldsymbol{T}_\sigma^2\mathbf{0}(x) = \boldsymbol{T}_\sigma\,(\boldsymbol{T}_\sigma\mathbf{0})\,(x) \\ &= \boldsymbol{T}_\sigma\,(r_\sigma(\cdot))\,(x) \\ &= r_\sigma(x) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ r_\sigma[F(x,\sigma(x),z)]\mathscr{P}(\mathrm{d}z)\right). \end{aligned}$$

Using operator notation, the non-expected total discounted utility J_1 under policy σ can also be expressed as $J_1(\cdot,\sigma) = r_\sigma + \beta \boldsymbol{M}_\sigma r_\sigma$.

Under Assumptions 6.4.3 and 6.4.4, it can be seen easily that the sequence $\{J_t(x,\sigma)\}_{t\in\mathbb{N}_0}$ is monotone increasing in $t \in \mathbb{N}_0$ and bounded from below by $\mathbf{0}$ for each $x \in \mathbb{S}$ and $\sigma \in \Sigma$. As a result, *the non-expected total discounted utility in the infinite time horizon* (if it exists) is naturally defined by

$$J(x,\sigma) := \lim_{t\to\infty} J_t(x,\sigma) \qquad (x \in \mathbb{S},\, \sigma \in \Sigma). \tag{6.13}$$

Remark 6.4.3. Since the sequence $\{J_t(x,\sigma)\}_{t\in\mathbb{N}_0}$ is monotone increasing in $t \in \mathbb{N}_0$ and bounded from below by $\mathbf{0}$ for each $x \in \mathbb{S}$ and $\sigma \in \Sigma$, the (pointwise) limit in (6.13) always exists and is well defined, although it may be infinite $(+\infty)$.

Furthermore, under Assumption 6.4.5, for any $t \in \mathbb{N}_0$, the t-stage total discounted utility has a uniform bound such as[12]

$$J_t(x,\sigma) \leqslant \frac{\phi^{-1}(R \cdot \kappa(x))}{1 - \phi(\beta)N_\kappa} \qquad (x \in \mathbb{S},\, \sigma \in \Sigma),$$

[12] The proof of this argument is analogous to that in Pages.184-185 of Bäuerle and Jaśkiewicz (2018). As ϕ is increasing and concave and by Jensen's inequality, we have the same result as that in Bäuerle and Jaśkiewicz (2018).

and hence, taking limits on t gives

$$\boldsymbol{J}(x,\sigma) \leqslant \frac{\phi^{-1}(R\cdot\kappa(x))}{1-\phi(\beta)N_\kappa} \qquad (x\in\mathbb{S},\ \sigma\in\Sigma).$$

This uniform bound implies that for each fixed $x \in \mathbb{S}$ and $\sigma \in \Sigma$, our non-expected discounted utility in the infinite time horizon is well defined and finite valued (since the function κ is real valued).

6.4.2 Statement of the Problem

We aim to find an optimal value (the so-called *value function*) $\boldsymbol{J}^*$ of the non-expected total discounted utility associated with the generic, recursive preference in the infinite time horizon and an optimal policy $\sigma^* \in \Sigma$ satisfying[13]

$$\boldsymbol{J}^*(x) := \boldsymbol{J}(x,\sigma^*) = \sup_{\sigma\in\Sigma} \boldsymbol{J}(x,\sigma) \qquad (x\in\mathbb{S}).$$

6.5 Main Results

The dynamic programming technique typically rests upon Bellman's principle of optimality which states that an optimal policy possesses the property that whatever the initial state and initial decision are, the decisions that follow must constitute an optimal policy starting from the state resulting from the first decision.[14]

To formulate the principle of optimality, we consider a *Bellman operator* $\boldsymbol{T}$ associated with generic nonlinear preferences that is defined through

$$\boldsymbol{T}w(x) = \sup_{a\in\Gamma(x)} \left\{ r(x,a) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi\circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z) \right) \right\} \qquad (x\in\mathbb{S}) \tag{6.14}$$

for any $w \in m(\mathbb{S})_+$.

Definition 6.5.1. Given $w \in m(\mathbb{S})_+$, we define $\sigma \in \Sigma$ to be *w-greedy* if

$$\sigma(x) \in \operatorname*{argmax}_{a\in\Gamma(x)} \left\{ r(x,a) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi\circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z) \right) \right\} \tag{6.15}$$

for each $x \in \mathbb{S}$.

We use the term "greedy" here to describe the case when the agent seeks to maximize the immediate next payoffs irrespective of the possibility that such an action may not provide access to better alternatives in the future.

[13] Recall that Σ is not empty, so that $\boldsymbol{J}^*$ will be well defined as soon as $\boldsymbol{J}$ is well defined.

[14] Here, we utilize the term "decision" to indicate a choice of action at a particular time, and the term "policy" to indicate the entire decision rule (i.e., action sequence) or action function.

In order to make a clear comparison with our fixed point theorem in Section 6.2, we choose $m(\mathbb{S})$ and $c(\mathbb{S})$ as the underlying Riesz spaces $\mathbb{E}$ and let $\|\cdot\|_\kappa$ be the weighted supremum norm (cf. subsection 6.3.1). By construction, equipped with the finite weighted supremum norm, the associated Banach lattices $\mathbb{B}$ are the set $b_\kappa m(\mathbb{S})$ and $b_\kappa c(\mathbb{S})$, respectively.

Under Assumption 6.4.4, as can be seen from Example 6.3.1, a map $\Phi\colon m(\mathbb{S})_+ \to m(\mathbb{S})_+$ that is defined by $(\Phi f)(x) = \phi \circ f(x) \equiv \phi[f(x)]$ for each $x \in \mathbb{S}$ is an attitude-adjusted transformation on $m(\mathbb{S})_+$.

In this connection, it is clear that the corresponding candidate function space $\mathbb{F}_m$ in this context is defined through

$$\mathbb{F}_m := \{f \in m(\mathbb{S})_+ : \ \phi \circ f \text{ is } \kappa\text{-bounded}\}.$$

Analogously, denote by $\mathbb{F}_c$ the collection of all continuous functions in $\mathbb{F}_m$; namely,

$$\mathbb{F}_c := \{f \in c(\mathbb{S})_+ : \ \phi \circ f \text{ is } \kappa\text{-bounded}\}.$$

Now we are ready to state the main results.

Theorem 6.5.1. *Under Assumptions 6.4.1 to 6.4.5, we have the following results.*

(i) There exist a unique function $v^* \in \mathbb{F}_c$ satisfying

$$v^*(x) = \sup_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ v^* \left[F(x,a,z)\right] \mathscr{P}(\mathrm{d}z) \right) \right\}$$

for all $x \in \mathbb{S}$.

(ii)

$$v^*(x) = \sup_{\sigma \in \Sigma} \boldsymbol{J}(x,\sigma)$$

for all $x \in \mathbb{S}$.

(iii) A feasible policy is optimal if and only if it is v^*-greedy. Moreover, at least one such policy exists.

6.6 Properties of the Value Function

In this section, we aim to explore some further properties of the continuation value function. To this end, the state space $\mathbb{S}$ is additionally assumed to be *partially ordered and convex*.

6.6.1 Monotonicity

Our first result gives sufficient conditions for the value function to be monotone increasing on $\mathbb{S}$.

Proposition 6.6.1. *The value function v^* is monotone increasing on $\mathbb{S}$ whenever Γ is increasing on $\mathbb{S}$ and, for any $x, x' \in \mathbb{S}$ with $x \leqslant x'$, we have*

1. $r(x,a) \leqslant r(x',a)$ *for all* $a \in \Gamma(x)$*; and*
2. $F(x,a,z) \leqslant F(x',a,z)$ *for all* $a \in \Gamma(x)$ *and for each* $z \in \mathbb{Z}$.

6.6.2 Concavity

The following result gives conditions under which the value function v^* is concave.

Proposition 6.6.2. *Let the conditions of Proposition 6.6.1 hold. If, in addition,*

1. $\operatorname{gr}\Gamma$ *is convex;*
2. r *is concave on* $\operatorname{gr}\Gamma$*;*
3. *the operator* $g \mapsto \phi^{-1}(\int \phi \circ g \ \mathrm{d}\mathscr{P})$ *is concave; and*
4. $(x,a) \mapsto F(x,a,z)$ *is concave on* $\operatorname{gr}\Gamma$ *for each* $z \in \mathbb{Z}$,

then the value function v^* is concave.

6.6.3 Differentiability and Euler Equation

Next let us turn to the differentiability of the value function v^*. Before doing so, we need to strengthen our assumptions; in particular, we need to ensure that our primitives are smooth. To simplify the analysis, we treat the reward function and the transition function in a form of $r(x,a) = u(x-a)$ and $F(x,a,z) = f(a,z)$, respectively.

Assumption 6.6.1. The function $u \colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $(0,\infty)$, continuous, strictly increasing, and strictly concave, satisfying $u(0) = 0$ and $\lim_{c \downarrow 0} u'(c) = \infty$.

Assumption 6.6.2. The function $f \colon \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}_+$ satisfies the following.

(i) For each $z \in \mathbb{Z}$, $f(0,z) = 0$, and $f(\cdot, z) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $(0,\infty)$, continuous, concave, and strictly increasing.

(ii) For each $a \in \mathbb{R}_+$, $f(a,\cdot) \colon \mathbb{Z} \to \mathbb{R}_+$ is a Borel-measurable function.

(iii) There exists an (interior) feasible action $a > 0$ such that $\int_{\mathbb{Z}} f'(a,z)\mathscr{P}(\mathrm{d}z) > 0$, where $f'(a,z) := \frac{\partial f(a,z)}{\partial a}$ denotes the partial derivative of f with respect to a.[15]

It is worth noting that if $\int_{\mathbb{Z}} f'(a,z)\mathscr{P}(\mathrm{d}z) = 0$ for all $a > 0$, then $f(a,z) = 0$ $\mathscr{P}$-almost surely for all $a > 0$. Thus, the role of condition (iii) together with condition (i) in Assumption 6.6.2 is to rule out the trivial case that $f(a,z) = 0$ $\mathscr{P}$-almost surely for all $a \in \mathbb{R}_+$.

[15] Note that by Theorem 7.4 in Stokey et al. (1989), it follows that the function $z \to f'(a,z)$ is Borel-measurable, since it is the pointwise limit of the sequence $\{\left[f(a+\frac{1}{n},\cdot) - f(a,\cdot)\right] n\}_{n=1}^{\infty}$.

Under the conditions of Assumptions 6.6.1 and 6.6.2, it is not hard to see that the value function v^* is strictly concave and strictly increasing, while the optimal policy is unique, increasing, continuous and single-valued. In the following statement of the result, $\sigma := \sigma^*$ is the optimal policy and $c(x) := x - \sigma(x)$ is optimal consumption.

Proposition 6.6.3. *The value function v^* is differentiable on $(0, \infty)$.*

Moreover, $(v^*)'(x) = u'(x - \sigma^*(x))$ for all $x > 0$.

Proposition 6.6.4. *Under Assumptions 6.6.1 and 6.6.2, we have the following.*

(i) For any $x > 0$, the Euler equation holds

$$u' \circ c(x) = \beta \left(\phi^{-1}\right)' \left(\int_{\mathbb{Z}} \phi \circ v^* \left[f(\sigma(x), z) \right] \mathscr{P}(\mathrm{d}z) \right) \times \\ \times \int_{\mathbb{Z}} u' \circ c \left[f(\sigma(x), z) \right] f' \left(\sigma(x), z \right) \, \phi' \circ v^* \left[f(\sigma(x), z) \right] \mathscr{P}(\mathrm{d}z).$$

(ii) The functions $x \mapsto \sigma(x)$ and $x \mapsto c(x)$ are increasing.

6.6.4 Convergence of Value Iteration

In the following, we confine ourselves to the bounded reward case and discuss some interesting results. To do so, observe that the set $\mathbb{F}$ in this context is

$$\mathbb{F} = \{ f \in m(\mathbb{S})_+ : \ \phi \circ f \text{ is bounded} \} = bm(\mathbb{S})_+.$$

The last equality follows from elementary reasoning.

Recall that the positive cone $bm(\mathbb{S})_+$ is a complete metric space under the induced metric

$$d_\infty(f, g) := \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|, \qquad (\forall f, g \in bm(\mathbb{S})_+).$$

At the same time, we define an alternative metric d_∞^ϕ on $bm(\mathbb{S})_+$ through

$$d_\infty^\phi(f, g) := \|\phi \circ f - \phi \circ g\|_\infty = \sup_{x \in X} | \phi[f(x)] - \phi[g(x)] |, \qquad (\forall f, g \in bm(\mathbb{S})_+).$$

It is not hard to see that these two metric spaces $(bm(\mathbb{S})_+, d_\infty)$ and $(bm(\mathbb{S})_+, d_\infty^\phi)$ are isometrically isomorphic, and hence both are complete.

The following lemma establishes an interesting result of the convergence in different metric spaces but on the same positive cone $bm(\mathbb{S})_+$.

Lemma 6.6.1. *Given two complete metric spaces $(bm(\mathbb{S})_+, d_\infty)$ and $(bm(\mathbb{S})_+, d_\infty^\phi)$,*

$$d_\infty^\phi(f_n, f^*) \to 0 \quad \textit{implies} \quad d_\infty(f_n, f^*) \to 0, \qquad (n \to \infty).$$

Remark 6.6.1. Note that the convergence in supremum norm implies the uniform convergence.

Proposition 6.6.5. *Let $v_0 \in bc(\mathbb{S})_+$. Fix $n \in \mathbb{N}$, and let $v_n := \boldsymbol{T}^n v_0$, where $\boldsymbol{T}$ is the Bellman operator. If $\sigma \in \Sigma$ is v_n-greedy, then*

$$\|\phi \circ v^* - \phi \circ v_\sigma\|_\infty \leqslant \frac{2\phi(\beta)}{1-\phi(\beta)} \|\phi \circ v_n - \phi \circ v_{n-1}\|_\infty. \tag{6.16}$$

Proof. The proof is omitted here, since the proof of the standard case can also be applied to our framework without modification. The interested reader may refer to Theorem 10.2.1 in Stachurski (2009). □

The next corollary is a consequence of Proposition 6.6.5, which measures how close to being optimal is the v_n-greedy policy σ_n that the algorithm produces.

Corollary 6.6.1. *Let $\{v_n\}_{n\geqslant 0}$ be as in Proposition 6.6.5. If $\{\sigma_n\}_{n\geqslant 0}$ is a sequence in Σ such that σ_n is v_n-greedy for each $n \geqslant 0$, then $\|\phi \circ v^* - \phi \circ v_{\sigma_n}\|_\infty \to 0$ as $n \to \infty$.*

6.7 Applications

For an illustrative application of Epstein–Zin recursive utility, we consider a classic stochastic optimal growth model in the context of a stationary Markov environment with independent and identically distributed (i.i.d.) shocks $\{z_t\}_{t\in\mathbb{N}}$ and distribution $\mathscr{P} \in \Delta(\mathbb{Z})$. Preferences are represented by a constant-discounting and constant-elasticity aggregator and a general linear homogeneous certainty equivalent. A dynamic consumption/investment problem for this environment is characterized by the Bellman equation which implicitly defines the value function:

$$\mathcal{J}(x) = \max_{a\in\Gamma(x)} \left\{ (1-\beta)c^{1-\rho} + \beta \left(\int [\mathcal{J}(y)]^{1-\gamma} P(x,a;\mathrm{d}y) \right)^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}, \tag{6.17}$$

subject to the consumption constraint $c = x - a$, where a denotes investment.

Invoking Lemma A.4.1, we can put Bellman Equation (6.17) into an additive form by using the monotone increasing transformation $\hat{\mathcal{J}}(x) := [\mathcal{J}(x)]^{1-\rho}$ as follows

$$\hat{\mathcal{J}}(x) = \max_{a\in\Gamma(x)} \left\{ (1-\beta)c^{1-\rho} + \beta \left(\int [\hat{\mathcal{J}}(y)]^{\frac{1-\gamma}{1-\rho}} P(x,a;\mathrm{d}y) \right)^{\frac{1-\rho}{1-\gamma}} \right\}.$$

Setting $\theta := (1-\gamma)/(1-\rho)$, the modified (scaled) Bellman equation becomes

$$\hat{\mathcal{J}}(x) = \max_{a\in\Gamma(x)} \left\{ (1-\beta)c^{1-\rho} + \beta \left(\int [\hat{\mathcal{J}}(y)]^{\theta} P(x,a;\mathrm{d}y) \right)^{1/\theta} \right\},$$

which is identical to the Bellman operator given in (6.14), except perhaps for the definition of the reward function.

6.7.1 Examples

We now provide some classical examples of the reward and transition functions that meet our assumptions used in Section 6.4.1.

Example 6.7.1. (*A model with Cobb-Douglas production function with depreciation*)

Consider a stochastic optimal growth model. At the start of time t, an agent has income y_t, which is divided between consumption c_t and savings k_t.

From consumption c the agent receives utility $u(c)$, where $u\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a (fixed) function that is invariant over time. Savings are added to the existing capital stock.

After time t, an investment decision is made, and an income shock W_{t+1} is observed. Production then takes place, yielding

$$y_{t+1} = k_t^\alpha W_{t+1} + (1-\delta)k_t, \qquad (W_t)_{t\in\mathbb{N}} \overset{\text{IID}}{\sim} \mathscr{P} \in \Delta(\mathbb{Z}),\ \mathbb{Z} \in \mathscr{B}(\mathbb{R}).$$

where $\alpha \in (0,1)$ and the depreciation rate is $\delta \in (0,1]$.

This fits our model framework with the state variable $y \in \mathbb{S} := [0,\infty)$ and the control $k \in \mathbb{A} := [0,\infty)$. The constraint Γ is the map $\mathbb{S} \ni y \mapsto [0,y] \subset \mathbb{A}$ that defines feasible savings given income y. Thus, Assumption 6.4.1 holds true. Assumption 6.4.4 is satisfied whenever θ lies in $(0,1]$. The reward function $r(y,k)$ on $\operatorname{gr}\Gamma$ is $u(y-k)$. Let the period utility function $u(\cdot)$ be given by $u(c) = c^{1-\rho}$ with $\rho \in (0,1)$. Moreover, the transition function in this case is $F(y,k,z) = f(k,z) = k^\alpha z + (1-\delta)k$. Evidently, Assumptions 6.4.3 and 6.4.2 are satisfied.

Suppose that $\hat{z} := \int_{\mathbb{Z}} z\mathscr{P}(\mathrm{d}z)$ is finite, Condition (6.6) in Assumption 6.4.5 holds for $\kappa(x) = (x+d)^{1-\gamma}$ with arbitrary $d \geqslant 1$ and $\gamma \in [\rho, 1)$.[16] Meanwhile, Condition (6.7) can be guaranteed by

$$N_\kappa := \left(1 + \frac{(\hat{z}/\delta)^{\frac{1}{1-\alpha}}}{d}\right)^{1-\rho}$$

where $d \geqslant 1$ must be chosen sufficiently large so that $N_\kappa \beta^\theta < 1$. The proof of Condition (6.7) in Assumption 6.4.5 can be found in Example 2 in Jaśkiewicz and Nowak (2011) or Example 1 in Bäuerle and Jaśkiewicz (2018). Since all conditions of Theorem 6.5.1 are verified, the statement of that theorem now follows.

[16] In fact, the parameterizations with $0 < \rho < 1$ and $0 < \theta = \frac{1-\gamma}{1-\rho} \leqslant 1$ imply that $0 < \rho \leqslant \gamma < 1$. This means that early resolution of uncertainty is preferred, i.e., RRA $> 1/EIS$.

Example 6.7.2. (*An income fluctuation problem*) We now consider an optimal savings problem in our framework. In the problem, an agent chooses a consumption plan $\{c_t\}_{t \geqslant 0}$ to maximize the objective function of the form (6.17) subject to the constraints

$$c_t + a_{t+1} \leqslant (1+r)a_t + z_t, \quad c_t \geqslant 0, \quad a_t \geqslant -b, \quad (t \in \mathbb{N}_0).$$

Here, a_t is asset holdings at time t, c_t is consumption, $r > 0$ is the exogenous interest rate, b is an exogenous borrowing constraint, and z_t is non-capital income (wages, unemployment compensation, etc.). For simplicity, the non-capital income $\{z_t\}$ is assumed to be taking values in $\mathbb{Z} \subset (0, \infty)$ with i.i.d. distribution $\mathscr{P}$. In addition, we assume a no-borrowing constraint by setting $b = 0$, because it simplifies the exposition and costs no generality. In this context, we write the optimal saving problem in a dynamic programming form with cash-in-hand as a state variable, i.e., $x = (1+r)a + z$. That is, for each $x \in \mathbb{S}$,

$$\mathcal{J}(x) = \max_{c \in \Gamma(x)} \left\{ (1-\beta)c^{1-\rho} + \beta \left(\int \left[\mathcal{J}(x') \right]^{1-\gamma} P(x, c; \mathrm{d}x') \right)^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}},$$

subject to

$$x' = (1+r)(x-c) + z, \qquad x' \geqslant z.$$

The term x' denotes cash in hand next period given that today's cash in hand is x and given the current income realization is z. One-period assets carried $a = x - c$ over into next period must be nonnegative, so the no-borrowing constraint is $x \geqslant c$.

Obviously, it also fits our model framework with the state variable $x \in \mathbb{S} := [0, \infty)$ and the control $c \in \mathbb{A} := [0, \infty)$. The constraint Γ is the map $\mathbb{S} \ni x \mapsto [0, x] \subset \mathbb{A}$ that defines feasible savings given income x. Thus, Assumption 6.4.1 holds true. The reward function $r(x, c)$ on $\operatorname{gr} \Gamma$ is $u(c)$. Let the period-utility function $u(\cdot)$ be given by $u(c) = c^{1-\rho}$ with $\rho \in (0, 1)$. Since the law of motion for the new state variable is

$$x_{t+1} = F(x_t, c_t, z_{t+1}) = (1+r)(x_t - c_t) + z_{t+1}, \qquad (z_t)_{t \in \mathbb{N}} \overset{\text{IID}}{\sim} \mathscr{P} \in \Delta(\mathscr{B}(\mathbb{R}_{++})),$$

the transition function is $F(x, c, z) = (1+r)(x-c) + z$. Evidently, Assumptions 6.4.3 and 6.4.2 are satisfied.

Assume that $\hat{z} := \int_{\mathbb{Z}} z \mathscr{P}(\mathrm{d}z)$ is finite. Let the weight function $\kappa(\cdot)$ be as in Example 6.7.1. According to the preceding discussion in Example 6.7.1, we know that Assumption 6.4.3 holds. Regarding Condition (6.7) in Assumption 6.4.5, invoking Jaśkiewicz and Nowak's result, it holds for all $\beta \in (0, 1)$ satisfying the inequality $\beta^{\theta}(1+r)^{1-\rho} < 1$ (since in this case $N_{\kappa} = (1+r)^{1-\rho}$), which essentially requires that impatience dominates the "mean" growth. We can now conclude that Assumptions 6.4.1 to 6.4.5 are all satisfied, especially, Assumption 6.4.4 is satisfied whenever θ lies in $(0, 1]$. Hence, the results of Theorem 6.5.1 hold true.

Appendix A

Chapter Appendixes

A.1 Appendix to Chapter 2

Proof of Lemma 2.2.3. Consider the map $\overline{\boldsymbol{B}^+}$ defined in (2.11). To see that this map $\overline{\boldsymbol{B}^+}\colon \mathbb{E} \to \mathbb{E}$ is well defined, pick any $w \in \mathbb{E}$ with $w = u_1 - v_1 = u_2 - v_2$ for some $u_1, u_2, v_1, v_2 \in \mathbb{K}$. Clearly, by the convexity of the positive cone $\mathbb{K}$, $u_1 + v_2 = u_2 + v_1 \in \mathbb{K}$. Then, making use of the additivity of $\boldsymbol{B}^+$, we have the following relations

$$\begin{aligned}
\boldsymbol{B}^+(u_1 + v_2) = \boldsymbol{B}^+(u_2 + v_1) &\iff \boldsymbol{B}^+(u_1) + \boldsymbol{B}^+(v_2) = \boldsymbol{B}^+(u_2) + \boldsymbol{B}^+(v_1) \\
&\iff \boldsymbol{B}^+(u_1) - \boldsymbol{B}^+(v_1) = \boldsymbol{B}^+(u_2) - \boldsymbol{B}^+(v_2) \\
&\iff \overline{\boldsymbol{B}^+}(u_1 - v_1) = \overline{\boldsymbol{B}^+}(u_2 - v_2) \equiv \overline{\boldsymbol{B}^+}(w)
\end{aligned}$$

Thus, $\overline{\boldsymbol{B}^+}$ is a well defined mapping on $\mathbb{E}$.

By the construction of $\overline{\boldsymbol{B}^+}$, positivity of the extension is obvious, and $\overline{\boldsymbol{B}^+}|_{\mathbb{K}} = \boldsymbol{B}^+$.

Regarding the linearity of $\overline{\boldsymbol{B}^+}$, we first show the additivity of $\overline{\boldsymbol{B}^+}$. To this end, recall that $\mathbb{E}$ is a Banach lattice, and thus it is a Riesz space. Then, for every $f \in \mathbb{E}$, we have a decomposition $f = f^+ - f^-$ where $f^+ := f \vee \mathbf{0} \in \mathbb{K}$ and $f^- := (-f) \vee \mathbf{0} \in \mathbb{K}$. Now pick any f, g in $\mathbb{E}$. It then follows from $f = f^+ - f^-$ and $g = g^+ - g^-$ with $f^+, f^-, g^+, g^- \in \mathbb{K}$ that

$$\begin{aligned}
\overline{\boldsymbol{B}^+}(f + g) &= \overline{\boldsymbol{B}^+}\left(f^+ - f^- + g^+ - g^-\right) = \overline{\boldsymbol{B}^+}\left[(f^+ + g^+) - (f^- + g^-)\right] \\
&= \boldsymbol{B}^+(f^+ + g^+) - \boldsymbol{B}^+(f^- + g^-) \\
&= \left[\boldsymbol{B}^+(f^+) - \boldsymbol{B}^+(f^-)\right] + \left[\boldsymbol{B}^+(g^+) - \boldsymbol{B}^+(g^-)\right] \\
&= \overline{\boldsymbol{B}^+}(f^+ - f^-) + \overline{\boldsymbol{B}^+}(g^+ - g^-) = \overline{\boldsymbol{B}^+}f + \overline{\boldsymbol{B}^+}g.
\end{aligned}$$

where the second and last lines follow directly from the definition of $\overline{\boldsymbol{B}^+}$ in (2.11), and the third line is derived by the additivity of $\boldsymbol{B}^+$.

Next we show that for each $f \in \mathbb{E}$ and every $\alpha \in \mathbb{R}$, $\overline{\boldsymbol{B}^+}(\alpha f) = \alpha\overline{\boldsymbol{B}^+}(f)$ holds. By the additivity of $\overline{\boldsymbol{B}^+}$, we observe that $\overline{\boldsymbol{B}^+}(\alpha f) = \alpha\overline{\boldsymbol{B}^+}(f)$ holds for any rational number $\alpha \geqslant 0$ and each

$f \in \mathbb{K}$.[1] In this connection, given any real number $\alpha \geqslant 0$ and $f \in \mathbb{K}$, we take two sequences of nonnegative rational numbers $\{r_n\}$ and $\{s_n\}$ such that $r_n \uparrow \alpha$ and $s_n \downarrow \alpha$. Hence, we have that $r_n\overline{\boldsymbol{B}^+}(f) \leqslant \overline{\boldsymbol{B}^+}(r_n f) \leqslant \overline{\boldsymbol{B}^+}(\alpha f) \leqslant \overline{\boldsymbol{B}^+}(s_n f) \leqslant s_n\overline{\boldsymbol{B}^+}(f)$. Since $\mathbb{E}$ is Archimedean, by the squeeze theorem, it follows that $\overline{\boldsymbol{B}^+}(\alpha f) = \alpha\overline{\boldsymbol{B}^+}(f)$ for any $\alpha \geqslant 0$ and $f \in \mathbb{K}$.

Further, it follows that for any $\alpha \geqslant 0$ and $f \in \mathbb{E}$, $\overline{\boldsymbol{B}^+}(\alpha f) = \overline{\boldsymbol{B}^+}(\alpha f^+ - \alpha f^-) = \boldsymbol{B}^+(\alpha f^+) - \boldsymbol{B}^+(\alpha f^-) = \overline{\boldsymbol{B}^+}(\alpha f^+) - \overline{\boldsymbol{B}^+}(\alpha f^-) = \alpha\overline{\boldsymbol{B}^+}(f^+) - \alpha\overline{\boldsymbol{B}^+}(f^-) = \alpha\boldsymbol{B}^+(f^+) - \alpha\boldsymbol{B}^+(f^-) = \alpha\overline{\boldsymbol{B}^+}(f)$.

