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The Oberbeck–Boussinesq problem modified by a thermo-absorption term

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ABSTRACT

We consider the Oberbeck–Boussinesq problem with an extra coupling, establishing a suitable relation between the velocity and the temperature. Our model involves a system of equations given by the transient Navier–Stokes equations modified by introducing the thermo-absorption term. The model involves also the transient temperature equation with nonlinear diffusion. For the obtained problem, we prove the existence of weak solutions for any $N \geqslant 2$ and its uniqueness if $N = 2$. Then, considering a low range of temperature, but upper than the phase changing one, we study several properties related with vanishing in time of the velocity component of the weak solutions. First, assuming the buoyancy forces field extinct after a finite time, we prove the velocity component will extinct in a later finite time, provided the thermo-absorption term is sublinear. In this case, considering a suitable buoyancy forces field which vanishes at some instant of time, we prove the velocity component extinct at the same instant. We prove also that for non-zero buoyancy forces, but decaying at a power time rate, the velocity component decay at analogous power time rates, provided the thermo-absorption term is superlinear. At last, we prove that for a general non-zero bounded buoyancy force, the velocity component exponentially decay in time whether the thermo-absorption term is sub or superlinear.

1. Introduction

In general, the motion of a fluid driven by buoyancy forces is compressible. But, for many convective motions the system may be considerably simplified by assuming the motion is isochoric, *i.e.* the flow is essentially incompressible except in the body forcing term. Such fluids are said to be, roughly speaking, mechanically incompressible but thermally compressible. This simplification of the problem is known in the literature as the Oberbeck–Boussinesq (OB), or only Boussinesq, approximation. In thermal convection problems, the density changes are caused by temperature changes alone. For such problems the compressibility effects are small and, consequently, the density may be regarded as constant. Into the light of the OB approximation, this corresponds to assume the density is constant except in the body force term. In consequence, when all of these simplifying features are present, the equations for a linearly viscous, heat conducting, homogeneous, incompressible fluid reduce to the following OB equations

$$\operatorname{div}\mathbf{u} = 0, \tag{1.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\theta) - \frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u}, \tag{1.2}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \Delta\varphi(\theta). \tag{1.3}$$

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Here $\mathbf{u}$ is the velocity field, p is the pressure, θ is the absolute temperature, ν is the constant kinematics viscosity and φ is a nonlinear function which usually expresses the thermal conductivity. In (1.2), ρ stands for a reference density constant corresponding to a reference temperature, which can be taken to be the mean temperature in the flow or the temperature at the boundary. The forcing term is given by $\mathbf{f}(\theta) = -(\rho(\theta)/\rho)\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity and $\rho(\theta)$ is the varying density in the forcing term. For a detailed discussion on the OB approximation, see *e.g.* Joseph [13], Rajagopal et al. [20] and the references cited therein.

During the last two decades, considerable progress has been made in the mathematical analysis of the OB equations (1.1)–(1.3). First Cannon and DiBenedetto [7] proved the existence of a unique, local in time, weak solution in $\mathbb{R}^N \times (0,T]$, where the advection for the temperature equation is satisfied with an extra term which is given. These results were carried out by transforming the problem into a singular integral equation which in turn was solved by using a contraction argument. They also proved a global existence theorem for small initial data, if the exterior forces field depends on the temperature in a suitable Lipschitz way. Later Morimoto [18] and Kagei [14] proved the existence of weak solutions to the classical OB problem, by using the Galerkin method, and their uniqueness in some Lebesgue spaces. Goncharova [11] and Díaz and Galiano [8] proved the existence of weak solutions for a generalization of the classical OB problem with temperature-depending viscosity and with nonlinear thermal diffusion by using also the Galerkin method. There, it was also proved the uniqueness of weak solutions in the special case of $N = 2$ and regularity results as well.

With respect to the qualitative properties of the weak solutions of the OB problems, it should be remarked that questions of time and spatial behavior have been studied by many authors. Hishida [12] proved that, when some parameters are small enough, a strong solution near a steady state exists globally in time and uniformly goes to the steady state, as $t \to \infty$, with exponential rate. In Rajopadhye et al. [21] is considered a generalized OB problem with dissipation and are established algebraic bounds, in the L^2-norm, for the decay rate of the associated energy. For the OB problem introduced in Díaz and Galiano [8], Galiano [10] proved the existence and spatial localization of the free boundaries $\theta = 0$. He has proved also the extinction in a finite time property but only for the temperature component of the weak solutions of the considered problem. As for the Navier–Stokes problem, it seems to be very difficult to obtain better qualitative time results for the velocity component of the OB problem weak solutions. In this paper we give a step towards this direction by extending the results established in Antontsev and Oliveira [4], for the Navier–Stokes problem, to the Oberbeck–Boussinesq problem. We consider the following modified OB equations

$$\operatorname{div}\mathbf{u} = 0, \tag{1.4}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\theta) - \alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u} - \nabla p + \nu\Delta\mathbf{u}, \tag{1.5}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \Delta\varphi(\theta) \tag{1.6}$$

in a general cylinder

$$Q_T := \Omega \times (0,T) \subset \mathbb{R}^N \times \mathbb{R}^+, \quad \text{with } \Gamma_T := \partial\Omega \times (0,T),$$

where Ω is a bounded domain with a compact boundary $\partial\Omega$ and $N \geqslant 2$ accounts for a general dimension. Eqs. (1.4)–(1.6) are supplemented by the initial and boundary conditions

$$\mathbf{u} = \mathbf{u}_0 \quad \text{and} \quad \theta = \theta_0 \quad \text{when } t = 0, \tag{1.7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \varphi(\theta) = \varphi_* \quad \text{on } \Gamma_T, \tag{1.8}$$

