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Probability of ruin in discrete insurance risk model with dependent Pareto claims

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Abstract: We present basic properties and discuss potential insurance applications of a new class of probability distributions on positive integers with power law tails. The distributions in this class are zero-inflated discrete counterparts of the Pareto distribution. In particular, we obtain the probability of ruin in the compound binomial risk model where the claims are zero-inflated discrete Pareto distributed and correlated by mixture.

Keywords: Actuarial science, compound binomial risk model, dependence by mixture, discrete Pareto distribution, geometric distribution, heavy tail, mixture, power law, ruin probability, zero-altered distribution, zero-inflated distribution, zero-modified distribution

1 Introduction

Discrete heavy-tailed distributions are an important and active area in non-life insurance research and practice (see, e.g., [4, 5, 21, 29]). It is well-known that Pareto and Weibull distributions are used in insurance practice for modelling claim sizes. However, their theoretical implementation in collective risk models is nontrivial. We consider the compound binomial risk model

$$U_t = u + t - \sum_{i=1}^{t} X_i, \quad t \in \mathbb{N}_0 = \{0, 1, \dots\}, \tag{1}$$

introduced in [9]. The probability of ruin,

$$\psi(u) = \mathbb{P}(U_t < 0 \text{ for some } t > 0 | U_0 = u), \tag{2}$$

admits an explicit form when the claim amounts $\{X_i\}$ have *zero-modified geometric* (ZMG) distribution $ZMG(q, \rho)$. The latter is given by the probability mass function (PMF) $\mathbb{P}(X_i = k) = g(k)$, where

$$g(k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q)\rho(1 - \rho)^{k-1}, \quad k \in \mathbb{N}_0, \tag{3}$$

and δ_{kj} is the Kronecker delta function. In this case we have

$$\psi(u) = \min\left\{\frac{1 - q}{\rho}\left(\frac{1 - \rho}{q}\right)^{u+1}, 1\right\}, \tag{4}$$

see [34].

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In [8] the authors extended the formula (4) by using a mixing approach as in [1] and [6], and assuming that given $\Theta = \theta$, where Θ is a “mixing" random variable on $\mathbb{R}_+$, the claim amounts $\{X_i\}$ are independent, identically distributed (IID) zero-modified geometric $ZMG(q, \rho)$ with the *success probability*

$$\rho = e^{-\theta}. \tag{5}$$

In this set-up, [8] derived the probability of ruin (2) for three particular cases:

(i) For Θ having exponential distribution with parameter λ, given by the probability density function (PDF)

$$f(x) = \lambda e^{-\lambda x}, \quad x \in \mathbb{R}_+, \tag{6}$$

in which case the claim amounts have a zero-modified Yule distribution.
(ii) For Θ having gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$, given by the PDF

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \in \mathbb{R}_+. \tag{7}$$

In this case the claim amounts have the PMF

$$\mathbb{P}(X = k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q) \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j \lambda^\alpha}{(\lambda + j)^\alpha}, \quad k \in \mathbb{N}_0,$$

and the probability of ruin can be expressed in terms of incomplete gamma function.
(iii) For Θ having positive stable distribution with index $1/2$ (Lévy distribution), given by the PDF

$$f(x) = \frac{\tau}{2\sqrt{\pi x^3}} e^{-\frac{\tau^2}{4x}}, \quad x \in \mathbb{R}_+. \tag{8}$$

In this case the claim amounts have the PMF

$$\mathbb{P}(X = k) = (1 - q) \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-\tau\sqrt{j}}, \quad k \in \mathbb{N}_0,$$

and the ruin probability can be expressed in terms of complementary error special function.

The purpose of this note is two-fold. First, we point out that in the above set-up with discrete claims correlated by mixture and, conditionally on $\Theta = \theta$, having $ZMG(q, \rho)$ distribution, it is more convenient to assume that

$$\rho = 1 - e^{-\theta} \tag{9}$$

rather than (5) as in [8]. Thus, while in the set-up above the geometric *probability of success* is taken as $e^{-\theta}$, we use this expression for the *probability of failure*. Let us note that a geometric distribution with the probability of success given by (9) is a discrete version of an exponential one, since the geometric PMF can be derived as the difference of two consecutive exponential tails with parameter θ:

$$\mathbb{P}(X = k) = e^{-(k-1)\theta} - e^{-k\theta} = \left(1 - e^{-\theta}\right) \left(e^{-\theta}\right)^{k-1}, \quad k \in \mathbb{N} = \{1, 2, \ldots\}.$$

As shown below, this modification of the approach leads to convenient formulas for the probability of ruin as well for the tail probabilities (which were considered in Section 4.2 of [8]). As in [8], the mixing variable Θ will still be taken as exponential, gamma, or positive stable. However, with this choice of Θ, the resulting distributions of the claim amounts are generally quite different than those obtained by [8]:

(i) For Θ having exponential distribution with parameter λ, the claim amounts are zero-modified *discrete Pareto* (10) with tail index $\alpha = 1$, which is different than the Yule distribution (unless Θ is standard exponential with $\lambda = 1$).
(ii) For Θ having gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$, the claim amounts are zero-modified discrete Pareto (10).

(iii) For Θ having the Lévy stable distribution (8), the claim amounts have a zero-modified *discrete Weibull* distribution.

This brings us to the second motivation for this paper, which is the introduction of new classes of discrete probability models resulting from this mixing scheme. Namely, as shown in the sequel, when Θ is gamma distributed with the PDF (7) and, given $\Theta = \theta$, the claim amounts are IID $ZMG(q, \rho)$ with ρ as in (9), the PMF of the unconditional distribution of the claim amount X becomes

$$\mathbb{P}(X = k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q)\left\{\left(\frac{1}{1 + \frac{k-1}{\lambda}}\right)^{\alpha} - \left(\frac{1}{1 + \frac{k}{\lambda}}\right)^{\alpha}\right\}, \quad k \in \mathbb{N}_0. \tag{10}$$

We obtain a mixture of a point mass at zero with probability q and a heavy-tail, *discrete Pareto* (DP) distribution of [3], given by the PMF

$$\mathbb{P}(X = k) = \left(\frac{1}{1 + \frac{k-1}{\lambda}}\right)^{\alpha} - \left(\frac{1}{1 + \frac{k}{\lambda}}\right)^{\alpha}, \quad k \in \mathbb{N}, \tag{11}$$

with probability $1 - q$. Similarly, when Θ has a positive stable distribution with index $\alpha \in (0, 1)$, given by the Laplace transform (LT)

$$\phi(t) = e^{-\tau t^{\alpha}}, \quad t \in \mathbb{R}_+, \tag{12}$$

and, given $\Theta = \theta$, the claim amounts are IID $ZMG(q, \rho)$ with ρ as in (9), then the PMF of the claim amount X becomes

$$\mathbb{P}(X = k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q)\left\{e^{-\tau(k-1)^{\alpha}} - e^{-\tau k^{\alpha}}\right\}, \quad k \in \mathbb{N}_0. \tag{13}$$

We again obtain a mixture, this time involving a discrete version of Weibull distribution with parameter $\alpha \in (0, 1)$. Let us note that theory and applications of such zero-modified discrete distributions is an important area in distribution theory, with applications in manufacturing (see, e.g., [20]), econometrics (see, e.g., [24]), economics (see, e.g., [2, 16, 31]), and accident analysis (see, e.g., [22, 30]), among others. Such modifications, also known as *zero-adjusted*, *zero-altered*, or *zero-inflated* discrete distributions, have been developed for many standard discrete distributions to account for disproportionally large (or small) frequencies of zeroes observed in empirical data, compared with the standard models (see, e.g., [17], pp. 312-318). Popular models of this type include those based upon Poisson distribution (see, e.g., [10, 11, 13, 14, 23–26, 32]), generalized Poisson distribution (see, e.g., [12]), binomial distribution (see, e.g., [13]), geometric and negative binomial distributions (see, e.g., [2, 11, 14–16, 23, 31]), and logarithmic distribution (see, e.g., [18, 27]).