Invoking the preceding result, for any $\alpha \leqslant 0$ and $f \in \mathbb{E}$, we obtain $\overline{\boldsymbol{B}^+}(\alpha f) = \overline{\boldsymbol{B}^+}(-\alpha(-f)) = -\alpha\overline{\boldsymbol{B}^+}(-f)$, since $-f \in \mathbb{E}$ and $-\alpha \geqslant 0$. On the other hand, $\overline{\boldsymbol{B}^+}(-f) = \overline{\boldsymbol{B}^+}(f^- - f^+) = \boldsymbol{B}^+(f^-) - \boldsymbol{B}^+(f^+) = -[\boldsymbol{B}^+(f^+) - \boldsymbol{B}^+(f^-)] = -\overline{\boldsymbol{B}^+}(f)$. It follows that $\overline{\boldsymbol{B}^+}(\alpha f) = \alpha\overline{\boldsymbol{B}^+}(f)$ for any $\alpha \leqslant 0$ and each $f \in \mathbb{E}$.

Combining our results so far, the linearity of $\overline{\boldsymbol{B}^+}$ is proved.

Regarding the uniqueness, we assume that there are two distinct linear extensions $\overline{\boldsymbol{B}_1^+}$ and $\overline{\boldsymbol{B}_2^+}$. By the fact that $\overline{\boldsymbol{B}_1^+}|_{\mathbb{K}} = \overline{\boldsymbol{B}_2^+}|_{\mathbb{K}} = \boldsymbol{B}^+$, we have that for any $f \in \mathbb{K}$, $(\overline{\boldsymbol{B}_1^+} - \overline{\boldsymbol{B}_2^+})(f) = \boldsymbol{B}^+(f) - \boldsymbol{B}^+(f) = \mathbf{0}$. On the other hand, for any $-f \in \mathbb{K}$, $(\overline{\boldsymbol{B}_1^+} - \overline{\boldsymbol{B}_2^+})(f) = (\overline{\boldsymbol{B}_2^+} - \overline{\boldsymbol{B}_1^+})(-f) = \boldsymbol{B}^+(-f) - \boldsymbol{B}^+(-f) = \mathbf{0}$. It then follows that $\overline{\boldsymbol{B}_1^+}(f) = \overline{\boldsymbol{B}_2^+}(f)$ for each $f \in \mathbb{E}$, which proves the uniqueness.

Turning to boundedness, for any $f \in \mathbb{E}$, we have that

$$\begin{aligned}
\|\overline{\boldsymbol{B}^+}(f)\| = \|\overline{\boldsymbol{B}^+}(f^+ - f^-)\| &= \|\boldsymbol{B}^+(f^+) - \boldsymbol{B}^+(f^-)\| \\
&\leqslant \|\boldsymbol{B}^+(f^+)\| + \|\boldsymbol{B}^+(f^-)\| \\
&\leqslant \|\boldsymbol{B}^+\| \cdot \|f^+\| + \|\boldsymbol{B}^+\| \cdot \|f^-\| \\
&\leqslant 2\|\boldsymbol{B}^+\| \cdot \|\,|f|\,\| = 2\|\boldsymbol{B}^+\| \cdot \|f\|.
\end{aligned}$$

The first inequality follows from the norm's triangle inequality and the second one follows from the boundedness of $\boldsymbol{B}^+$. The third inequality is derived from the fact that by the property of a lattice norm, the conditions $\mathbf{0} \leqslant f^+ \leqslant |f| = f^+ + f^-$ and $\mathbf{0} \leqslant f^- \leqslant |f|$ imply $\|f^+\| \leqslant \||f|\|$ and $\|f^-\| \leqslant \||f|\|$, respectively. The last equality follows from the fact that $\|f\| = \|\,|f|\,\|$ for all f in a Banach lattice $\mathbb{E}$.[2] Hence, the boundedness of $\overline{\boldsymbol{B}^+}$ has been proved, given that $\|\boldsymbol{B}^+\|$ is finite.

This completes the proof. □

Proof of Theorem 2.2.1. **Part (I)** We first prove claims (1)–(3) together.

[1] To see this, we first note by the additivity of $\overline{\boldsymbol{B}^+}$ that $\overline{\boldsymbol{B}^+}(\mathbf{0}) = \overline{\boldsymbol{B}^+}(\mathbf{0}+\mathbf{0}) = \overline{\boldsymbol{B}^+}(\mathbf{0}) + \overline{\boldsymbol{B}^+}(\mathbf{0})$, which implies $\overline{\boldsymbol{B}^+}(\mathbf{0}) = \mathbf{0}$. Next, for any positive integer α, making use of the additivity of $\overline{\boldsymbol{B}^+}$ again yields $\overline{\boldsymbol{B}^+}(\alpha f) = \alpha\overline{\boldsymbol{B}^+}(f)$. Therefore, for each positive integer α, it follows that $\overline{\boldsymbol{B}^+}(f) = \overline{\boldsymbol{B}^+}(\alpha \cdot \frac{1}{\alpha} f) = \alpha \cdot \overline{\boldsymbol{B}^+}(\frac{1}{\alpha} f)$, and thus, $\frac{1}{\alpha}\overline{\boldsymbol{B}^+}(f) = \overline{\boldsymbol{B}^+}(\frac{1}{\alpha} f)$. Finally, for any nonnegative integer β, we have $\overline{\boldsymbol{B}^+}(\frac{\beta}{\alpha} f) = \overline{\boldsymbol{B}^+}(\beta \cdot \frac{1}{\alpha} f) = \beta \cdot \overline{\boldsymbol{B}^+}(\frac{1}{\alpha} f) = \frac{\beta}{\alpha}\overline{\boldsymbol{B}^+}(f)$. That is, $\overline{\boldsymbol{B}^+}(rf) = r\overline{\boldsymbol{B}^+}(f)$ for each $r \in \mathbb{Q} \cap \mathbb{R}_+$.

[2] More precisely, given any normed Riesz space $\mathbb{E}$ (not necessarily complete), the equality $\|f\| = \|\,|f|\,\|$ holds for all $f \in \mathbb{E}$.

By virtue of Assumption 2.2.1 and Lemma 2.2.1, we know that A is a monotone increasing and order concave operator on $\mathbb{K}$. In addition, as the solid cone $\mathbb{K}$ possesses the property of reproducing. Hence, by Assumption 2.2.3, Corollary 2.2.1 applies and gives us that $g^* = (I - B^+)^{-1}h$ is an upper solution of (2.8) (i.e., $g^* \geqslant Ag^*$), and g^* is distinct from $\mathbf{0}$ (i.e., $g^* > \mathbf{0}$).

Then, in order to apply Du's Theorem 3.1 (cf. Corollary 2.1.1 in the present thesis), it suffices to show that the nonlinear operator $A\colon [\mathbf{0}, g^*] \subset \mathbb{K} \to \mathbb{K}$ satisfies the strong lower-solution condition. To see this, observe that $A\mathbf{0} = h + B\mathbf{0} \geqslant h \gg \mathbf{0}$ by the interior condition of h. Thus, $\mathbf{0}$ is indeed a strong lower solution of (2.8) (i.e., $\mathbf{0} \ll A\mathbf{0}$).[3]

As a result, by virtue of Du's Theorem 3.1, we get the conclusion of (1)–(3) given that $r = 1 - \varepsilon$. In addition, $f^* = Af^* \geqslant A\mathbf{0} \gg \mathbf{0}$ implies that the fixed point f^* must lie in $\mathring{\mathbb{K}}$. It is worth noting that by making use of Du's Theorem, we only establish the existence and uniqueness result of the fixed point of A in $\left[\mathbf{0},\, (I - B^+)^{-1}h\right]$.

In this connection, we aim to obtain a much stronger result that the fixed point f^* of A is unique and globally attracting in the positive cone $\mathbb{K}$.

Regarding the *uniqueness* of the fixed point f^* in $\mathbb{K}$, suppose to the contrary that there is another (second) fixed point f^{**} of A in $\mathbb{K}$. Since the fixed point f^* of A in $[\mathbf{0},\, g^*]$ is unique, the second distinct fixed point f^{**} must lie in $\mathbb{K} \setminus [\mathbf{0},\, g^*]$.

Observe that, invoking Corollary 2.2.1 (or Lemma 2.2.2), we have

$$f^{**} = Af^{**} = h + Bf^{**} \leqslant h + B^+ f^{**}.$$

Recall that $B^+\colon \mathbb{E} \to \mathbb{E}$ is a continuous, linear, positive operator on the ordered real Banach space $\mathbb{E}$ with the spectral radius condition $r(B^+) < 1$. Since $f^{**}, g^* \in \mathbb{K} \subset \mathbb{E}$ and $h \in \mathring{\mathbb{K}}$, by virtue of the abstract Gronwall lemma (see, e.g., Proposition 7.15 of Zeidler (1986)), the conditions

$$f^{**} \leqslant h + B^+ f^{**} \qquad \text{and} \qquad g^* = h + B^+ g^*$$

always imply $f^{**} \leqslant g^*$.[4]

This contradicts our hypothesis $f^{**} \in \mathbb{K} \setminus [\mathbf{0},\, g^*]$. Hence, we conclude that the positive fixed point f^* of A indeed lies in $(\mathbf{0},\, g^*]$ and, more importantly, is unique in the positive cone $\mathbb{K}$.

Part (II) We turn next to prove the global attractivity of f^* in the positive cone $\mathbb{K}$.

Step 1. (Lower convergence) Consider a positive number $\underline{t} < 1$ first and let

$$v_0 = \underline{t} f^*, \qquad \text{and} \qquad v_{n+1} = Av_n \qquad (n \in \mathbb{N}_0).$$

[3] In fact, we can also apply Du's Theorem 2.1 (cf. Theorem 2.1.1 in the present thesis), since according to $A\mathbf{0} \in \mathring{\mathbb{K}}$, there must exist an $\varepsilon \in (0, 1)$ sufficiently small so as to satisfy $A\mathbf{0} \geqslant \varepsilon g^*$.

[4] To see this, we define an operator $A^+\colon \mathbb{E} \to \mathbb{E}$ through $A^+ f := h + B^+ f$. Clearly, A^+ is also linear and positive, and hence monotone increasing. It then follows from the hypothesis $f^{**} \leqslant A^+ f^{**}$ that $f^{**} \leqslant A^+ f^{**} \leqslant (A^+)^2 f^{**} \leqslant \cdots \leqslant (A^+)^n f^{**}$ for each $n \in \mathbb{N}$. Since $r(B^+) < 1$, the Neumann series converges, and thus $\| (B^+)^n \| \to 0$ as $n \to \infty$. This implies that $f^{**} \leqslant (A^+)^n f^{**} = \sum_{k=0}^{n-1} (B^+)^k h + (B^+)^n f^{**} \to (I - B^+)^{-1} h = g^*$ (as $n \to \infty$), as was to be shown.

Clearly, by the monotonicity of $\boldsymbol{A}$, we have

$$\mathbf{0} < \underline{t} f^* = v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant f^*.$$

Define $\tau_n = \sup\{t > 0 \colon t f^* \leqslant v_n\}$, then it yields

$$\tau_n f^* \leqslant v_n \qquad (n \in \mathbb{N}_0), \tag{A.1}$$

and

$$0 < \underline{t} = \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant \cdots \leqslant 1. \tag{A.2}$$

We claim that

$$\lim_{n\to\infty} \tau_n = 1. \tag{A.3}$$

To see this, if otherwise, then there exists a positive number $\gamma \in [\underline{t}, 1)$ such that $\lim_{n\to\infty} \tau_n = \gamma < 1$. As $0 < \underline{t} \leqslant \gamma < 1$, we have

$$\begin{aligned} \boldsymbol{A}(\gamma f^*) &\geqslant \gamma \boldsymbol{A} f^* + (1-\gamma)\boldsymbol{A}\mathbf{0} \\ &\geqslant \gamma \boldsymbol{A} f^* + (1-\gamma)\varepsilon g^* \\ &\geqslant \gamma \boldsymbol{A} f^* + (1-\gamma)\varepsilon f^* = \gamma \boldsymbol{A} f^* + (1-\gamma)\varepsilon \boldsymbol{A} f^* \\ &= \gamma \left[1 + \frac{1-\gamma}{\gamma}\varepsilon\right] \boldsymbol{A} f^*. \end{aligned} \tag{A.4}$$

The first inequality follows from the order concavity of $\boldsymbol{A}$, the second is derived from the interior condition of h, and the third from the fact $f^* \leqslant g^*$ by the Gronwall lemma.

Now, pick any positive number t satisfying $0 < t \leqslant \gamma$,

$$\begin{aligned} \boldsymbol{A}(t f^*) &= \boldsymbol{A}\left(\frac{t}{\gamma} \cdot \gamma f^*\right) \\ &\geqslant \frac{t}{\gamma}\boldsymbol{A}(\gamma f^*) + \left(1 - \frac{t}{\gamma}\right)\boldsymbol{A}\mathbf{0} && \text{(by concavity of } \boldsymbol{A}) \\ &\geqslant \frac{t}{\gamma} \cdot \gamma \left[1 + \frac{1-\gamma}{\gamma}\varepsilon\right] \boldsymbol{A} f^* + \left(1 - \frac{t}{\gamma}\right)\varepsilon g^* && \text{(by (A.4))} \\ &\geqslant \frac{t}{\gamma} \cdot \gamma \left[1 + \frac{1-\gamma}{\gamma}\varepsilon\right] f^* + \left(1 - \frac{t}{\gamma}\right)\varepsilon f^* && \text{(because } g^* \geqslant f^*) \\ &= \left\{t\left[1 + \frac{1-\gamma}{\gamma}\varepsilon\right] + \left(1 - \frac{t}{\gamma}\right)\varepsilon\right\} f^* \\ &= t f^* \cdot \left\{1 + \frac{1-t}{t}\varepsilon\right\}. \end{aligned}$$

Hence,

$$\boldsymbol{A}(\tau_n f^*) \geqslant \tau_n f^* \cdot \left\{1 + \frac{1-\tau_n}{\tau_n}\varepsilon\right\} \qquad (n \in \mathbb{N}_0). \tag{A.5}$$

It follows from (A.1) and (A.5) that

$$v_{n+1} = Av_n \geqslant A\left(\tau_n f^*\right) \geqslant \tau_n f^* \cdot \left\{1 + \frac{1-\tau_n}{\tau_n}\varepsilon\right\},$$

and hence, from the definition of τ_{n+1}, we get the following result by induction

$$\begin{aligned}\tau_{n+1} &\geqslant \tau_n \left[1 + \frac{1-\tau_n}{\tau_n}\varepsilon\right] \geqslant \tau_{n-1}\left[1 + \frac{1-\tau_{n-1}}{\tau_{n-1}}\varepsilon\right] \cdot \left[1 + \frac{1-\tau_n}{\tau_n}\varepsilon\right] \\ &\vdots \\ &\geqslant \tau_0 \prod_{i=0}^{n}\left[1 + \frac{1-\tau_i}{\tau_i}\varepsilon\right] = \tau_0 \prod_{i=0}^{n}\left[1 + \left(\frac{1}{\tau_i} - 1\right)\varepsilon\right].\end{aligned}$$

Invoking (A.2), we know the sequence $\{\tau_n\}_{n\geqslant 0} \uparrow \gamma < 1$. Therefore, $\{\tau_n^{-1}\}_{n\geqslant 0} \downarrow \gamma^{-1}$, which in turn yields that

$$\begin{aligned}\tau_{n+1} &\geqslant \tau_0 \prod_{i=0}^{n}\left[1 + \left(\frac{1}{\tau_i} - 1\right)\varepsilon\right] \\ &\geqslant \tau_0 \prod_{i=0}^{n}\left[1 + \left(\frac{1}{\gamma} - 1\right)\varepsilon\right] = \tau_0\left[1 + \left(\frac{1}{\gamma} - 1\right)\varepsilon\right]^n.\end{aligned}$$

Evidently, with a sufficiently large number n, $\tau_{n+1} > 1$ in contradiction with (A.2). Thus, (A.3) holds.

Step 2. (Upper convergence) Next, consider a positive number $\bar{t} > 1$, and let

$$u_0 = \bar{t} f^*, \qquad \text{and} \qquad u_{n+1} = Au_n \qquad (n \in \mathbb{N}_0).$$

Then, we use the result from Lemma 2.1.1 that the order concave operator B and hence the operator A is subhomogeneous. Clearly, it follows from this subhomogeneity of B with $\bar{t} > 1$ by (2.5) that

$$u_1 = A\left(\bar{t} f^*\right) = h + B\left(\bar{t} f^*\right) \leqslant h + \left(\bar{t} \cdot Bf^*\right) \leqslant \bar{t} \cdot Af^* = \bar{t} f^* = u_0,$$

which yields by induction that

$$\bar{t} f^* = u_0 \geqslant u_1 \geqslant \cdots \geqslant u_n \geqslant \cdots \geqslant f^*.$$

Define $s_n = \inf\{t > 0 \colon t f^* \geqslant u_n\}$, then it yields

$$s_n f^* \geqslant u_n \qquad (n \in \mathbb{N}_0), \tag{A.6}$$

and

$$\bar{t} = s_0 \geqslant s_1 \geqslant \cdots \geqslant s_n \geqslant \cdots \geqslant 1. \tag{A.7}$$

We claim that

$$\lim_{n\to\infty} s_n = 1. \tag{A.8}$$

If otherwise, there exists a constant δ such that $\lim_{n\to\infty} s_n = \delta > 1$. By the concavity and subhomogeneity of $\boldsymbol{A}$, it then follows that

$$\begin{aligned} f^* = \boldsymbol{A}f^* = \boldsymbol{A}\left(\frac{1}{\delta}\cdot \delta f^*\right) &\geqslant \frac{1}{\delta}\boldsymbol{A}\left(\delta f^*\right) + \left(1-\frac{1}{\delta}\right)\boldsymbol{A}\mathbf{0} \\ &\geqslant \frac{1}{\delta}\boldsymbol{A}\left(\delta f^*\right) + \left(1-\frac{1}{\delta}\right)\varepsilon g^* \geqslant \frac{1}{\delta}\boldsymbol{A}\left(\delta f^*\right) + \left(1-\frac{1}{\delta}\right)\varepsilon \boldsymbol{A}f^* \\ &= \frac{1}{\delta}\boldsymbol{A}\left(\delta f^*\right) + \frac{1}{\delta}\left[\left(1-\frac{1}{\delta}\right)\varepsilon\right]\delta \boldsymbol{A}f^* \\ &\geqslant \frac{1}{\delta}\boldsymbol{A}\left(\delta f^*\right) + \frac{1}{\delta}\left[\left(1-\frac{1}{\delta}\right)\varepsilon\right]\boldsymbol{A}\left(\delta f^*\right) \\ &= \frac{1}{\delta}\left[1+\left(1-\frac{1}{\delta}\right)\varepsilon\right]\boldsymbol{A}\left(\delta f^*\right). \end{aligned}$$

The first inequality follows from the concavity of $\boldsymbol{A}$ and the last inequality follows from the subhomogeneity of $\boldsymbol{A}$ in the version of $\delta > 1$ (i.e., $\delta \boldsymbol{A}f^* \geqslant \boldsymbol{A}\left(\delta f^*\right)$).

After some arrangement, we obtain

$$\boldsymbol{A}\left(\delta f^*\right) \leqslant \delta f^* \cdot \left[1+\left(1-\frac{1}{\delta}\right)\varepsilon\right]^{-1}.$$

Evidently, for any $t \geqslant \delta$, we conclude that

$$\boldsymbol{A}\left(t f^*\right) \leqslant t f^* \cdot \left[1+\left(1-\frac{1}{\delta}\right)\varepsilon\right]^{-1}.$$

To see this, consider

$$\boldsymbol{A}\left(\delta f^*\right) = \boldsymbol{A}\left(\frac{\delta}{t}\cdot t f^*\right) \geqslant \frac{\delta}{t}\boldsymbol{A}\left(t f^*\right) + \left(1-\frac{\delta}{t}\right)\boldsymbol{A}\mathbf{0} \geqslant \frac{\delta}{t}\boldsymbol{A}\left(t f^*\right),$$

which in turn yields

$$\boldsymbol{A}\left(t f^*\right) \leqslant \frac{t}{\delta}\boldsymbol{A}\left(\delta f^*\right) \leqslant t f^* \cdot \left[1+\left(1-\frac{1}{\delta}\right)\varepsilon\right]^{-1},$$

as was to be shown.

Since $s_n \geqslant \delta$, we know that

$$\boldsymbol{A}\left(s_n f^*\right) \leqslant s_n f^* \cdot \left[1+\left(1-\frac{1}{\delta}\right)\varepsilon\right]^{-1} \qquad (n \in \mathbb{N}_0). \tag{A.9}$$

It follows from (A.6) and (A.9) that

$$u_{n+1} = \boldsymbol{A} u_n \leqslant \boldsymbol{A}\left(s_n f^*\right) \leqslant s_n f^* \cdot \left[1 + \left(1 - \frac{1}{\delta}\right)\varepsilon\right]^{-1},$$

and therefore, from the definition of u_{n+1}, we obtain

$$s_{n+1} \leqslant s_n \left[1 + \left(1 - \frac{1}{\delta}\right)\varepsilon\right]^{-1} \qquad (n \in \mathbb{N}_0).$$

Continuing to iterate in this way, an inductive argument confirms that

$$s_{n+1} \leqslant s_0 \left[1 + \left(1 - \frac{1}{\delta}\right)\varepsilon\right]^{-n}.$$

Clearly, the right-hand side of the above inequality can be made arbitrarily small by choosing n sufficiently large; that is, for $n \to \infty$, $s_{n+1} \leqslant \bar{t}\left[1 + (1 - \frac{1}{\delta})\varepsilon\right]^{-n} \to 0$, which contradicts (A.7). Thus, (A.8) holds.

Step 3. (Global attractivity) Now we are ready to show that for any initial point $f_0 \in \mathbb{K}$,

$$\|\boldsymbol{A}^n f_0 - f^*\| = \|f_n - f^*\| \to 0 \qquad (\text{as } n \to \infty).$$

To this end, pick and fix an arbitrary element w in $\mathring{\mathbb{K}}$. Let

$$\mathbb{E}_w := \{f \in \mathbb{E} \colon \text{there exists } \lambda > 0 \text{ such that } -\lambda w \leqslant f \leqslant \lambda w\}, \text{ and}$$
$$\|f\|_w := \inf\{\lambda > 0 \colon -\lambda w \leqslant f \leqslant \lambda w\}, \qquad (f \in \mathbb{E}_w).$$

By virtue of the fact that $\mathbb{K}$ is a normal solid cone in the Banach space $\mathbb{E}$ with $w \in \mathring{\mathbb{K}}$, $\mathbb{E}_w = \mathbb{E}$ and the norm $\|\cdot\|_w$ is equivalent to the original norm $\|\cdot\|$ (see, e.g., Theorem 2.5.7 in Guo et al. (2004)).[5] In addition, it further follows that $\mathbb{K}_w = \mathbb{K} \cap \mathbb{E}_w = \mathbb{K} \cap \mathbb{E} = \mathbb{K}$ is a normal solid cone in the Banach space $\mathbb{E}_w$ and

$$\mathring{\mathbb{K}}_w = \{f \in \mathbb{E}_w \colon \text{there exists } \xi > 0 \text{ such that } f \geqslant \xi w\}$$
$$= \{f \in \mathbb{E} \colon \text{there exist } \lambda > \xi > 0 \text{ such that } \xi w \leqslant f \leqslant \lambda w\}.$$

This implies that $\mathring{\mathbb{K}}$ can also be expressed as

$$\mathring{\mathbb{K}} = \mathring{\mathbb{K}}_w = \{f \in \mathbb{E} \colon \text{there exist } \lambda > \xi > 0 \text{ such that } \xi w \leqslant f \leqslant \lambda w\}.$$

Invoking $f^* \in \mathring{\mathbb{K}}$ in part (I), we then know from $f^* \in \mathring{\mathbb{K}} = \mathring{\mathbb{K}}_w$ that there exist positive constants

[5] Recall that the norms $\|\cdot\|$ and $\|\cdot\|_w$ on the space $\mathbb{E}$ are said to be *equivalent* if there exist positive constants m and M such that $m\|f\| \leqslant \|f\|_w \leqslant M\|f\|$ for each $f \in \mathbb{E}$.

$a_0, b_0 > 0$ such that

$$a_0 w \leqslant f^* = \boldsymbol{A} f^* \leqslant b_0 w. \tag{A.10}$$

In the meantime, for any initial point $f_0 \in \mathbb{K}$, $\boldsymbol{A} f_0 = h + \boldsymbol{B} f_0 \geqslant h \gg \mathbf{0}$, and thus $f_1 = \boldsymbol{A} f_0 \in \mathring{\mathbb{K}} = \mathring{\mathbb{K}}_w$. This means that there also exist positive constants $a_1, b_1 > 0$ such that

$$a_1 w \leqslant f_1 = \boldsymbol{A} f_0 \leqslant b_1 w. \tag{A.11}$$

Then making use of (A.10) and (A.11), we have

$$\frac{f^*}{b_0} \leqslant w \leqslant \frac{f^*}{a_0}, \quad \text{and hence} \quad \frac{a_1}{b_0} f^* \leqslant f_1 = \boldsymbol{A} f_0 \leqslant \frac{b_1}{a_0} f^*.$$

Now we choose the real numbers $\underline{t}$ and $\bar{t}$ such that

$$0 < \underline{t} < \min\{1,\ a_1/b_0\}, \qquad \text{and} \qquad \max\{1,\ b_1/a_0\} < \bar{t}.$$

Clearly, $0 < \underline{t} < 1$ and $\bar{t} > 1$.

Set $v_0 := \underline{t} f^*$ and $u_0 := \bar{t} f^*$, we then have

$$v_0 = \underline{t} f^* < a_1 b_0^{-1} f^* \leqslant f_1 = \boldsymbol{A} f_0 \leqslant b_1 a_0^{-1} f^* < \bar{t} f^* = u_0.$$

On continuing this process of acting the operator $\boldsymbol{A}$ successively on the above chain and using induction, we obtain that

$$v_n \leqslant f_{n+1} \leqslant u_n, \qquad (n \in \mathbb{N}_0).$$

Invoking (A.1) and (A.6) now yields $\tau_n f^* \leqslant v_n \leqslant f_{n+1} \leqslant u_n \leqslant s_n f^*$. It then follows that $(\tau_n - 1) f^* \leqslant f_{n+1} - f^* \leqslant (s_n - 1) f^*$, and hence invoking (A.10) yields

$$(\tau_n - 1) b_0 w \leqslant f_{n+1} - f^* \leqslant (s_n - 1) b_0 w.$$

Recall from (A.3) with (A.2) and (A.8) with (A.7) that $\tau_n \uparrow 1$ and $s_n \downarrow 1$, we therefore obtain that $\lim_{n\to\infty} \|f_{n+1} - f^*\|_w = \lim_{n\to\infty} \|f_{n+1} - f^*\| = 0$, as was to be shown.

This completes the proof of Theorem 2.2.1. □

Proof of Lemma 2.2.4. Observe first that $\kappa \in b_\kappa m(\mathbb{X})_+$, since $\kappa \geqslant 1$ for all $x \in \mathbb{X}$ and $\|\kappa\|_\kappa = 1$.

To prove that κ indeed lies in the interior of $b_\kappa m(\mathbb{X})_+$, it suffices to show that there is some $\varepsilon > 0$ such that a neighborhood $N_\varepsilon(\kappa)$ of w is contained in $b_\kappa m(\mathbb{X})_+$. To see this, let $\varepsilon \in (0, 1)$. Suppose there exists $f \in b_\kappa m(\mathbb{X})$ such that $\|f - \kappa\|_\kappa < \varepsilon$, and there exists $x_0 \in \mathbb{X}$ such that

$f(x_0) < 0$. Then, we have

$$|f(x_0) - \kappa(x_0)| = \kappa(x_0) - f(x_0) > \kappa(x_0),$$

and hence,

$$\|f - \kappa\|_\kappa \geqslant \frac{|f(x_0) - \kappa(x_0)|}{\kappa(x_0)} > \frac{\kappa(x_0)}{\kappa(x_0)} = 1 > \varepsilon.$$

However, this contradicts our hypothesis, and therefore

$$N_\varepsilon(w) = \{f \in b_\kappa m(\mathbb{X})\colon \|f - \kappa\|_\kappa < \varepsilon\} \subset b_\kappa m(\mathbb{X})_+,$$

which is what we wish to show. □

Proof of Lemma 2.2.5. We first prove one direction that $b_\kappa \mathring{m}(\mathbb{X})_+ \subset \{f \in b_\kappa m(\mathbb{X})_+ : [f]_\kappa > 0\}$. To do this, pick any f_0 in $b_\kappa \mathring{m}(\mathbb{X})_+$. Then there exists a constant $r > 0$ such that a closed ball

$$\overline{B(f_0, r)} = \{f \in b_\kappa m(\mathbb{X}) : \|f - f_0\|_\kappa \leqslant r\} \subset b_\kappa m(\mathbb{X})_+,$$

and thus any element $h := f_0 \pm rg$ of $b_\kappa m(\mathbb{X})$ also lies in $b_\kappa m(\mathbb{X})_+$ for all $g \in b_\kappa m(\mathbb{X})$ with $\|g\|_\kappa \leqslant 1$. In particular, it holds for $g \equiv \kappa$. It then follows that $f_0 - r\kappa \geqslant \mathbf{0}$, and hence for each $x \in \mathbb{X}$,

$$\frac{f_0(x) - r\kappa(x)}{\kappa(x)} \geqslant 0,$$

from which we obtain $[f_0]_\kappa \geqslant r > 0$.