where $\mathbf{u}_0$, θ_0 and φ_* are given functions. In (1.5), α is a positive constant and σ is a temperature-depending function such that $\sigma(\theta) > 1$ for any $\theta \in \mathbb{R}$. Notice that, for the sake of simplicity, we have assumed $\rho \equiv 1$ in (1.5) and if we let $\alpha = 0$, then we fall in the usual OB problem. In the sequel, we will refer to the problem (1.4)–(1.8) as the modified Oberbeck–Boussinesq problem, or abbreviating the modified OB problem. For the motivation and some physical justification for considering the term $\alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u}$ in the momentum equation (1.5), we address the reader to Antontsev and Oliveira [4], where was studied the modified Navier–Stokes problem. Notice that, in the present case, the power σ is a temperature-depending function which brings much more difficulty to our model. By virtue of that, we will denote $\alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u}$ as a thermo-absorption term. A possible physical justification for the thermo-absorption term in (1.5), is the consideration, in the momentum equation, of a forcing term like

$$\mathbf{h}(\theta,\mathbf{u}) = \mathbf{f} - \alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u},$$

where here $\mathbf{f}$ is a given vector field. With a similar writing, we already have considered in Antontsev et al. [2] a two-dimensional stationary version of the OB problem, where the forcing term was assumed to satisfy

$$\mathbf{f}(\theta,\mathbf{u})\cdot\mathbf{u} \leqslant -\alpha|\mathbf{u}|^{\sigma(\theta)} \quad \forall(\theta,\mathbf{u}) \in \mathbb{R}\times\mathbb{R}^2.$$

For the modified OB problem (1.4)–(1.8) we will study the existence of weak solutions, their uniqueness and the asymptotic behavior in time of the velocity component. The outline of the paper is the following. In Section 2, we introduce the main notation used throughout this paper, as well some auxiliary results which will be very important in the sequel. The existence of weak solutions for the modified OB problem (1.4)–(1.8) will be proved, in Section 3, by introducing an iterative scheme to uncouple the system and by using well-known facts about the Navier–Stokes and the nonlinear diffusion problems. To handle the thermo-absorption term, we shall work in the context of the Lebesgue spaces with variable exponents to obtain the *a priori* estimates and the convergence of the correspondingly Galerkin approximations. In this section it is also proved the uniqueness result for $N=2$ by using the fact that the natural spaces for velocity and temperature are the same. The qualitative properties of the velocity component of the weak solutions as the extinction in a finite time or the asymptotic stability for large t are made in Section 4 by using some relations about the generalized Lebesgue norms together with a suitable energy method. The different obtained properties will depend on the range of the σ function.

2. Preliminaries

The notation used throughout this paper is largely standard in Mathematical Analysis and in particular in Mathematical Fluid Mechanics – see, *e.g.*, Lions [17] and Joseph [13]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C will denote a generic constant – generally a positive one, whose value will not be specified; it can change from one inequality to another. The dependence of C on other constants or parameters will always be clear from the exposition. Sometimes we will use subscripted letters attached to C to relate a constant with the result where it is derived from. In this paper, the notation Ω stands always for a domain, *i.e.*, a connected open subset of $\mathbb{R}^N$.

Let $1 \leqslant p \leqslant \infty$ and $\Omega \subset \mathbb{R}^N$, with $N \geqslant 1$, be a domain. We will use the classical Lebesgue spaces $\mathrm{L}^p(\Omega)$, whose norm is denoted by $\|\cdot\|_{\mathrm{L}^p(\Omega)}$. For any nonnegative k, $\mathrm{W}^{k,p}(\Omega)$ denotes the Sobolev space of all functions $u \in \mathrm{L}^p(\Omega)$ such that the weak derivatives $\mathrm{D}^\alpha u$ exist, in the generalized sense, and are in $\mathrm{L}^p(\Omega)$ for any multi-index α such that $0 \leqslant |\alpha| \leqslant k$. The norm in $\mathrm{W}^{k,p}(\Omega)$ is denoted by $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. The associated trace spaces are denoted by $\mathrm{W}^{k-1/p,p}(\partial\Omega)$. Given $T>0$ and a Banach space X, $\mathrm{L}^p(0,T;X)$ and $\mathrm{W}^{k,p}(0,T;X)$ denote the usual Lebesgue and Sobolev spaces used in evolutive problems, with norms denoted by $\|\cdot\|_{\mathrm{L}^p(0,T;X)}$ and $\|\cdot\|_{\mathrm{W}^{k,p}(0,T;X)}$. The corresponding spaces of vector-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $p=2$. In the last case, we use the abbreviations $\mathrm{W}^{k,2}=\mathrm{H}^k$ and $\mathrm{W}^{k-1/2,2}=\mathrm{H}^{k-1/2}$. By $\mathrm{C}^{0,\lambda}(\overline{\Omega})$, with $0<\lambda\leqslant 1$, we shall denote the Banach space of Hölder-continuous functions. In the special case of $\lambda=1$, it is called the Banach space of Lipschitz-continuous functions.