In the ruin theory literature, the binomial risk model has been developed in different directions (see, e.g., [7, 28, 35, 36]). Our new, zero-modified discrete Pareto and Weibull distributions may provide a useful addition to an actuary's statistical toolbox, going beyond modeling claim amounts of discrete type. We note that this mixing approach introduces a dependence structure that produces tractable results in a few instances that we analyze in this paper. Specifically, starting from classical ruin theory results for independent light-tail claims, we explore heavy-tailed scenarios with *conditionally* independent claims. In fact, the zero-modified DP model with the PMF (10) may be a useful heavy-tail model for the frequency of claims as well, as it can be extended to a continuous-time, discrete-valued stochastic process in the spirit of the classical Poisson process due to its fundamental property of infinite divisibility, established in the sequel.

The rest of the paper is organized as follows. In Section 2 we derive the probability of ruin in the above set-up within a compound binomial risk model with mixed zero-modified geometric claims, including the case where the claims are conditionally independent, zero-modified discrete Pareto. We exemplify our theory with a concrete example with real data from an insurance-reinsurance company. In turn, in Section 3, we focus on the zero-modified discrete Pareto model, which provided the best fit to the data. Here, we present basic information on this new stochastic model and develop its important properties, which should provide a useful reference for actuaries and others who use discrete stochastic models in their work.

2 Compound binomial risk model with mixed zero-modified geometric claims

Consider again the compound binomial risk model (1) where, given $\Theta = \theta$, the $\{X_i\}$ have ZMG distribution given by the PMF (3) with the success probability as in (9). To see why the latter condition is more convenient than the one given by (5), we first derive the PDF of the claim amount X. Let F_Θ be the cumulative distribution function (CDF) of the mixing variable Θ and let f_Θ be the corresponding PDF (if it exists). Clearly, $\mathbb{P}(X = 0) = q$, while for $k \geq 1$, we have:

$$\mathbb{P}(X = k) = \int_0^\infty \mathbb{P}(X = k|\Theta = \theta)dF_\Theta(\theta) = \int_0^\infty (1-q)(1-e^{-\theta})(e^{-\theta})^{k-1}dF_\Theta(\theta)$$

$$= (1-q)\left\{\int_0^\infty e^{-\theta(k-1)}dF_\Theta(\theta) - \int_0^\infty e^{-\theta k}dF_\Theta(\theta)\right\} = (1-q)\left\{\phi_\Theta(k-1) - \phi_\Theta(k)\right\},$$

where ϕ_Θ is the Laplace transform (LT) of the variable Θ. This leads to a convenient, general formula for the PMF of X:

$$\mathbb{P}(X = k) = q\delta_{k0} + (1-\delta_{k0})(1-q)\left\{\phi_\Theta(k-1) - \phi_\Theta(k)\right\}, \quad k \in \mathbb{N}_0. \tag{14}$$

Note that when Θ has a gamma distribution with the PDF (7), then the LT is given by

$$\phi_\Theta(t) = \left(\frac{1}{1+t/\lambda}\right)^\alpha, \quad t \in \mathbb{R}_+, \tag{15}$$

and the PMF of the claim amount X turns into that of the zero-modified discrete Pareto (ZMP) distribution, given by (10). Similarly, when Θ is positive stable with the LT (12), the claim amounts become conditionally

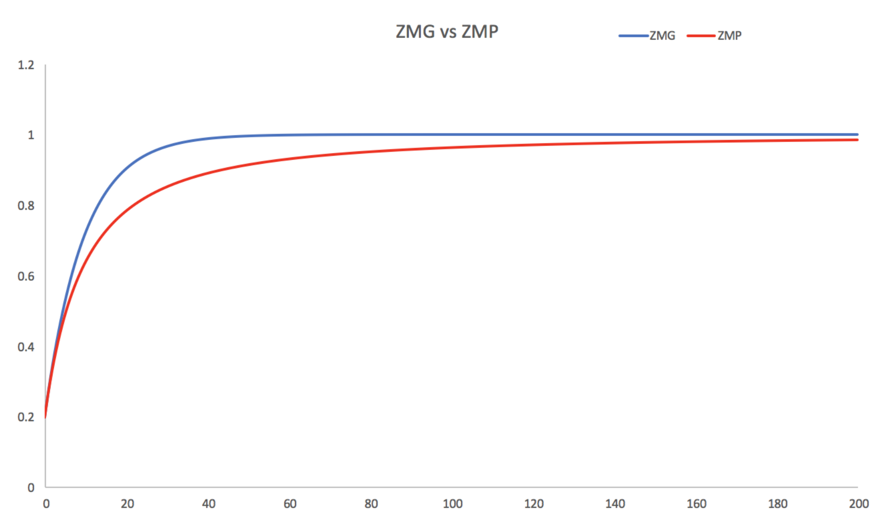

Figure 1: The CDFs under the ZMG and the ZMP models.

independent zero-modified discrete Weibull (13).

When comparing the ZMP and the ZMG models (see in Figures 1 and 2), we notice that for the same expectation of claims, the PMFs for both models have the same value of q when the zero claims occured, however, the PMF drops faster under ZMG model, displaying the heavier tail of the ZMP distribution.

Similar calculations show that that the CDF of the claim distribution in our set-up is given by

$$\mathbb{P}(X \leq x) = 1 - (1-q)\phi_\Theta(\lfloor x \rfloor), \quad x \in \mathbb{R}_+,$$

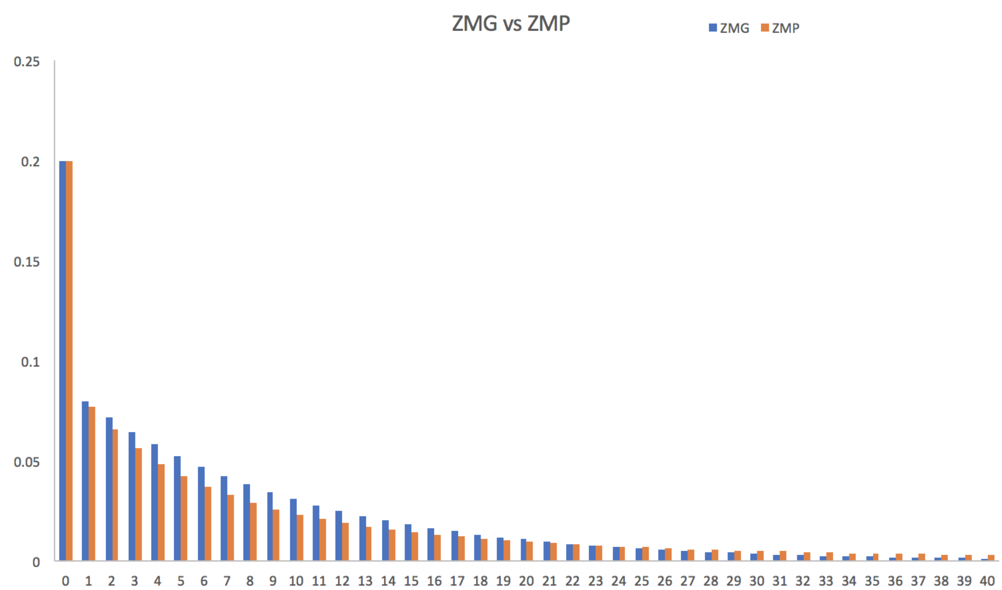

Figure 2: The PMFs under the ZMG and the ZMP models.

while the survival probability becomes

$$\mathbb{P}(X > x) = (1-q)\phi_{\Theta}(\lfloor x \rfloor), \quad x \in \mathbb{R}_{+},$$

where $\lfloor x \rfloor$ denotes the integer part of x (the floor function). When Θ is either gamma distributed with the PDF (7) or is positive stable with the LT (12), then the tail probabilities take on particularly simple forms, given by

$$\mathbb{P}(X > x) = (1-q)\left(\frac{1}{1+\lfloor x \rfloor/\lambda}\right)^{\alpha} \quad \text{and} \quad \mathbb{P}(X > x) = (1-q)e^{-\tau(\lfloor x \rfloor)^{\alpha}},$$

respectively. The above formulas should be contrasted with the rather inconvenient integral that appears in the first paragraph of Section 4.2 in [8].