Conversely, let $f \in b_\kappa m(\mathbb{X})_+$ and $[f]_\kappa > 0$. Then, for any $g \in b_\kappa m(\mathbb{X})$ with $\|g\|_\kappa \leqslant 1$, it follows from $|g| \leqslant \kappa$ that

$$\begin{aligned} h_1(x) &:= f(x) - [f]_\kappa g(x) \geqslant f(x) - [f]_\kappa \kappa(x) \geqslant 0 \qquad (g \geqslant \mathbf{0}), \\ h_2(x) &:= f(x) + [f]_\kappa g(x) \geqslant f(x) - [f]_\kappa \kappa(x) \geqslant 0 \qquad (g \leqslant \mathbf{0}) \end{aligned}$$

for all $x \in \mathbb{X}$. This implies that an open ball $B(f, [f]_\kappa) \subset b_\kappa m(\mathbb{X})$ centered on f with radius $[f]_\kappa$ is contained in $b_\kappa m(\mathbb{X})_+$. Therefore, f is an interior point of $b_\kappa m(\mathbb{X})_+$, i.e. $f \in b_\kappa \mathring{m}(\mathbb{X})_+$.

This completes the proof. □

A.2 Appendix to Chapter 3

Proof of Proposition 3.3.1. Define the operator $\boldsymbol{T}$

$$(\boldsymbol{T} f)(x) := h(x) + \beta \mathcal{M}_\theta f(x), \qquad (x \in \mathbb{X})$$

where $\mathcal{M}_\theta$ is as given in (3.1) and $h(x) := (1 - \beta)c(x)^{1-\rho}$.

Then, the solution to (3.6) becomes the fixed-point problem corresponding to the operator $\boldsymbol{T}$. Observe that the operator $\boldsymbol{T}$ defined above is identical to the operator given in (2.9). Recalling from Lemma 2.2.4 that the positive cone $b_\kappa m(\mathbb{X})_+$ is solid, we now aim to apply Theorem 2.2.1 to solve (3.6).

Evidently, $b_\kappa m(\mathbb{X})$ is a real Banach lattice and its positive cone $b_\kappa m(\mathbb{X})_+$ is reproducing and normal.[6] Then, it is sufficient to show the following conditions:

(i) The nonlinear operator $\mathcal{M}_\theta$ is order concave.

(ii) There exists a majorant $\mathcal{M}_\theta^+$ for $\mathcal{M}_\theta$ such that $\mathcal{M}_\theta^+ \colon b_\kappa m(\mathbb{X}) \to b_\kappa m(\mathbb{X})$ is a bounded linear operator with the operator norm $\|\mathcal{M}_\theta^+\| = d$ for some constant $d \in (0, 1/\beta)$.

(iii) $h = (1-\beta)c^{1-\rho}$ is an interior point of $b_\kappa m(\mathbb{X})_+$.

Regarding condition (i), as $\theta < 1$ and $\theta \neq 0$, by Lemma 3.2.2, it follows that $\mathcal{M}_\theta$ is order concave on $b_\kappa m(\mathbb{X})_+$.

Regarding condition (ii), one can verify that by making use of Jensen's inequality, $\mathcal{M}_\theta$ is dominated by an additive and bounded operator $\mathcal{M}_\theta^+$ on $b_\kappa m(\mathbb{X})_+$, where $\mathcal{M}_\theta^+$ is defined by $\mathcal{M}_\theta^+ f(x) := \int f(x')Q(x, \mathrm{d}x')$ for all $f \in b_\kappa m(\mathbb{X})_+$ and each $x \in \mathbb{X}$. By Lemma 2.2.3, we obtain that $\mathcal{M}_\theta^+$ has a unique positive extension to a bounded linear operator from $b_\kappa m(\mathbb{X})$ to itself. It remains to show such operator $\mathcal{M}_\theta^+$ on $b_\kappa m(\mathbb{X})$ is bounded with $\|\mathcal{M}_\theta^+\| = d$. To see this, fix any $x \in \mathbb{X}$ and pick any $f \in b_\kappa m(\mathbb{X})$. We then have,

$$\begin{aligned}
\left|\left(\mathcal{M}_\theta^+ f\right)(x)\right| &= \left|\int f(x')Q(x, \mathrm{d}x')\right| && \text{(by the definition of } \mathcal{M}_\theta^+\text{)} \\
&\leqslant \int \left|f(x')\right| Q(x, \mathrm{d}x') \\
&\leqslant \int \|f\|_\kappa \cdot \kappa(x')Q(x, \mathrm{d}x') && (\because f \in b_\kappa m(\mathbb{X})\,) \\
&\leqslant \|f\|_\kappa \cdot d_\kappa \cdot \kappa(x).
\end{aligned}$$

Dividing $\kappa(x)$ for both sides of the preceding inequality and taking the supremum over $x \in \mathbb{X}$ gives $\|\mathcal{M}_\theta^+\| = \sup_{\|f\|_\kappa = 1} \|\mathcal{M}_\theta^+ f\|_\kappa = d_\kappa \in (0, 1/\beta)$, as was to be shown.

Regarding condition (iii), invoking Lemma 2.2.5, it then follows that the interiority of h in $b_\kappa m(\mathbb{X})_+$ is immediate and guaranteed by Assumption 3.3.2.

Hence all conditions of Theorem 2.2.1 are verified and the conclusions of that theorem now follow. □

Proof of Proposition 3.3.2. Regarding claim (i), we note that the operator $\boldsymbol{T}$ maps $\mathscr{L}_1(\tilde{\pi})_+$ into itself. In particular, the positive cone $\mathscr{L}_1(\tilde{\pi})_+$ is regular (see, e.g., Section 1.5 in Krasnosel'skiĭ (1964)). In this connection, by virtue of Theorem 3.1.4 in Guo et al. (2004), it suffices to show that

[6] It is worth noting that the normality constant of $b_\kappa m(\mathbb{X})$ is 1.

(1) $\boldsymbol{T}$ is monotone increasing on $\mathscr{L}_1(\tilde{\pi})_+$.

(2) $\boldsymbol{T}$ has a lower solution and an upper solution in $\mathscr{L}_1(\tilde{\pi})_+$.

To this end, invoking Lemma 3.2.2, it follows that the non-additive Markov operator $\tilde{\mathcal{M}}_\theta$ for $0 \neq \theta \leqslant 1$ is order concave and hence, so is $\boldsymbol{T}$. Since $\boldsymbol{T}$ leaves the positive cone $\mathscr{L}_1(\tilde{\pi})_+$ invariant and since $\boldsymbol{T}$ is order concave, Lemma 3.4 of Du (1990) applies and implies that $\boldsymbol{T}$ is monotone increasing on $\mathscr{L}_1(\tilde{\pi})_+$.

Turning to the existence of a lower solution for $\boldsymbol{T}$, we observe that a natural choice of a lower solution is the zero element $\mathbf{0}$ in $\mathscr{L}_1(\tilde{\pi})_+$. Indeed, since $\tilde{\mathcal{M}}_\theta \mathbf{0} = \mathbf{0}$ and since $\xi^{1-\rho} > 0$ and $e(x) > 0$ together imply $h(x) = \xi^{1-\rho} e^{-1/\theta}(x) > 0$, we then have that $\boldsymbol{T}\mathbf{0} = h > \mathbf{0}$, as desired.

In order to find an upper solution for $\boldsymbol{T}$, we shall apply Corollary 2.2.1, since $\mathscr{L}_1(\tilde{\pi})_+$ is not solid. To do so, it is sufficient to verify the conditions in Assumption 2.2.3; that is, to show that there exists a majorant $\tilde{\mathcal{M}}_\theta^+$ for $\tilde{\mathcal{M}}_\theta$ such that $\tilde{\mathcal{M}}_\theta^+$ is a bounded linear operator on $\mathscr{L}_1(\tilde{\pi})$ with the operator norm $\|\tilde{\mathcal{M}}_\theta^+\| = d$, for some number $d \in (0, \exp(\xi))$.

By virtue of Jensen's inequality, one can verify that a linear and positive operator $\tilde{\mathcal{M}}_\theta^+$ that is defined by $(\tilde{\mathcal{M}}_\theta^+ g)(x) := \int g(x')\tilde{Q}(x, \mathrm{d}x')$, dominates $\tilde{\mathcal{M}}_\theta$ on $\mathscr{L}_1(\tilde{\pi})_+$. In this connection, it remains only to show that the majorant $\tilde{\mathcal{M}}_\theta^+$ on $\mathscr{L}_1(\tilde{\pi})$ is bounded with $\|\tilde{\mathcal{M}}_\theta^+\| = d$ for some constant $0 < d < \exp(\xi)$.[7]

To see this, pick any $g \in \mathscr{L}_1(\tilde{\pi})$. We have that

$$
\begin{aligned}
\|\tilde{\mathcal{M}}_\theta^+ g\| &= \int \left| \left(\tilde{\mathcal{M}}_\theta^+ g\right)(x) \right| \tilde{\pi}(\mathrm{d}x) \\
&= \int \left| \int g(x')\tilde{Q}(x, \mathrm{d}x') \right| \tilde{\pi}(\mathrm{d}x) \qquad \text{(by the definition of } \tilde{\mathcal{M}}_\theta^+\text{)} \\
&\leqslant \int \int |g(x')| \, \tilde{Q}(x, \mathrm{d}x')\tilde{\pi}(\mathrm{d}x) \\
&= \int |g(x')| \int \tilde{Q}(x, \mathrm{d}x')\tilde{\pi}(\mathrm{d}x) \\
&= \int |g(x')| \, \tilde{\pi}(\mathrm{d}x') = \|g\|
\end{aligned}
$$

where the forth equality, by virtue of Assumption 3.3.4, follows from the fact that $\tilde{\pi}$ is the stationary distribution of $\tilde{Q}$ (i.e., $\int \tilde{Q}(x, B)\tilde{\pi}(\mathrm{d}x) = \tilde{\pi}(B)$ for all $B \in \mathscr{B}$). Therefore, the above equality implies that $\|\tilde{\mathcal{M}}_\theta^+\| = \sup_{\|g\|=1} \|\tilde{\mathcal{M}}_\theta^+ g\| \leqslant 1$. As a consequence, $\|\tilde{\mathcal{M}}_\theta^+\| \leqslant 1 < \exp(\xi)$, because $\xi > 0$ by Assumption 3.3.5 and thus $\exp(\xi) > 1$, which is what we needed to show.

Regarding claim (ii), we follow the setting of $\zeta > 0$ in Hansen and Scheinkman (2012) and let $\zeta = [1 - \exp(-\xi)]^{1/(1-\rho)}$, and hence the element h in (3.12) becomes $h = [1 - \exp(-\xi)]e^{-1/\theta}$. In this connection, making use of the convexity and bijectivity of $\phi(s) := s^\theta$ for $\theta \geqslant 1$ on $\mathbb{R}_+$ and applying Jensen's inequality yield

$$
\boldsymbol{S}\ell(x) \leqslant [1 - \exp(-\xi)]e(x)^{-1} + \exp(-\xi) \int \ell(x')\tilde{Q}(x, \mathrm{d}x') \tag{A.12}
$$

[7] More precisely, the operator norm $\|\tilde{\mathcal{M}}_\theta^+\|$ is equal to 1.

for every $\ell \in \mathscr{L}_1(\tilde{\pi})_+$ and all $x \in \mathbb{X}$, in which case we obtain

$$\int S\ell(x)\tilde{\pi}(\mathrm{d}x) \leqslant [1-\exp(-\xi)]\int e(x)^{-1}\tilde{\pi}(\mathrm{d}x) + \exp(-\xi)\int\int \ell(x')\tilde{Q}(x,\mathrm{d}x')\tilde{\pi}(\mathrm{d}x)$$
$$= [1-\exp(-\xi)]\int e(x)^{-1}\tilde{\pi}(\mathrm{d}x) + \exp(-\xi)\int \ell(x')\tilde{\pi}(\mathrm{d}x').$$

Since ℓ is chosen arbitrarily from $\mathscr{L}_1(\tilde{\pi})_+$ and by Assumption 3.3.7, we have $S\ell \in \mathscr{L}_1(\tilde{\pi})_+$ and thus the operator S indeed maps $\mathscr{L}_1(\tilde{\pi})_+$ into itself.

Evidently, $\mathbf{0} \in \mathscr{L}_1(\tilde{\pi})_+$ is a natural choice of a lower solution for the operator S. In the sequel, we consider a linear operator S^+ defined on $\mathscr{L}_1(\tilde{\pi})$ given by

$$S^+\ell(x) := [1-\exp(-\xi)]e(x)^{-1} + \exp(-\xi)\int \ell(x')\tilde{Q}(x,\mathrm{d}x') \quad (x \in \mathbb{X}).$$

It is clear that S^+ is a self-map on $\mathscr{L}_1(\tilde{\pi})$ and, by virtue of Assumption 3.3.5, Banach's Contraction Mapping Principle applies to S^+ and gives us a unique fixed point ℓ^{+*} in $\mathscr{L}_1(\tilde{\pi})$. Since $[1-\exp(-\xi)]e(x)^{-1} > 0$ for all $x \in \mathbb{X}$, the fixed point ℓ^{+*} must lie in $\mathscr{L}_1(\tilde{\pi})_+ \setminus \{\mathbf{0}\}$. Hence, invoking (A.12), we have $S\ell^{+*} \leqslant S^+\ell^{+*} = \ell^{+*}$, which in turn implies that the fixed point $\ell^{+*} \in \mathscr{L}_1(\tilde{\pi})_+ \setminus \{\mathbf{0}\}$ of S^+ is an upper solution of S.

Hence, applying Theorem 3.1.4 in Guo et al. (2004), we obtain that the operator S has a minimal fixed point and a maximal fixed point in $[\mathbf{0}, \ \ell^{+*}]$. Since there is one-to-one correspondence between fixed points of S and fixed points of T for $\theta \geqslant 1$, it follows that the operator T has a minimal fixed point and a maximal fixed point in $\mathscr{L}_1(\tilde{\pi})_+$ accordingly.

This completes the proof of Proposition 3.3.2. □

Proof of Proposition 3.3.3. It is clear that $\mathscr{L}_1(\tilde{\pi})$ is a real Banach lattice and its positive cone $\mathscr{L}_1(\tilde{\pi})_+$ is reproducing and normal. Hence, we aim to apply Corollary 2.2.2 to obtain the stated results.

To this end, we need to show that

(a) The operator $\mathcal{M}_\theta$ is order concave on $\mathscr{L}_1(\tilde{\pi})_+$.

(b) There exists a majorant $\mathcal{M}_\theta^+$ for $\mathcal{M}_\theta$ such that $\mathcal{M}_\theta^+$ is a bounded linear operator on $\mathscr{L}_1(\tilde{\pi})$ with the operator norm $\|\mathcal{M}_\theta^+\| = d$, for some number $d \in (0, \exp(\xi))$.

(c) $h = \zeta^{1-\rho}e^{-1/\theta}$ lies in $\mathscr{L}_1(\tilde{\pi})_+ \setminus \{\mathbf{0}\}$, and $h \geqslant \varepsilon g$ ($\tilde{\pi}$-a.e.) for some upper solution g of T and some arbitrarily small positive number $\varepsilon \in (0,1)$.

The proofs regarding points (a) and (b) have been shown in Proposition 3.3.2, and hence are omitted here. The first part of point (c) is obvious: since $\zeta^{1-\rho}$ is strictly positive and the principle eigenvector e is also strictly positive, it then follows from Assumption 3.3.6 that $h \in \mathscr{L}_1(\tilde{\pi})_+ \setminus \{\mathbf{0}\}$. Regarding the remaining part of (c), recall the proof of Lemma 2.2.2 first. Since $\|\exp(-\xi)\mathcal{M}_\theta^+\| \leqslant \exp(-\xi) < 1$, the Neumann series theorem applies, and we then have that

$$g^{+*}(x) = [1 - \exp(-\xi)]\, \hat{g}_1(x) \qquad (x \in \mathbb{X}),$$

which is an upper solution of $\boldsymbol{T}$. As the Perron-Frobenius eigenvector e and hence h and $\hat{g}_1$ are strictly positive, it is not hard to see that $\mathbf{0} < g^{+*}$ $\tilde{\pi}$-a.e.. In addition, the inequality condition in point (c) is guaranteed by Assumption 3.3.8. Therefore, Corollary 2.2.2 applies now and gives us the stated results of Proposition 3.3.3. □

Proof of Proposition 3.3.4. Invoking the proof of Proposition 3.3.2, we have that the auxiliary operator $\boldsymbol{S}\colon \mathscr{L}_1(\tilde{\pi})_+ \to \mathscr{L}_1(\tilde{\pi})_+$ has a lower solution $\mathbf{0}$ and an upper solution

$$\ell^{+*}(x) = [1 - \exp(-\xi)]\, \hat{g}_2(x) \qquad (x \in \mathbb{X}).$$

Evidently, the operator $\boldsymbol{S}$ is order concave on $\mathscr{L}_1(\tilde{\pi})_+$, which is derived from the linearity of the integral operator and the concavity of the scalar function

$$\psi(t) = \left[b + \exp(-\xi) t^{\frac{1}{\theta}}\right]^{\theta} \qquad (t > 0).$$

Further, by virtue of Lemma 3.4 of Du (1990), it follows from concavity of $\boldsymbol{S}$ that it is monotone increasing. Furthermore, by Assumption 3.3.9, it follows that $\boldsymbol{S}\mathbf{0} = [1 - \exp(\xi)]e^{-1} \geqslant \ell^{+*}$ ($\tilde{\pi}$-a.e.), in which case Corollary 2.2.2 applies and yields the stated results of Proposition 3.3.4 for the operator $\boldsymbol{S}$. □

Lemma A.2.1. *If $\theta \geqslant 1$, then the operator (3.10) is a strict contraction on $\mathscr{L}_\theta(\tilde{\pi})_+$.*

As $\mathscr{L}_\theta(\tilde{\pi})_+$ is a closed subset of the Banach space $\mathscr{L}_\theta(\tilde{\pi})$, by virtue of Lemma A.2.1, Banach contraction mapping theorem applies.

Proof of Lemma A.2.1. Recall that the operator $\boldsymbol{T}$ is defined through

$$g(x) = (\boldsymbol{T}g)\,(x) = h(x) + \exp(-\xi) \left\{ \int g(x')^{\theta} \tilde{Q}(x, \mathrm{d}x') \right\}^{1/\theta} \qquad (x \in \mathbb{X}),$$

Pick any two functions g_1 and g_2 in $\mathscr{L}_\theta(\tilde{\pi})_+$. Observe that, for fixed $x \in \mathbb{X}$, we have

$$\begin{aligned} |\boldsymbol{T}g_1(x) - \boldsymbol{T}g_2(x)| &= \exp(-\xi) \left| \left\{ \int g_1(x')^{\theta} \tilde{Q}(x, \mathrm{d}x') \right\}^{1/\theta} - \left\{ \int g_2(x')^{\theta} \tilde{Q}(x, \mathrm{d}x') \right\}^{1/\theta} \right| \\ &\leqslant \exp(-\xi) \left\{ \int |g_1 - g_2|\,(x')^{\theta} \tilde{Q}(x, \mathrm{d}x') \right\}^{1/\theta} \end{aligned}$$

where the last inequality follows from the reverse triangle inequality with respect to the norm $\|g\|_{\tilde{Q}} := \{\int g(x')^{\theta} \tilde{Q}(x, \mathrm{d}x')\}^{1/\theta}$.

Raising the exponent θ on both sides of the above inequality, we obtain

$$|\boldsymbol{T}g_1(x) - \boldsymbol{T}g_2(x)|^\theta \leqslant \exp(-\theta\xi) \int |g_1 - g_2|\,(x')^\theta \tilde{Q}(x, \mathrm{d}x').$$

It then follows that

$$\begin{aligned}\int |\boldsymbol{T}g_1(x) - \boldsymbol{T}g_2(x)|^\theta \tilde{\pi}(\mathrm{d}x) &\leqslant \exp(-\theta\xi) \int \left[\int |g_1 - g_2|\,(x')^\theta \tilde{Q}(x, \mathrm{d}x')\right] \tilde{\pi}(\mathrm{d}x) \\ &= \exp(-\theta\xi) \int |g_1 - g_2|\,(x')^\theta \tilde{\pi}(\mathrm{d}x')\end{aligned}$$

where the last equality follows from the fact that $\tilde{\pi}$ is the stationary distribution of $\tilde{Q}$. Further, raising the exponent $1/\theta$ on both sides of the above inequality yields

$$\begin{aligned}\|\boldsymbol{T}g_1 - \boldsymbol{T}g_2\|_\theta &= \left\{\int |\boldsymbol{T}g_1(x) - \boldsymbol{T}g_2(x)|^\theta \tilde{\pi}(\mathrm{d}x)\right\}^{1/\theta} \\ &\leqslant \exp(-\xi) \left\{\int |g_1 - g_2|\,(x)^\theta \tilde{\pi}(\mathrm{d}x)\right\}^{1/\theta} = \exp(-\xi)\|g_1 - g_2\|_\theta.\end{aligned}$$

Since $\exp(-\xi) < 1$ by assumption, we complete the proof of Lemma A.2.1. □

Proof of Proposition 3.4.1. Define an operator $\boldsymbol{T}$ on $c(\mathbb{X})$ through

$$\boldsymbol{T}\tilde{g}(x) = h(x) + \frac{\beta}{1-\gamma} \ln\left(\int \exp\left[(1-\gamma)\tilde{g}(x')\right] \tilde{Q}(x, \mathrm{d}x')\right) \tag{A.13}$$

with

$$h(x) := \frac{\beta}{1-\gamma} \ln r(\boldsymbol{K}) + \frac{\beta - 1}{1-\gamma} \ln e(x) \quad \text{for each } x \in \mathbb{X}.$$

In this connection, we can express (3.17) as an operator equation $\tilde{g} = \boldsymbol{T}\tilde{g}$, and obviously, a solution $\tilde{g}^*$ solves (3.17) if and only if $\tilde{g}^*$ is a fixed point of $\boldsymbol{T}$.

Evidently, the operator $\boldsymbol{T}$ maps $c(\mathbb{X})$ into itself. In addition, we note that an operator $\mathcal{M}$ defined on $c(\mathbb{X})$ through

$$\mathcal{M}\tilde{g}(x) := \frac{1}{1-\gamma} \ln\left(\int \exp\left[(1-\gamma)\tilde{g}(x')\right] \tilde{Q}(x, \mathrm{d}x')\right)$$

is monotone increasing and order concave.[8] It then follows that the operator $\boldsymbol{T}$ is monotone increasing and order concave on $c(\mathbb{X})$.

In the sequel, we seek for an upper solution of $\boldsymbol{T}$. To this end, we consider an operator $\boldsymbol{T}^+$ defined on $c(\mathbb{X})$ through

$$\boldsymbol{T}^+\tilde{g}(x) = h(x) + \beta \int \tilde{g}(x')\tilde{Q}(x, \mathrm{d}x') \qquad (x \in \mathbb{X}).$$

[8] Regarding the concavity of $\mathcal{M}$ defined above, interested reader may refer to Föllmer and Schied (2004), or Bäuerle and Jaśkiewicz (2018).

Since $c(\mathbb{X})$ is a Banach space (a complete normed vector space), and since $\beta \in (0,1)$, by virtue of Banach's Fixed Point Theorem, it follows that $\boldsymbol{T}^+$ has a unique fixed point $\tilde{g}^{+*}$ in $c(\mathbb{X})$. In addition, by making use of Jensen's inequality and properties of monotone decreasing and convex with respect to the function $\phi(t) = \exp(-\theta t)$ for $t \in \mathbb{R}$ with $\theta > 0$, we observe that $\boldsymbol{T}^+$ is a majorant operator for $\boldsymbol{T}$ on $c(\mathbb{X})$, i.e., $\boldsymbol{T}\tilde{g} \leqslant \boldsymbol{T}^+\tilde{g}$ for all $\tilde{g} \in c(\mathbb{X})$. Hence, we have $\boldsymbol{T}\tilde{g}^{+*} \leqslant \boldsymbol{T}^+\tilde{g}^{+*} = \tilde{g}^{+*}$, which in turn implies that $\tilde{g}^{+*}$ is an upper solution of $\boldsymbol{T}$.

Turning to the existence of a strong lower solution of $\boldsymbol{T}$. Since the Krein-Rutman eigenvector e lies in $c(\mathbb{X})$, and since $\mathbb{X}$ is compact, the maximum and minimum of e must exist, and henceforth are denoted by $\hat{e}$ and $\check{e}$, respectively. We claim that for any $\varepsilon \in \mathbb{R}$ satisfying $\varepsilon < \min\{\beta \ln r(\boldsymbol{K})/(1-\beta)(1-\gamma) + \ln \check{e}/(\gamma - 1),\ \min \tilde{g}^{+*}\}$, a function $\varepsilon \mathbb{1}_{\mathbb{X}}$ is a strong lower solution of $\boldsymbol{T}$.

To see that it is so, pick and fix a constant ε such that $\varepsilon < \beta \ln r(\boldsymbol{K})/[(1-\beta)(1-\gamma)] + \ln \check{e}/(\gamma - 1)$. Evidently, the function $\varepsilon \mathbb{1}_{\mathbb{X}}$ is in $c(\mathbb{X})$ and further, we obtain

$$
\begin{aligned}
\boldsymbol{T}\left(\varepsilon \mathbb{1}_{\mathbb{X}}\right)(x) &= h(x) + \beta \varepsilon \mathbb{1}_{\mathbb{X}}(x) \\
&\geqslant \frac{\beta}{1-\gamma} \ln r(\boldsymbol{K}) + \frac{\beta - 1}{1-\gamma} \ln \check{e} + \beta \varepsilon \mathbb{1}_{\mathbb{X}}(x) \\
&> \varepsilon \mathbb{1}_{\mathbb{X}}(x)
\end{aligned}
$$

for each $x \in \mathbb{X}$. To see that the last strict inequality holds true, we observe that together with $1 - \beta > 0$, the condition $\varepsilon < \beta \ln r(\boldsymbol{K})/[(1-\beta)(1-\gamma)] + \ln \check{e}/(\gamma - 1)$ implies $(1-\beta)\varepsilon < \beta \ln r(\boldsymbol{K})/(1-\gamma) + (1-\beta)\ln \check{e}/(\gamma - 1)$, and so, with some rearranging, we obtain $\varepsilon < \beta \ln r(\boldsymbol{K})/(1-\gamma) + (\beta - 1)\ln \check{e}/(1-\gamma) + \beta\varepsilon$.

Since $\boldsymbol{T}\left(\varepsilon \mathbb{1}_{\mathbb{X}}\right)(x) > \varepsilon \mathbb{1}_{\mathbb{X}}(x)$ for all $x \in \mathbb{X}$, it is clear that $\boldsymbol{T}\varepsilon \mathbb{1}_{\mathbb{X}} \gg \varepsilon \mathbb{1}_{\mathbb{X}}$ in $c(\mathbb{X})$, which is what we wish to show.

Finally, for such a strong lower solution $\varepsilon \mathbb{1}_{\mathbb{X}}$ with $\varepsilon < \min\{\beta \ln r(\boldsymbol{K})/(1-\beta)(1-\gamma) + \ln \check{e}/(\gamma - 1),\ \min \tilde{g}^{+*}\}$, we know that $\varepsilon \mathbb{1}_{\mathbb{X}} < \tilde{g}^{+*}$. Since $\boldsymbol{T}$ is monotone increasing and order concave, Du's Theorem applies and gives us the stated results of Proposition 3.4.1. □

Proof of Lemma 3.4.1. Recall the definition of the non-additive Markov operator, we have

$$
\left(\mathcal{M}_{1-\gamma} g\right)(x) := [\int g(x')^{1-\gamma} \tilde{Q}(x, \mathrm{d}x')]^{1/(1-\gamma)}
$$

for each $g \in c(\mathbb{X})_{++}$.

Monotonicity of $\mathcal{M}_{1-\gamma}$ is obvious. Since $1-\gamma$ is strictly less than 1 and distinct from zero, together with the positive homogeneity of $\mathcal{M}_{1-\gamma}$, Lemma 3.2.2 applies and yields that $\mathcal{M}_{1-\gamma}$ is order concave.