Let us denote by $\mathcal{P}(\Omega)$ the set of all measurable functions $p:\Omega \to [1,\infty]$ and define

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

Given $p \in \mathcal{P}(\Omega)$, we denote by $\mathrm{L}^{p(\cdot)}(\Omega)$ the space of all measurable functions u in Ω such that its semimodular is finite:

$$A_{p(\cdot)}(u) := \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty. \tag{2.9}$$

The space $\mathrm{L}^{p(\cdot)}(\Omega)$ is called Lebesgue space with variable exponent, or generalized Lebesgue space. Equipped with the norm

$$\|u\|_{\mathrm{L}^{p(\cdot)}(\Omega)} := \inf\left\{\lambda>0:\ A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}, \tag{2.10}$$

$\mathrm{L}^{p(\cdot)}(\Omega)$ becomes a Banach space. Note that the infimum in (2.10) is attained if $A_{p(\cdot)}(u)>0$. Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects. If $p^+<\infty$, $\mathrm{L}^{p(\cdot)}(\Omega)$ is separable and the space $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}^{p(\cdot)}(\Omega)$. Moreover, if

$$1 < p^- \leqslant p^+ < \infty, \tag{2.11}$$

$\mathrm{L}^{p(\cdot)}(\Omega)$ is reflexive. One problem in variable exponent Lebesgue spaces is the relation between the semimodular (2.9) and the norm (2.10). If (2.11) is satisfied, one can shows that

$$\|u\|_{L^{p(\cdot)}}^{p^-} - 1 \leqslant A_{p(\cdot)}(u) \leqslant \|u\|_{L^{p(\cdot)}}^{p^+} + 1. \tag{2.12}$$

For every $f \in \mathrm{L}^{p(\cdot)}(\Omega)$ and $g \in \mathrm{L}^{q(\cdot)}(\Omega)$, with p and q satisfying to (2.11) and $1/q(\cdot)+1/p(\cdot)=1$, the following generalized Hölder's inequality is valid

$$\int_\Omega uv\,dx \leqslant C\|u\|_{L^{p(\cdot)}}\|v\|_{L^{q(\cdot)}}, \quad C = 1+\frac{1}{p^-}+\frac{1}{q^-}. \tag{2.13}$$

If Ω is bounded, $p(\cdot) \leqslant q(\cdot)$ a.e. in Ω and $q^+<\infty$, then it hold the following continuous imbeddings:

$$L^\infty(\Omega) \hookrightarrow L^{q^+}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \hookrightarrow L^{p^-}(\Omega) \hookrightarrow L^1(\Omega). \tag{2.14}$$

An important result that will be used in the sequel is the famous Gagliardo–Nirenberg–Sobolev inequality.

Lemma 2.1. *Let Ω be a domain of $\mathbb{R}^N$, $N \geqslant 1$, with a locally compact boundary $\partial\Omega$. Assume that $u \in \mathrm{W}_0^{1,p}(\Omega)$. Then, for every fixed number $r \geqslant 1$ there exists a constant C depending only on N, p, r such that*

$$\|u\|_{\mathrm{L}^q(\Omega)} \leqslant C\|\nabla u\|^{\gamma}_{\mathrm{L}^p(\Omega)}\|u\|^{1-\gamma}_{\mathrm{L}^r(\Omega)}, \tag{2.15}$$

where $p, q \geqslant 1$, are linked by

$$\gamma = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}, \tag{2.16}$$

and their admissible range is:

(1) *If $N = 1$, $q \in [r, \infty]$, $\gamma \in [0, \frac{p}{p+r(p-1)}]$, $C = [1 + (p-1)/pr]^{\gamma}$;*
(2) *If $p < N$, $q \in [\frac{Np}{N-p}, r]$ if $r \geqslant \frac{Np}{N-p}$ and $q \in [r, \frac{Np}{N-p}]$ if $r \leqslant \frac{Np}{N-p}$, $\gamma \in [0, 1]$ and $C = [(N-1)p/(N-p)]^{\gamma}$;*
(3) *If $p \geqslant N > 1$, $q \in [r, \infty)$, $\gamma \in [0, \frac{Np}{Np+r(p-N)})$ and $C = \max\{q(N-1)/N, 1 + (p-1)pr\}^{\gamma}$.*

When $\gamma = 1$, (2.15) is known as the Sobolev inequality and, in this case, if $q = p = 2$, then (2.15) is usually denominated as the Poincaré inequality. This result is valid whether the domain Ω is bounded or not and notice the constant C does not depend on Ω. See the proof in Ladyzhenskaya et al. [16, p. 62]. The extension of Gagliardo–Nirenberg inequality (2.15) to generalized Lebesgue spaces was proved by Kopaliani and Chelidze [15] for $\gamma \equiv j/k = (1/q(\cdot) - 1/p(\cdot))/(1/r(\cdot) - 1/p(\cdot))$ and $0 < k/m < 1$ under the assumption that the exponents $p(\cdot)$ and $r(\cdot)$ are in $\mathcal{B}(\Omega)$. $\mathcal{B}(\Omega)$ denotes there the class of all exponents $p(\cdot)$ for which the Hardy–Littlewood operator is bounded in $\mathrm{L}^{p(\cdot)}(\Omega)$ (see [15]). Without this assumption on the exponents, it is possible to prove the following Sobolev generalized inequality

$$\|u\|_{\mathrm{L}^{q(\cdot)}(\Omega)} \leqslant C\|\nabla u\|_{\mathrm{L}^{p(\cdot)}(\Omega)}, \qquad p(\cdot) \leqslant q(\cdot) \leqslant p^*(\cdot) := \frac{Np(\cdot)}{N - kp(\cdot)}, \tag{2.17}$$

where $p \in \mathcal{P}(\Omega)$ and $1 < p^- \leqslant p(\cdot) \leqslant p^+ < k/N$ and $C = C(\Omega, N, p, q)$.

In this work we shall make use of two other well-known inequalities written in the following lemma.