2.1 The probability of ruin

Let us now derive the probability of ruin under our set-up. First, let us note that the probability of ruin in (4) becomes

$$\psi(u) = \frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1}$$

if and only if $\rho \geq 1-q$ (the net profit condition). To see this, observe that the above holds if and only if

$$\frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1} \leq 1,$$

which is equivalent to

$$\frac{(1-\rho)^{u+1}}{\rho} \leq \frac{q^{u+1}}{1-q}. \tag{16}$$

Consider the function $h(\rho) = (1-\rho)^{u+1}/\rho$, $\rho \in (0,1)$. Since

$$\frac{dh(\rho)}{d\rho} = -(1-\rho)^{u}\frac{(u+1)\rho + 1 - \rho}{\rho^2} < 0,$$

the function h is decreasing on the interval $(0,1)$, and so (16) is equivalent to $\rho \geq 1-q$ as desired.

Now, if we set $1 - \rho = e^{-\theta}$, then the net profit condition becomes $\theta > \theta^\star$, where $\theta^\star = -\log q \in (0, \infty)$. Then, analogously to (10) in [8], the probability of ruin can be written as

$$\psi(u) = F_\Theta(\theta^\star) + J(u, \theta^\star), \tag{17}$$

where

$$J(u, \theta^\star) = \frac{1-q}{q^{u+1}} \int_{\theta^\star}^{\infty} \frac{e^{-\theta(1+u)}}{1-e^{-\theta}} dF_\Theta(\theta). \tag{18}$$

One can obtain a compact formula for the above probability in terms of a geometric random variable $N \sim Geo(p)$, given by the PMF

$$\mathbb{P}(N = k) = p(1-p)^{k-1}, \quad k \in \mathbb{N}, \tag{19}$$

and the probability generating function (PGF)

$$\mathbb{E}\left(s^N\right) = \frac{sp}{1 - s(1-p)}, \quad s \in (0, 1), \tag{20}$$

and the *excess* random variable

$$\Theta^\star \stackrel{d}{=} \Theta - \theta^\star | \Theta \geq \theta^\star. \tag{21}$$

If Θ is absolutely continuous, then the PDF of the latter is

$$f_{\Theta^\star}(\theta) = \frac{f_\Theta(\theta + \theta^\star)}{1 - F_\Theta(\theta^\star)}, \quad \theta \in \mathbb{R}_+. \tag{22}$$

The following result provides relevant details.

Proposition 2.1. *Let Θ have an absolutely continuous distribution on $\mathbb{R}_+$ with the CDF and the PDF denoted by F_Θ and f_Θ, respectively, and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ of the discrete time risk model (1) are IID modified geometric $ZMG(q, \rho)$ with the PMF (3) and $\rho = 1 - e^{-\theta}$. Then, the probability of ruin is given by*

$$\psi(u) = F_\Theta(\theta^\star) + [1 - F_\Theta(\theta^\star)]\mathbb{E}\left\{e^{-(u+N)\Theta^\star}\right\}, \tag{23}$$

where $\theta^\star = -\log q$, $\Theta^\star$ is the excess random variable given by the PDF (22), and N is a geometric random variable (19) with parameter $p = 1 - q$, independent of $\Theta^\star$.

Proof. Let us work with the quantity $J(u, \theta^\star)$ given by (18). We have

$$J(u, \theta^\star) = [1 - F_\Theta(\theta^\star)] \frac{1-q}{q} \int_{\theta^\star}^{\infty} \frac{e^{-\theta u} e^{-\theta} q^{-u}}{1 - e^{-\theta}} \frac{f_\Theta(\theta)}{[1 - F_\Theta(\theta^\star)]} d\theta. \tag{24}$$

Note that

$$q^{-u} = e^{-u \log q} = e^{\theta^\star u},$$

so that

$$J(u, \theta^\star) = [1 - F_\Theta(\theta^\star)] \frac{1-q}{q} \int_{\theta^\star}^{\infty} \frac{e^{-u(\theta - \theta^\star)} e^{-\theta}}{1 - e^{-\theta}} \frac{f_\Theta(\theta)}{[1 - F_\Theta(\theta^\star)]} d\theta.$$

Upon the substitution $x = \theta - \theta^\star$ in (24) we obtain

$$J(u, \theta^\star) = [1 - F_\Theta(\theta^\star)] \int_0^\infty e^{-ux} \frac{(1-q)e^{-x}}{1 - qe^{-x}} f_{\Theta^\star}(x) dx. \tag{25}$$

We now recognize the term

$$\frac{(1-q)e^{-x}}{1 - qe^{-x}}$$

under the integral in (25) as the PGF of geometric variable N with the PMF (19) and $p = 1 - q$, evaluated at $s = e^{-x}$ (so this is actually the LT of N), so that we can write the above integral as

$$\mathbb{E}\left\{e^{-u\Theta^\star}\mathbb{E}\left(e^{-\Theta^\star N}|\Theta^\star\right)\right\} = \mathbb{E}\left\{\mathbb{E}\left(e^{-u\Theta^\star}e^{-\Theta^\star N}|\Theta^\star\right)\right\} = \mathbb{E}\left\{e^{-(u+N)\Theta^\star}\right\},$$

as desired. This completes the proof. □

Routine calculations lead to the following result, describing the special case with gamma-distributed Θ and zero-modified discrete Pareto (10) correlated claim amounts. Note that the probability of ruin given below involves the (upper) incomplete gamma function,

$$\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt, \tag{26}$$

as it does in an analogous problem considered by [8].

Corollary 2.1. *Let Θ have a gamma distribution with the PDF (7) and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ in (1) be IID modified geometric $ZMG(q, \rho)$ with the PMF (3) and $\rho = 1 - e^{-\theta}$. Then, the probability of ruin $\psi(u)$ is given by*

$$\psi(u) = 1 - \frac{\Gamma(\alpha, -\lambda\log q)}{\Gamma(\alpha)} + \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{1-q}{q^{u+1}}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha, -(k+u+\lambda)\log q)}{(k+u+\lambda)^\alpha}.$$

Below we present a special case with exponential mixing distribution, where the probability of ruin may take on an explicit form.

Corollary 2.2. *Let Θ have an exponential distribution with parameter $\lambda > 0$ and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ in (1) are IID modified geometric $ZMG(q, \rho)$ with the PMF (3) and $\rho = 1 - e^{-\theta}$. Then, if $\lambda \in \mathbb{N}$, the probability of ruin is given by*

$$\psi(u) = (1-q)\left\{1 - \frac{\lambda}{q^{u+1}}\left[\log(1-q) + \sum_{k=1}^{u+\lambda}\frac{q^k}{k}\right]\right\}.$$

Remark 2.1. Figure 3 shows a comparison of the ruin probabilities under two different settings with conditional ZMG claims, where, given $\Theta = \theta$, the geometric probability of success is given either by (5) as in [8] or by (9), as proposed in this paper. Moreover, in each cases Θ has gamma distribution (7), with parameters α_i, λ_i, $i = 1, 2$, satisfying the condition

$$1 - \left(\frac{\lambda_1}{\lambda_1 + 1}\right)^{\alpha_1} = \left(\frac{\lambda_2}{\lambda_2 + 1}\right)^{\alpha_2},$$

so that the expected geometric probabilities of success coincide, $\mathbb{E}(\rho_1) = \mathbb{E}(\rho_2)$. As can be seen in Figure 3, the ruin probability curves under our model drop faster then those under the model of [8]. Note that the settings for the parameter λ affect the position of the ruin probability curves. In addition, according to the expression of the ruin probability given by [8], the equations only accept the integer initial capitals.