Regarding the first part, monotone increasing property and concavity of $\boldsymbol{N}$ is a direct consequence of the monotonicity and concavity of $\mathcal{M}_{1-\gamma}$.

Regarding the second part, fix $g \in c(\mathbb{X})_{++}$. Since g is strictly positive and continuous on the compact metric space $\mathbb{X}$, so is $g^{1-\gamma}$. Thus, $g^{1-\gamma}$ is bounded and strictly positive everywhere, from which it then follows from the continuity of the transition kernel $\tilde{Q}$ that the function $\int g(x')^{1-\gamma}\tilde{Q}(x, \mathrm{d}x')$ is strictly positive and continuous on $\mathbb{X}$.[9] Therefore, we can conclude that $\mathcal{M}_{1-\gamma}g$ is strictly positive and continuous. By Assumption 3.4.4, the stated result follows immediately. □

Proof of Lemma 3.4.2. To see that this is so, we consider an auxiliary scalar function ψ defined by

$$\psi(t) = \left\{a + dt^{1-\rho}\right\}^{\frac{1}{1-\rho}} \qquad (t > 0) \tag{A.14}$$

where $a > 0$ and $d \in (0, 1)$ are fixed constants.

Evidently, such a function ψ is monotone increasing, continuous and, more importantly is concave whenever $0 < \rho \neq 1$.

Since h is positive everywhere and $\beta r(\boldsymbol{K})^{1/\theta} < 1$, these aforementioned properties of ψ imply that $\tilde{\boldsymbol{T}}$ possesses the analogous properties. Hence, it completes the proof of Lemma 3.4.2. □

Proof of Lemma 3.4.3. Observe that $\boldsymbol{T}$ is a composition operator of $\boldsymbol{N}$ and $\tilde{\boldsymbol{T}}$. Combining the results of Lemmas 3.4.1 and 3.4.2, we obtain that $\boldsymbol{T}$ is monotone increasing and concave on $c(\mathbb{X})_{++}$. In addition, it is easy to see that $\boldsymbol{T}g$ is strictly positive and continuous on $\mathbb{X}$ whenever $g \in c(\mathbb{X})_{++}$, as was to be shown. □

Proof of Lemma 3.4.4. **Case I :** $0 < \rho < 1$. Regarding the existence of a strong lower solution to (3.19), let

$$w_1 := \left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}$$

be a constant function defined on $\mathbb{X}$, where $\delta := \beta r(\boldsymbol{K})^{1/\theta}$.

We claim that the preceding constant function w_1 satisfies that, for any given $x \in \mathbb{X}$, we have

$$\begin{aligned} \boldsymbol{T}w_1(x) = \tilde{\boldsymbol{T}}\left(\boldsymbol{N}w_1\right)(x) = \tilde{\boldsymbol{T}}(w_1 + j)(x) &= \left\{h(x) + \delta\left[w_1(x) + j(x)\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &\geqslant \left\{\check{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &> w_1(x), \end{aligned} \tag{A.15}$$

where the second equality follows from the elementary property of the operator $\mathcal{M}_{1-\gamma}$ such that $\mathcal{M}_{1-\gamma}(d) = d$ for any nonnegative constant function d.

[9] To see that it is strictly positive, please refer to Paper 3. Regarding the continuity, please refer to Aliprantis and Border (2006).

It is clear that the last uniformly strict inequality (A.15) implies that such a constant function w_1 is a strong lower solution of (3.19). Hence, to show the existence of a strong lower solution, it is sufficient to show that (A.15) holds for the function w_1 defined above.

To see that this is so, observe that, since $0 < 1-\rho < 1$, we have the following equivalence relations

$$\left\{\check{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} > w_1 \iff \check{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho} > w_1^{1-\rho}$$
$$\iff \left(\frac{w_1^{1-\rho} - \check{h}}{\delta}\right)^{\frac{1}{1-\rho}} - w_1 - \check{j} < 0.$$

Setting

$$\check{\varphi}(w_1) := \left(\frac{w_1^{1-\rho} - \check{h}}{\delta}\right)^{\frac{1}{1-\rho}} - w_1 - \check{j},$$

and invoking the definition of w_1, we have

$$\check{\varphi}(w_1) = \check{\varphi}\left[\left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}\right] = \left(\frac{\frac{\delta^{1/\rho}}{1-\delta^{1/\rho}}\check{h}}{\delta}\right)^{\frac{1}{1-\rho}} - \left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \check{j}$$
$$= \left[\left(\delta^{\frac{1}{\rho}-1}\right)^{\frac{1}{1-\rho}} - 1\right]\left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \check{j}$$
$$= \left[\delta^{\frac{1}{\rho}} - 1\right]\left(\frac{\check{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \check{j}.$$

Since $\delta \in (0,1)$ and $\rho \in (0,1)$, we have $\delta^{1/\rho} - 1 < 0$. And since $\check{h} > 0$, the term $[\check{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}$ is strictly positive. As a result, it is clear that $\check{\varphi}(w_1) < 0$, which is what we needed to show. Therefore, according to the uniformly strict inequality (A.15), we can see that $\boldsymbol{T}w_1 \gg w_1$, as desired.

It remains to show the existence of an upper solution w_2 to (3.19). Given a positive constant function d defined on $\mathbb{X}$, observe first that

$$\boldsymbol{T}d(x) = \tilde{\boldsymbol{T}}\left(\boldsymbol{N}d\right)(x) = \tilde{\boldsymbol{T}}(d+j)(x) = \left\{h(x) + \delta\left[d(x) + j(x)\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}}$$
$$\leqslant \left\{\hat{h} + \delta\left[d + \hat{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}},$$

for all $x \in \mathbb{X}$. Evidently, to show the existence of an upper solution w_2, it suffices to show that there is a positive constant d satisfying

$$\left\{\hat{h} + \delta\left[d + \hat{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \leqslant d. \tag{A.16}$$

Since $0 < 1-\rho < 1$, after some rearrangement, showing that (A.16) holds is identical to showing that

$$\left(\frac{d^{1-\rho}-\hat{h}}{\delta}\right)^{\frac{1}{1-\rho}}-d-\hat{j}\geqslant 0.$$

Again, we utilize an auxiliary scalar function $\hat{\varphi}$ defined by

$$\hat{\varphi}(s):=\left(\frac{s^{1-\rho}-\hat{h}}{\delta}\right)^{\frac{1}{1-\rho}}-s-\hat{j}\qquad(s>0).$$

Consider its first derivative $\varphi'(s)$ as follows

$$\hat{\varphi}'(s)=\frac{1}{\delta}\left(\frac{s^{1-\rho}-\hat{h}}{\delta}\right)^{\frac{\rho}{1-\rho}}s^{-\rho}-1,$$

from which we obtain the following equivalence relations

$$\hat{\varphi}'(s)>0\iff\left(\frac{s^{1-\rho}-\hat{h}}{\delta}\right)^{\frac{\rho}{1-\rho}}>\delta s^{\rho}\iff\frac{s^{1-\rho}-\hat{h}}{\delta}>\delta^{\frac{1-\rho}{\rho}}s^{1-\rho}$$
$$\iff(1-\delta^{\frac{1}{\rho}})s^{1-\rho}>\hat{h}\iff s>\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}.$$

Analogous to the previous result of $\check{\varphi}$, we now have

$$\hat{\varphi}\left[\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}\right]=\left[\delta^{\frac{1}{\rho}}-1\right]\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}-\hat{j}<0.$$

In addition, one can verify that both the first and the second derivatives of $\hat{\varphi}$ on the interval $([\hat{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)},\infty)\subset\mathbb{R}_{++}$ are strictly positive. Hence the graph of the scalar function $\hat{\varphi}$ is concave upward on $([\hat{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)},\infty)$. This means that $\hat{\varphi}(s)$ goes to ∞, as $s\to\infty$. Now this combined with the fact that $\hat{\varphi}$ attains a negative value at $s_*=[\hat{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}$ assures us that $\hat{\varphi}$ must cross the horizontal x-axis, by the intermediate value theorem, and thus has a root s_0 which is apparently greater than $s_*(\geqslant w_1>0)$.[10] In other words, since $\hat{\varphi}$ is strictly increasing for all $s>s_*$, we conclude that for any $s\geqslant s_0$, $\hat{\varphi}(s)\geqslant 0$. This, in turn, implies that there exists a positive constant d satisfying (A.16). For such d, we can take w_2 to be the constant function $w_2\equiv d$ on $\mathbb{X}$, in which case it follows directly that $\boldsymbol{T}w_2\leqslant w_2$, as was to be shown.

Case II : $\rho>1$. Regarding the existence of a strong lower solution to (3.19), let

$$w_1:=\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}$$

be a constant function defined on $\mathbb{X}$, where $\delta:=\beta r(\boldsymbol{K})^{1/\theta}$.

[10] A point $s_0\in\mathbb{R}$ is called a *root* of a scalar function φ if $\varphi(s_0)=0$.

We claim that the preceding constant function w_1 satisfies that, for any given $x \in \mathbb{X}$, we have

$$\begin{aligned} \boldsymbol{T}w_1(x) = \tilde{\boldsymbol{T}}\left(\boldsymbol{N}w_1\right)(x) = \tilde{\boldsymbol{T}}(w_1 + j)(x) &= \left\{h(x) + \delta\left[w_1(x) + j(x)\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &\geqslant \left\{\hat{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &> w_1(x), \end{aligned} \tag{A.17}$$

where the second equality follows from the elementary property of the operator $\mathcal{M}_{1-\gamma}$ such that $\mathcal{M}_{1-\gamma}(d) = d$ for any nonnegative constant function d.

Evidently, (A.17) implies that such w_1 is a strong lower solution of (3.19). Hence, to show the existence of a strong lower solution, it suffices to show that (A.17) holds for the function w_1 defined above.

To see that this is so, observe that, since $1-\rho$ is negative now, we have the following equivalence relations

$$\begin{aligned} \left\{\hat{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} > w_1 &\iff \hat{h} + \delta\left[w_1 + \check{j}\right]^{1-\rho} < w_1^{1-\rho} \\ &\iff \left(\frac{w_1^{1-\rho} - \hat{h}}{\delta}\right)^{\frac{1}{1-\rho}} - w_1 - \check{j} < 0. \end{aligned}$$

Set

$$\check{\varphi}(w_1) := \left(\frac{w_1^{1-\rho} - \hat{h}}{\delta}\right)^{\frac{1}{1-\rho}} - w_1 - \check{j}.$$

Then, invoking the definition of w_1, we have

$$\begin{aligned} \check{\varphi}(w_1) = \check{\varphi}\left[\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}\right] &= \left(\frac{\frac{\delta^{1/\rho}}{1-\delta^{1/\rho}}\hat{h}}{\delta}\right)^{\frac{1}{1-\rho}} - \left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \check{j} \\ &= \left[\delta^{\frac{1}{\rho}} - 1\right]\left(\frac{\hat{h}}{1-\delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \check{j}. \end{aligned}$$

Since $\delta \in (0,1)$ and $\rho > 1$, we have $\delta^{1/\rho} - 1 < 0$. And since $\hat{h} \geqslant \check{h} > 0$, the term $[\hat{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}$ is strictly positive. As a result, it is clear that $\check{\varphi}(w_1) < 0$, which is what we needed to show. Therefore, according to the uniformly strict inequality (A.17), we can see that $\boldsymbol{T}w_1 \gg w_1$, as desired.

It remains to show the existence of an upper solution w_2 to (3.19). Given a positive constant function d defined on $\mathbb{X}$, observe first that

$$\begin{aligned} \boldsymbol{T}d(x) = \tilde{\boldsymbol{T}}\left(\boldsymbol{N}d\right)(x) = \tilde{\boldsymbol{T}}(d + j)(x) &= \left\{h(x) + \delta\left[d(x) + j(x)\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \\ &\leqslant \left\{\check{h} + \delta\left[d + \hat{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}}, \end{aligned}$$

for all $x \in \mathbb{X}$. Evidently, to show the existence of an upper solution w_2, it suffices to show that there is a positive constant d satisfying

$$\left\{\check{h} + \delta\left[d + \hat{j}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}} \leqslant d. \tag{A.18}$$

Since $1 - \rho < 0$, after some rearrangement, showing that (A.18) holds is identical to showing that

$$\left(\frac{d^{1-\rho} - \check{h}}{\delta}\right)^{\frac{1}{1-\rho}} - d - \hat{j} \geqslant 0.$$

Again, we utilize an auxiliary scalar function $\hat{\varphi}$ defined by

$$\hat{\varphi}(s) := \left(\frac{s^{1-\rho} - \check{h}}{\delta}\right)^{\frac{1}{1-\rho}} - s - \hat{j} \qquad (s > 0).$$

Consider its first derivative $\varphi'(s)$ as follows

$$\hat{\varphi}'(s) = \frac{1}{\delta}\left(\frac{s^{1-\rho} - \check{h}}{\delta}\right)^{\frac{\rho}{1-\rho}} s^{-\rho} - 1,$$

from which we obtain the following equivalence relations

$$\begin{aligned}
\hat{\varphi}'(s) > 0 \iff & \left(\frac{s^{1-\rho} - \check{h}}{\delta}\right)^{\frac{\rho}{1-\rho}} > \delta s^{\rho} \quad \text{and} \quad s^{1-\rho} - \check{h} > 0 \\
\iff & \frac{s^{1-\rho} - \check{h}}{\delta} < \delta^{\frac{1-\rho}{\rho}} s^{1-\rho} \quad \text{and} \quad s < \check{h}^{\frac{1}{1-\rho}} \\
\iff & (1 - \delta^{\frac{1}{\rho}}) s^{1-\rho} < \check{h} \quad \text{and} \quad s < \check{h}^{\frac{1}{1-\rho}} \\
\iff & \left(\frac{\check{h}}{1 - \delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} < s < \check{h}^{\frac{1}{1-\rho}}.
\end{aligned}$$

Analogous to the previous result of $\check{\varphi}$, we now have

$$\hat{\varphi}\left[\left(\frac{\check{h}}{1 - \delta^{1/\rho}}\right)^{\frac{1}{1-\rho}}\right] = \left[\delta^{\frac{1}{\rho}} - 1\right]\left(\frac{\check{h}}{1 - \delta^{1/\rho}}\right)^{\frac{1}{1-\rho}} - \hat{j} < 0.$$

In addition, one can easily verify that both the first and the second derivatives of $\hat{\varphi}$ on the interval $([\check{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}, \check{h}^{1/(1-\rho)}) \subset \mathbb{R}_{++}$ are strictly positive. Hence $\hat{\varphi}$ is concave upward on $([\check{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}, \check{h}^{1/(1-\rho)})$. This means that $\hat{\varphi}(s)$ goes to ∞, as $s \to \check{h}^{1/(1-\rho)}$. Now this combined with the fact that $\hat{\varphi}$ attains a negative value at $s_* = [\check{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}$ assures us that $\hat{\varphi}$ must cross the horizontal x-axis, by the intermediate value theorem, and thus has a root s_0 which is apparently greater than $s_*(\geqslant w_1 > 0)$. In other words, since $\hat{\varphi}$ is strictly increasing for all $s \in (s_*, \check{h}^{1/(1-\rho)})$, we conclude that $\hat{\varphi}(s) \geqslant 0$, for any $s \in [s_0, \check{h}^{1/(1-\rho)})$. This, in turn, implies that there exists a positive constant d satisfying (A.18). For such d, it then follows from taking $w_2 \equiv d$ that $\boldsymbol{T}w_2 \leqslant w_2$, which is what we needed to show. □

Proof of Proposition 3.4.2. From Lemma 3.4.3, the operator $\boldsymbol{T}$ is monotone increasing and order concave on $c(\mathbb{X})_{++}$. In addition to Lemma 3.4.3, by virtue of Lemma 3.4.4, $\boldsymbol{T}$ maps an order interval $[w_1, w_2] \subset c(\mathbb{X})_{++}$ into itself, along with w_1 being a strong lower solution and w_2 being an upper solution of $\boldsymbol{T}$. The desired results then stem from Du's fixed point theorem (see, e.g., Theorem 3.1 in Du (1990)). □

Proof of Proposition 3.4.3. In order to obtain the stated results, we aim to apply Theorem 3.1 in Du (1990).

We first consider **the case of** $0 < \rho < 1$. As done in the corresponding proof of Lemma 3.4.4, we take a lower solution w_1 to (3.19) to be

$$w_1 = \left(\frac{\check{h}}{1 - \delta^{1/\rho}} \right)^{\frac{1}{1-\rho}}.$$

For the sake of simplicity, given such a fixed function w_1, let $\mathscr{C}_{w_1}(\mathbb{X}) := \{w \in c(\mathbb{X})_+ : w \geqslant w_1\}$. Clearly, $\mathscr{C}_{w_1}(\mathbb{X})$ is a convex subset of the positive cone $c(\mathbb{X})_+$, and more precisely, it is also a convex subset of $c(\mathbb{X})_{++}$.

By Assumption 3.4.6 and hence invoking Remark 3.4.3, we have $w_1 > -\check{j}/(1-\delta^{1/\rho}) > -\check{j}$, which in turn implies that for all functions $w \in \mathscr{C}_{w_1}(\mathbb{X})$, $w(x) + j(x) > 0$ for each $x \in \mathbb{X}$. In addition, this implies that the operator $\boldsymbol{N}$ defined in (3.22) is a well-defined map from $\mathscr{C}_{w_1}(\mathbb{X})$ to $c(\mathbb{X})_{++}$, and hence $\tilde{\boldsymbol{T}}$ defined in (3.23) is well defined on $c(\mathbb{X})_{++}$.

In this connection, Lemmas 3.4.1 and 3.4.2 apply and yield Lemma 3.4.3, which implies that the corresponding operator $\boldsymbol{T}$ defined in (3.21), mapping from $\mathscr{C}_{w_1}(\mathbb{X})$ to $c(\mathbb{X})_{++}$, is well defined, and is monotone increasing and order concave on $\mathscr{C}_{w_1}(\mathbb{X})$.

To apply Du's theorem, it now remains to show that

(a) the function w_1 defined above is a strong lower solution, and

(b) there exists an upper solution w_2.

Regarding part (a), the proof is almost identical to that of Lemma 3.4.4. In particular, for such w_1 defined above, we have

$$\check{\varphi}(w_1) = \left[\delta^{\frac{1}{\rho}} - 1\right] \left(\frac{\check{h}}{1 - \delta^{1/\rho}} \right)^{\frac{1}{1-\rho}} - \check{j} = \left[\delta^{\frac{1}{\rho}} - 1\right] \left[\left(\frac{\check{h}}{1 - \delta^{1/\rho}} \right)^{\frac{1}{1-\rho}} - \frac{\check{j}}{\delta^{\frac{1}{\rho}} - 1} \right].$$

By Assumption 3.4.6, and invoking the chaining inequalities (3.24) in Remark 3.4.3, we obtain $\check{\varphi}(w_1) < 0$. Following the analogous arguments in Lemma 3.4.4, part (a) is proved.

Regarding part (b), the proof is also essentially identical to that of Lemma 3.4.4. Similarly, making use of the chaining inequalities (3.24), $\hat{\varphi}$ attains a negative value at $s_* = [\hat{h}/(1-\delta^{1/\rho})]^{1/(1-\rho)}$. Then, following the same arguments as in Lemma 3.4.4 proves part (b).

Therefore, all conditions of theorem 3.1 of Du (1990) are verified, and the conclusions of that theorem now yield the stated results in Proposition 3.4.3.

Next we consider **the case of** $\rho > 1$. As done in the corresponding proof of Lemma 3.4.4, we take a lower solution w_1 to (3.19) to be

$$w_1 = \left(\frac{\hat{h}}{1 - \delta^{1/\rho}} \right)^{\frac{1}{1-\rho}}.$$

By Assumption 3.4.6 and hence invoking Remark 3.4.3, we note that $w_1 > -\check{j}/(1-\delta^{1/\rho}) > -\check{j}$, which in turn implies that for any function w in the corresponding set $\mathscr{C}_{w_1}(\mathbb{X})$, it follows that $w(x) + j(x) > 0$ for each $x \in \mathbb{X}$. Using the same arguments mentioned above, we have that the operator $\boldsymbol{T}$ mapping from $\mathscr{C}_{w_1}(\mathbb{X})$ to $c(\mathbb{X})_{++}$ is well defined, monotone increasing and order concave.

Recalling the preceding arguments for the case of $0 < \rho < 1$ and making use of the chaining inequalities (3.25), the remaining proofs regarding the existence of a strong lower solution and an upper solution are essentially identical to that of Lemma 3.4.4, and hence omitted here. □

A.3 Appendix to Chapter 4

Let $m\mathsf{X}$ represent all Borel-measurable functions in $\mathbb{R}^{\mathsf{X}}$ and let $c\mathsf{X}$ denote all continuous functions in $m\mathsf{X}$. Let $bm\mathsf{X}$ be the bounded functions in $m\mathsf{X}$ and let $bc\mathsf{X}$ be the continuous functions in $bm\mathsf{X}$. Let $m\mathsf{X}_+$ and $m\mathsf{X}_{++}$ be the nonnegative and positive functions in $m\mathsf{X}$, respectively. Recall that a self-map A on a convex subset M of $bm\mathsf{X}$ is called

- *asymptotically stable* on M if A has a unique fixed point v^* in M and $A^n v \to v^*$ as $n \to \infty$ whenever $v \in M$,
- *isotone* if $Av \leqslant Av'$ whenever $v, v' \in M$ with $v \leqslant v'$,
- *convex* if $A(\lambda v + (1-\lambda)v') \leqslant \lambda Av + (1-\lambda)Av'$ whenever $v, v' \in M$ and $0 \leqslant \lambda \leqslant 1$, and
- *concave* if $A(\lambda v + (1-\lambda)v') \geqslant \lambda Av + (1-\lambda)Av'$ whenever $v, v' \in M$ and $0 \leqslant \lambda \leqslant 1$.

For $f, g \in bm\mathsf{X}$, the statement $f \ll g$ means that there exists an $\varepsilon > 0$ such that $f(x) \leqslant g(x) - \varepsilon$ for all $x \in \mathsf{X}$.

For each $\sigma \in \Sigma$, we define the *σ-value operator* T_σ on $\mathscr{V}$ by

$$T_\sigma v(x) := Q(x, \sigma(x), v) \quad \text{for all } x \in \mathsf{X},\ v \in \mathscr{V}. \tag{A.19}$$

Stating that $v_\sigma \in \mathscr{V}$ solves (4.3) is equivalent to stating that v_σ is a fixed point of T_σ. By Lemma 4.2.1, the operator T_σ is a well-defined self-map on $\mathscr{V}$.

Lemma A.3.1. *If Assumption 4.2.2 holds, then, for each $\sigma \in \Sigma$, the operator T_σ is asymptotically stable on $\mathscr{V}$.*

Proof of Lemma A.3.1. Fix $\sigma \in \Sigma$. We aim to apply theorem 3.1 of Du (1990). To this end, it is sufficient to show that

(i) T_σ is isotone and convex on $\mathscr{V}$.

(ii) $T_\sigma w_1 \geqslant w_1$ and $T_\sigma w_2 \ll w_2$.

Regarding condition (i), pick any $v, v' \in \mathscr{V}$ with $v \leqslant v'$. For fixed $x \in \mathsf{X}$, we have

$$T_\sigma v(x) = Q(x, \sigma(x), v) \leqslant Q(x, \sigma(x), v') = T_\sigma v'(x),$$

by (4.1). Hence, isotonicity of T_σ holds.

To see that T_σ is convex, fix $v, v' \in \mathscr{V}$ and $\lambda \in [0, 1]$. For any given $x \in \mathsf{X}$, we have

$$\begin{aligned} T_\sigma(\lambda v + (1-\lambda)v')(x) &= Q(x, \sigma(x), \lambda v + (1-\lambda)v') \\ &\leqslant \lambda Q(x, \sigma(x), v) + (1-\lambda) Q(x, \sigma(x), v') \\ &= \lambda T_\sigma v(x) + (1-\lambda) T_\sigma v'(x), \end{aligned}$$

where the inequality directly follows from part (a) of Assumption 4.2.2. Since $x \in \mathsf{X}$ was arbitrary, the convexity of T_σ follows.

The first part of condition (ii) follows directly from (4.2), since, for each $x \in \mathsf{X}$,

$$T_\sigma w_1(x) = Q(x, \sigma(x), w_1) \geqslant w_1(x).$$

To see that the second part of condition (ii) is satisfied, it follows from part (b) of Assumption 4.2.2 that

$$T_\sigma w_2(x) = Q(x, \sigma(x), w_2) \leqslant w_2(x) - \varepsilon$$

for each $x \in \mathsf{X}$ and for some $\varepsilon > 0$. Hence $w_2 \gg T_\sigma w_2$, as was to be shown. □

Proof of Proposition 4.2.1. This follows directly from Lemma A.3.1. □

Given $v \in \mathscr{V}$, a policy σ in Σ will be called *v-maximal-greedy* if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} Q(x, a, v) \quad \text{for all } x \in \mathsf{X}. \tag{A.20}$$

Lemma A.3.2. *If $v \in \mathscr{C}$, then there exists at least one v-maximal-greedy policy.*

Proof. Fixing $v \in \mathscr{C}$ and using the compactness and continuity conditions in Assumption 4.2.1, we can choose for each $x \in \mathsf{X}$ an action $\sigma(x) \in \Gamma(x)$ such that (A.20) holds. The map σ constructed in this manner can be chosen to be Borel-measurable by Theorem 18.19 of Aliprantis and Border (2006). □

Lemma A.3.3. *If Assumption 4.2.2 holds, then T is asymptotically stable on $\mathscr{C}$.*

Proof of Lemma A.3.3. In order to apply theorem 3.1 of Du (1990), it suffices to show that

(i) T is isotone and convex on $\mathscr{C}$.

(ii) $Tw_1 \geqslant w_1$ and $Tw_2 \ll w_2$.

The isotonicity of T on $\mathscr{C}$ is obvious, since, by the monotonicity restriction (4.1),

$$v \leqslant v' \implies \max_{a \in \Gamma(x)} Q(x,a,v) \leqslant \max_{a \in \Gamma(x)} Q(x,a,v) \quad \text{for all } x \in \mathsf{X}.$$

In other words, by definition of T, $v \leqslant v'$ implies $Tv \leqslant Tv'$.

To show the convexity of T, fix $v, v' \in \mathscr{C}$ and $\lambda \in [0,1]$. For any given $(x,a) \in \mathbb{G}$, we have, by part (a) of Assumption 4.2.2,

$$\begin{aligned} Q(x,a,\lambda v + (1-\lambda)v') &\leqslant \lambda Q(x,a,v) + (1-\lambda) Q(x,a,v') \\ &\leqslant \lambda \max_{a \in \Gamma(x)} Q(x,a,v) + (1-\lambda) \max_{a \in \Gamma(x)} Q(x,a,v') \\ &= \lambda Tv(x) + (1-\lambda) Tv'(x). \end{aligned}$$

Since $(x,a) \in \mathbb{G}$ was arbitrary, the above inequality implies

$$\max_{a \in \Gamma(x)} Q(x,a,\lambda v + (1-\lambda)v') \leqslant \lambda Tv(x) + (1-\lambda) Tv'(x)$$

for each $x \in \mathsf{X}$, which in turn means that $T[\lambda v + (1-\lambda)v'] \leqslant \lambda Tv + (1-\lambda)Tv'$.

The first part of condition (ii) follows directly from (4.2), since, for each $x \in \mathsf{X}$,

$$Tw_1(x) = \max_{a \in \Gamma(x)} Q(x,a,w_1) \geqslant Q(x,a,w_1) \geqslant w_1(x).$$

To see that the second part of condition (ii) is satisfied, it follows from part (b) of Assumption 4.2.2 that

$$Tw_2(x) = \max_{a \in \Gamma(x)} Q(x,a,w_2) \leqslant w_2(x) - \varepsilon$$

for each $x \in \mathsf{X}$ and for some $\varepsilon > 0$. Hence, $Tw_2 \ll w_2$, as was to be shown. □

Theorem A.3.1. *If T_σ is asymptotically stable on $\mathscr{V}$ for all $\sigma \in \Sigma$ and T is asymptotically stable on $\mathscr{C}$, then the conclusions of Theorem 4.2.1 hold.*

Proof. Let v^* be the maximum value function and let $\bar{v}$ be the unique fixed point of T in $\mathscr{C}$. To see that $\bar{v} = v^*$, first observe that $\bar{v} \in \mathscr{C}$ and hence $\bar{v}$ has at least one maximal-greedy policy σ. For this policy we have, by definition, $T_\sigma \bar{v}(x) = T\bar{v}(x)$ at each x, from which it follows that $\bar{v} = T\bar{v} = T_\sigma \bar{v}$. Since T_σ is asymptotically stable on $\mathscr{V}$, we know that its unique fixed point is v_σ, so $\bar{v} = v_\sigma$. But then $\bar{v} \leqslant v^*$, by the definition of v^*.