Lemma 2.2. *For all $p \in \mathcal{P}(\Omega)$ satisfying to (2.11) and for all constant $\delta \geqslant 0$, there exist constants C_1 and C_2, depending on p^-, p^+ and N, such that for all $\xi, \eta \in \mathbb{R}^N$, $N \geqslant 1$,*

$$\left||\xi|^{p(\mathbf{x})-2}\xi - |\eta|^{p(\mathbf{x})-2}\eta\right| \leqslant C_1|\xi - \eta|^{1-\delta}\big(|\xi| + |\eta|\big)^{p(\mathbf{x})-2+\delta} \tag{2.18}$$

and

$$\big(|\xi|^{p(\mathbf{x})-2}\xi - |\eta|^{p(\mathbf{x})-2}\eta\big) \cdot (\xi - \eta) \geqslant C_2|\xi - \eta|^{2+\delta}\big(|\xi| + |\eta|\big)^{p(\mathbf{x})-2-\delta}. \tag{2.19}$$

Proof. The proof can be easily adapted from the one given, for constant p, in Barret and Liu [6]. □

For a detailed exposition of the theory of Lebesgue (and Sobolev) spaces with variable exponents, we address the reader to the monograph by Diening et al. [9].

3. Weak formulation

In this section, we will prove the existence of weak solutions for the modified OB problem (1.4)–(1.8) and, in the case $N = 2$, its uniqueness. If $\alpha = 0$ in (1.5), then we fall in the classical OB problem and it is well known that the corresponding problem has a weak solution which is unique if $N = 2$ (see *e.g.* Morimoto [18] and Kagei [14]). In order to define the notion of a weak solution to the modified problem (1.4)–(1.8), let us introduce the free divergence function spaces:

$$\mathcal{V} := \big\{\mathbf{v} \in \mathbf{C}_0^{\infty}(\Omega)\colon\ \operatorname{div}\mathbf{v} = 0\big\};$$
$$\mathbf{H}^{r-1} := \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^r(\Omega), \quad r \geqslant 1;$$
$$\mathbf{V}^s := \text{closure of } \mathcal{V} \text{ in } \mathbf{H}^s(\Omega), \quad s \geqslant 1;$$

where, for simplicity, we can assume s as the smaller integer not lesser than $N/2$, to avoid the complicated Sobolev spaces with s non-integer. When $r = 2$, we denote as usual $\mathbf{H}^1$ simply by $\mathbf{H}$ and when $s = 1$, which happens when the dimension is $N = 2$, we simply denote, as usual, $\mathbf{V}^1$ by $\mathbf{V}$. For the theory of all these spaces, we address the reader to the monograph by Lions [17]. We only want to note that

$$\mathbf{V}^s \hookrightarrow \mathbf{V} \hookrightarrow \mathbf{H} = \mathbf{H}' \hookrightarrow \mathbf{V}' \hookrightarrow \mathbf{V}^{s'}, \quad s > 1. \tag{3.20}$$

Moreover the compact embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ implies that the embedding $\mathbf{V}^s \hookrightarrow \mathbf{H}$ is also compact for any $s \geqslant 1$.

In order to deal with the thermo-absorption term, let us set

$$q := \sigma \circ \theta, \tag{3.21}$$

where $\circ$ denotes the composition operator. Given $\sigma \in \mathcal{P}(\mathbb{R})$ satisfying to (2.11) and $\theta \in L^\infty(Q_T)$, we can readily see that $q \in \mathcal{P}(Q_T)$ and also satisfies to (2.11). Moreover, $q(\cdot,t) \in \mathcal{P}(\Omega)$ for all $t \in [0,T]$ and still satisfies to (2.11).

The notion of weak solution for the modified OB problem (1.4)–(1.8) follows in a standard manner.

Definition 3.1. Let us assume that $\sigma \in \mathcal{P}(\mathbb{R})$ and satisfies to (2.11). The pair $(\mathbf{u},\theta)$ is a weak solution of the modified OB problem (1.4)–(1.8), if:

(1) $\mathbf{u} \in \mathrm{L}^2(0,T;\mathbf{V}) \cap \mathrm{L}^\infty(0,T;\mathbf{H}) \cap \mathbf{L}^{q(\cdot,\cdot)}(Q_T)$;
(2) $\theta \in \mathrm{L}^\infty(Q_T)$ and $\varphi(\theta) \in \{\varphi_* + \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega))\}$;
(3) $\mathbf{u}(\cdot,0) = \mathbf{u}_0$ a.e. in Ω, and for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^N(\Omega) \cap \mathbf{L}^{q(t)}(\Omega)$, $q(t) = q(\cdot,t)$,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{u}(t):\nabla\mathbf{v}\,d\mathbf{x} + \int_\Omega \big[\big(\mathbf{u}(t)\cdot\nabla\big)\mathbf{u}(t)\big]\cdot\mathbf{v}\,d\mathbf{x}\\ &\quad + \alpha\int_\Omega \big|\mathbf{u}(t)\big|^{q(t)-2}\mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\big(\theta(t)\big)\cdot\mathbf{v}\,d\mathbf{x}\end{aligned} \tag{3.22}$$

for a.a. $t \geqslant 0$;
(4) for every $\zeta \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)) \cap \mathrm{W}^{1,1}(0,T;\mathrm{L}^2(\Omega))$ with $\zeta(T) = 0$

$$\int_{Q_T} \theta\zeta_t\,d\mathbf{x}\,dt + \int_{Q_T} \nabla\big(\varphi(\theta) - \theta\mathbf{u}\big)\cdot\nabla\zeta\,d\mathbf{x}\,dt = \int_\Omega \theta_0\zeta(0)\,d\mathbf{x}.$$