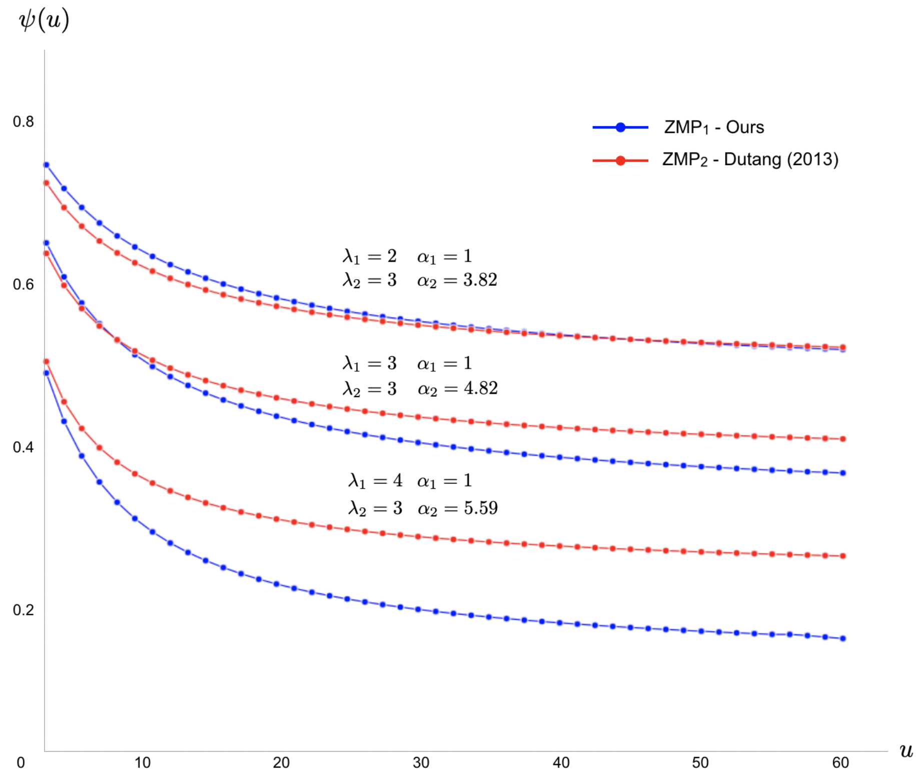

Figure 3: Ruin probabilities under our settings and [8]'s settings for ZMP model.

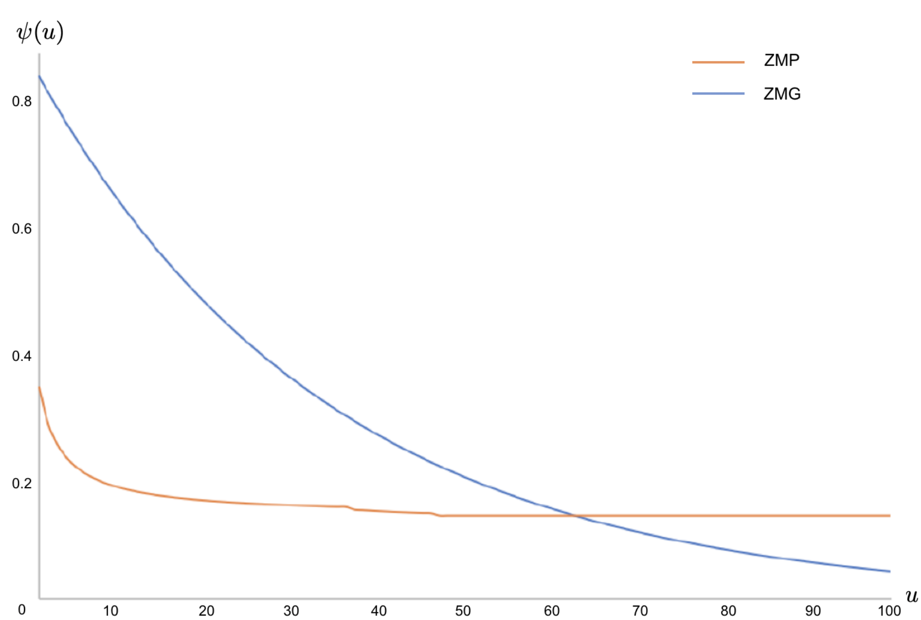

Figure 4: Ruin probabilities under the ZMG and the ZMP models.

Remark 2.2. As can be seen from the ruin probability formula in the ZMP case, the probability of ruin converges to a non-zero level as $u \to \infty$, which is due to the net profit condition being violated. Therefore, in the ZMP model the ruin probability is more stable for large u compared with its behavior under the ZMG model. Furthermore, the rate of convergence can vary with the parameters, as can be seen in the example given in Table 2, by the parameters 1-4 provided in Table 1 below. When comparing Set 1 with Set 2, and Set 2 with Set 3, one can notice that larger λ and smaller α lead to a larger probability of ruin and faster convergence (the difference in ruin probabilities between $u = n$ and $u = n + 1$ is smaller than 10^{-8}). In other words, larger λ and lower α flatten the ruin probability. According to Set 4, one can see that as the probability q of no claims increases, the ruin probability decreases. Moreover, starting with $u = 53$, the probability is already convergent to the level where the net profit condition is violated. We also notice that the decrease is of 9.719% (from $\psi(0) = 54.1\%$ to $\psi(53) = 44.39\%$). This decrease is larger than the one in the case of Set 1, which was only 0.028% (from $\psi(0) = 86.6\%$ to $\psi(20) = 86.36\%$). Thus, the larger the q, the lower the ruin probability, the steeper the decrease, and the slower the convergence.

Table 1: Parameters' coefficients.

Set	1	2	3	4
α	2	2	4	2
λ	5	10	5	5
q	0.2	0.2	0.2	0.5

Table 2: Results for the speed of convergence.

Set	1	2	3	4
$\psi(0)$	0.86584	0.99264	0.49289	0.54108
$\psi(\infty)$	0.86356	0.99263	0.46225	0.44389
convergent after $u =$	20	15	24	53

The result below provides the ruin probability for the special case where Θ is Levy stable with index $\alpha = 1/2$ and PDF (8), in which case we have conditionally independent zero-modified discrete Weibull (ZMW) claim amounts, with the PMF (13) and $\alpha = 1/2$. As in the analogous problem considered by [8], the probability of ruin can be expressed in terms of the complementary error special function

$$erfc(z) = 1 - erf(z) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2}\, dt. \tag{27}$$

Corollary 2.3. *Let Θ be a Lévy stable random variable with the PMF (8) and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ in (1) are IID modified geometric $ZMG(q, \rho)$ with the PMF (3) and $\rho = 1 - e^{-\theta}$. Then the probability of ruin is given by*

$$\psi(u) = erfc\left(\frac{\tau}{2\sqrt{-\log q}}\right) + \frac{1-q}{q^{u+1}} \sum_{k=1}^{\infty} \left\{ q^{u+k} erf\left(\frac{\tau}{2\sqrt{-\log q}}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{2n+1} (u+k)^{n+\frac{1}{2}}}{n! \sqrt{\pi} (2n+1) 4^n} \Gamma\left(-\frac{2n-1}{2}, -(u+k)\log q\right) \right\},$$

where $\Gamma(\cdot, \cdot)$ and $erfc(\cdot)$ are given by (26) and (27), respectively.

