# Domain decomposition methods for nonlinear transmission conditions

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**Abstract**

We consider the transformation of quasilinear partial differential equations to a coupled system of linear equations, but with nonlinear transmission conditions on the interfaces. After deriving a variational formulation, we will discuss Mortar finite element discretization strategies and present a numerical example.

## 1 Model problem

As a model problem we consider the following quasilinear boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\Gamma = \partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$ to find $p$ such that

$$\left.\begin{array}{rcll} -\nabla \cdot \Big(k_r\big(\theta(p)\big)\, \nabla p\Big) & = & f & \text{in } \Omega, \\ p & = & 0 & \text{on } \Gamma_D, \\ k_r\big(\theta(p)\big)\, \nabla p \cdot \mathbf{n} & = & g_N & \text{on } \Gamma_N. \end{array}\right\} \tag{1}$$

This model problem can be seen as a simplification of the stationary Richards equation without gravity, see [1].

If $k_r$ only depends on $\theta$, we can introduce

$$u(\mathbf{x}) := \kappa(p(\mathbf{x})) := \int_0^{p(\mathbf{x})} k_r(\theta(s))\, \mathrm{d}s$$

as the Kirchhoff transformation of $p$. Here we assume $k_r \circ \theta$ to be Lipschitz and bounded with

$$0 < c_l \le k_r(\theta(\cdot)) \le c_u < \infty.$$

Therefore we obtain

$$\nabla u = \kappa'(p)\, \nabla p = k_r(\theta(p))\, \nabla p$$

and problem (1) can be transformed to the following linear problem to find $u$ such that

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma_D, \qquad \nabla u \cdot \mathbf{n} = g_N \quad \text{on } \Gamma_N.$$

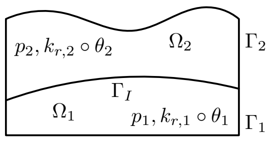


Figure 1: several domains

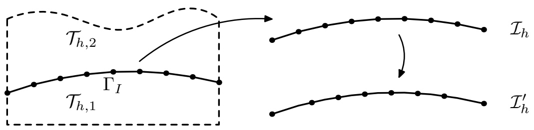


Figure 2: dual interface mesh

Obviously, such a simplification can not be achieved if $k_r$ depends in addition explicitly on $\mathbf{x} \in \Omega$. However, if this dependency is piecewise within non–overlapping subdomains $\Omega_i$, we can introduce local Kirchhoff transformations $u_i(\mathbf{x}) := \kappa_i(p_i(\mathbf{x}))$ for $\mathbf{x} \in \Omega_i$. So we can reformulate problem (1) and obtain the following nonlinear transmission problem, in the case of two subdomains, see Figure 1, to find $u_i$, $i = 1, 2$, such that

$$\left.\begin{array}{rcll} -\Delta u_i &=& f & \text{in } \Omega_i, \\ u_i &=& 0 & \text{on } \Gamma_{D,i}, \\ \nabla u_i \cdot \mathbf{n}_i &=& g_{N.i} & \text{on } \Gamma_{N,i}, \\ \nabla u_1 \cdot \mathbf{n} = \nabla u_2 \cdot \mathbf{n},\ \kappa_1^{-1}(u_1) &=& \kappa_2^{-1}(u_2) & \text{on } \Gamma_I, \end{array}\right\} \tag{2}$$

where $\Gamma_I = \partial\Omega_1 \cap \partial\Omega_2,\ \Gamma_{D,i} = \partial\Omega_i \cap \Gamma_D,\ \Gamma_{N,i} = \partial\Omega_i \cap \Gamma_N$ and $g_{N,i} = g_{N|_{\Gamma_{N,i}}}$. The second, nonlinear interface condition ensures the continuity of the solution $p$ of the original problem (1), where we started from. Next we will derive a variational formulation for the nonlinear transmission problem (2).

## 2 Primal–hybrid formulation

In [2,3], an equivalent primal–hybrid formulation for the Poisson equation is presented. The idea is to introduce Lagrange multipliers acting on interfaces which ensure global continuity of the solution. We use this approach for the nonlinear transmission problem (2) and obtain the following variational problem to find $(u, \lambda) \in X \times M$, such that