To see that the reverse inequality holds, pick any $\sigma \in \Sigma$. We have $T_\sigma \bar{v} \leqslant T\bar{v} = \bar{v}$. Iterating on this inequality and using the isotonicity of T_σ gives $T_\sigma^k \bar{v} \leqslant \bar{v}$ for all k. Taking the limit with respect to k and using the asymptotic stability of T_σ then gives $v_\sigma \leqslant \bar{v}$. Hence $v^* \leqslant \bar{v}$, and we can now conclude that $\bar{v} = v^*$.

Since $\bar{v} \in \mathscr{C}$, we have $v^* \in \mathscr{C}$. It follows that v^* is the unique solution to the Bellman maximization equation in $\mathscr{C}$, and that $T^n v \to v^*$ whenever $v \in \mathscr{C}$. Parts (a) and (b) of Theorem 4.2.1 are now established.

Regarding parts (c) and (d), by the definition of maximal-greedy policies, we know that σ is v^*-maximal-greedy iff $Q(x, \sigma(x), v^*) = \max_{a \in \Gamma(x)} Q(x, a, v^*)$ for all $x \in \mathsf{X}$. Since v^* satisfies the Bellman maximization equation, we then have

$$\sigma \text{ is } v^*\text{-maximal-greedy} \quad \iff \quad Q(x, \sigma(x), v^*) = v^*(x), \;\; \forall\, x \in \mathsf{X}.$$

But, by Proposition 4.2.1, the right-hand side is equivalent to the statement that $v^* = v_\sigma$. Hence, by this chain of logic and the definition of optimality,

$$\sigma \text{ is } v^*\text{-maximal-greedy} \iff v^* = v_\sigma \iff \sigma \text{ is optimal} \tag{A.21}$$

Moreover, the fact that v^* is in $\mathscr{C}$ combined with Lemma A.3.2 assures us that at least one v^*-maximal-greedy policy exists. Each such policy is optimal, so the set of optimal policies is non-empty. □

Proof of Proposition 4.2.2. Recalling the definitions of operators T_σ and T, we have

$$T_{\sigma^{k+1}} v_{\sigma^k} = T v_{\sigma^k} \geqslant T_{\sigma^k} v_{\sigma^k} = v_{\sigma^k}.$$

Making use of isotonicity of T_σ and applying $T_{\sigma^{k+1}}$ to the above chaining inequality, we obtain

$$T^2_{\sigma^{k+1}} v_{\sigma^k} = T_{\sigma^{k+1}} T v_{\sigma^k} \geqslant T_{\sigma^{k+1}} T_{\sigma^k} v_{\sigma^k} = T_{\sigma^{k+1}} v_{\sigma^k} = T v_{\sigma^k} \geqslant T_{\sigma^k} v_{\sigma^k} = v_{\sigma^k}.$$

Similarly, it follows that for all integers $m > 0$,

$$T^m_{\sigma^{k+1}} v_{\sigma^k} \geqslant T v_{\sigma^k} \geqslant v_{\sigma^k}.$$

Taking the limit as $m \to \infty$ and invoking Lemma A.3.1 yields

$$v_{\sigma^{k+1}} \geqslant T v_{\sigma^k} \geqslant v_{\sigma^k}, \quad \text{for all } k = 0, 1, \ldots. \tag{A.22}$$

If $v_{\sigma^{k+1}} = v_{\sigma^k}$, it follows that $v_{\sigma^{k+1}} = Tv_{\sigma^k} = v_{\sigma^k}$, and hence v_{σ^k} is a fixed point of the Bellman operator T. But we know from Theorem 4.2.1 that v^* is the unique fixed point of T, which in turn implies that v_{σ^k} must be equal to v^*.

In addition, by using induction on (A.22), we have

$$v_{\sigma^{k+1}} \geqslant Tv_{\sigma^k} \geqslant T^2 v_{\sigma^{k-1}} \geqslant T^3 v_{\sigma^{k-2}} \geqslant \dots,$$

in which case it follows that

$$v_{\sigma^{k+1}} \geqslant T^{k+1} v_{\sigma^0}, \quad \text{for all } k = 0, 1, \dots.$$

Since

$$v^* \geqslant v_{\sigma^k} \geqslant T^k v_{\sigma^0} \quad (k = 0, 1, , \dots) \quad \text{and} \quad \lim_{k \to \infty} \|T^k v_{\sigma^0} - v^*\| = 0,$$

by making use of the squeeze theorem, it follows that $\lim_{k\to\infty} \|v_{\sigma^k} - v^*\| = 0$.

Finally, if the number of policies is finite, (A.22) implies that there can be only a finite number of iterations for which $v_{\sigma^{k+1}}(x) < v_{\sigma^k}$ for some $x \in \mathsf{X}$, in which case we must have $v_{\sigma^{k+1}} = v_{\sigma^k}$ for some integer k, at which time $v_{\sigma^k} = v^*$, as was to be shown. □

Lemma A.3.4. *If Assumption 4.2.3 holds, then, for each $\sigma \in \Sigma$, the operator T_σ is asymptotically stable on $\mathscr{V}$.*

Proof of Lemma A.3.4. Fix $\sigma \in \Sigma$. We aim to apply theorem 3.1 of Du (1990). To this end, it is sufficient to show that

(i) T_σ is isotone and concave on $\mathscr{V}$, and

(ii) $T_\sigma w_1 \gg w_1$ and $T_\sigma w_2 \leqslant w_2$.

Clearly, T_σ is isotone, since, by the monotonicity restriction (4.1),

$$v \leqslant v' \implies Q(x, \sigma(x), v) \leqslant Q(x, \sigma(x), v') \quad \text{for all } x \in \mathsf{X}.$$

In other words, $v \leqslant v'$ implies $T_\sigma v \leqslant T_\sigma v'$.

Regarding the concavity of T_σ, fix $v, v' \in \mathscr{V}$ and $\lambda \in [0, 1]$. For any given $x \in \mathsf{X}$, by virtue of part (a) of Assumption 4.2.3, we obtain

$$\begin{aligned} T_\sigma(\lambda v + (1-\lambda)v')(x) &= Q(x, \sigma(x), \lambda v + (1-\lambda)v') \\ &\geqslant \lambda Q(x, \sigma(x), v) + (1-\lambda) Q(x, \sigma(x), v') \\ &= \lambda T_\sigma v(x) + (1-\lambda) T_\sigma v'(x). \end{aligned}$$

Since $x \in \mathsf{X}$ was arbitrary, the concavity of T_σ follows.

To see that the first part of condition (ii) is satisfied, it follows from part (b) of Assumption 4.2.3 that

$$T_\sigma w_1(x) = Q(x, \sigma(x), w_1) \geqslant w_1(x) + \varepsilon$$

for each $x \in \mathsf{X}$ and for some $\varepsilon > 0$. Hence, $T_\sigma w_1 \gg w_1$, as was to be shown.

The second part of condition (ii) follows directly from (4.2), since, for each $x \in \mathsf{X}$,

$$T_\sigma w_2(x) = Q(x, \sigma(x), w_2) \leqslant w_2(x).$$

This completes the proof. □

Proof of Proposition 4.2.3. This follows directly from Lemma A.3.4. □

Given $v \in \mathscr{V}$, a policy σ in Σ will be called *v-minimal-greedy* if

$$\sigma(x) \in \operatorname*{argmin}_{a \in \Gamma(x)} Q(x, a, v) \quad \text{for all } x \in \mathsf{X}. \tag{A.23}$$

Lemma A.3.5. *If $v \in \mathscr{C}$, then there exists at least one v-minimal-greedy policy.*

Proof. The proof of Lemma A.3.5 is essentially identical to that of Lemma A.3.2, and hence is omitted here. □

Lemma A.3.6. *If Assumption 4.2.3 holds, then S is asymptotically stable on $\mathscr{C}$.*

Proof of Lemma A.3.6. It follows from Berge's theorem of the minimum that, when v is in $\mathscr{C}$, we have

$$Sv(x) = \min_{a \in \Gamma(x)} Q(x, a, v)$$

and Sv is an element of $\mathscr{C}$.

In order to apply theorem 3.1 of Du (1990), it suffices to show that

(i) S is isotone and concave on $\mathscr{C}$, and

(ii) $Sw_1 \gg w_1$ and $Sw_2 \leqslant w_2$.

The isotonicity of S on $\mathscr{C}$ is obvious, since, by the monotonicity restriction (4.1),

$$v \leqslant v' \implies \min_{a \in \Gamma(x)} Q(x, a, v) \leqslant \min_{a \in \Gamma(x)} Q(x, a, v) \quad \text{for all } x \in \mathsf{X}.$$

In other words, by definition of S, $v \leqslant v'$ implies $Sv \leqslant Sv'$.

To show the concavity of S, fix $v, v' \in \mathscr{C}$ and $\lambda \in [0,1]$. For any given $(x,a) \in \mathbb{G}$, by part (a) of Assumption 4.2.3, we have

$$\begin{aligned} Q(x,a,\lambda v + (1-\lambda)v') &\geqslant \lambda Q(x,a,v) + (1-\lambda)Q(x,a,v') \\ &\geqslant \lambda \min_{a \in \Gamma(x)} Q(x,a,v) + (1-\lambda) \min_{a \in \Gamma(x)} Q(x,a,v') \\ &= \lambda S v(x) + (1-\lambda) S v'(x). \end{aligned}$$

Since $(x,a) \in \mathbb{G}$ was arbitrary, the above inequality implies

$$\min_{a \in \Gamma(x)} Q(x,a,\lambda v + (1-\lambda)v') \geqslant \lambda S v(x) + (1-\lambda) S v'(x)$$

for each $x \in \mathsf{X}$; namely, $S[\lambda v + (1-\lambda)v'] \geqslant \lambda S v + (1-\lambda) S v'$, as desired.

To see that the first part of condition (ii) is satisfied, it follows from part (b) of Assumption 4.2.3 that

$$S w_1(x) = \min_{a \in \Gamma(x)} Q(x,a,w_1) \geqslant w_1(x) + \varepsilon$$

for each $x \in \mathsf{X}$ and some $\varepsilon > 0$. Hence, $S w_1 \gg w_1$.

The second part of condition (ii) directly follows from (4.2), since, for each $x \in \mathsf{X}$,

$$S w_2(x) = \min_{a \in \Gamma(x)} Q(x,a,w_2) \leqslant Q(x,a,w_2) \leqslant w_2(x).$$

This finishes the proof. □

Theorem A.3.2. *If T_σ is asymptotically stable on $\mathscr{V}$ for all $\sigma \in \Sigma$ and S is asymptotically stable on $\mathscr{C}$, then the conclusions of Theorem 4.2.2 hold.*

Proof. Let v^* be the minimum cost function and let $\bar{v}$ be the unique fixed point of S in $\mathscr{C}$. To see that $\bar{v} = v^*$, first observe that $\bar{v} \in \mathscr{C}$ and hence $\bar{v}$ has at least one minimal-greedy policy σ. For this policy we have, by definition, $T_\sigma \bar{v}(x) = S\bar{v}(x)$ at each x, from which it follows that $\bar{v} = S\bar{v} = T_\sigma \bar{v}$. Since T_σ is asymptotically stable on $\mathscr{V}$, we know that its unique fixed point is v_σ, so $\bar{v} = v_\sigma$. But then $\bar{v} \geqslant v^*$, by the definition of v^* in (4.7).

To see that the reverse inequality holds, pick any $\sigma \in \Sigma$. We have $T_\sigma \bar{v} \geqslant S\bar{v} = \bar{v}$. Iterating on this inequality and using the isotonicity of T_σ gives $T_\sigma^k \bar{v} \geqslant \bar{v}$ for all k. Taking the limit with respect to k and using the asymptotic stability of T_σ then gives $v_\sigma \geqslant \bar{v}$. Hence $v^* \geqslant \bar{v}$, and we can now conclude that $\bar{v} = v^*$.

Since $\bar{v} \in \mathscr{C}$, we have $v^* \in \mathscr{C}$. Moreover, for v in $\mathscr{C}$ we can replace the inf in the definition of S with a min, and solutions to the Bellman equation in $\mathscr{C}$ exactly coincide with fixed points of S in that set. It follows that v^* is the unique solution to the Bellman equation in $\mathscr{C}$, and that $S^n v \to v^*$ whenever $v \in \mathscr{C}$. Parts (a) and (b) of Theorem 4.2.2 are now established.

Regarding parts (c) and (d), by the definition of minimal-greedy policies, we know that σ is v^*-minimal-greedy iff $Q(x, \sigma(x), v^*) = \min_{a \in \Gamma(x)} Q(x, a, v^*)$ for all $x \in \mathsf{X}$. Since v^* satisfies the Bellman equation, we then have

$$\sigma \text{ is } v^*\text{-minimal-greedy} \quad \iff \quad Q(x, \sigma(x), v^*) = v^*(x), \ \ \forall\, x \in \mathsf{X}.$$

But, by Proposition 4.2.3, the right-hand side is equivalent to the statement that $v^* = v_\sigma$. Hence, by this chain of logic and the definition of optimality,

$$\sigma \text{ is } v^*\text{-minimal-greedy} \iff v^* = v_\sigma \iff \sigma \text{ is optimal} \tag{A.24}$$

Moreover, the fact that v^* is in $\mathscr{C}$ combined with Lemma A.3.5 assures us that at least one v^*-minimal-greedy policy exists. Each such policy is optimal, so the set of optimal policies is non-empty. □

Proof of Proposition 4.2.4. The proof is essentially identical to that of Proposition 4.2.2 by reversing the direction of inequality. □

In the following, we prove some properties of the state-action aggregator Q defined in Section 4.3.4.

For the sake of exposition, fix $\theta \in \Theta$, we first define an operator R_θ on $bm(\mathsf{S} \times \mathsf{Z})_+$ by

$$(R_\theta w)(y, z) := \left[\int w(y, z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \right]^{1/\xi_1} \quad \text{for all } (y, z) \in \mathsf{S} \times \mathsf{Z}. \tag{A.25}$$

From this foundation, we then define an operator R that is a map sending w in $bm(\mathsf{S} \times \mathsf{Z})_+$ into

$$Rw(y, z, \theta) := (R_\theta w)(y, z) \quad \text{for all } (y, z, \theta) \in \mathsf{S} \times \mathsf{Z} \times \Theta. \tag{A.26}$$

The following lemma shows some useful properties of the operator R_θ.

Lemma A.3.7. *For fixed $\theta \in \Theta$, if ξ_1 lies in $(0,1)$, then the operator R_θ defined in (A.25) is isotone and concave on $bm(\mathsf{S} \times \mathsf{Z})_+$.*

Moreover, the function $R_\theta w$ is nonnegative, bounded, and Borel-measurable on $\mathsf{S} \times \mathsf{Z}$ whenever $w \in bm(\mathsf{S} \times \mathsf{Z})_+$ and continuous on $\mathsf{S} \times \mathsf{Z}$ whenever $w \in bc(\mathsf{S} \times \mathsf{Z})_+$.

Proof of Lemma A.3.7. Fix $\theta \in \Theta$. The isotonicity of R_θ is obvious, since the scalar function $\mathbb{R}_+ \ni t \mapsto t^{\xi_1} \in \mathbb{R}_+$ and its inverse are both strictly increasing on $\mathbb{R}_+$.

Since $\xi_1 \in (0,1)$, by virtue of Theorem 198 of Hardy et al. (1934), we know that R_θ is superadditive in the sense that for any $w, w' \in m(\mathsf{S} \times \mathsf{Z})_+$, $R_\theta(w + w') \geqslant R_\theta(w) + R_\theta(w')$.[11] As a

[11] This result can also be reviewed as the reverse Minkowski inequality, see, for example, Proposition 3.2 in page 225 of DiBenedetto (2002).

result, the super-additivity and the positive homogeneity of R_θ together yield the concavity of R_θ.[12] Indeed, pick any $\lambda \in [0,1]$ and $w, w' \in m(\mathsf{S} \times \mathsf{Z})_+$, by the convexity of $m(\mathsf{S} \times \mathsf{Z})_+$, we have

$$\begin{aligned} R_\theta[\lambda w + (1-\lambda)w'] &\geqslant R_\theta(\lambda w) + R_\theta((1-\lambda)w') \quad \text{(by super-additivity)} \\ &= \lambda R_\theta(w) + (1-\lambda)R_\theta(w') \quad \text{(by positive homogeneity)}, \end{aligned}$$

as was to be shown.

Regarding the second claim of Lemma A.3.7, nonnegativity and boundedness of $R_\theta w$ is immediate and it is easy to see that $R_\theta w$ is Borel-measurable on $\mathsf{S} \times \mathsf{Z}$ whenever $w \in bm(\mathsf{S} \times \mathsf{Z})_+$. Now fix $w \in bc(\mathsf{S} \times \mathsf{Z})_+$. We note that the function w^{ξ_1} also lies in $bc(\mathsf{S} \times \mathsf{Z})_+$. Then, by virtue of the Feller property of π_θ, the mapping $\mathsf{S} \times \mathsf{Z} \ni (y,z) \mapsto \int w(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z') \in \mathbb{R}_+$ is bounded and continuous on $\mathsf{S} \times \mathsf{Z}$. Furthermore, it follows that the mapping $\mathsf{S} \times \mathsf{Z} \ni (y,z) \mapsto [\int w(y,z')^{\xi_1} \pi_\theta(z, \mathrm{d}z')]^{1/\xi_1} \in \mathbb{R}_+$ is continuous on $\mathsf{S} \times \mathsf{Z}$, since the inverse of the map $t \mapsto t^{\xi_1}$ is also continuous on $\mathbb{R}_+$. Therefore, our claim follows. □

As an application of Lemma A.3.7, we now present the next result.

Lemma A.3.8. *The operator R defined in (A.26) is a well-defined map from $bm(\mathsf{S} \times \mathsf{Z})_+$ into $bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_+$.*

Proof of Lemma A.3.8. Fix w in $bm(\mathsf{S} \times \mathsf{Z})_+$. Since boundedness and nonnegativity of the function Rw are obvious, it remains to show that Rw is measurable on $\mathsf{S} \times \mathsf{Z} \times \Theta$.

On the one hand, for each $\theta \in \Theta$, it follows from Lemma A.3.7 that the function $Rw(\cdot,\cdot,\theta) = R_\theta w \colon \mathsf{S} \times \mathsf{Z} \to \mathbb{R}_+$ is Borel-measurable. On the other hand, for each $(y,z) \in \mathsf{S} \times \mathsf{Z}$, the function $Rw(y,z,\cdot) \colon \Theta \to \mathbb{R}_+$ is continuous, since Θ is a finite set (endowed with the discrete topology).

In this connection, we conclude that the function $Rw \colon \mathsf{S} \times \mathsf{Z} \times \Theta \to \mathbb{R}$ is a Carathéodory function, in the sense that

(1) for each $\theta \in \Theta$, the function $Rw(\cdot,\cdot,\theta) \colon \mathsf{S} \times \mathsf{Z} \to \mathbb{R}$ is Borel-measurable; and

(2) for each $(y,z) \in \mathsf{S} \times \mathsf{Z}$, the function $Rw(y,z,\cdot) \colon \Theta \to \mathbb{R}$ is continuous.

By virtue of Lemma 4.51 in Aliprantis and Border (2006), it follows that the Carathéodory function Rw is jointly measurable on $\mathsf{S} \times \mathsf{Z} \times \Theta$, as desired. □

In this connection, the state-action aggregator Q defined in (4.29) can be simply expressed as a composition of two operators R and $\tilde{Q}$ as follows

$$Q((s,z),y,\hat{v}) := \tilde{Q}((s,z),y,R\hat{v}), \tag{A.27}$$

[12] An operator A defined on the positive cone $bm\mathsf{X}_+$ of $bm\mathsf{X}$ is called *positively homogeneous* (of the first degree) if for any v in $bm\mathsf{X}_+$ and any real number $t \geqslant 0$, we have $A(tv) = tAv$.

with

$$\tilde{Q}((s,z),y,h) := \left\{ r(s,y,z) + \beta \left\{ \int h(y,z,\theta)\mu(z,\mathrm{d}\theta) \right\}^{1/\xi_2} \right\}^{\xi_2} \tag{A.28}$$

for all $((s,z),y) \in \mathbb{G}$ and $h \in bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$.

It is worth noting that the formula of $\tilde{Q}$ defined in (A.28) is almost identical to that of Q defined in (4.13). Hence, recalling the results associated with Q in Section 4.3.2, we have

Lemma A.3.9. *If $\xi_2 < 0$, then $\tilde{Q}$ defined in (A.28) is isotone and concave in its third argument on $bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$.*

In addition, the map $((s,z),y) \mapsto \tilde{Q}((s,z),y,h)$ is Borel-measurable on $\mathbb{G}$ whenever $h \in bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$, and continuous on $\mathbb{G}$ whenever the map $h(\cdot,\cdot,\theta)\colon \mathsf{S} \times \mathsf{Z} \to \mathbb{R}_{++}$ is continuous, for each $\theta \in \Theta$.

Proof of Lemma A.3.9. Analogous to the proofs in Section 4.3.2, for any fixed $b > 0$, we consider the scalar map $\psi(t) := (b + \beta t^{1/\xi_2})^{\xi_2}$ where $t > 0$. Since $\xi_2 < 0$, it is clear that the scalar function ψ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$ (cf. Section 4.3.2).

The first part of the claim is immediate from the monotonicity and concavity of ψ, along with the monotonicity and linearity of the integral.

For the remaining part, fix h in $bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$. Borel measurability of the map $((s,z),y) \mapsto \tilde{Q}((s,z),y,h)$ is obvious. Now fix a function h satisfying that the map $h(\cdot,\cdot,\theta)\colon \mathsf{S} \times \mathsf{Z} \to \mathbb{R}_{++}$ is continuous, for every $\theta \in \Theta$. By virtue of the continuity imposed on the distribution $\mu(\cdot,\cdot)$ and the finiteness of Θ, the map $\mathsf{S} \times \mathsf{Z} \ni (y,z) \mapsto \int h(y,z,\theta)\mu(z,\mathrm{d}\theta) \in \mathbb{R}_{++}$ is continuous on $\mathsf{S} \times \mathsf{Z}$.[13] It then follows from the continuity imposed on r and the continuity of ψ that $((s,z),y) \mapsto \tilde{Q}((s,z),y,h)$ is continuous on $\mathbb{G}$. □

Lemma A.3.10. *If $\xi_1 \in (0,1)$ and $\xi_2 < 0$, then the state-action aggregator Q defined in (4.29) is isotone and concave in its third argument on $bm(\mathsf{S} \times \mathsf{Z})_{++}$.*

In addition, the map $((s,z),y) \mapsto Q((s,z),y,v)$ is Borel-measurable on $\mathbb{G}$ whenever $v \in bm(\mathsf{S} \times \mathsf{Z})_{++}$ and continuous on $\mathbb{G}$ whenever $v \in bc(\mathsf{S} \times \mathsf{Z})_{++}$.

Proof of Lemma A.3.10. Since the aggregator Q is a composition of $\tilde{Q}$ and R, by Lemmas A.3.7 to A.3.9, the isotonicity, Borel measurability and continuity of Q immediately follow from those of $\tilde{Q}$ and R.

It only remains to show the concavity of Q. To see this, fix $((s,z),y) \in \mathbb{G}$, $\lambda \in [0,1]$ and w, w' in $bm(\mathsf{S} \times \mathsf{Z})_{++}$. For any given $\theta \in \Theta$, by concavity of R_θ and convexity of $bm(\mathsf{S} \times \mathsf{Z})_{++}$, we have

$$R_\theta[\lambda w + (1-\lambda)w'](y,z) \geqslant \lambda R_\theta w(y,z) + (1-\lambda) R_\theta w'(y,z);$$

[13] Since Θ is finite, this map becomes the sum of a finite number of functions that are continuous in (y,z), and thus it is continuous in (y,z) as well.

that is, for each $(y, z, \theta) \in \mathsf{S} \times \mathsf{Z} \times \Theta$,

$$R[\lambda w + (1-\lambda)w'](y, z, \theta) \geqslant \lambda Rw(y, z, \theta) + (1-\lambda)Rw'(y, z, \theta).$$

In operator notation, this translates to $R[\lambda w + (1-\lambda)w'] \geqslant \lambda Rw + (1-\lambda)Rw'$.

Observe that due to isotonicity and concavity of $\tilde{Q}$, we now obtain

$$\begin{aligned} Q((s,z), y, \lambda w + (1-\lambda)w') &= \tilde{Q}((s,z), y, R[\lambda w + (1-\lambda)w']) \\ &\geqslant \tilde{Q}((s,z), y, \lambda Rw + (1-\lambda)Rw') \\ &\geqslant \lambda \tilde{Q}((s,z), y, Rw) + (1-\lambda)\tilde{Q}((s,z), y, Rw') \\ &= \lambda Q((s,z), y, w) + (1-\lambda)Q((s,z), y, w'), \end{aligned}$$

where the first and last equalities follow immediately from the definition of Q in (A.27), while the first and second inequalities follow from isotonicity and concavity of $\tilde{Q}$, respectively. This completes the proof. □

Analogously, the state-action aggregator Q defined in (4.35) can be expressed as

$$Q((s,z), y, \hat{v}) = \tilde{Q}((s,z), y, R\hat{v}),$$

with the operator R defined as above, but

$$\tilde{Q}((s,z), y, h) := \exp\left((1-\eta)\left\{r(s,y,z) + \frac{\beta}{1-\eta}\ln\left[\int h(y,z,\theta)\mu(z, \mathrm{d}\theta)\right]\right\}\right) \tag{A.29}$$

for all $((s,z), y) \in \mathbb{G}$ and $h \in bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$.

Observe that the formula of $\tilde{Q}$ defined above is almost identical to that of Q defined in (4.22). In this connection, recalling the results associated with Q in Section 4.3.3, it is easy to see that

Lemma A.3.11. *If $\eta > 1$, then $\tilde{Q}$ defined in (A.29) is isotone and concave in its third argument on $bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$.*

In addition, the map $((s,z), y) \mapsto \tilde{Q}((s,z), y, h)$ is Borel-measurable on $\mathbb{G}$ whenever $h \in bm(\mathsf{S} \times \mathsf{Z} \times \Theta)_{++}$, and continuous on $\mathbb{G}$ whenever the map $h(\cdot, \cdot, \theta) \colon \mathsf{S} \times \mathsf{Z} \to \mathbb{R}_{++}$ is continuous, for each $\theta \in \Theta$.

Proof of Lemma A.3.11. Analogous to the proof of Lemma A.3.9, for fixed $b \in \mathbb{R}$, we consider the scalar map

$$\psi(t) := \exp\left[(1-\eta)\left(b + \frac{\beta}{1-\eta}\ln t\right)\right] \qquad (t > 0).$$

It is clear that this scalar function ψ is continuous, strictly increasing and strictly concave on $\mathbb{R}_{++}$.[14] As a consequence, the remaining proof of Lemma A.3.11 is identical to that of Lemma A.3.9, and thus omitted here. □

[14] For more details of the relevant results of such ψ, please refer to Section 4.3.3.

Lemma A.3.12. *If $\xi_1 \in (0,1)$ and $\eta > 1$, then the state-action aggregator Q defined in (4.35) is isotone and concave in its third argument on $bm(\mathsf{S}\times\mathsf{Z})_{++}$.*

In addition, the map $((s,z),y) \mapsto Q((s,z),y,v)$ is Borel-measurable on $\mathbb{G}$ whenever $v \in bm(\mathsf{S}\times\mathsf{Z})_{++}$ and continuous on $\mathbb{G}$ whenever $v \in bc(\mathsf{S}\times\mathsf{Z})_{++}$.

Proof of Lemma A.3.12. Invoking Lemmas A.3.7, A.3.8 and A.3.11, the proof is identical to that of Lemma A.3.10, and hence is omitted. □

Lemma A.3.13. *If it is either the case in which $\rho < 1 < \gamma$ or the case in which $1 < \rho < \gamma$, then the function ψ defined in (4.40) is monotone increasing and concave.*

Proof of Lemma A.3.13. Let us first consider the case where $\rho < 1 < \gamma$, from which we have that $1-\gamma < 0$, $0 < 1-\rho < 1$ and $\theta < 0$. Observe that the first derivative of ψ on $(0,\infty)$ is

$$\psi'(t) = \beta\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}^{\theta-1}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho} t^{\frac{\gamma}{1-\gamma}}.$$

According to the ranges of these parameters, it is clear that $\psi'(t) > 0$ for all $t > 0$, which in turn implies that ψ is strictly increasing on $(0,\infty)$.