Proof. Let $\theta^\star = -\log q$. Then, by taking into account the PDF of Θ given by (8) and Proposition 2.1, we obtain

$$\psi(u) = F_\Theta(\theta^\star) + (1 - F_\Theta(\theta^\star))\mathbb{E}\left\{e^{-(u+N)\theta^\star}\right\}$$

$$= erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + (1 - F_\Theta(\theta^\star))\sum_{k=1}^{\infty}\int_0^\infty \frac{f_\Theta(\theta+\theta^\star)}{1-F_\Theta(\theta^\star)}e^{-(u+k)\theta}(1-q)q^{k-1}d\theta$$

$$= erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + \sum_{k=1}^{\infty}\int_0^\infty f_\Theta(\theta+\theta^\star)e^{-(u+k)\theta}(1-q)q^{k-1}d\theta$$

$$= erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + \sum_{k=1}^{\infty}(1-q)q^{k-1}e^{(u+k)\theta^\star}\int_{\theta^\star}^\infty f_\Theta(t)e^{-(u+k)t}dt$$

$$= erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + \sum_{k=1}^{\infty}(1-q)q^{k-1}q^{-(u+k)}\int_{\theta^\star}^\infty f_\Theta(t)e^{-(u+k)t}dt$$

$$= erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + \frac{1-q}{q^{u+1}}\sum_{k=1}^{\infty}\int_{\theta^\star}^\infty f_\Theta(t)e^{-(u+k)t}dt,$$

where in the last equality we used

$$\int_{\theta^\star}^\infty f_\Theta(t)e^{-(u+k)t}dt = \int_{\theta^\star}^\infty e^{-(u+k)\theta}d\left\{erfc\left(\frac{\tau}{2\sqrt{\theta}}\right)\right\}$$

$$= e^{-(u+k)\theta^\star}erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + (u+k)\int_{\theta^\star}^\infty e^{-(u+k)\theta}erfc\left(\frac{\tau}{2\sqrt{\theta}}\right)d\theta.$$

Finally, the substitution

$$erfc\left(\frac{\tau}{2\sqrt{\theta}}\right) = 1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\tau}{2})^{2n+1}}{n!(2n+1)}\theta^{-n-\frac{1}{2}}$$

leads to

$$\int_{\theta^\star}^\infty f_\Theta(t)e^{-(u+k)t}dt =$$

$$e^{-(u+k)\theta^\star}erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) + (u+k)\int_{\theta^\star}^\infty e^{-(u+k)\theta}\left(1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\tau}{2})^{2n+1}}{n!(2n+1)}\theta^{-n-\frac{1}{2}}\right)d\theta$$

$$= e^{-(u+k)\theta^\star}erfc\left(\frac{\tau}{2\sqrt{\theta^\star}}\right) - \sum_{n=0}^{\infty}\frac{(-1)^n\tau^{2n+1}(u+k)^{n+\frac{1}{2}}}{n!\sqrt{\pi}(2n+1)4^n}\Gamma\left(-\frac{2n-1}{2}, (u+k)\theta^\star\right),$$

and the result follows. □

Remark 2.3. Let $L = F_\Theta(\theta^\star)$ be the level at which the net profit condition is violated. In Figure 5, one can set up the same level L of $\psi(u)$ as $u \to \infty$ for both, zero modified Pareto and Weibull models (denoted, respectively, by ZMP and ZMW). From Figure 5, one can see that the ruin probability curve is steeper under the ZMP model and it starts from a higher initial ruin probability $\psi(0)$.

Table 3 below shows that, when we increase the value of τ (the parameter in the ZMW model) from 1 to 1.1, the ruin probability curve decreases by 3% at given level L. This can be observed by increasing the expectation of the claims. Additionally, a smaller τ corresponds to a larger ruin probability and faster convergence to level L.

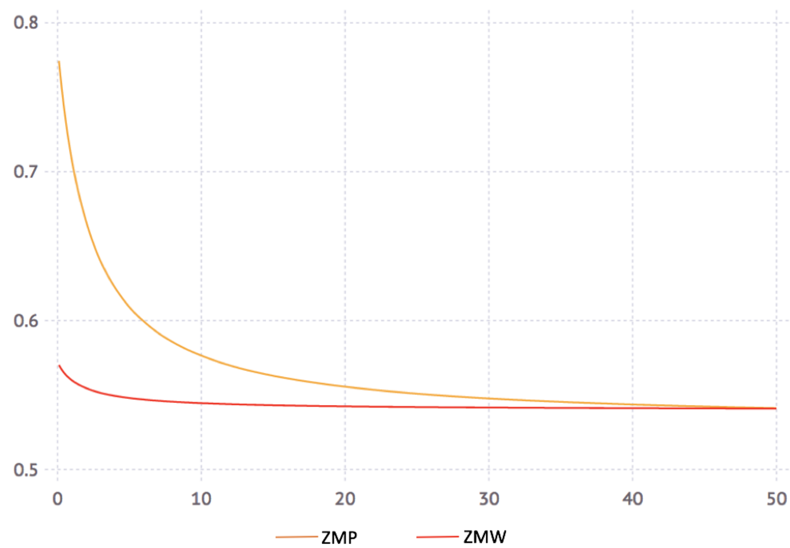

Figure 5: Ruin probabilities under the ZMP and ZMW models with a same level of $\lim_{u\to\infty} \psi(u)$.

Table 3: Results for the speed of convergence under the ZMW model.

Set	$\alpha = 1$	$\alpha = 1.1$
$\psi(0)$	0.60338	0.57028
$\psi(\infty)$	0.57776	0.54037
convergent after u =	50	70

2.2 Illustrative data example

As an illustration, we fit the three zero-modified models, ZMG, ZMP and ZMW, to data from a non-life reinsurance company. The data were skewed and scaled for confidentiality reasons. Claims data span the time period of 11 years, with claims recorded on a monthly basis. The zero and the non-zero frequencies are shown in Table 4 given below. Zero claims refer to accidents that the company paid nothing for, due to deductibles or other contracts considerations. The model frequency q of zero claims is estimated by the corresponding sample frequency, $\hat{q}$, resulting in $\hat{q} = 0.218$. The parameters of all three models are estimated by the method of moments, and are provided in Table 5 below. Figure 6 illustrates the ruin probabilities under the three models.

Table 4: The structure of the analyzed reinsurance data set.

	Zero claims	Non-zero claims	Total claims
Number	97	348	445

Remark 2.4. Note that while fitting the data, we will keep the same net profit condition, meaning the same θ^* in (23). In the Figure 5, the levels of convergence $F(\theta^*)$ are different due to different distributions F.

Table 5: Estimated parameters of the three considered models.

	ZMG	ZMP	ZMW	St.Er
$\hat{q}$	0.218	0.218	0.218	N/A
$\hat{p}$	0.480	N/A	N/A	0.0053
$\hat{\lambda}$	N/A	2.110	N/A	0.0052
$\hat{\alpha}$	N/A	2.012	N/A	0.0055
$\hat{\tau}$	N/A	N/A	0.968	0.0020
SSE	0.028	0.005	0.002	N/A

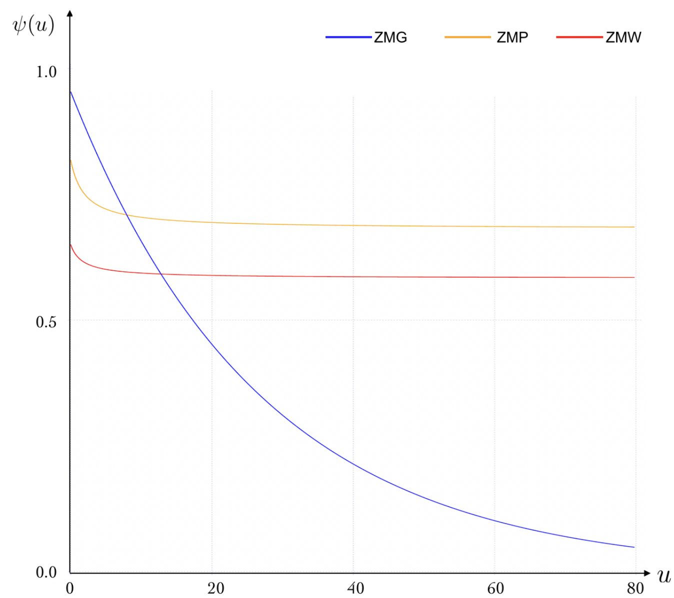

Figure 6: Ruin probabilities for the three considered models.