With respect to the problem data, in the sequel we shall make the following assumptions:

$$\theta_0 \geqslant 0 \quad \text{and} \quad \theta_0 \in \mathrm{L}^\infty(\Omega); \tag{3.23}$$

$$\varphi_* \in \mathrm{L}^2\big(0,T;\mathrm{H}^{1/2}(\partial\Omega)\big); \tag{3.24}$$

$$\varphi \in \mathrm{C}^1(0,\infty), \qquad \varphi(0) = 0 \quad \text{and} \quad \varphi \text{ is non-decreasing}; \tag{3.25}$$

$$\varphi^{-1} \in \mathrm{C}^{0,\lambda}\big([0,\infty)\big), \quad 0 < \lambda < 1 \ \big(\text{needed only if } |\nabla\mathbf{f}| \neq \mathbf{0} \text{ or } \sigma' \neq 0\big); \tag{3.26}$$

$$\mathbf{u}_0 \in \mathbf{H}; \tag{3.27}$$

$$\mathbf{f} \in \mathbf{C}^{0,1}\big([0,\infty),\mathbb{R}^N\big); \tag{3.28}$$

$$\sigma \in \mathrm{C}^{0,1}\big([0,\infty)\big); \tag{3.29}$$

$$\exists\sigma^-,\sigma^+ \in (1,\infty): \quad 1 < \sigma^- \leqslant \sigma(\theta) \leqslant \sigma^+ < \infty \quad \forall\theta \in \mathbb{R}. \tag{3.30}$$

Remark 3.1. For $N \leqslant 4$, (3.22) holds for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^{q(t)}(\Omega)$, because, due to Sobolev's inequality, $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^N(\Omega)$ for $N \leqslant 4$. Analogously for $\sigma(\cdot)$ satisfying to (3.30) with $\sigma^+ \leqslant 4$, (2.13)–(2.14) and Sobolev's inequality, and still (3.21), imply that $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^{q(t)}(\Omega)$ for all $t \geqslant 0$. In this case, (3.22) holds for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^N(\Omega)$. If both $N, \sigma^+ \leqslant 4$, then (3.22) holds only for every $\mathbf{v} \in \mathbf{V}$.

Let us denote by θ_* the function which simultaneously extends φ_* and θ_0 to the entire domain Q_T. This function can be defined as the unique weak solution of the following initial–boundary value problem

$$\begin{cases} \dfrac{\partial\theta}{\partial t} = \Delta\varphi(\theta) & \text{in } Q_T,\\ \theta = \theta_0 & \text{for } t = 0,\\ \varphi(\theta) = \varphi_* & \text{on } \Gamma_T. \end{cases}$$

Its unique existence is proved under the assumptions (3.23)–(3.25) (see *e.g.* Alt and Luckhaus [1]). Therefore it is reasonable to assume that the extension function θ_* satisfies to

$$\theta_* \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big) \cap \mathrm{H}^1\big(0,T;\mathrm{L}^2(\Omega)\big) \cap \mathrm{L}^\infty(Q_T). \tag{3.31}$$

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz-continuous boundary $\partial\Omega$, and assume that* (3.23)–(3.31) *are fulfilled. Then, there exists, at least, a weak solution of the modified OB problem* (1.4)–(1.8) *in the sense of Definition* 3.1.

To prove Theorem 3.1 we proceed as in Díaz and Galiano [8] and adapt well-known results for the Navier–Stokes problem (see *e.g.* Lions [17]) and for nonlinear diffusion equations (see *e.g.* Alt and Luckhaus [1]). However, it is worth to notice that, in (1.5), additionally to the usual nonlinear term for the classical Navier–Stokes equations, $(\mathbf{u}\cdot\nabla)\mathbf{u}$, we have another one, the thermo-absorption term $\alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u}$. To prove the convergence of the correspondingly Galerkin approximations, we will need to work in the context of generalized Lebesgue spaces.

Proof. We will split the proof of Theorem 3.1 into several steps.

Step 1. *We introduce an iterative scheme to uncouple the problem.* For each $n \in \mathbb{N}$, we set

$$\operatorname{div}\mathbf{u}_n = 0 \quad \text{in } Q_T, \tag{3.32}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n = \mathbf{f}(\theta_{n-1}) - \alpha|\mathbf{u}_n|^{\sigma(\theta_{n-1})-2}\mathbf{u}_n - \nabla p + \nu\Delta\mathbf{u}_n \quad \text{in } Q_T, \tag{3.33}$$

$$\frac{\partial \theta_n}{\partial t} + \mathbf{u}_{n-1}\cdot\nabla\theta_n = \Delta\varphi(\theta_n) \quad \text{in } Q_T, \tag{3.34}$$

$$\mathbf{u}_n = \mathbf{u}_0 \quad \text{and} \quad \theta_n = \theta_0 \quad \text{when } t=0, \tag{3.35}$$

$$\mathbf{u}_n = \mathbf{0} \quad \text{and} \quad \theta_n = \varphi_* \quad \text{on } \Gamma_T. \tag{3.36}$$

Step 2. *Given a temperature, to prove the existence of a velocity.* Let us consider the problem

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } Q_T, \tag{3.37}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\omega) - \alpha|\mathbf{u}|^{q(\cdot,\cdot)-2}\mathbf{u} - \nabla p + \nu\Delta\mathbf{u} \quad \text{in } Q_T, \tag{3.38}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{when } t=0, \tag{3.39}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_T, \tag{3.40}$$

where, for simplifying the notation, $\mathbf{u}$ and ω stay for $\mathbf{u}_n$ and θ_{n-1}, respectively, and, according to (3.21), $q = \sigma\circ\omega$. Problem (3.37)–(3.40) corresponds to the modified Navier–Stokes problem studied in Antontsev and Oliveira [4], but with q depending upon the spatial and time variables. To emphasize this dependence, we shall write $q(\cdot,\cdot)$, instead of q, or $q(\cdot,t)$ instead of the usual written $q(t)$.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz-continuous boundary $\partial\Omega$, and assume that conditions* (3.27)–(3.30) *are fulfilled. Assume also that $\omega \in \mathrm{L}^\infty(Q_T)$. Then, there exists, at least, a weak solution of the modified Navier–Stokes problem* (3.37)–(3.40) *satisfying to* (1) *and* (3) *of Definition* 3.1 *and such that $\mathbf{u}_t \in \mathrm{L}^2(0,T;\mathbf{V}^{s'})$.*