Further, we consider the second derivative of ψ on $(0,\infty)$ as follows

$$\begin{aligned}\psi''(t) =& \left\{\beta\frac{\theta-1}{\theta}\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}^{-1}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho}+\frac{-\rho}{1-\gamma}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-1}\right.\\ &\left.+\frac{\gamma}{1-\gamma}t^{\frac{-1}{1-\gamma}}\right\}\times\left\{\beta\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}^{\theta-1}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho}t^{\frac{2\gamma}{1-\gamma}}\right\},\end{aligned}$$

and then, after rearrangement with some subtle manipulation, we have

$$\begin{aligned}\psi''(t) &= \left\{\beta\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}^{\theta-2}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho-1}t^{\frac{2\gamma-1}{1-\gamma}}\right\}\times\\ &\quad\times\left\{\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}\left[t^{\frac{1}{1-\gamma}}+b\right]t^{\frac{1}{1-\gamma}}\right\}\times\\ &\quad\times\left\{\frac{1}{\theta}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-1}\left\{1-\frac{\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}}{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}}\right\}-\left[t^{\frac{1}{1-\gamma}}\right]^{-1}+\right.\\ &\quad\left.+\frac{1}{1-\gamma}\left(\left[t^{\frac{1}{1-\gamma}}\right]^{-1}-\left[t^{\frac{1}{1-\gamma}}+b\right]^{-1}\right)+\frac{\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho}}{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}}\right\}\\ &= \left\{\beta\left\{c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right\}^{\theta-2}\left[t^{\frac{1}{1-\gamma}}+b\right]^{-\rho-1}t^{\frac{2\gamma-1}{1-\gamma}}\right\}\times \qquad \text{(A.30)}\\ &\quad\times\left\{\frac{c(1-\theta)}{\theta}t^{\frac{1}{1-\gamma}}+\frac{b\gamma}{1-\gamma}\left(c+\beta\left[t^{\frac{1}{1-\gamma}}+b\right]^{1-\rho}\right)\right\}. \qquad \text{(A.31)}\end{aligned}$$

Evidently, the term in (A.30) is always positive, while invoking the ranges of those parameters (i.e., $\theta < 0$ implied by $\rho < 1 < \gamma$), the term in (A.31) is always negative for all $t > 0$.

Therefore, the fact that $\psi'' < 0$ implies the function ψ is (strictly) concave on $(0, \infty)$, as was to be shown.[15]

Analogously, when $\theta > 1$ implied by $1 < \rho < \gamma$, we obtain that the corresponding first derivative of ψ is positive and the second derivative is negative for all $t > 0$. These results imply directly that such a function ψ is also monotone increasing and (strictly) concave on $(0, \infty)$.

This completes the proof of Lemma A.3.13. □

Proof of Lemma 4.3.1. Let constants m and M be as defined in (4.14). As B is continuous on a compact set, there exists a finite constant

$$l := \min_{((s,z),y)\in\mathbb{G}} B(s,y,z) \quad \text{and} \quad L := \max_{((s,z),y)\in\mathbb{G}} B(s,y,z).$$

Case I : $\rho < 1 < \gamma$. To show condition (SL) of Lemma 4.3.1, we first claim that there exists a positive constant function w_1 such that for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_1) &= \left\{ r(s,y,z) + \beta \left[w_1^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta} \\ &\geqslant \left\{ M + \beta \left[w_1^{\frac{1}{1-\gamma}} + L \right]^{1-\rho} \right\}^{\theta} \\ &> w_1(s,z). \end{aligned} \tag{A.32}$$

Evidently, the uniformly strict inequality (A.32) implies that such a positive constant function w_1 satisfies condition (SL).

To prove our claim that there exists a positive constant function w_1 satisfying (A.32), we note that, since $0 < 1 - \rho < 1$ and $\theta < 0$, the following equivalence relation holds

$$\left\{ M + \beta \left[w_1^{\frac{1}{1-\gamma}} + L \right]^{1-\rho} \right\}^{\theta} > w_1 \iff \left(\frac{w_1^{\frac{1}{\theta}} - M}{\beta} \right)^{\frac{1}{1-\rho}} - w_1^{\frac{1}{1-\gamma}} - L > 0.$$

Let $d := w_1^{\frac{1}{1-\gamma}}$ and set

$$\varphi(d) := \left(\frac{d^{1-\rho} - M}{\beta} \right)^{\frac{1}{1-\rho}} - d - L \qquad (d > 0),$$

Showing that (A.32) holds is equivalent to showing that there exists a positive constant d^* such that $\varphi(d^*) > 0$. To show that the latter holds true, one can verify that both the first and

[15] To be precise, one can check that $\psi''(t) < 0$ for all $t > 0$, which in turn means that ψ is strictly concave on $(0, \infty)$.

the second derivatives of φ on the interval $(\underline{d}, \infty) \subset \mathbb{R}_{++}$ are positive, where $\underline{d} := [M/(1-\beta^{1/\rho})]^{1/(1-\rho)}$. (We have $\underline{d} > 0$, since $M \geqslant m > 0$.) Hence φ is concave upward on $(\underline{d}, \infty)$. This means that $\varphi(d)$ goes to ∞, as $d \to \infty$, which in turn implies that there exists a positive constant $d^* > \underline{d}$ such that $\varphi(d^*) > 0$. Letting $w_1 := (d^*)^{1-\gamma}$ finishes the proof of condition (SL).

Regarding condition (U) of Lemma 4.3.1, we claim first that there is a positive constant function w_2 such that for fixed $((s,z),y) \in \mathbb{G}$, we have

$$\begin{aligned} Q((s,z),y,w_2) &= \left\{ r(s,y,z) + \beta \left[w_2^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta} \\ &\leqslant \left\{ m + \beta \left[w_2^{\frac{1}{1-\gamma}} + l \right]^{1-\rho} \right\}^{\theta} \\ &\leqslant w_2(s,z). \end{aligned} \tag{A.33}$$

Evidently, to show the existence of an upper solution w_2, it is sufficient to show that there exists a positive constant function w_2 satisfying (A.33). Further, after some rearrangement, we note showing that (A.33) holds is equivalent to showing that

$$\left(\frac{w_2^{\frac{1}{\theta}} - m}{\beta} \right)^{\frac{1}{1-\rho}} - w_2^{\frac{1}{1-\gamma}} - l \leqslant 0.$$

Let $w_2 := [m/(1-\beta^{1/\rho})]^{\theta}$. Then the left-hand side of the preceding inequality equals

$$\left(\frac{\frac{\beta^{1/\rho}}{1-\beta^{1/\rho}} m}{\beta} \right)^{\frac{1}{1-\rho}} - \left(\frac{m}{1-\beta^{\frac{1}{\rho}}} \right)^{\frac{1}{1-\rho}} - l = \left[\beta^{\frac{1}{\rho}} - 1 \right] \left(\frac{m}{1-\beta^{\frac{1}{\rho}}} \right)^{\frac{1}{1-\rho}} - l.$$

Since $\beta \in (0,1)$ and $\rho \in (0,1)$, we have $\beta^{1/\rho} - 1 < 0$. Further, it follows from $m > 0$ and $l \geqslant 0$ that the right-hand side of the above equality is negative. This, in turn, implies that for w_2 defined above, (A.33) is satisfied, which proves condition (U).

To see that $w_1 < w_2$, observe that $0 < \underline{d} < d^*$ and $1-\gamma < 0$ imply $0 < w_1 \equiv (d^*)^{1-\gamma} < (\underline{d})^{1-\gamma}$. In addition, since $m \leqslant M$ and $\theta < 0$, we have $(\underline{d})^{1-\gamma} \equiv [M/(1-\beta^{1/\rho})]^{\theta} \leqslant [m/(1-\beta^{1/\rho})]^{\theta} \equiv w_2$. We can now conclude that $w_1 < w_2$, as desired.

Case II : $1 < \rho < \gamma$. For this case, the proof is similar. Regarding condition (SL) of Lemma 4.3.1, we claim first that there exists a positive constant function w_1 such that for fixed $((s,z),y) \in \mathbb{G}$, we have

$$
\begin{aligned}
Q((s,z),y,w_1) &= \left\{ r(s,y,z) + \beta \left[w_1^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta} \\
&\geqslant \left\{ m + \beta \left[w_1^{\frac{1}{1-\gamma}} + L \right]^{1-\rho} \right\}^{\theta} \\
&> w_1(s,z). \qquad \text{(A.34)}
\end{aligned}
$$

The uniformly strict inequality (A.34) implies that w_1 satisfies condition (SL).

To show that there exists a positive constant function w_1 satisfying (A.34), we note that, since $1-\rho < 0$ and $\theta > 1$, the following equivalence relation holds

$$
\left\{ m + \beta \left[w_1^{\frac{1}{1-\gamma}} + L \right]^{1-\rho} \right\}^{\theta} > w_1 \iff \left(\frac{w_1^{\frac{1}{\theta}} - m}{\beta} \right)^{\frac{1}{1-\rho}} - w_1^{\frac{1}{1-\gamma}} - L > 0.
$$

Let $d \equiv w_1^{\frac{1}{1-\gamma}}$ and set

$$
\phi(d) := \left(\frac{d^{1-\rho} - m}{\beta} \right)^{\frac{1}{1-\rho}} - d - L \qquad (d > 0),
$$

Showing that (A.34) holds is equivalent to showing that there exists a positive constant d^* such that $\phi(d^*) > 0$. To show the latter holds, one can check that both the first and second derivatives of ϕ on the interval $(\underline{d},\, m^{1/(1-\rho)}) \subset \mathbb{R}_{++}$ are positive, where $\underline{d} := [m/(1-\beta^{1/\rho})]^{1/(1-\rho)}$. Hence, the graph of ϕ on $(\underline{d},\, m^{1/(1-\rho)})$ is concave upward. Hence $\phi(d)$ approaches $+\infty$ as d approaches $m^{1/(1-\rho)}$. It follows that there exists a positive constant $d^* \in (\underline{d},\, m^{1/(1-\rho)})$ satisfying $\phi(d^*) > 0$. Finally, for such d^*, letting $w_1 \equiv (d^*)^{1-\gamma}$ finishes the proof of condition (SL).

Next, to show condition (U), we claim first that there is a positive constant function w_2 such that for fixed $((s,z),y) \in \mathbb{G}$, we have

$$
\begin{aligned}
Q((s,z),y,w_2) &= \left\{ r(s,y,z) + \beta \left[w_2^{\frac{1}{1-\gamma}} + B(s,y,z) \right]^{1-\rho} \right\}^{\theta} \\
&\leqslant \left\{ M + \beta \left[w_2^{\frac{1}{1-\gamma}} + l \right]^{1-\rho} \right\}^{\theta} \\
&\leqslant w_2(s,z). \qquad \text{(A.35)}
\end{aligned}
$$

To show the existence of an upper solution w_2, it suffices to show that there exists a positive constant function w_2 satisfying (A.35), or equivalently,

$$
\left(\frac{w_2^{\frac{1}{\theta}} - M}{\beta} \right)^{\frac{1}{1-\rho}} - w_2^{\frac{1}{1-\gamma}} - l \leqslant 0.
$$

Let $w_2 := [M/(1-\beta^{1/\rho})]^\theta$. Then the left-hand side of the preceding inequality equals

$$\left(\frac{\frac{\beta^{1/\rho}}{1-\beta^{1/\rho}}M}{\beta}\right)^{\frac{1}{1-\rho}} - \left(\frac{M}{1-\beta^{\frac{1}{\rho}}}\right)^{\frac{1}{1-\rho}} - l = \left[\beta^{\frac{1}{\rho}} - 1\right]\left(\frac{M}{1-\beta^{\frac{1}{\rho}}}\right)^{\frac{1}{1-\rho}} - l < 0.$$

This, in turn, implies that for such w_2 defined above, (A.35) is naturally satisfied, which is what we needed to show for condition (U).

Our choices of w_1 and w_2 satisfy $w_1 < w_2$. To see that this is so, observe that $w_1 \equiv (d^*)^{1-\gamma} < (\underline{d})^{1-\gamma} \equiv [m/(1-\beta^{1/\rho})]^\theta$. Furthermore, it follows from $\theta > 1$ and $m \leqslant M$ that $[m/(1-\beta^{1/\rho})]^\theta \leqslant [M/(1-\beta^{1/\rho})]^\theta \equiv w_2$, from which we conclude that $w_1 < w_2$, as was to be shown. □

A.4 Appendix to Chapter 6

Proof of Lemma 6.2.1. We first prove that d_Φ is a metric on $\mathbb{F}$. To see this, observe that d_Φ is nonnegative due to the nonnegativity of the norm $\|\cdot\|$. Also, given any $u, v \in \mathbb{F}$, we have

$$d_\Phi(u,v) = 0 \;\Leftrightarrow\; \|\Phi u - \Phi v\| = 0 \;\Leftrightarrow\; \Phi u - \Phi v = 0 \;\Leftrightarrow\; u = v,$$

where the second equivalent relation follows from the elementary property of the norm (i.e., $\|w\| = 0$ if and only if $w = 0$), and the third equivalent relation from the fact that Φ is injective.

Symmetry of d_Φ is immediately derived from the symmetry property of the norm $\|\cdot\|$. Regarding the triangle inequality, it holds for d_Φ because, for any u, v, w in $\mathbb{F}$,

$$\begin{aligned} d_\Phi(u,v) = \|\Phi u - \Phi v\| &= \|(\Phi u - \Phi w) + (\Phi w - \Phi v)\| \\ &\leqslant \|\Phi u - \Phi w\| + \|\Phi w - \Phi v\| = d_\Phi(u,w) + d_\Phi(w,v). \end{aligned}$$

Hence, d_Φ is indeed a well-defined metric on $\mathbb{F}$ and it remains only to show its completeness.

To this end, we consider the space $(\mathbb{B}_+, d_0)$ with the metric $d_0(u,v) := \|u - v\|$ induced by its complete lattice norm $\|\cdot\|$ on $\mathbb{B}$. Since the positive cone $\mathbb{B}_+$ is closed in $\mathbb{B}$ and $\mathbb{B}$ is a Banach lattice, the metric space $(\mathbb{B}_+, d_0)$ is complete (see theorem 15.1 in Zaanen (1997)). In this connection, in order to show the completeness of the metric space $(\mathbb{F}, d_\Phi)$, it suffices to show that $(\mathbb{F}, d_\Phi)$ and $(\mathbb{B}_+, d_0)$ are isometrically isomorphic.

Observe that for each u, v in $\mathbb{F}$, we have

$$d_\Phi(u,v) = \|\Phi u - \Phi v\| = d_0(\Phi u, \Phi v).$$

Thus, the restricted map $\Phi|_{\mathbb{F}}$ is an isometry from $\mathbb{F}$ to $\mathbb{B}_+$, and hence it is automatically injective. Moreover, the restricted map $\Phi|_{\mathbb{F}} \colon \mathbb{F} \to \mathbb{B}_+$ is surjective, i.e., $\Phi|_{\mathbb{F}}(\mathbb{F}) = \mathbb{B}_+$. Indeed, the facts that $\Phi\colon \mathbb{E}_+ \to \mathbb{E}_+$ is surjective and that $\mathbb{B}_+$ is a subset of the codomain $\mathbb{E}_+$ imply that $\Phi(\Phi^{-1}(\mathbb{B}_+)) = \mathbb{B}_+$.[16]

Therefore, the restricted map $\Phi|_{\mathbb{F}}$ is a bijective isometry from $\mathbb{F}$ to $\mathbb{B}_+$.[17] This means that $(\mathbb{F}, d_\Phi)$ is isometrically isomorphic to the complete metric space $(\mathbb{B}_+, d_0)$, as was to be shown.

This completes the proof. □

Proof of Theorem 6.2.1. In order to obtain the above stated result, we aim to apply the Banach Fixed-Point theorem. By virtue of Lemmas 6.2.1 and 6.2.2, we know that the operator $\boldsymbol{A}$ defined in (6.3) is a self-map on the complete metric space $(\mathbb{F},\, d_\Phi)$. Therefore, it suffices to show that the operator $\boldsymbol{A}$ is strictly contractive on $(\mathbb{F},\, d_\Phi)$ with the modulus $\|\boldsymbol{K}\|$.

To this end, take any f, g in $\mathbb{F}$ and consider

$$\begin{aligned}
|\Phi(\boldsymbol{A}f) - \Phi(\boldsymbol{A}g)| &= \left|\Phi\left(h + \Phi^{-1}\boldsymbol{K}\Phi f\right) - \Phi\left(h + \Phi^{-1}\boldsymbol{K}\Phi g\right)\right| \\
&\leqslant |\boldsymbol{K}\Phi f - \boldsymbol{K}\Phi g| = |\boldsymbol{K}(\Phi f - \Phi g)| \\
&\leqslant \boldsymbol{K}\,|\Phi f - \Phi g|
\end{aligned}$$

where the first inequality follows from (6.1), and the second follows from the linearity and monotonicity of $\boldsymbol{K}$.

By virtue of the lattice norm property of $\mathbb{B}$, it follows that

$$\|\Phi(\boldsymbol{A}f) - \Phi(\boldsymbol{A}g)\| \leqslant \|\boldsymbol{K}\,|\Phi f - \Phi g|\| \leqslant \|\boldsymbol{K}\| \cdot \|\Phi f - \Phi g\|,$$

and thus,

$$d_\Phi(\boldsymbol{A}f,\, \boldsymbol{A}g) \leqslant \|\boldsymbol{K}\|\, d_\Phi(f, g).$$

This is what we wish to show, given that $\|\boldsymbol{K}\| < 1$. Now applying the Banach Fixed-Point theorem gives us the desired result. □

Proof of Theorem 6.2.2. As a first step, we claim that the inequality

$$|\Phi\boldsymbol{A}^n f - \Phi\boldsymbol{A}^n g| \leqslant \boldsymbol{K}^n|\Phi f - \Phi g| \qquad (\forall f,\, g \in \mathbb{F}) \tag{A.36}$$

holds for all $n \in \mathbb{N}_0$. Evidently it holds for $n = 0$, since $\boldsymbol{A}^0$ and $\boldsymbol{K}^0$ are by definition identity maps. Now suppose that it holds for some fixed n'. We claim it also holds at $n' + 1$. To see this,

[16] That is, if the map $\Phi\colon \mathbb{E}_+ \to \mathbb{E}_+$ is surjective, then the subset $\mathbb{B}_+$ of the codomain $\mathbb{E}_+$ of Φ can be recovered from its preimage $\Phi^{-1}(\mathbb{B}_+) \equiv \mathbb{F}$.

[17] A bijective isometry is also referred to as an *isometric isomorphism*.

pick any $f,\ g \in \mathbb{F}$. We have

$$\begin{aligned}\left|\Phi \boldsymbol{A}^{n'+1} f - \Phi \boldsymbol{A}^{n'+1} g\right| &= \left|\Phi\left(h + \Phi^{-1} K \Phi \boldsymbol{A}^{n'} f\right) - \Phi\left(h + \Phi^{-1} \boldsymbol{K} \Phi \boldsymbol{A}^{n'} g\right)\right| \\ &\leqslant \left|\boldsymbol{K} \Phi \boldsymbol{A}^{n'} f - \boldsymbol{K} \Phi \boldsymbol{A}^{n'} g\right|,\end{aligned}$$

where the inequality is due to (6.1). Using this bound and the linearity and isotonicity of $\boldsymbol{K}$ leads us to

$$\left|\Phi \boldsymbol{A}^{n'+1} f - \Phi \boldsymbol{A}^{n'+1} g\right| \leqslant \boldsymbol{K}\left|\Phi \boldsymbol{A}^{n'} f - \Phi \boldsymbol{A}^{n'} g\right| \leqslant \boldsymbol{K}^{n'+1} |\Phi f - \Phi g|.$$

Here the last inequality uses the induction hypothesis combined with isotonicity of $\boldsymbol{K}$. We have now shown that (A.36) holds for all $n \in \mathbb{N}_0$ as claimed.

Together, (A.36) and the lattice norm property yield

$$\|\Phi \boldsymbol{A}^n f - \Phi \boldsymbol{A}^n g\| \leqslant \|\boldsymbol{K}^n\, |\Phi f - \Phi g|\,\| \qquad (\forall f,\, g \in \mathbb{F},\, n \in \mathbb{N}_0).$$

From the definition of the induced operator norm,

$$\|\boldsymbol{K}^n\, |\Phi f - \Phi g|\,\| \leqslant \|\boldsymbol{K}^n\| \cdot \|\, |\Phi f - \Phi g|\,\| = \|\boldsymbol{K}^n\| \cdot \|\Phi f - \Phi g\|.$$

We now have $d_\Phi(\boldsymbol{A}^n f,\, \boldsymbol{A}^n g) \leqslant \|\boldsymbol{K}^n\| \cdot d_\Phi(f, g)$ for all $n \in \mathbb{N}_0$. Applying Gelfand's formula $r(\boldsymbol{K}) = \lim_{n\to\infty} \|\boldsymbol{K}^n\|^{1/n}$ and using $r(\boldsymbol{K}) < 1$ guarantee the existence of an $N \in \mathbb{N}$ such that $\|\boldsymbol{K}^N\| < 1$. For this N, the map $\boldsymbol{A}^N$ is a strict contraction map on $(\mathbb{F}, d_\Phi)$ with modulus $\|\boldsymbol{K}^N\|$.

This completes the proof. □

Proof of Lemma 6.3.1. To see this, we first consider the case of ϕ being increasing and concave, and assume without loss of generality that $a \leqslant b$. Since ϕ is increasing, showing (6.4) is equivalent to proving that $\phi(b+c) - \phi(a+c) \leqslant \phi(b) - \phi(a)$.

To this end, fix constant $c \in \mathbb{R}_+$, we define a function $f(x) := \phi(x+c) - \phi(x)$ for all $x \in \mathbb{R}$. Since ϕ is concave, it is absolutely continuous on any closed subinterval of $\mathbb{R}$. Hence, applying the fundamental theorem of Lebesgue integral calculus (FTLIC), the function ϕ has a derivative ϕ' almost everywhere (a.e.) which is Lebesgue integrable. It then follows that so as is f with $f'(x) = \phi'(x+c) - \phi'(x)$ for a.e. x.

By virtue of the concavity of ϕ again, we know that $f'(x) = \phi'(x+c) - \phi'(x) \leqslant 0$ for a.e. x. Now applying FTLIC again yields $f(b) - f(a) = \int_a^b f'(x)\mathrm{d}x \leqslant 0$. That is, $\phi(b+c) - \phi(b) \equiv f(b) \leqslant f(a) \equiv \phi(a+c) - \phi(a)$, which is what we wish to show.

Regarding the case of ϕ being decreasing and convex, observe that $-\phi$ is increasing and concave. Hence, applying (6.4) to the function $-\phi$ and noticing that minus signs cancel out in absolute value show that (6.4) still holds for convex and decreasing functions. □

In all what follows, Assumptions 6.4.1 to 6.4.5 are assumed to be satisfied. Two kinds of contraction mappings are used to study the optimality results.

Existence and Uniqueness of the fixed point of the intertemporal recursion operator $\boldsymbol{T}_\sigma$

First, we consider the operator $\boldsymbol{T}_\sigma$ defined in (6.11). It is not hard to see that solutions to this operator $\boldsymbol{T}_\sigma$ will line up with fixed points of the operator $\boldsymbol{A}$ defined in (6.3), in a form to be discussed. In this connection, we establish the existence and uniqueness result of the fixed point of the operator $\boldsymbol{T}_\sigma$ by applying our generalized fixed point theorem.

Proposition A.4.1. *For every $\sigma \in \Sigma$, the operator $\boldsymbol{T}_\sigma$ has a unique fixed point w_σ in $\mathbb{F}_m$, and $\|\phi \circ (\boldsymbol{T}_\sigma^n w) - \phi \circ w_\sigma\|_\kappa \to 0$ (as $n \to \infty$) for any initial point w in $\mathbb{F}_m$.*

Proof of Proposition A.4.1. Recall that the operator $\boldsymbol{T}_\sigma \colon \mathbb{F}_m \subset m(\mathbb{S})_+ \to m(\mathbb{S})_+$ in (6.11) can be expressed as

$$\begin{aligned}\boldsymbol{T}_\sigma w(x) &= r_\sigma(x) + \phi^{-1}\left(\phi(\beta)\int_{\mathbb{Z}} \phi \circ w[F(x,\sigma(x),z)]\mathscr{P}(\mathrm{d}z)\right) \qquad (x \in \mathbb{S}, \sigma \in \Sigma)\\ &= r_\sigma(x) + \phi^{-1}\left(\phi(\beta)\int_{\mathbb{Z}} \phi \circ w(y) P_\sigma(x,\mathrm{d}y)\right)\\ &= r_\sigma(x) + \Phi^{-1}\left(\boldsymbol{K}_\sigma \Phi w\,(x)\right),\end{aligned}$$

where an operator $\boldsymbol{K}_\sigma \colon b_\kappa m(\mathbb{S}) \to b_\kappa m(\mathbb{S})$ is defined by

$$\begin{aligned}\boldsymbol{K}_\sigma g(x) &:= \phi(\beta)\int g[F(x,\sigma(x),z)]\mathscr{P}(\mathrm{d}z) \qquad (x \in \mathbb{S}, \sigma \in \Sigma)\\ &= \phi(\beta)\int g(y) P_\sigma(x,\mathrm{d}y). \end{aligned} \tag{A.37}$$

As a result, to apply Corollary 6.2.1 to the operator $\boldsymbol{T}_\sigma$, we need to verify those conditions in Theorem 6.2.2 as follows:

(i) r_σ is in $\mathbb{F}_m$;

(ii) Φ is sub-additive on $m(\mathbb{S})_+$ (i.e., Assumption 6.2.1 holds); and

(iii) $\boldsymbol{K}_\sigma$ defined in (A.37) is linear, monotone increasing and satisfies $r(\boldsymbol{K}_\sigma) < 1$ on $b_\kappa m(\mathbb{S})$.

Regarding Condition (i), the measurability and nonnegativity of r_σ are obvious. Invoking Condition (6.6) in Assumption 6.4.5, it follows that $\phi \circ r_\sigma$ is κ-bounded and also measurable, and hence, by the definition of $\mathbb{F}_m$, r_σ indeed belongs to $\mathbb{F}_m$. Condition (ii) was established in Example 6.3.3. Regarding Condition (iii), $\boldsymbol{K}_\sigma$ is linear and isotone by standard properties of the integral. In addition, we obtain

$$
\begin{aligned}
|\boldsymbol{K}_\sigma g(x)| = \left|\phi(\beta)\int g(y)P_\sigma(x,\mathrm{d}y)\right| &\leqslant \phi(\beta)\int |g(y)|\, P_\sigma(x,\mathrm{d}y)\\
&\leqslant \phi(\beta)\int \|g\|_\kappa\, \kappa(y)P_\sigma(x,\mathrm{d}y)\\
&\leqslant \phi(\beta)\|g\|_\kappa N_\kappa \cdot \kappa(x)
\end{aligned}
$$

for any $g \in b_\kappa m(\mathbb{S})$ and $x \in \mathbb{S}$. The second inequality follows from the fact that $|g(y)| \leqslant \|g\|_\kappa \kappa(y)$ for each $y \in \mathbb{S}$ (since $g \in b_\kappa m(\mathbb{S})$), and the third follows from Condition (6.7) in Assumption 6.4.5. Hence, dividing both sides of the above inequality by the term $\kappa(x)$ and taking the supremum over $x \in \mathbb{S}$ yields that

$$
\|\boldsymbol{K}_\sigma g\|_\kappa \leqslant \phi(\beta)N_\kappa\|g\|_\kappa
$$

and hence, by Assumption 6.4.5,

$$
\|\boldsymbol{K}_\sigma\| := \sup_{\|g\|_\kappa=1} \|\boldsymbol{K}_\sigma g\|_\kappa \leqslant \phi(\beta)N_\kappa < 1.
$$

Since $r(\boldsymbol{K}_\sigma) \leqslant \|\boldsymbol{K}_\sigma\|$, we have $r(\boldsymbol{K}_\sigma) < 1$. Therefore, all the conditions of Theorem 6.2.2 are satisfied, and hence Corollary 6.2.1 applies and implies that the operator $\boldsymbol{T}_\sigma$ has a unique fixed point w_σ in $\mathbb{F}_m$ and $\|\phi \circ (\boldsymbol{T}_\sigma^n w) - \phi \circ w_\sigma\|_\kappa \to 0$ as $n \to \infty$ for any $w \in \mathbb{F}_m$.

This completes the proof. □

Remark A.4.1. We note that the convergence in the weighted supremum norm implies the pointwise convergence, because if $\|g_n - g^*\|_\kappa < \varepsilon$ for some $\varepsilon > 0$, then $|g_n(x) - g^*(x)| < \varepsilon \cdot \kappa(x)$ for each $x \in \mathbb{S}$.