Table 6: Ruin probabilities for three considered models.

u	$\psi(u)_{ZMG}$	$\psi(u)_{ZMP}$	$\psi(u)_{ZMW}$
0	0.954	0.818	0.650
1	0.919	0.772	0.625
2	0.885	0.749	0.614
3	0.852	0.736	0.608
4	0.821	0.727	0.603
5	0.791	0.720	0.601
10	0.656	0.704	0.593
15	0.544	0.698	0.590
25	0.374	0.692	0.588
30	0.311	0.690	0.587
40	0.214	0.688	0.586
50	0.147	0.687	0.585
51-100	0.146-0.005	0.687-0.685	0.585-0.584

To measure the goodness-of-fit, we use P-P plots and the sum of the squared errors (SSE), shown in Figure 7 and Table 5, respectively. Based on the results ZMW and ZMP present a much better fit than ZMG. Furthermore, our data analysis leads to the same conclusion as that provided by our theoretical results. Namely, while the ZMG model has the largest ruin probability when $u = 0$, it decays very quickly as the initial investment increases. As far as the ZMP and ZMW models, the ruin probability under the ZMP model is always larger than that under the ZMW model.

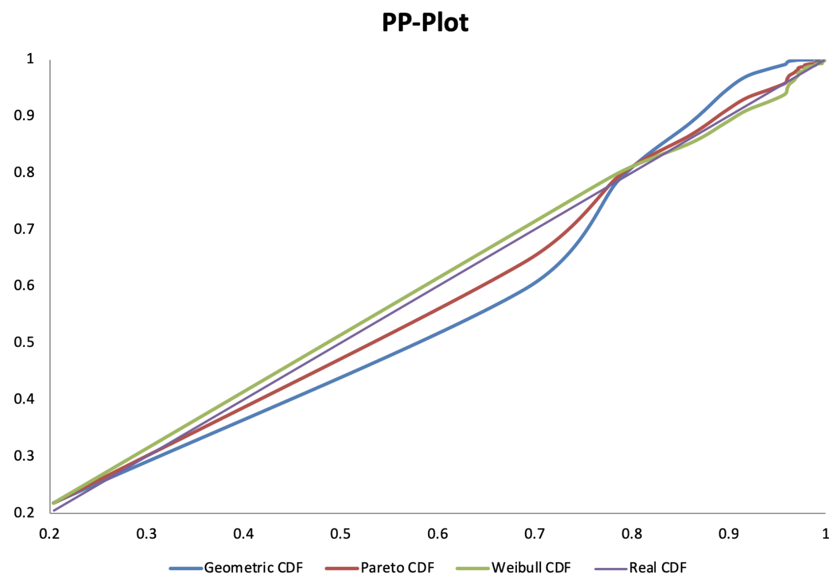

Figure 7: PP-Plots for the three considered models.

3 A zero-modified discrete Pareto distribution

In this section we present basic properties of zero-modified Pareto distribution, given by the PMF (10). We shall use the notation ZMDP(α, λ, q), or in short ZMDP, for this distribution. Some of our results presented below shall be stated in an alternative parameterization, which conveniently accounts for the special special case $\alpha = \infty$, corresponding to the zero-modified geometric distribution given by (3). Namely, as in [3], we replace α with its reciprocal and instead of λ we set $\rho = 1 - \exp(-1/(\alpha\lambda))$, so that $1/\lambda = -\alpha \log(1-\rho)$ and the PMF (10) takes on the form

$$\mathbb{P}(X = k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q)\left\{\left(\frac{1}{1 - \alpha(k-1)\log(1-\rho)}\right)^{\frac{1}{\alpha}} - \left(\frac{1}{1 - \alpha k \log(1-\rho)}\right)^{\frac{1}{\alpha}}\right\},$$

with $k \in \mathbb{N}_0$. We use ZMDP$^*(\alpha, \rho, q)$ for the zero-modified discrete Pareto distribution with the above PMF. As shown below, the parameter $\alpha \geq 0$ is a tail parameter, $\rho \in [0, 1]$ has to do with the “size" of X, while the parameter $q \in [0, 1]$ controls the point mass at zero.

The main motivation for the re-parameterization is that the distribution can be defined at the boundary case $\alpha = 0$, which is understood as the limit of the ZMDP$^*(\alpha, \rho, q)$ distribution with $\rho \in (0, 1)$ as α converges to zero. It follows that in the limit we obtain the zero-modified geometric distribution (3). On the other hand, we do not get a proper distribution when $\alpha \to \infty$. We also have a few other special cases as follows:

(i) If $q = 1$ the distribution is a point mass at $k = 0$.

(ii) If $q = 0$, we get the discrete Pareto distribution.

(iii) If $q \in (0, 1)$ and $\rho = 1$, the distribution is a point mass at $k = 1$.
As mentioned above, the parameter α controls the tails of the ZMDP distributions, which follow a power law just as they do in the case of DP distribution. The following result, which is straightforward to prove using the ZMDP survival function, makes this more precise.

Proposition 3.1. *If $X \sim ZMDP(\alpha, \lambda, q)$ then*

$$\mathbb{P}(X > x) \sim (1-q)\lambda^{\alpha} x^{-\alpha}, \quad as \ \ x \to \infty.$$

Next, we argue that in some sense the parameter $\lambda > 0$ controls the "size" of the ZMDP random variable, although it is not a scale parameter in the usual sense. As we show below, as λ is increasing, the distribution is increasing in a stochastic sense. Recall that a random variable X_2 is said to be *stochastically larger* than a random variable X_1 if $F_2(x) \leq F_1(x)$ for all x, where F_1 and F_2 are the CDFs of X_1 and X_2, respectively. The following result, which is an extension of an analogous property of DP distribution, is a simple consequence of the particular form of the CDF of ZMDP distribution given in Proposition 3.3 below.

Proposition 3.2. *If $X_1 \sim ZMDP(\alpha, \lambda_1, q)$ and $X_2 \sim ZMDP(\alpha, \lambda_2, q)$, where $\lambda_1 < \lambda_2$, then X_2 is stochastically larger than X_1.*

3.1 The CDF and the quantile functions

In order to describe the CDF, the survival function (SF), and the quantile function connected with the ZMDP model, it is convenient to use the standard floor and ceiling functions. Recall that, for $x \in \mathbb{R}$, the floor function, often denoted by $\lfloor x \rfloor$, is the largest integer that is less than or equal to x. Similarly, the ceiling function, often denoted by $\lceil x \rceil$, is the smallest integer that is larger than or equal to x. With this notation, the CDF and the SF of a ZMDP model admit the expressions given in the following result, whose elementary proof shall be omitted.

Proposition 3.3. *The CDF and the SF of $X \sim ZMDP^{\star}(\alpha, \rho, q)$ are given by*

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 1 - (1-q)\left(\frac{1}{1-\alpha\log(1-\rho)\lfloor x \rfloor}\right)^{\frac{1}{\alpha}}, & \text{for } x \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$S(x) = \mathbb{P}(X > x) = \begin{cases} (1-q)\left(\frac{1}{1-\alpha\log(1-\rho)\lfloor x \rfloor}\right)^{\frac{1}{\alpha}}, & \text{for } x \geq 0, \\ 1, & \text{otherwise}, \end{cases} \tag{28}$$

respectively.