Proof. 1. *Existence of approximate solutions.* We proceed as in Antontsev and Oliveira [4] by searching, for each $m \in \mathbb{N}$, for an approximate solution $\mathbf{u}_m$ of (3.22) in the form

$$\mathbf{u}_m(t) = \sum_{k=1}^{m} c_{km}(t)\mathbf{v}_k, \tag{3.41}$$

where $\mathbf{v}_k \in \mathbf{V}^m$, $\mathbf{V}^m$ is the m-dimensional space spanned by m elements of the basis of $\mathbf{V}^s :=$ closure of $\mathcal{V}$ in $\mathbf{H}^s(\Omega)$ and $c_{km}(t)$ are the functions we look for. Note that the exponent s is chosen such that $\mathbf{V}^s \hookrightarrow \mathbf{L}^\infty(\Omega)$, *i.e.* $s > N/2$. In particular, we have $\mathbf{V}^s \hookrightarrow \mathbf{V}$. The functions $c_{km}(t)$ are found by solving the following system of ordinary differential equations obtained from (3.22):

$$\frac{d}{dt}\int_\Omega \mathbf{u}_m(t)\cdot\mathbf{v}_k\,d\mathbf{x} + \nu\int_\Omega \nabla\big(\mathbf{u}_m(t)\big):\nabla\mathbf{v}_k\,d\mathbf{x} + \int_\Omega \big(\mathbf{u}_m(t)\cdot\nabla\big)\mathbf{u}_m(t)\cdot\mathbf{v}_k\,d\mathbf{x} + \alpha\int_\Omega \big|\mathbf{u}_m(t)\big|^{q(\cdot,t)-2}\mathbf{u}_m(t)\cdot\mathbf{v}_k\,d\mathbf{x}$$
$$= \int_\Omega \mathbf{f}\big(\omega(t)\big)\cdot\mathbf{v}_k\,d\mathbf{x}; \tag{3.42}$$

$$c_{km}(0) = \int_\Omega \mathbf{u}_{0m}\cdot\mathbf{v}_k\,d\mathbf{x}; \tag{3.43}$$

for $k = 1, \dots, m$ and where $\mathbf{u}_{0m} = \mathbf{u}_m(0) \in \mathbf{V}^m$ is such that

$$\mathbf{u}_{0m} \to \mathbf{u}_0 \quad \text{strongly in } \mathbf{H} \text{ as } m \to \infty.$$

Since $\omega \in \mathrm{L}^\infty(Q_T)$ and $\mathbf{f}$ is Lipschitz-continuous, then

$$\int_\Omega \mathbf{f}\big(\omega(t)\big) \cdot \mathbf{v}_k \, d\mathbf{x} \in \mathrm{L}^\infty(0, T) \quad \text{for all } k.$$

On the other hand, since σ (and consequently q) is Lipschitz-continuous, using (2.12) and Hölder's generalized inequality (2.13), we can prove that also

$$\int_\Omega \big|\mathbf{u}_m(t)\big|^{q(\cdot,t)-2} \mathbf{u}_m(t) \cdot \mathbf{v}_k \, d\mathbf{x} \in \mathrm{L}^\infty(0, T) \quad \text{for all } k.$$

From the elementary theory of ordinary differential equations, problem (3.41)–(3.43) has a unique solution $c_{km} \in \mathrm{C}^1[0, T_m]$, for some small interval of time $[0, T_m] \subset [0, T]$.

2. *A priori estimates.* We multiply (3.42) by $c_{km}(t)$, add these equations from $k = 1$ to $k = m$ and argue as in Lions [17] to obtain

$$\big\|\mathbf{u}_m(t)\big\|^2_{\mathbf{L}^2(\Omega)} + \nu \int_0^t \big\|\nabla \mathbf{u}_m(s)\big\|^2_{\mathbf{L}^2(\Omega)} \, ds + 2\alpha \int_{Q_t} \big|\mathbf{u}_m(t)\big|^{q(\cdot,s)} \, d\mathbf{x} \, ds \leqslant \|\mathbf{u}_{0m}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu} \int_0^t \big\|\mathbf{f}\big(\omega(s)\big)\big\|^2_{V'} \, ds \tag{3.44}$$

for a.a. $t < T_m$. Since $\|\mathbf{u}_{0m}\|_{\mathbf{L}^2(\Omega)} \leqslant \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}$, the assumptions (3.27) and (3.28), and also $\omega \in \mathrm{L}^\infty(Q_T)$, justify that the right-hand side of (3.44) is finite. In particular, we deduce that $T_m = T$ for all $m \in \mathbb{N}$. On the other hand, one can readily see that, from (3.44),

$$\mathbf{u}_m \text{ remains bounded in } \mathrm{L}^\infty(0, T; \mathbf{H}), \tag{3.45}$$

$$\mathbf{u}_m \text{ remains bounded in } \mathrm{L}^2(0, T; \mathbf{V}). \tag{3.46}$$