In general, the convergence in $(b_\kappa m(\mathbb{S}), \|\cdot\|_\kappa)$ does not imply the uniform convergence. But the convergence in $(b_\kappa m(\mathbb{S}), \|\cdot\|_\kappa)$ does imply the uniform convergence on any compact subset of $\mathbb{S}$. To see this, pick an arbitrary compact set $\mathbb{Y}$ in $\mathbb{S}$ and thus the continuous weight function κ has a maximum on $\mathbb{Y}$. For simplicity, denote by $M_{\mathbb{Y}} := \max_{x\in\mathbb{Y}} \kappa(x)$ the maximum of κ on $\mathbb{Y}$. It then follows that

$$
\begin{aligned}
\sup_{x\in\mathbb{Y}} |g_n(x) - g^*(x)| = M_{\mathbb{Y}} \sup_{x\in\mathbb{Y}} \frac{|g_n(x) - g^*(x)|}{M_{\mathbb{Y}}}
&\leqslant M_{\mathbb{Y}} \sup_{x\in\mathbb{Y}} \frac{|g_n(x) - g^*(x)|}{\kappa(x)}\\
&\leqslant M_{\mathbb{Y}} \sup_{x\in\mathbb{S}} \frac{|g_n(x) - g^*(x)|}{\kappa(x)} = M_{\mathbb{Y}}\|g_n - g^*\|_\kappa.
\end{aligned}
$$

As a result, $\|g_n - g^*\|_\kappa \to 0$ as $n \to \infty$ implies that $\sup_{x\in\mathbb{Y}} |g_n(x) - g^*(x)| \to 0$; that is, the sequence $\{g_n\}$ converges to g^* uniformly on $\mathbb{Y}$.

From Proposition A.4.1, it is worth noticing that the global attractivity of the unique fixed point

w_σ of $\boldsymbol{T}_\sigma$ on $\mathbb{F}_m$ is a very powerful property that allows us to find the solution iteratively by starting from any possible initial point in $\mathbb{F}_m$. In particular, observe that the zero constant function $\mathbf{0}$ belongs to $\mathbb{F}_m$ and hence, the unique fixed point w_σ of $\boldsymbol{T}_\sigma$ corresponding to σ can be easily computed through[18]

$$w_\sigma = \lim_{n\to\infty} \boldsymbol{T}_\sigma^n(\mathbf{0}) \qquad (\forall \sigma \in \Sigma).$$

Existence and Uniqueness of the fixed point of the Bellman operator

In the sequel, we consider the Bellman operator $\boldsymbol{T}$ defined in (6.14).

Proposition A.4.2. *The Bellman operator $\boldsymbol{T}$ has a unique fixed point w^* in $\mathbb{F}_c$ and $\|\phi \circ (\boldsymbol{T}^n w) - \phi \circ w^*\|_\kappa \to 0$ as $n \to \infty$ for any $w \in \mathbb{F}_c$.*

The contraction property of the Bellman operator $\boldsymbol{T}$ that will be proved later gives a globally convergent algorithm to compute the value function. In addition, it allows us to formalize the intuitive result that the solution to an infinite-horizon problem is the limit of that for a finite-horizon problem.

In order to prove Proposition A.4.2, we need some auxiliary lemmas.

Lemma A.4.1. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing and surjective function. If g is a continuous nonnegative function defined on a compact metric space $\mathbb{X}$, then we have*

$$f\left(\sup_{x\in\mathbb{X}} g(x)\right) = \sup_{x\in\mathbb{X}} f\circ g(x). \tag{A.38}$$

Proof of Lemma A.4.1. We first show the inequality $f\left(\sup_{x\in\mathbb{X}} g(x)\right) \geqslant \sup_{x\in\mathbb{X}} f\circ g(x)$. It is obvious that $g(x) \leqslant \sup_{x\in\mathbb{X}} g(x)$ for all $x \in \mathbb{X}$. Since f is a strictly increasing function, we have $f\circ g(x) \leqslant f\left(\sup_{x\in\mathbb{X}} g(x)\right)$ for all $x \in \mathbb{X}$, and hence $\sup_{x\in\mathbb{X}} f\circ g(x) \leqslant f\left(\sup_{x\in\mathbb{X}} g(x)\right)$.

Next we prove the reverse direction of the above inequality, i.e., $f\left(\sup_{x\in\mathbb{X}} g(x)\right) \leqslant \sup_{x\in\mathbb{X}} f\circ g(x)$. It is clear that $(f\circ g)(x) \leqslant \sup_{x\in\mathbb{X}} (f\circ g)(x)$ for all $x \in \mathbb{X}$. Since f is a strictly increasing function and hence injective, together with the surjective property of f, we know that f is a bijection on $\mathbb{R}_+$. As a result, its inverse function $f^{-1}\colon \mathbb{R}_+ \to \mathbb{R}_+$ exists naturally and possesses the same strict monotonicity. Furthermore, the bijectivity and strict monotonicity of f together yield the continuity of f. This implies that $\sup_{x\in\mathbb{X}} f\circ g(x)$ is finite, as $f\circ g$ is continuous on the compact set $\mathbb{X}$. It then follows that $g(x) \leqslant f^{-1}\left(\sup_{x\in\mathbb{X}} f\circ g(x)\right)$ for all $x \in \mathbb{X}$, and hence, $\sup_{x\in\mathbb{X}} g(x) \leqslant f^{-1}\left(\sup_{x\in\mathbb{X}} f\circ g(x)\right)$ holds. Making use of the monotonicity of f again, we obtain $f\left(\sup_{x\in\mathbb{X}} g(x)\right) \leqslant \sup_{x\in\mathbb{X}} f\circ g(x)$, as desired. □

[18] To be precise, as we noted before, a sequence $\{f_n\}$ converges to f^* in the metric of $\mathbb{F}_m$ (i.e., $\|\phi\circ f_n - \phi\circ f^*\|_\kappa \to 0$ as $n \to \infty$) implies the pointwise convergence in the sense that $\lim_{n\to\infty} \phi\circ f_n(x) = \phi\circ f^*(x)$ for each $x \in \mathbb{S}$. It then follows from the continuity of ϕ that $\phi\circ[\lim_{n\to\infty} f_n(x)] = \lim_{n\to\infty} \phi\circ f_n(x) = \phi\circ f^*(x)$, and hence the strict monotonicity of ϕ yields $\lim_{n\to\infty} f_n(x) = f^*(x)$ (pointwise) for every $x \in \mathbb{S}$.

The following lemma is crucial for solving Bellman equations.

Lemma A.4.2. *The function*

$$\operatorname{gr}\Gamma \ni (x,a) \mapsto \phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right) \in \mathbb{R}_+$$

is continuous on $\operatorname{gr}\Gamma$ *whenever* $w \in \mathbb{F}_c$*.*

Proof of Lemma A.4.2. Observe first from Assumption 6.4.4 that the inverse function ϕ^{-1} is continuous on $\mathbb{R}_+$. Thus, showing the continuity of the function $(x,a) \mapsto \phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right)$ is equivalent to showing the continuity of the function $(x,a) \mapsto \int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)$.

Since w lies in $\mathbb{F}_c$, the function $\phi \circ w$ is in $b_\kappa c(\mathbb{S})_+$. It then follows from Assumption 6.4.5 that the mapping $(x,a) \mapsto \int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)$ is continuous on $\operatorname{gr}\Gamma$ (see, e.g., Lemma 12.2.20 in Stachurski (2009)), which finishes the proof. □

Lemma A.4.3. *The Bellman operator* $\boldsymbol{T}$ *maps* $\mathbb{F}_c$ *to itself.*

Proof of Lemma A.4.3. To do so, pick any $w \in \mathbb{F}_c$ and for any $x \in \mathbb{S}$. We first prove that $\phi \circ (\boldsymbol{T}w)$ is κ-bounded. Invoking Lemma A.4.1, we have

$$\begin{aligned}
\phi \circ (\boldsymbol{T}w(x)) &= \phi\left(\sup_{a\in\Gamma(x)}\left\{r(x,a) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right)\right\}\right) \\
&= \sup_{a\in\Gamma(x)}\left\{\phi\left(r(x,a) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right)\right)\right\} \\
&\leqslant \sup_{a\in\Gamma(x)}\left\{\phi \circ r(x,a) + \phi(\beta)\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right)\right\} \\
&\leqslant \sup_{a\in\Gamma(x)}\{\phi \circ r(x,a)\} + \phi(\beta)\sup_{a\in\Gamma(x)}\left\{\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right\} \\
&\leqslant \sup_{a\in\Gamma(x)}\{\phi \circ r(x,a)\} + \phi(\beta)\sup_{a\in\Gamma(x)}\left\{\int_{\mathbb{Z}} \|\phi \circ w\|_\kappa \cdot \kappa[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right\} \\
&\leqslant R \cdot \kappa(x) + \phi(\beta)\|\phi \circ w\|_\kappa N_\kappa \cdot \kappa(x) = (R + \phi(\beta)\|\phi \circ w\|_\kappa N_\kappa) \cdot \kappa(x).
\end{aligned}$$

The first inequality is derived from the subadditivity of the function ϕ (see Example 6.3.3). The third inequality follows from the fact that $w \in \mathbb{F}_c$ and hence $\phi \circ w \in b_\kappa c(\mathbb{S})_+$. The last inequality directly follows from the two conditions in Assumption 6.4.5.

Hence, it is clear that the function $\phi \circ (\boldsymbol{T}w)$ is κ-bounded. More precisely, we have $\|\phi \circ (\boldsymbol{T}w)\|_\kappa \leqslant R + \phi(\beta)\|\phi \circ w\|_\kappa N_\kappa < +\infty$.

It remains to show the continuity of $\boldsymbol{T}w$. To see this, invoking Lemma A.4.2 and Assumption 6.4.3, we know that the objective function

$$\operatorname{gr}\Gamma \ni (x,a) \mapsto r(x,a) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a,z)]\mathscr{P}(\mathrm{d}z)\right)$$

is continuous on $\operatorname{gr}\Gamma$. As a result, by virtue of Assumption 6.4.1, Berge's maximum theorem applies and gives us that $\boldsymbol{T}w$ is continuous on $\mathbb{S}$.[19]

To sum up, the Bellman operator $\boldsymbol{T}$ transforms $\mathbb{F}_c$ invariant. This completes the proof. □

Proof of Proposition A.4.2. Invoking the content of Section 6.2.1, we note that the metric space $(\mathbb{F}_c, d_\kappa^\phi)$ is complete, where the metric is defined by $d_\kappa^\phi(f,g) := \|\phi \circ f - \phi \circ g\|_\kappa$.[20] Invoking Lemma A.4.3, it remains to show that $\boldsymbol{T}$ is a strict contraction on $\mathbb{F}_c$.

To this end, pick any two elements w_1, w_2 in $\mathbb{F}_c$ and for each $x \in \mathbb{S}$. Observe that the deviation $|\phi \circ [\boldsymbol{T}w_1(x)] - \phi \circ [\boldsymbol{T}w_2(x)]|$ is equal to

$$\left|\phi \circ \left(\sup_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_1[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right) \right.$$
$$\left. - \phi \circ \left(\sup_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_2[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right) \right|.$$

Now invoking Lemma A.4.1, the operation order of ϕ and the operator "sup" can be interchanged and hence, the preceding expression becomes

$$\begin{aligned}
&\left| \sup_{a \in \Gamma(x)} \phi \circ \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_1[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right. \\
&\qquad \left. - \sup_{a \in \Gamma(x)} \phi \circ \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_2[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right| \\
&\leqslant \sup_{a \in \Gamma(x)} \left| \phi \circ \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_1[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right. \\
&\qquad \left. - \phi \circ \left\{ r(x,a) + \beta \phi^{-1} \left(\int_{\mathbb{Z}} \phi \circ w_2[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right) \right\} \right| \\
&\leqslant \sup_{a \in \Gamma(x)} \phi(\beta) \left| \int_{\mathbb{Z}} \phi \circ w_1[F(x,a,z)] \mathscr{P}(\mathrm{d}z) - \int_{\mathbb{Z}} \phi \circ w_2[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \right| \\
&\leqslant \sup_{a \in \Gamma(x)} \phi(\beta) \int_{\mathbb{Z}} \left| \phi \circ w_1[F(x,a,z)] - \phi \circ w_2[F(x,a,z)] \right| \mathscr{P}(\mathrm{d}z) \\
&\leqslant \sup_{a \in \Gamma(x)} \phi(\beta) \int_{\mathbb{Z}} \|\phi \circ w_1 - \phi \circ w_2\|_\kappa \kappa[F(x,a,z)] \mathscr{P}(\mathrm{d}z) \\
&\leqslant \phi(\beta) \|\phi \circ w_1 - \phi \circ w_2\|_\kappa N_\kappa \cdot \kappa(x).
\end{aligned}$$

The first inequality follows from the fact that $|\sup_{x \in \mathbb{X}} f - \sup_{x \in \mathbb{X}} g| \leqslant \sup_{x \in \mathbb{X}} |f - g|$ for any continuous functions f, g and for any compact set $\mathbb{X}$.[21] The second inequality follows directly

[19] The reader may be referred to Berge's theorem, see, e.g., pages 115-116 in Berge (1963) or Proposition 10.2 of Schäl (1975).

[20] Evidently, recalling Lemma 6.2.1, the metric space $(\mathbb{F}_c, d_\kappa^\phi)$ is isometrically isomorphic to the complete metric space $(b_\kappa c(\mathbb{S})_+, d_\kappa)$.

[21] To see this, take an arbitrary compact set $\mathbb{X}$ and any two continuous functions f and g. We have $\sup_{x \in \mathbb{X}} f = \sup_{x \in \mathbb{X}} (f - g + g) \leqslant \sup_{x \in \mathbb{X}} (f - g) + \sup_{x \in \mathbb{X}} g \leqslant \sup_{x \in \mathbb{X}} |f - g| + \sup_{x \in \mathbb{X}} g$, and thus $\sup_{x \in \mathbb{X}} f - \sup_{x \in \mathbb{X}} g \leqslant \sup_{x \in \mathbb{X}} |f - g|$. The same argument exchanging the roles of f and g finishes the proof.

from our crucial inequality (6.4). Hence, taking the supremum with respect to the deviation $|\phi \circ [Tw_1(x)] - \phi \circ [Tw_2(x)]| / \kappa(x)$ over $x \in \mathbb{S}$ yields

$$\begin{aligned} d_\kappa^\phi(Tw_1, Tw_2) &= \|\phi \circ (Tw_1) - \phi \circ (Tw_2)\|_\kappa \\ &\leqslant \phi(\beta) N_\kappa \|\phi \circ w_1 - \phi \circ w_2\|_\kappa = \phi(\beta) N_\kappa d_\kappa^\phi(w_1, w_2). \end{aligned}$$

By Assumption 6.4.5 that $\phi(\beta)N_\kappa < 1$, it follows that the operator T is a strict contraction on $\mathbb{F}_c$ with modulus $\phi(\beta)N_\kappa$. Thus, Banach Fixed-Point theorem applies and gives us that the Bellman operator T has exactly one fixed point w^* in $\mathbb{F}_c$ and $d_\kappa^\phi(T^n w, w^*) = \|\phi \circ (T^n w) - \phi \circ w^*\|_\kappa \to 0$ as $n \to \infty$ for any $w \in \mathbb{F}_c$.

This completes the proof. □

Proposition A.4.2 proves part (i) of Theorem 6.5.1. Following that, we shall show the optimality result.

Lemma A.4.4. *If $w \in \mathbb{F}_c$, then the objective function on the right-hand side of* (6.15) *is continuous in a for every $x \in \mathbb{S}$, and Σ contains at least one w-greedy policy.*

Proof. By virtue of Assumption 6.4.3 and Lemma A.4.2, it follows that the objective function on the right-hand side of (6.15) is continuous on $\operatorname{gr}\Gamma$. Clearly, the objective function is continuous with respect to a for each x.

By virtue of Assumption 6.4.1, the constraint set $\Gamma(x)$ is compact, which implies that a solution to the maximization problem of the objective function exists. Hence, for each $x \in \mathbb{S}$ we can find at least one a_x^* that attains the maximum, and the map $x \mapsto a_x^*$ certainly defines a function $\sigma \colon \mathbb{S} \to \mathbb{A}$ satisfying (6.15).

The existence of a Borel-measurable mapping $\sigma \colon \mathbb{S} \to \mathbb{A}$ satisfying (6.15) follows from corollary 1 of Brown and Purves (1973). Thus, such policy function σ being Borel-measurable indeed lies in Σ, which finishes the proof. □

Proposition A.4.3. *The value function $J^* = \sup_{\sigma \in \Sigma} J(\cdot, \sigma)$ is the unique fixed point of the Bellman operator T in $\mathbb{F}_c$.*

Proof. It follows from Proposition A.4.2 that the Bellman operator T has exactly one positive fixed point w^* in $\mathbb{F}_c$. Then, in order to show the stated result in Proposition A.4.3, it is sufficient to verify that $J^* = w^*$.

Since $w^* \in \mathbb{F}_c$, Lemma A.4.4 applies and hence, there exists a w^*-greedy policy $\hat{\sigma} \in \Sigma$ satisfying

$$Tw^* = T_{\hat{\sigma}} w^*.$$

For this w^*-greedy policy $\hat{\sigma}$, we have $w^* = Tw^* = T_{\hat{\sigma}} w^*$.

On the other hand, invoking Proposition A.4.1, for such policy $\hat{\sigma}$, we know that $w_{\hat{\sigma}} \in \mathbb{F}_m$ is the unique fixed point of $\boldsymbol{T}_{\hat{\sigma}}$. Thanks to the fact that $\mathbb{F}_c \subset \mathbb{F}_m$, we have

$$w^* = w_{\hat{\sigma}}. \tag{A.39}$$

Invoking Remark A.4.1 with (6.12) and (6.13) yields that

$$w_{\hat{\sigma}} = \lim_{n\to\infty} \boldsymbol{T}^n_{\hat{\sigma}}(\boldsymbol{0}) = \boldsymbol{J}(\cdot, \hat{\sigma}). \tag{A.40}$$

Combining (A.39) and (A.40), we can conclude that

$$w^* = \boldsymbol{J}(\cdot, \hat{\sigma}) \leqslant \sup_{\sigma\in\Sigma} \boldsymbol{J}(\cdot, \sigma) = \boldsymbol{J}^*.$$

To check the reverse inequality, we first observe from (6.5.1) that

$$\begin{aligned} w^*(x) = \boldsymbol{T}w^*(x) &= \sup_{a\in\Gamma(x)} \left\{ r(x,a) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w^* \left[F(x,a,z)\right] \mathscr{P}(\mathrm{d}z) \right) \right\} \\ &\geqslant r(x,\sigma(x)) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w^* \left[F(x,\sigma(x),z)\right] \mathscr{P}(\mathrm{d}z) \right) \\ &= \boldsymbol{T}_\sigma w^*(x) \qquad\qquad (x \in \mathbb{S}) \end{aligned}$$

for any $\sigma \in \Sigma$.

In light of that, pick and fix an arbitrary $\tilde{\sigma} \in \Sigma$, we have $w^* = \boldsymbol{T}w^* \geqslant \boldsymbol{T}_{\tilde{\sigma}} w^*$. Making use of the monotonicity of $T_{\tilde{\sigma}}$ and iterating on this inequality $w^* \geqslant \boldsymbol{T}_{\tilde{\sigma}} w^*$ give us that

$$w^* \geqslant \boldsymbol{T}_{\tilde{\sigma}} w^* \geqslant \boldsymbol{T}^2_{\tilde{\sigma}} w^* \geqslant \cdots \geqslant \boldsymbol{T}^n_{\tilde{\sigma}} w^* \geqslant \cdots$$

for every $n \in \mathbb{N}$.

As $w^* \in \mathbb{F}_c \subset \mathbb{F}_m$, taking limits of the above sequence and using the global attractivity of the fixed point of $\boldsymbol{T}_{\tilde{\sigma}}$ from Proposition A.4.1 that

$$\boldsymbol{T}^n_{\tilde{\sigma}} w^* \to w_{\tilde{\sigma}} = \boldsymbol{J}(\cdot, \tilde{\sigma}) \quad \text{pointwise,} \qquad (\text{as } n \to \infty),$$

we then obtain $w^* \geqslant w_{\tilde{\sigma}} = J(\cdot, \tilde{\sigma})$. Since $\tilde{\sigma}$ is arbitrary, it follows that

$$w^* \geqslant \sup_{\sigma\in\Sigma} \boldsymbol{J}(\cdot, \sigma) = \boldsymbol{J}^*.$$

In summary, $\boldsymbol{J}^* = w^* \in \mathbb{F}_c$, as was to be shown. This completes the proof. □

Lemma A.4.5. *A feasible policy $\sigma \in \Sigma$ is optimal if and only if it is w^*-greedy.*

Proof. Observe from Definition 6.5.1 that a policy $\sigma \in \Sigma$ is w^*-greedy if and only if

$$w^*(x) = r(x, \sigma(x)) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w^*[F(x, \sigma(x), z)]\mathscr{P}(\mathrm{d}z)\right) \qquad (x \in \mathbb{S}).$$

Invoking the definition of the operator $\boldsymbol{T}_\sigma$, we can express the above functional equation as $w^* = \boldsymbol{T}_\sigma w^*$ (in operator notation). Meanwhile, invoking from Proposition A.4.1, $w_\sigma \in \mathbb{F}_m$ is the unique fixed point of $\boldsymbol{T}_\sigma$. This implies that $w^* = w_\sigma$, which in functional form becomes

$$\sup_{a \in \Gamma(x)} \left\{ r(x, a) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w^*[F(x, a, z)]\mathscr{P}(\mathrm{d}z)\right)\right\} = w^*(x)$$
$$= w_\sigma(x) = r(x, \sigma(x)) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w^*[F(x, \sigma(x), z)]\mathscr{P}(\mathrm{d}z)\right)$$

for each $x \in \mathbb{S}$. This equality says precisely that σ is optimal, since it attains the supremum of the objective function. □

Proposition A.4.3 proves part (ii) of Theorem 6.5.1, and Lemmas A.4.4 and A.4.5 prove the stated result of part (iii) in Theorem 6.5.1.

In the sequel, we denote by $i\mathbb{F}_c$ the collection of all increasing functions in $\mathbb{F}_c$.

Lemma A.4.6. *The set $i\mathbb{F}_c$ is a closed subset of $(\mathbb{F}, d_\kappa^\phi)$.*

Proof. To see this, pick an arbitrary sequence $\{f_n\}_{n \in \mathbb{N}_0} \subset i\mathbb{F}_c$ such that $d_\kappa^\phi(f_n, f^*) \to 0$ as $n \to \infty$. Recalling that the metric space $(\mathbb{F}, d_\kappa^\phi)$ is complete, so f^* in $\mathbb{F}_c$. Invoking the Footnote 18 in Remark A.4.1, the convergence in $\mathbb{F}_c$ implies pointwise convergence; that is, $f_n(x) \to f^*(x)$ for each $x \in \mathbb{S}$.

Take and fix any two points x and x' in $\mathbb{S}$ with $x \leqslant x'$, we have

$$f^*(x) - f^*(x') = (f^*(x) - f_n(x)) + (f_n(x) - f_n(x')) + (f_n(x') - f^*(x')).$$

By the increasing monotonicity of f_n, we know that $f_n(x) - f_n(x') \leqslant 0$ for every $n \in \mathbb{N}_0$. In this connection, taking the limit on both sides of the aforementioned inequality yields

$$\lim_{n\to\infty} (f^*(x) - f^*(x')) = \lim_{n\to\infty} (f_n(x) - f_n(x')) \leqslant 0.$$

This means that f^* is an increasing function and hence, f^* is in $i\mathbb{F}_c$. It then follows that $i\mathbb{F}_c$ is a closed subset of $\mathbb{F}_c$. □

Proof of Proposition 6.6.1. Recalling the proof of Proposition A.4.2, the Bellman operator $\boldsymbol{T}$ maps $\mathbb{F}_c$ to itself and has a unique fixed point v^* in $\mathbb{F}_c$. By Lemma A.4.6, since $i\mathbb{F}_c$ is a closed subset of $\mathbb{F}_c$, we only need to show that $\boldsymbol{T}$ leaves $i\mathbb{F}_c$ invariant (i.e., $\boldsymbol{T}(i\mathbb{F}_c) \subset i\mathbb{F}_c$).

To do so, take any x and x' in $\mathbb{S}$ with $x \leqslant x'$ and fix $w \in i\mathbb{F}_c$. Let σ be w-greedy policy and let $a^* = \sigma(x)$.

Then, we have

$$\begin{aligned}
\boldsymbol{T}w(x) &= r(x,a^*) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x,a^*,z)]\mathscr{P}(\mathrm{d}z)\right) \\
&\leqslant r(x',a^*) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x',a^*,z)]\mathscr{P}(\mathrm{d}z)\right) \\
&\leqslant \sup_{a\in\Gamma(x')}\left\{(x',a) + \beta\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w[F(x',a,z)]\mathscr{P}(\mathrm{d}z)\right)\right\} =: \boldsymbol{T}w(x'),
\end{aligned}$$

where the first inequality is derived from the fact that the mappings $r(\cdot,a)$, $F(\cdot,a,z)$, w, integral operator, ϕ and its inverse ϕ^{-1} are all monotone increasing, and the second inequality follows from the assumption that Γ is increasing (i.e., $\Gamma(x) \subset \Gamma(x')$ whenever $x \leqslant x'$). According to the above inequality, we conclude that $\boldsymbol{T}w$ is monotone increasing and in $i\mathbb{F}_c$, and hence so is v^*. □

In the following, we denote by $\mathcal{C}i\mathbb{F}_c$ the set of all concave functions in $i\mathbb{F}_c$, where the latter is endowed as before with the complete metric d_κ^ϕ.

Lemma A.4.7. *The set $\mathcal{C}i\mathbb{F}_c$ is a closed subset of $i\mathbb{F}_c$.*

Proof. To do so, take any sequence $\{f_n\}_{n\in\mathbb{N}_0} \subset \mathcal{C}i\mathbb{F}_c$ converging to some arbitrary $f^* \in i\mathbb{F}_c$. Invoking Footnote 18, the convergence in $(i\mathbb{F}_c, d_\kappa^\phi)$ implies the pointwise convergence; that is, $\lim_{n\to\infty} \|\phi \circ f_n - \phi \circ f^*\|_\kappa = 0$ implies $\lim_{n\to\infty} f_n(x) = f^*(x)$ for each $x \in \mathbb{S}$.

Now, pick any two points x and x' in $\mathbb{S}$, and any $\lambda \in [0,1]$. Since $\{f_n\}_{n\in\mathbb{N}_0} \subset \mathcal{C}i\mathbb{F}_c$, we have

$$f_n\left(\lambda x + (1-\lambda)x'\right) \geqslant \lambda f_n(x) + (1-\lambda)f_n(x')$$

for each $n \in \mathbb{N}_0$. Alternatively, $f_n\left(\lambda x + (1-\lambda)x'\right) - \lambda f_n(x) - (1-\lambda)f_n(x') \geqslant 0$ hold for all $n \in \mathbb{N}_0$. By virtue of the pointwise convergence, taking the limit on the above inequality yields

$$\begin{aligned}
&f^*\left(\lambda x + (1-\lambda)x'\right) - \lambda f^*(x) - (1-\lambda)f^*(x') \\
&= \lim_{n\to\infty}\left[f_n\left(\lambda x + (1-\lambda)x'\right) - \lambda f_n(x) - (1-\lambda)f_n(x')\right] \geqslant 0,
\end{aligned}$$

namely, $f^*\left(\lambda x + (1-\lambda)x'\right) \geqslant \lambda f^*(x) + (1-\lambda)f^*(x')$. This means that f^* is concave and in $\mathcal{C}i\mathbb{F}_c$, and hence, $\mathcal{C}i\mathbb{F}_c$ is closed in $i\mathbb{F}_c$. □

Proof of Proposition 6.6.2. By Proposition 6.6.1, $\boldsymbol{T}\colon i\mathbb{F}_c \to i\mathbb{F}_c$ and v^* is in $i\mathbb{F}_c$. We wish to show additionally that $v^* \in \mathcal{C}i\mathbb{F}_c$. Analogous to the proof of Proposition 6.6.1, since $\mathcal{C}i\mathbb{F}_c$ is a closed subset of $i\mathbb{F}_c$, it suffices to show that $\boldsymbol{T}$ maps $\mathcal{C}i\mathbb{F}_c$ into itself. To do so, take and fix any $w \in \mathcal{C}i\mathbb{F}_c$. Since $\boldsymbol{T}w \in i\mathbb{F}_c$, it remains only to show that $\boldsymbol{T}w$ is also concave.