In turn, the quantile function

$$Q(u) = \inf\{x : F(x) \geq u\}, \quad u \in (0, 1),$$

of the ZMDP model is obtained by inverting its CDF, leading to the result below.

Proposition 3.4. *The quantile function of $X \sim ZMDP^{\star}(\alpha, \rho, q)$ is given by*

$$Q(q) = \begin{cases} \left\lceil -\frac{1}{\alpha}\frac{1}{\log(1-\rho)}\frac{(1-q)^{\alpha}-(1-u)^{\alpha}}{(1-u)^{\alpha}} \right\rceil, & \text{for } q < u < 1, \\ 0, & \text{for } 0 < u \leq q. \end{cases}$$

3.2 Moments and related parameters

We start with probability generating function (PGF) of a ZMDP random variable X, defined as

$$G(s) = \mathbb{E}s^X = \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n), \quad s \in (0, 1).$$

Perhaps the most convenient way to derive it is through the mixture representation (3.10) coupled with the formula for the PGF of the DP distribution (see [3], Proposition 2.6). This immediately produces the result below.

Proposition 3.5. *The PGF of $X \sim ZMDP^*(\alpha, \rho, q)$ is given by*

$$G(s) = s + (1-s)\left\{q - (1-q)\sum_{n=1}^{\infty}\left(\frac{1}{1-\alpha\log(1-\rho)n}\right)^{\frac{1}{\alpha}} s^n\right\}, \quad s \in (0, 1). \tag{29}$$

The formulas for the moments connected with the ZMDP distribution are straightforward to derive when we take into account mixture representation on Proposition (3.10) and results on the moments of the DP distribution (see [3], Proposition 2.7). Note that according to Proposition 3.1, the moments $\mathbb{E}X^r$ of $X \sim ZMDP^*(\alpha, \rho, q)$, where $r > 0$, are finite if and only if $r < 1/\alpha$. The following result, which is straightforward to derive, provides further details.

Proposition 3.6. *Let $X \sim ZMDP^*(\alpha, \rho, q)$ and $r > 0$. Then $\mathbb{E}X^r$ exists if and only if $r < 1/\alpha$, in which case we have*

$$\mathbb{E}X^r = (1-q)\left\{1 + \sum_{n=1}^{\infty}\left(\frac{n^{\alpha r}}{1-\alpha\log(1-\rho)n}\right)^{\frac{1}{\alpha}}\left[\left(1+\frac{1}{n}\right)^r - 1\right]\right\}.$$

In particular, the mean exists whenever $\alpha < 1$, and simplifies to

$$\mathbb{E}X = (1-q)\sum_{n=0}^{\infty}\left(\frac{1}{1-\alpha\log(1-\rho)n}\right)^{\frac{1}{\alpha}} = c_{\alpha\rho,q}\zeta\left(\frac{1}{\alpha}, -\frac{1}{\alpha\log(1-\rho)}\right),$$

where

$$c_{\alpha,p,q} = (1-q)\left(\frac{1}{-\alpha\log(1-\rho)}\right)^{\frac{1}{\alpha}}$$

and

$$\zeta(s, p) = \sum_{n=0}^{\infty}\left(\frac{1}{n+p}\right)^s$$

is the Hurwitz-zeta function. Note that in the special case of $ZMDP^*(\alpha, \rho, q)$ distribution with $-\alpha\log(1-\rho) = 1$ the r-the moment takes on the form

$$\mathbb{E}X^r = (1-q)\sum_{n=0}^{\infty}\frac{(n+1)^r - n^r}{(1+n)^{\frac{1}{\alpha}}}.$$

Further, in this case the mean is given by $EX = (1-q)\xi(1/\alpha)$, where

$$\xi(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}, \quad s > 1,$$

is the Riemann-zeta special function.

3.3 Stability properties

Due to the close connection between ZMDP and DP distributions, it is not surprising that the stability properties of the letter (see, e.g, Section 3.1 of [3]) carry over, with some modifications, to the former.

3.3.1 Stability connected with minima

Our first result is related to the minimum of independent ZMDP variables. Due to the particular form of ZMDP survival function, it can be seen that the minimum $M_n = \min_{1\le i\le n}\{X_i\}$ of n IID ZMDP variables $\{X_i\}$ will also have ZMDP distribution, but with different parameters. Indeed, if the SF of the $\{X_i\}$ is given by $S(x)$ as in (28), then the SF of M_n is of the form

$$S_n(x) = \mathbb{P}(M_n > x) = [S(x)]^n = (1-q_n)\left(\frac{1}{1-\alpha_n \log(1-\rho_n)\lfloor x\rfloor}\right)^{\frac{1}{\alpha_n}}, \quad x \ge 0, \tag{30}$$

where

$$\alpha_n = \alpha/n, \qquad \rho_n = 1-(1-\rho)^n, \qquad q_n = 1-(1-q)^n, \tag{31}$$

which is seen to be a SF of the ZMDP distribution. In turn, if the SF of M_n is of the form (30), then it follows that the SF of the X_i must be given by (28). This leads to the following result, which is an extension of similar property of DP distributions [3, 19]).

Proposition 3.7. *Let $X_1, X_2, \dots, X_n$ be non negative IID integer-valued random variables and let $M_n = \min_{1\le i\le n} X_i$, $n \in \mathbb{N}$. Then M_n is ZMDP if and only if the $\{X_i\}$ are ZMDP, in which case we have $M_n \sim$ ZMDP$^*(\alpha_n, \rho_n, q_n)$ and $X_i \sim$ ZMDP$^*(\alpha, \rho, q)$, with the parameters connected through (31).*

This result can be extended to the case of independent but not necessarily identically distributed ZMDP variables, as long as they have a common “scale" parameter.

Proposition 3.8. *Let $X_i \sim ZMDP(\alpha_i, \lambda, q_i)$ for $i = 1, 2, \dots, n$, and let $M_n = \min_{1\le i\le n}\{X_i\}$. Then $M_n \sim ZMDP(\alpha, \lambda, q)$, where $\alpha = \sum_{i=1}^n \alpha_i$ and $q = 1 - \prod_{i=1}^n q_i$.*

3.3.2 Stability of the conditional tail

We now consider the “tail" random variable X_u, which is also known as the excess, defined as $X - u$ given that $X \ge u$, where $u \in \mathbb{N}_0$ is interpreted as a threshold beyond which we have an observation. Recall that the geometric distribution (supported on $\mathbb{N}_0$) is stable, in the sense that the variables X_u and X have the same distribution for each $u \in \mathbb{N}_0$ when X is geometric. As shown below, if X is ZMDP then X_u is also ZMDP for each $u \in \mathbb{N}_0$, although their distributions have different parameters. This result extends similar property of DP distribution to the ZMDP case [3] .

Proposition 3.9. *Let $X \sim ZMDP(\alpha, \lambda, q)$. Then for any $u \in N$ the random variable X_u, defined as $X - u$ given $X \ge u$, has $ZMDP(\alpha, \lambda_u, q_u)$ distribution, where*

$$\lambda_u = \lambda + u \quad \textit{and} \quad q_u = 1 - \left(1 - \frac{1}{\lambda + u}\right)^{\alpha}.$$

3.4 Stochastic representations

Here, we present several useful stochastic representations of the ZMDP distribution. We start with its basic relation to the DP model of [3].

Proposition 3.10. *If $X \sim ZMDP(\alpha, \lambda, q)$ then*

$$X \stackrel{d}{=} IN,$$

where the variables I and N are independent, I has a Bernoulli distribution with parameter $1-q$, and $N \sim DP(\alpha, \lambda)$ with the PMF (11).