Moreover, using also (3.44) and the relation between the semimodular $A_{q(\cdot,\cdot)}(\mathbf{u})$ and the norm $\|\mathbf{u}\|_{\mathbf{L}^{q(\cdot,\cdot)}(Q_T)}$ (see (2.12)), we can prove that

$$\mathbf{u}_m \text{ remains bounded in } \mathbf{L}^{q(\cdot,\cdot)}(Q_T), \tag{3.47}$$

$$|\mathbf{u}_m|^{q(\cdot,\cdot)-2}\mathbf{u}_m \text{ remains bounded in } \mathbf{L}^{q'(\cdot,\cdot)}(Q_T). \tag{3.48}$$

Proceeding in a similar way, and using, in addition, generalized Holder's inequality (2.13), we can prove that

$$|\mathbf{u}_m|^{q(\cdot,\cdot)-2}\mathbf{u}_m \text{ remains bounded in } L^2\big(0, T; \mathbf{V}_s'\big). \tag{3.49}$$

Arguing as for the classical Navier–Stokes problem (see Lions [17]) and using in addition (3.49), we can prove that

$$\frac{\partial \mathbf{u}_m}{\partial t} \text{ remains bounded in } \mathrm{L}^2\big(0, T; \mathbf{V}_s'\big). \tag{3.50}$$

3. *Passing to the limit.* From (3.45)–(3.50), there exist functions $\mathbf{u}$ and $\mathbf{Z}$, and there exists a subsequence, which we still denote by $\mathbf{u}_m$, such that

$$\mathbf{u}_m \to \mathbf{u} \quad \text{weak-star in } \mathrm{L}^\infty(0, T; \mathbf{H}) \text{ as } m \to \infty, \tag{3.51}$$

$$\mathbf{u}_m \to \mathbf{u} \quad \text{weakly in } \mathrm{L}^2(0, T; \mathbf{V}) \text{ as } m \to \infty, \tag{3.52}$$

$$\mathbf{u}_m \to \mathbf{u} \quad \text{weakly in } \mathbf{L}^{q(\cdot,\cdot)}(Q_T) \text{ as } m \to \infty, \tag{3.53}$$

$$|\mathbf{u}_m|^{q(\cdot,\cdot)-2}\mathbf{u}_m \to \mathbf{Z} \quad \text{weakly in } \mathbf{L}^{q'(\cdot,\cdot)}(Q_T) \text{ as } m \to \infty \tag{3.54}$$

and

$$\frac{\partial \mathbf{u}_m}{\partial t} \to \frac{\partial \mathbf{u}}{\partial t} \quad \text{weakly in } \mathrm{L}^2\big(0, T; \mathbf{V}_s'\big) \text{ as } m \to \infty. \tag{3.55}$$

Then, due to (3.20), (3.52) and (3.55), and according to a well-known compactness result (see Lions [17, p. 58]),

$$\mathbf{u}_m \to \mathbf{u} \quad \text{strongly in } \mathrm{L}^2(0, T; \mathbf{H}) \text{ and a.e. in } Q_T, \text{ as } m \to \infty. \tag{3.56}$$

Now, we multiply (3.42) by $\psi \in C^1([0,T])$, with $\psi(T)=0$, integrate the resulting equations from 0 to T and pass to the limit $m \to \infty$ by using the results (3.52)–(3.54) and (3.56). Due to the arbitrariness of ψ, we obtain for every $\mathbf{v} \in \mathbf{V}_s$ and for a.a. $t \in (0,T)$

$$\frac{d}{dt}\int_\Omega \mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega \big(\mathbf{u}(t)\cdot\nabla\big)\mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{u}(t):\nabla\mathbf{v}\,d\mathbf{x} + \alpha\int_\Omega \mathbf{Z}(t)\cdot\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f}\big(\omega(t)\big)\cdot\mathbf{v}\,d\mathbf{x}. \tag{3.57}$$

4. *Use of monotonicity.* To finish the proof of Lemma 3.1, it remains to prove that $\mathbf{u}=\mathbf{Z}$. Proceeding as in Lions [17, pp. 212–215], using (3.57), we can prove that for a.a. $t \in (0,T)$

$$\frac{1}{2}\int_\Omega \big|\mathbf{u}(t)\big|^2\,d\mathbf{x} + \nu\int_{Q_t} \big|\nabla\mathbf{u}(s)\big|^2\,d\mathbf{x}\,ds + \int_{Q_t} \mathbf{Z}(s)\cdot\mathbf{u}(s)\,d\mathbf{x}\,ds \geqslant \int_{Q_t} \mathbf{f}\big(\omega(s)\big)\cdot\mathbf{u}(s)\,d\mathbf{x}\,ds + \frac{1}{2}\int_\Omega |\mathbf{u}_0|^2\,d\mathbf{x}. \tag{3.58}$$

On the other hand, using (2.19), the operator defined by the thermo-absorption term satisfies to the following monotonicity property

$$\big(|\mathbf{u}_1|^{q(\cdot,\cdot)-2}\mathbf{u}_1 - |\mathbf{u}_2|^{q(\cdot,\cdot)-2}\mathbf{u}_2\big)\cdot(\mathbf{u}_1-\mathbf{u}_2) \geqslant 0 \quad \forall \mathbf{u}_1,\mathbf{u}_2 \in \mathbb{R}^N. \tag{3.59}$$

Finally, using (3.58) and (3.59), we can prove that

$$\int_{Q_T} \big(\mathbf{Z} - |\mathbf{v}|^{q(\cdot,\cdot)-2}\mathbf{v}\big)\cdot(\mathbf{u}-\mathbf{v})\,d\mathbf{x}\,dt \geqslant 0 \quad \forall \mathbf{v} \in L^2(0,T;\mathbf{V}).$$