Let x and x' in $\mathbb{S}$, and let $\lambda \in [0,1]$. Set $x'' = \lambda x + (1-\lambda)x'$. Let σ be a w-greedy policy, let $a := \sigma(x)$ and $a' = \sigma(x')$. We now define $a'' := \lambda a + (1-\lambda)a'$. By the convexity of $\operatorname{gr}\Gamma$, $a'' \in \Gamma(x'')$. It follows that

$$\boldsymbol{T}w(x'') \geqslant r(x'', a'') + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x'', a'', z)\right] \mathscr{P}(\mathrm{d}z)\right).$$

Consider the two terms on the right-hand side. Regarding the first term, by Condition 2, we know that

$$r(x'', a'') \geqslant \lambda r(x, a) + (1-\lambda) r(x', a').$$

Regarding the second term, by Condition 4, it follows that

$$F(x'', a'', z) \geqslant \lambda F(x, a, z) + (1-\lambda) F(x', a', z), \qquad (z \in \mathbb{Z}),$$

and by the fact that $w \in \mathcal{C}i\mathbb{F}_c$ is increasing and concave, we have

$$\begin{aligned} w\left[F(x'', a'', z)\right] &\geqslant w\left[\lambda F(x, a, z) + (1-\lambda) F(x', a', z)\right] \\ &\geqslant \lambda w\left[F(x, a, z)\right] + (1-\lambda) w\left[F(x', a', z)\right], \qquad (z \in \mathbb{Z}). \end{aligned}$$

It then follows that

$$\begin{aligned} &\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x'', a'', z)\right] \mathscr{P}(\mathrm{d}z)\right) \\ &\geqslant \phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ \left(\lambda w\left[F(x, a, z)\right] + (1-\lambda) w\left[F(x', a', z)\right]\right) \mathscr{P}(\mathrm{d}z)\right) \\ &\geqslant \lambda\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x, a, z)\right] \mathscr{P}(\mathrm{d}z)\right) + (1-\lambda)\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x', a', z)\right] \mathscr{P}(\mathrm{d}z)\right), \end{aligned}$$

where the first inequality follows from the monotonicity of ϕ and ϕ^{-1}, and the second follows from Condition 3.

To sum up, we obtain that

$$\begin{aligned} \boldsymbol{T}w(x'') &\geqslant \lambda\left\{r(x, a) + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x, a, z)\right] \mathscr{P}(\mathrm{d}z)\right)\right\} \\ &\qquad + (1-\lambda)\left\{r(x', a') + \beta\,\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ w\left[F(x', a', z)\right] \mathscr{P}(\mathrm{d}z)\right)\right\} \\ &= \lambda \boldsymbol{T}w(x) + (1-\lambda)\boldsymbol{T}w(x'). \end{aligned}$$

Hence, $\boldsymbol{T}w$ is concave on $\mathbb{S}$ and $\boldsymbol{T}w \in \mathcal{C}i\mathbb{F}_c$, which implies that v^* is concave.

This completes the proof. □

For $a \in \mathbb{R}_+$, we define

$$W(a) := \phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ v^* \left[f(a,z)\right] \mathscr{P}(\mathrm{d}z)\right). \tag{A.41}$$

In what follows, we assume that all conditions in Assumptions 6.6.1 and 6.6.2 hold true.

Lemma A.4.8. *The function $W(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined in (A.41) is continuous, increasing and concave. In addition, it satisfies:*

$$W'_+(0) := \lim_{\Delta x \to 0^+} \frac{W(0+\Delta x) - W(0)}{\Delta x} = \infty, \tag{A.42}$$

where W'_+ denotes the right-hand derivative of W which exists everywhere by concavity.

Proof of Lemma A.4.8. Given that the value function $v*$ lies in $\mathcal{C}i\mathbb{F}_c$, the first part follows from Lemma A.4.2 and the monotonicity and concavity of $\boldsymbol{M}$, along with the proofs of Propositions 6.6.1 and 6.6.2.

It remains to verify (A.42). To do so, take any sequence $x_n \to 0^+$ as $n \to \infty$. By the facts that $u(0) = 0$ and $f(0,z) = 0$ for all $z \in \mathbb{Z}$, we obtain that $v^*(0) = 0$ and hence $W(0) = 0$. For $x_n > 0$ ($\forall n \in \mathbb{N}$), we thus have

$$\begin{aligned}\frac{W(x_n) - W(0)}{x_n} &= \frac{\phi^{-1}\left(\int_{\mathbb{Z}} \phi \circ v^* \left[f(x_n,z)\right] \mathscr{P}(\mathrm{d}z)\right)}{x_n} \\ &= \phi^{-1}\left(\int_{\mathbb{Z}} \phi\left(\frac{v^*\left[f(x_n,z)\right]}{x_n}\right) \mathscr{P}(\mathrm{d}z)\right) \\ &\geqslant \phi^{-1}\left(\int_{\mathbb{Z}} \phi\left(\frac{u\left[f(x_n,z)\right]}{x_n}\right) \mathscr{P}(\mathrm{d}z)\right).\end{aligned}$$

Since $x_n > 0$ and $f(x_n,z) > f(0,z) = 0$ by the strictly increasing property of $f(\cdot,z)$, we then obtain from the chain rule that for each $z \in \mathbb{Z}$,

$$\frac{u\left[f(x_n,z)\right]}{x_n} = \frac{u\left[f(x_n,z)\right] - u(0)}{f(x_n,z) - f(0,z)} \cdot \frac{f(x_n,z) - f(0,z)}{x_n},$$

and taking the limit on the above equality yields

$$\lim_{n\to\infty} \frac{u\left[f(x_n,z)\right]}{x_n} = u'_+(f(0,z)) f'_+(0,z). \tag{A.43}$$

Notice that the convergence in (A.43) is monotone increasing as $x_n \downarrow 0$, since $u(\cdot)$ and $f(\cdot,z)$ are concave and $f(\cdot,z)$ is strictly increasing for all $z \in \mathbb{Z}$.

In light of that, by the monotone convergence theorem, we get

$$\lim_{n\to\infty}\phi^{-1}\left(\int_{\mathbb{Z}}\phi\left(\frac{u\left[f(x_n,z)\right]}{x_n}\right)^{\theta}\mathscr{P}(\mathrm{d}z)\right)=\phi^{-1}\left(\int_{\mathbb{Z}}\phi\left(\lim_{n\to\infty}\frac{u\left[f(x_n,z)\right]}{x_n}\right)\mathscr{P}(\mathrm{d}z)\right)$$
$$=\phi^{-1}\left(\int_{\mathbb{Z}}\phi\left[u'_+(f(0,z))f'_+(0,z)\right]\mathscr{P}(\mathrm{d}z)\right),$$

and it follows that

$$W'_+(0)=\lim_{n\to\infty}\frac{W(x_n)-W(0)}{x_n}\geqslant\phi^{-1}\left(\int_{\mathbb{Z}}\phi\left[u'_+(f(0,z))f'_+(0,z)\right]\mathscr{P}(\mathrm{d}z)\right).$$

From Assumption 6.6.2, we know that $f'_+(0,z)=\lim_{x_n\downarrow 0}f'(x_n,z)>0$. Hence, this together with the fact that $u'_+(f(0,z))=\lim_{c\downarrow 0}u'(c)=\infty$ yields the stated result. □

Lemma A.4.9. *If $\sigma\in\Sigma$ is v^*-greedy, then $\sigma(x)\in(0,x)$ for every $x>0$.*

Proof of Lemma A.4.9. Let $x>0$. Define

$$\hat{W}(x,y):=u(x-y)+\beta W(y),\qquad (x\in(0,\infty),y\in[0,x]).$$

Since σ is v^*-greedy, we have

$$\hat{W}(x,y)=u(x-y)+\beta W(y)\leqslant u(x-\sigma(x))+\beta W(\sigma(x))=:\hat{W}(x,\sigma(x))\tag{A.44}$$

for every $y\in[0,x]$.

If $\sigma(x)=x$, then $\sigma(x)>0$. So, suppose that $\sigma(x)<x$. Let $y\in(\sigma(x),x)$. It then follows that

$$\beta\frac{W(y)-W(\sigma(x))}{y-\sigma(x)}\leqslant\frac{u(x-\sigma(x))-u(x-y)}{y-\sigma(x)}.$$

Letting $y\downarrow\sigma(x)$ yields

$$\beta W'_+(\sigma(x))\leqslant u'_+(x-y)=u'(x-y),$$

which implies $\sigma(x)>0$ by (A.42) and the differentiability of u on $(0,\infty)$ (namely, $u'(x-y)<\infty$ since $y<x$).

It remains to check $\sigma(x)<x$. Let $y\in(0,\sigma(x))$. Making use of (A.44) again gives

$$\beta\frac{W(y)-W(\sigma(x))}{y-\sigma(x)}\geqslant\frac{u(x-y)-u(x-\sigma(x))}{\sigma(x)-y},$$

and letting $y\uparrow\sigma(x)$ yields

$$\beta W'_-(\sigma(x))\geqslant u'_+(x-\sigma(x)),$$

where W'_- denotes the left-hand derivative of W, which exists everywhere by concavity. By virtue of the assumption $u'(0) = \infty$, it follows that $x - \sigma(x) > 0$, as was to be shown. □

Proof of Proposition 6.6.3. Given Lemmas A.4.8 and A.4.9, the way of showing Proposition 6.6.3 proceeds along the same lines as the proofs of Proposition 12.1.18 and Corollary 12.1.19 in Stachurski (2009). □

Given a sufficiently large constant $\eta > 0$, we define a function $w\colon (0, \eta) \to \mathbb{R}_+$ through

$$w(a) := \int_{\mathbb{Z}} \phi \circ v^* \left[f(a, z) \right] \mathscr{P}(\mathrm{d}z). \tag{A.45}$$

Lemma A.4.10. *The function $w(\cdot)$ defined in (A.45) is differentiable on $(0, \eta)$ for some positive constant η, and the derivative of the function $w(\cdot)$ is given by*

$$w'(a) = \int_{\mathbb{Z}} \phi' \circ v^* \left[f(a, z) \right] (v^*)' \left[f(a, z) \right] f'(a, z) \mathscr{P}(\mathrm{d}z) \qquad (0 < a < \eta). \tag{A.46}$$

Proof of Lemma A.4.10. Fix $\eta > 0$. For the sake of simplicity, let $v := v^*$ be the value function. Denote by $F(a, z) := \phi \circ v^* \left[f(a, z) \right]$ for each $a \in (0, \eta)$ and every $z \in \mathbb{Z}$. By the continuity of ϕ and v and condition (ii) in Assumption 6.6.2, we know that $F(a, \cdot)$ is measurable for each $a \geqslant 0$.

Step 1. Take and fix any $a \in (0, \eta)$. Let us assume for now that

(i) $w(a) = \int_{\mathbb{Z}} F(a, z) \mathscr{P}(\mathrm{d}z) < +\infty$;

(ii) for each $z \in \mathbb{Z}$, the function $F(\cdot, z)$ is differentiable at a;

(iii) there exists a positive valued measurable function $\ell\colon \mathbb{Z} \to (0, \infty)$ such that $\int_{\mathbb{Z}} \ell(z) \mathscr{P}(\mathrm{d}z) < +\infty$, and that for all a_1, a_2 in a neighborhood of a and each $z \in \mathbb{Z}$, the following inequality holds

$$|F(a_1, z) - F(a_2, z)| \leqslant \ell(z) \, |a_1 - a_2| \,. \tag{A.47}$$

In the sequel, invoking (A.47) gives us

$$|w(a_1) - w(a_2)| \leqslant \int_{\mathbb{Z}} |F(a_1, z) - F(a_2, z)| \, \mathscr{P}(\mathrm{d}z) \leqslant L \cdot |a_1 - a_2| \,,$$

for all a_1, a_2 in a neighborhood of a, where $L := \int_{\mathbb{Z}} \ell(z) \mathscr{P}(\mathrm{d}z) > 0$.[22]

[22] In fact, given a measurable function $g\colon \mathbb{Z} \to \mathbb{R}$, if integrand g is strictly positive (i.e., $g(z) > 0$ for all $z \in \mathbb{Z}$), then its integral is still strictly positive, i.e., $\int_{\mathbb{Z}} g(z) \mathscr{P}(\mathrm{d}z) > 0$. To see that it is so, observe that $g(z) > 0$ for each $z \in \mathbb{Z}$ and then $\mathbb{Z} = \cup_{n=1}^{\infty} \{z \in \mathbb{Z}\colon g(z) > \frac{1}{n}\}$. Hence, there must exist some positive integer $N > 0$ such that $A_N := \{z \in \mathbb{Z}\colon g(z) > \frac{1}{N}\}$ has positive probability measure (i.e., $\mathscr{P}(A_N) > 0$). We thus have $\int_{\mathbb{Z}} g(z) \mathscr{P}(\mathrm{d}z) \geqslant \int_{A_N} g(z) \mathscr{P}(\mathrm{d}z) \geqslant \int_{A_N} \frac{1}{N} \mathscr{P}(\mathrm{d}z) = \frac{\mathscr{P}(A_N)}{N} > 0$, as desired. Alternatively, by virtue of Chebyshev's inequality (i.e., $\mathscr{P}(\{z \in \mathbb{Z}\colon g(z) > t\}) \leqslant \frac{1}{t} \int_{\mathbb{Z}} g(z) \mathscr{P}(\mathrm{d}z)$ for any real number $t > 0$), we also obtain the identical result by setting $t = 1/N$.

Since $w(a)$ is finite by condition (i), it follows that $w(\cdot)$ is well defined, finite valued and Lipschitz continuous in the neighborhood of a.

For $\tau \neq 0$, we consider the ratio

$$G_\tau(a, z; h) := \frac{F(a + \tau h, z) - F(a, z)}{\tau}$$

for all $h \in \mathbb{R}$.

It follows from (A.47) that $|G_\tau(a, z; h)| \leqslant L \cdot |h|$ for any fixed $h \in \mathbb{R}$, and hence the differentiability of $F(\cdot, z)$ in the sense of Fréchet guaranteed by condition (ii) yields

$$\lim_{\tau \to 0} G_\tau(a, z; h) = d_G F(a, z; h)$$

for each $z \in \mathbb{Z}$, where $d_G F(a, z; h)$ denotes the Gâteaux differential of $F(\cdot, z)$ at the point $a \in (0, \eta)$ in the direction $h \in \mathbb{R}$ and is defined by $d_G F(a, z; h) = F'(a, z)h$. Since $F(\cdot, z)$ is (Fréchet) differentiable at a, $d_G F(a, z; h)$ is linear in h, and $F'(a, z) := \frac{\partial}{\partial a} F(a, z)$ is well defined and exists as the partial Fréchet (or Gâteaux) derivative of $F(\cdot, z)$ at the point $a \in (0, \eta)$.

Hence, applying the Lebesgue Dominated Convergence theorem gives us

$$\lim_{\tau \to 0} \int_{\mathbb{Z}} G_\tau(a, z; h) \mathscr{P}(\mathrm{d}z) = \int_{\mathbb{Z}} \lim_{\tau \to 0} G_\tau(a, z; h) \mathscr{P}(\mathrm{d}z) = \int_{\mathbb{Z}} d_G F(a, z; h) \mathscr{P}(\mathrm{d}z);$$

that is, $d_G w(a; h) = \int_{\mathbb{Z}} d_G F(a, z; h) \mathscr{P}(\mathrm{d}z)$ for all $h \in \mathbb{R}$.

In order to prove that $w(\cdot)$ is Fréchet differentiable at a, it suffices to show that $d_G w(a; h)$ is linear in h, since $w(a)$ is locally Lipschitz continuous.[23] Evidently, the linearity of $d_G F(a, z; h)$ implies the linearity of $d_G w(a; \cdot)$.

It then follows that $w(\cdot)$ is Fréchet differentiable at a with its Fréchet (or Gâteaux) derivative is given by

$$\begin{aligned} w'(a) := \frac{\partial}{\partial a} w(a) &= \int_{\mathbb{Z}} \frac{\partial}{\partial a} F(a, z) \mathscr{P}(\mathrm{d}z) \\ &= \int_{\mathbb{Z}} \phi' \circ v^* \left[f(a, z) \right] (v^*)' \left[f(a, z) \right] f'(a, z) \mathscr{P}(\mathrm{d}z) \end{aligned}$$

where the last equality is derived from the chain rule.

Since a is chosen arbitrarily from $(0, \eta)$, we obtain the stated result.

Step 2. Now, let us turn to check those conditions that we assumed before.

[23] Clearly, Fréchet differentiability implies Gâteaux differentiability. The converse of that is not necessarily true. However, for locally Lipschitz continuous mappings with the domain being finite dimensional, the concepts of Fréchet and Gâteaux differentiability do coincide (see, e.g., Proposition 3.4.2 of Schirotzek (2007) Nonsmooth Analysis). That is, given that $\mathbb{X}$ is finite dimensional, if $g \colon \mathbb{X} \to \mathbb{Y}$ is Gâteaux differentiable at x_0 and Lipschitz continuous in a neighborhood of x_0, then $g(x)$ is differentiable at x_0 in the sense of Fréchet, or simply differentiable at x_0.

Regarding condition (i), observe that the value function v lies in $\mathcal{C}i\mathbb{F}_c$, and hence $\phi \circ v$ is κ-bounded. It follows that for each $a \in (0, \eta)$,

$$\begin{aligned}
\int_{\mathbb{Z}} F(a, z)\mathscr{P}(\mathrm{d}z) &= \int_{\mathbb{Z}} \phi \circ v^* \left[f(a,z)\right] \mathscr{P}(\mathrm{d}z) \\
&\leqslant \int_{\mathbb{Z}} \kappa \left[f(a,z)\right] \mathscr{P}(\mathrm{d}z) \\
&\leqslant \sup_{a \in [0,\eta]} \int_{\mathbb{Z}} \kappa \left[f(a,z)\right] \mathscr{P}(\mathrm{d}z) \\
&\leqslant N_\kappa \cdot \kappa(\eta)
\end{aligned}$$

for some constant N_κ, where the last inequality follows from (6.7) in Assumption 6.4.5.

Since η is fixed and $\kappa(\cdot)$ is real-valued, the term $\int_{\mathbb{Z}} F(a, z)\mathscr{P}(\mathrm{d}z)$ is finite, as was to be shown.

Regarding condition (ii), we first observe that $f(\cdot, z)$ is positive and differentiable on $(0, \infty)$ for each $z \in \mathbb{Z}$ by Assumption 6.6.2, $v(\cdot)$ is differentiable on $(0, \infty)$ by Proposition 6.6.3, and ϕ is also differentiable on $(0, \infty)$. From the fact that the composition of differentiable functions is differentiable, we thus know that $F(\cdot, z)$ is differentiable on $(0, \infty)$ for each $z \in \mathbb{Z}$, as desired.

It remains to show that condition (iii) holds true. Let $B(a, r)$ denote an open neighborhood (ball) of a with a radius $r > 0$ such that $B(a, r) \subset (0, \infty)$.

Now pick any a_1, a_2 in a neighbourhood $B(a, r) \subset (0, \infty)$ of a.

By virtue of Mean Value theorem (MVT), it follows from condition (ii) that there exists an $a' \in (\min\{a_1, a_2\}, \max\{a_1, a_2\})$ such that

$$|F(a_1, z) - F(a_2, z)| = \frac{\partial}{\partial a} F(a', z) \cdot |a_1 - a_2| \,.$$

Note that the function $F(\cdot, z)$ is concave for each z, since the maps $f(\cdot, z)$, $v(\cdot)$ and $\phi(\cdot)$ are all increasing and concave, and since the composition of increasing concave functions is concave. Hence, by the concavity of $F(\cdot, z)$, we know that for each $z \in \mathbb{Z}$, $F(\cdot, z)$ has a monotonic decreasing derivative, i.e., $\frac{\partial}{\partial a} F(a', z) \leqslant \frac{\partial}{\partial a} F(a'', z)$ whenever $a'' \leqslant a'$.

In this connection, we then have

$$\sup_{a' \in B(a,r)} \frac{\partial}{\partial a} F(a', z) = \frac{\partial}{\partial a} F(a'', z) =: \ell(z)$$

for each $z \in \mathbb{Z}$, with $a'' := a - r$. It then follows that

$$|F(a_1, z) - F(a_2, z)| = \frac{\partial}{\partial a} F(a', z) \cdot |a_1 - a_2| \leqslant \ell(z) \cdot |a_1 - a_2|$$

for any $a_1, a_2 \in B(a, r)$.

Finally, we need only check that $\ell(\cdot)$ is positive and $\int_{\mathbb{Z}} \ell(z)\mathscr{P}(\mathrm{d}z) < +\infty$.

The positivity of $\ell(z)$ is obvious. Indeed, $f(\cdot, z)$, $v(\cdot)$ and $\phi(\cdot)$ are all strictly increasing, and so is $F(\cdot, z)$, which implies that $\frac{\partial}{\partial a}F(a, z) > 0$ for any $a > 0$ and each $z \in \mathbb{Z}$.

To show that $\int_{\mathbb{Z}} \ell(z)\mathscr{P}(\mathrm{d}z)$ is finite, it is sufficient to show that the term

$$\int_{\mathbb{Z}} \frac{\partial}{\partial a} F(a'', z)\mathscr{P}(\mathrm{d}z)$$

is finite for all $a'' \in (0, a) \subset (0, \eta)$.

To this end, we claim that if a strictly increasing and concave function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $(0, \infty)$, then we have

$$g'(x) := \frac{\mathrm{d}}{\mathrm{d}x} g(x) \leqslant \frac{g(x)}{x} \qquad (x > 0).$$

To see that this is the case, consider the right-hand side term of the preceding inequality. Making use of Lagrange's MVT, for a given fixed $x > 0$, we know that there must exist a $\xi \in (0, x)$ such that

$$g'(\xi) = \frac{g(x) - g(0)}{x - 0},$$

and it then follows that $g'(\xi) = [g(x) - g(0)]/[x - 0] \leqslant g(x)/x$, since $g(0) \geqslant 0$. On the other hand, by virtue of the concavity of g, we have that $g'(x) \leqslant g'(\xi)$, since g has the monotonic decreasing derivative and $\xi < x$. Hence, combining these two facts yields $g'(x) \leqslant g(x)/x$ for all $x > 0$, as was to be shown.

Since the functions $f(\cdot, z)$ and $\phi \circ v(\cdot)$ do possess the aforementioned properties of g, invoking the above claim now yields

$$\frac{\partial}{\partial a} f(a, z) \leqslant \frac{f(a'', z)}{a''} \qquad (z \in \mathbb{Z}) \tag{A.48}$$

for all $a'' > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} \phi \circ v(x) \leqslant \frac{\phi \circ v(x)}{x} \leqslant \frac{\|\phi \circ v\|_\kappa \cdot \kappa(x)}{x}$$

for all $x > 0$, where the last inequality follows from the fact that $\phi \circ v$ is κ-bounded. In other words, for each $x > 0$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}x} \phi \circ v(x) \cdot x \leqslant \|\phi \circ v\|_\kappa \cdot \kappa(x). \tag{A.49}$$

As a result, for each $a'' \in (0, \eta)$,

$$\begin{aligned}\frac{\partial}{\partial a}F(a'', z) = \frac{\partial}{\partial a}\phi \circ v\left[f(a'', z)\right] &= \frac{\partial}{\partial f}\phi \circ v\left[f(a'', z)\right] \cdot \frac{\partial}{\partial a}f(a'', z)\\ &\leqslant \frac{\partial}{\partial f}\phi \circ v\left[f(a'', z)\right] \cdot \frac{f(a'', z)}{a''}\\ &\leqslant \frac{\|\phi \circ v\|_\kappa \cdot \kappa\left[f(a'', z)\right]}{a''} \qquad (z \in \mathbb{Z}),\end{aligned}$$

where the last two inequalities follow directly from (A.48) and (A.49), respectively.

Therefore, for any $a'' \in (0, \eta)$ with fixed sufficiently large η, invoking (6.7), we obtain

$$\begin{aligned}\int_{\mathbb{Z}} \frac{\partial}{\partial a}F(a'', z)\mathscr{P}(\mathrm{d}z) &\leqslant \int_{\mathbb{Z}} \frac{\|\phi \circ v\|_\kappa \cdot \kappa\left[f(a'', z)\right]}{a''}\mathscr{P}(\mathrm{d}z)\\ &\leqslant \frac{\|\phi \circ v\|_\kappa}{a''} \cdot N_\kappa \cdot \kappa(\eta) < +\infty.\end{aligned}$$

This completes the proof. □

Lemma A.4.11. *The function $W(\cdot)$ defined in (A.41) is differentiable on $(0, \eta)$ for some positive constant η, and the derivative of the function $W(\cdot)$ is given by*

$$\begin{aligned}W'(a) = \left(\phi^{-1}\right)' &\left(\int_{\mathbb{Z}} \phi \circ v^* \left[f(a,z)\right] \mathscr{P}(\mathrm{d}z)\right)\\ &\cdot \int_{\mathbb{Z}} \phi' \circ v^* \left[f(a,z)\right] (v^*)' \left[f(a,z)\right] f'(a,z)\mathscr{P}(\mathrm{d}z) \qquad (0 < a < \eta).\end{aligned}$$

Proof of Lemma A.4.11. Observe that $W(a) = \phi^{-1}\left[w(a)\right]$ for each $a \in (0, \eta)$. Invoking Footnote 22, we know that $w(a) > 0$ whenever $a > 0$. Since the function ϕ^{-1} is differentiable on $(0, \infty)$, it follows from Lemma A.4.10 that $W(\cdot) = \phi^{-1}\left[w(\cdot)\right]$ is differentiable on $(0, \eta)$. In addition, making use of the chain rule now, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}a}W(a) = \frac{\mathrm{d}}{\mathrm{d}w}\phi^{-1}\left[w(a)\right] \cdot \frac{\mathrm{d}}{\mathrm{d}a}w(a)$$

together with (A.46), gives us the desired result. □

Proof of Proposition 6.6.4. Claim (i) follows directly from Proposition 6.6.3 and Lemma A.4.11.

The proof of claim (ii) is omitted, because the way of showing that proceeds along the same lines as the proof of part (b) in Theorem 2 of Bäuerle and Jaśkiewicz (2018). □

Proof of Lemma 6.6.1. To see this, take any sequence $\{f_n\}_{n\in\mathbb{N}} \subset bm(\mathbb{S})_+$ converging to some arbitrary $f^* \in bm(\mathbb{S})_+$.

In order to prove the stated result of Lemma 6.6.1, we first need to show that

$$|f_n(x) - f^*(x)| \leqslant L_n \cdot |\phi[f_n(x)] - \phi[f^*(x)]|$$

for each $x \in \mathbb{S}$ and some constant $L_n \geqslant 0$. Note that the value of L_n depends on the function f_n.

We can treat the constant L_n as a Lipschitz constant, and hence by the bijectivity of ϕ, we observe that showing the above inequality is equivalent to proving

$$|f_n(x) - f^*(x)| = \left| \phi^{-1}\left(\phi[f_n(x)]\right) - \phi^{-1}\left(\phi[f^*(x)]\right) \right| \leqslant L_n \cdot |\,\phi[f_n(x)] - \phi[f^*(x)]|\,.$$

In light of that, making use of Mean Value theorem (MVT), there is a constant $c_n \in (0, M_n)$ such that

$$\begin{aligned} |f_n(x) - f^*(x)| &= \left| \phi^{-1}\left(\phi \circ f_n(x)\right) - \phi^{-1}\left(\phi \circ f^*(x)\right) \right| \\ &= (\phi^{-1})'(c_n) \cdot |\phi \circ f_n(x) - \phi \circ f^*(x)|\,. \end{aligned}$$

By the boundedness of f_n and f, the constant $M_n > 0$ can be taken such that $M_n := 1 + \sup\{\|\phi \circ f_n\|_\infty,\ \|\phi \circ f^*\|_\infty\}$. Thus, such a Lipschitz constant L_n exists and can be taken for any constant $L_n > (\phi^{-1})'(c_n) > 0$.

In addition, since the sequence $\{\phi \circ f_n\} \subset bm(\mathbb{S})_+$ converges to $\phi \circ f^*$ uniformly (as $d^{\phi}_{\infty}(f_n, f^*) \to 0$), we then have a positive constant $\bar{M} > 0$ such that $\bar{M} = \sup_{n \in \mathbb{N}} M_n = 1 + \sup_{n \in \mathbb{N}}\{\sup\{\|\phi \circ f_n\|_\infty,\ \|\phi \circ f^*\|_\infty\}\}$, and hence the corresponding $\bar{c} = \sup_{n \in \mathbb{N}} c_n$ is also bounded.

It then follows from the convexity of ϕ^{-1} and hence the monotonicity of $(\phi^{-1})'$ that

$$|f_n(x) - f^*(x)| \leqslant (\phi^{-1})'(\bar{c}) \cdot |\phi \circ f_n(x) - \phi \circ f^*(x)|$$

for each $x \in \mathbb{S}$. As x is chosen arbitrarily, taking the supremum over $x \in \mathbb{S}$ yields

$$\|f_n - f^*\|_\infty \leqslant L \cdot \|\phi \circ f_n - \phi \circ f^*\|_\infty$$

for some Lipschitz constant L such that $L \geqslant (\phi^{-1})'(\bar{c})$; namely, $d_\infty(f_n, f^*) \leqslant L \cdot d^{\phi}_{\infty}(f_n, f^*)$.

This completes the proof. $\square$

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