Since, as shown in Proposition 3.4 of [3], the variable N from Proposition 3.10 is (conditionally) geometric with parameter $\rho = 1 - e^{\theta}$ given that $\Theta = \theta$, where Θ is a gamma variable given by the LT (15), we obtain the following representation.

Proposition 3.11. *Suppose that Θ has gamma distribution with the LT (15) and the PDF (7). Further, given $\Theta = \theta$, let X have a ZMG distribution with the PMF (3) where $\rho = 1 - e^{-\theta}$. Then $X \sim ZMDP(\alpha, \lambda, q)$.*

One can also relate the ZMDP distribution to randomly stopped Poisson process. Indeed, it is well-known that if $\{N(t),\ t \in \mathbb{R}_+\}$ is a standard Poisson process and Z is standard exponential variable, independent of Z, then $N(Z/\beta)$ has a geometric distribution (supported on $\mathbb{N}_0$) with parameter $\rho = \beta/(\beta + 1)$. In particular, when $\beta = e^{\theta} - 1$, then $\rho = 1 - e^{-\theta}$. Consequently, in view of Proposition 3.11, we obtain the following result.

Proposition 3.12. *If $X \sim ZMDP(\alpha, \lambda, q)$, then*

$$X \stackrel{d}{=} I[N(T) + 1],$$

where all the variables on the right-hand-side of (3.12) are independent, I has Bernoulli distribution with parameter $1 - q$,

$$T \stackrel{d}{=} \frac{Z}{e^{\Theta} - 1},$$

the variable Z is standard exponential, Θ has gamma distribution with the PDF (7), and $\{N(t),\ t \in \mathbb{R}_+\}$ is a standard Poisson process.

3.5 Divisibility properties

Recall that the probability distribution of a random variable X is *infinitely divisible* (ID) if for each $n \in \mathbb{N}$ we have the equality in distribution

$$X \stackrel{d}{=} X_{n,1} + \cdots + X_{n,n}, \tag{32}$$

where the $\{X_{n,j}\}$ $(1 \le j \le n)$ are IID random variables. Additionally, if the distribution of X is supported on $\mathbb{N}_0$, then it is *discrete infinitely divisible* if it is ID and the variables $\{X_{n,j}\}$ in (32) are supported on $\mathbb{N}_0$ as well. As shown in [3], the DP distribution is ID (and its shifted version, supported on $\mathbb{N}_0$, is discrete ID). However, as shown below, the infinite divisibility of zero-modified DP distribution depends on its parameters. Generally speaking, if $X \sim ZMDP(\alpha, \lambda, q)$ then the discrete ID property holds when the values of q are near 1 and does not hold if q is near zero. The following result summarizes these facts.

Proposition 3.13. *Let $X \sim ZMDP(\alpha, \lambda, q)$. Then the distribution of X is discrete ID (and thus ID) when*

$$\frac{1}{1 + d_{\alpha,\lambda}} \le q \le 1 \tag{33}$$

and it is not discrete ID when

$$0 \le q < \frac{1}{1 + 2d_{\alpha,\lambda}}, \tag{34}$$

where

$$d_{\alpha,\lambda} = \left[\left(\frac{\lambda}{\lambda+1}\right)^{\alpha} - \left(\frac{\lambda}{\lambda+2}\right)^{\alpha}\right]\left[1 - \left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\right]^{-2}. \tag{35}$$

Proof. To prove discrete ID we shall use a sufficient condition for this property, stating that the sequence of probabilities $(p_k)_{k \in \mathbb{N}_0}$, where $p_k = \mathbb{P}(X = k)$, is log-convex, that is $p_k > 0$ for all k and the sequence $(p_{k+1}/p_k)_{k \in \mathbb{N}_0}$ is non-decreasing (see, e.g., [33], Theorem 10.1, p. 60). We use this condition to establish discrete ID of ZMDP distribution with parameters satisfying (33) and $q < 1$, as for $q = 1$ the distribution reduces to a point mass at zero, which is clearly discrete ID. In this case the probabilities are positive, so it remains to show the inequality

$$\frac{p_k}{p_{k-1}} \le \frac{p_{k+1}}{p_k}, \quad k \in \mathbb{N}, \tag{36}$$

where the $\{p_k\}$ are given by the right-hand-side of (10). For $n = 1$, this inequality produces $(1-q)/q \le d_{\alpha,\lambda}$ with $d_{\alpha,\lambda}$ as in (35), and results in (33) upon solving for q. Next, we establish (36) for any $k \ge 2$, which we accomplish by showing that the function p_{k+1}/p_k of *real* argument k is increasing on $(1,\infty)$. To this end, consider the function

$$g(x) = \frac{1-\left(\frac{\lambda+x}{\lambda+x+1}\right)^{\alpha}}{\left(\frac{\lambda+x}{\lambda+x-1}\right)^{\alpha}-1}, \quad x > 1, \tag{37}$$

which, according to (10), represents the ratio p_{x+1}/p_x of ZMDP probabilities (evaluated at real argument $x > 1$). By examining its derivative, we show that the function g is indeed increasing. Straightforward albeit rather lengthy algebra leads to the following expression for the derivative of g:

$$g'(x) = \frac{\alpha(\lambda+x)^{\alpha-1}(\lambda+x-1)^{\alpha-1}h(x)}{[(\lambda+x)^{\alpha}-(\lambda+x-1)^{\alpha}]^2(\lambda+x+1)^{\alpha+1}}, \quad x > 1,$$

where

$$h(x) = (\lambda+x-1)^{\alpha+1} + (\lambda+x+1)^{\alpha+1} - 2(\lambda+x)^{\alpha+1}, \quad x > 1.$$

Our objective is to show that $h(x) > 0$ ($x > 1$), in which case the derivative in (37) is positive and the function g is increasing. By setting $y = \lambda + x - 1$, we see that the condition $h(x) > 0$ ($x > 1$) is equivalent to $w(y) < w(y+1)$ ($y > \lambda$), where $w(y) = (y+1)^{\alpha+1} - y^{\alpha+1}$. However, the later inequality is true since the function w is increasing, as can be verified by taking its derivative. This completes the first part of the result.

We now move to the second part of the result, and show that the distribution of X is not discrete ID when q satisfies the inequality (34). This is clear when $q = 0$, since in this case the distribution is supported on $\mathbb{N}$ (as $p_0 = q = 0$) and consequently can not be discrete ID (see, e.g., [33], p. 23). Further, it is well-known that the characteristic sequence $(r_k)_{k\in\mathbb{N}_0}$ of a discrete ID distribution must be non-negative [33], Theorem 4.4, p. 36, where the elements of the sequence r_k are defined via the relations

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}, \quad n \in \mathbb{N}_0. \tag{38}$$

Solving (38) for r_0 and r_1 leads to $r_0 = p_1/p_0$ and $r_1 = (2p_2 - p_1 r_0)/p_0$, respectively, and the condition $r_1 \ge 0$ becomes $(1-q)/q \le 2d_{\alpha,\lambda}$ upon taking into account the particular form (10) of ZMDP probabilities. Since the last inequality is equivalent to $q \ge 1/(1+2d_{\alpha,\lambda})$, the distribution can not be discrete ID under (34). The proof is now complete. □

Remark 3.1. The property of discrete ID shown above allows one to construct a continuous-time, discrete-valued stochastic processes based on the ZMDP distribution with appropriate parameters. For example, if $1/(1+d_{\alpha,\lambda}) \le q < 1$, we can define a *Lévy motion* $\{X(t), \ t > 0\}$, a process with independent and stationary increments, where $X(1)$ is $ZMDP(\alpha,\lambda,q)$ with the PGF G given by (29) while for each $t > 0$ the PGF of $X(t)$ is G^t. Similar construction is possible for the un-modified, regular DP distribution as well. Such processes may prove to be useful tools for modeling the claim arrival processes of actuarial risk theory.

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