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \,\mathrm{d}x + \int_{\Gamma_I} \lambda\, [v]_{\Gamma_I} \,\mathrm{d}s_x \\ \displaystyle\qquad = \sum_{i=1}^{2} \int_{\Omega_i} f_i\, v_i \,\mathrm{d}x + \sum_{i=1}^{2} \int_{\Gamma_{N,i}} g_{N,i} v_i \,\mathrm{d}s_x \quad \text{for all } v \in X, \\ \displaystyle\int_{\Gamma_I} \mu\, (u)_{\Gamma_I} \,\mathrm{d}s_x = 0 \quad \text{for all } \mu \in M \end{array}\right\} \tag{3}$$

with the ansatz spaces

$$X := \{v \in L_2(\Omega) \mid v_i = v_{|\Omega_i} \in H^1_{0,\Gamma_{D,i}}(\Omega_i), i = 1, 2\}, \quad M := H_{00}^{\frac{1}{2}}(\Gamma_I)',$$

and with the jump terms

$$[v]_{\Gamma_I} := (v_1 - v_2)_{|\Gamma_I} \in M' \quad \text{and} \quad (u)_{\Gamma_I} := (\kappa_1^{-1}(u_1) - \kappa_2^{-1}(u_2))_{|\Gamma_I} \in M'.$$

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# 3 Discretization

For the discretization of (3) we introduce the finite dimensional subspaces

$$X_h := \{v \in L_2(\Omega) \mid v_i \in S_h^1(\mathcal{T}_{h,i}), i = 1, 2\} \subset X$$

and

$$M_h := \{\mu \in L_2(\Gamma_I) \mid \mu \in S_h^0(\mathcal{I}_h')\} \subset M,$$

where $\mathcal{T}_{h,i}$ is an admissible triangulation of $\Omega_i$, and $\mathcal{I}_h'$ is a modified dual mesh along the interface $\Gamma_I$ as depicted in Figure 2, see also [4]. For the solution of the nonlinear discrete problem we apply Newton's method by solving a sequence of the following linear problems. For given data $w_h \in X_h$, find $(u_h, \lambda_h) \in X_h \times M_h$, such that

$$\left.\begin{aligned}
&\sum_{i=1}^{2} \int_{\Omega_i} \nabla u_{h,i} \cdot \nabla v_{h,i} \, \mathrm{d}x + \int_{\Gamma_I} \lambda_h \, [v_h]_{\Gamma_I} \, \mathrm{d}s_x \\
&\qquad = \sum_{i=1}^{2} \int_{\Omega_i} f_i \, v_{h,i} \, \mathrm{d}x + \sum_{i=1}^{2} \int_{\Gamma_{N,i}} g_{N,i} v_{h,i} \, \mathrm{d}s_x \quad \forall v_h \in X_h, \\
&\int_{\Gamma_I} \mu_h \, (w_h, u_h)'_{\Gamma_I} \, \mathrm{d}s_x \\
&\qquad = \int_{\Gamma_I} \mu_h \, (w_h, w_h)'_{\Gamma_I} \, \mathrm{d}s_x - \int_{\Gamma_I} \mu_h \, (w_h)_{\Gamma_I} \, \mathrm{d}s_x \quad \forall \mu \in M_h
\end{aligned}\right\} \tag{4}$$

where

$$(w_h, u_h)'_{\Gamma_I} := \big((\kappa_1^{-1})'(w_{h,1}) \, u_{h,1} - (\kappa_2^{-1})'(w_{h,2}) \, u_{h,2}\big)_{|\Gamma_I}$$

is the linearization of the nonlinear jump $(u_h)_{\Gamma_I}$ and $w_h$ corresponds to the previous Newton iteration.

# 4 Numerical example

Consider $\mathbb{R}^2 \supset \overline{\Omega} = [0,1]^2 = \big([0, 0.5] \times [0, 1]\big) \cup \big([0.5, 1] \times [0, 1]\big) =: \overline{\Omega}_1 \cup \overline{\Omega}_2$, where $\Gamma_N = \{0\} \times (0, 1)$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. The local diffusion coefficients are chosen as in [1, Chapter 1]. The remaining data $f, g_{N,i}, g_{D,i}$ are given appropriately.

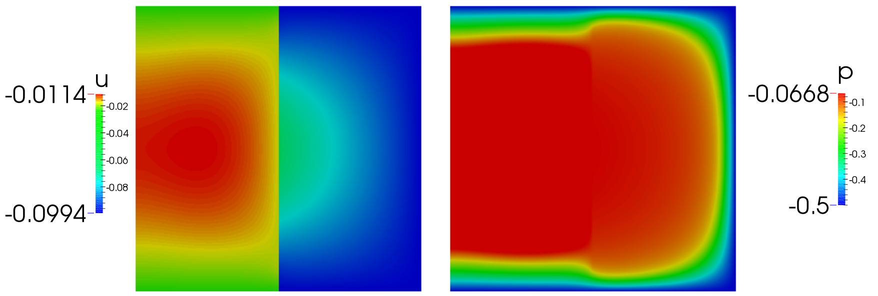


Figure 3: (discontinuous) solution $u$ of problem (4) and its (continuous) inverse Kirchhoff transformations $p_i = \kappa_i^{-1}(u_i)$, $i = 1, 2$.

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