Analogously we can obtain the reverse of this inequality and, in consequence, $\mathbf{Z} = |\mathbf{u}|^{q(\cdot,\cdot)-2}\mathbf{u}$. This concludes the proof of Lemma 3.1. □

Step 3. *Given a velocity field, to prove the existence of a temperature*. Here, we consider the problem for the nonlinear diffusion equation

$$\frac{\partial\theta}{\partial t} + \mathbf{w}\cdot\nabla\theta = \Delta\varphi(\theta) \quad \text{in } Q_T, \tag{3.60}$$

$$\theta(\cdot,0) = \theta_0 \quad \text{in } \Omega, \tag{3.61}$$

$$\theta = \varphi_* \quad \text{on } \Gamma_T, \tag{3.62}$$

where again, for simplifying the notation, $\mathbf{w}$ and θ stay for $\mathbf{u}_{n-1}$ and θ_n, respectively.

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz-continuous boundary $\partial\Omega$, and assume* (3.23)–(3.25) *are fulfilled. Assume also that $\mathbf{w} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$. Then, there exists a weak solution of the problem* (3.60)–(3.62) *satisfying to* (2) *and* (4) *of Definition* 3.1 *and such that $\theta \in C(0,T;H^{-1}(\Omega))$.*

Proof. The proof relies on an approximation argument. We consider the sequence of approximating problems (3.60)–(3.62), with $\mathbf{w}$ replaced by $\mathbf{w}_k$, $k \in \mathbb{N}$. It is assumed that for every $k \in \mathbb{N}$, $\mathbf{w}_k \in L^r(0,T;\mathbf{H}^{r-1})$. Moreover, we assume that

$$\|\mathbf{w}_k\|_{\mathbf{L}^\infty(Q_T)} \leqslant k \quad \forall k \in \mathbb{N} \tag{3.63}$$

and

$$\mathbf{w}_k \to \mathbf{w}, \quad \text{as } j \to \infty, \text{ in } \mathbf{L}^r(Q_T). \tag{3.64}$$

The exponent r is chosen in a way such that the embedding $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \hookrightarrow \mathbf{L}^r(Q_T)$ is verified. This happens for $r = 2+4/N$ if $1/2-1/N>0$, or $r = 4-1/s$ for any $s \geqslant 1$ if $1/2-1/N \leqslant 0$. Using (3.63) and the assumptions (3.23)–(3.25) and (3.31), we can use well-known results (see *e.g.* Alt and Luckhaus [1]) to prove that, for every $k \in \mathbb{N}$, there exists a unique weak solution θ_k to the approximating problem (3.60)–(3.62). Then we proceed to obtain some uniform estimates which allow us to extract subsequences θ_k such that

$$\theta_k \to \theta \quad \text{weak-star in } L^\infty(Q_T) \text{ as } k \to \infty, \tag{3.65}$$

$$\varphi(\theta_k) \to \psi \quad \text{weakly in } L^2\big(0,T;H^1(\Omega)\big) \text{ as } k \to \infty, \tag{3.66}$$

$$\frac{\partial\theta_k}{\partial t} \to \frac{\partial\theta}{\partial t} \quad \text{weakly in } L^2\big(0,T;H^{-1}(\Omega)\big) \text{ as } k \to \infty. \tag{3.67}$$

The proof of Lemma 3.2 finishes by a standard passing to the limit, where besides (3.64)–(3.67) it is used the convergence $\theta_k \to \theta$, as $k \to \infty$, in $C(0,T;H^{-1}(\Omega))$ and the equality $\psi = \varphi(\theta)$. See the details in Díaz and Galiano [8]. □

Step 4. *To extract convergent subsequences to the weak solution.* Lemmas 3.1 and 3.2 show us that, for each $n \in \mathbb{N}$, there exists a couple of functions $(\mathbf{u}_n, \theta_n)$ solution to the problem (3.32)–(3.36). Proceeding in a standard manner, we can extract subsequences, still denoted by $\mathbf{u}_n$ and θ_n, such that (3.51)–(3.56) hold with $n = m$ and $\mathbf{Z} = \mathbf{u}$, and also (3.65)–(3.67) hold, here with $n = k$. Then, we can pass to the limit in the following equations

$$-\int_{Q_T} \mathbf{u}_n \cdot \mathbf{v}\phi' \, d\mathbf{x}\, dt + \nu \int_{Q_T} \nabla \mathbf{u}_n : \nabla \mathbf{v}\phi \, d\mathbf{x}\, dt + \int_{Q_T} \big[(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n\big] \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt + \alpha \int_{Q_T} |\mathbf{u}_n|^{q_{n-1}-2}\mathbf{u}_n \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt$$
$$= \int_{Q_T} \mathbf{f}(\theta_{n-1}) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt, \quad q_{n-1} = \sigma \circ \theta_{n-1},$$
$$-\int_{Q_T} \theta_n \zeta_t \, d\mathbf{x}\, dt + \kappa \int_{Q_T} \nabla\varphi(\theta_n) \cdot \nabla\zeta \, d\mathbf{x}\, dt = \int_{Q_T} \theta_n \mathbf{u}_{n-1} \cdot \nabla\zeta \, d\mathbf{x}\, dt + \int_{\Omega} \theta_0 \zeta(0)\, d\mathbf{x},$$

where $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^N(\Omega) \cap \mathbf{L}^{q(t)}(\Omega)$ (recall that $q(t) = q(\cdot, t)$), $\phi \in \mathrm{C}_0^\infty(0, T)$ and $\zeta \in \mathrm{L}^2(0, T; \mathrm{H}_0^1(\Omega)) \cap \mathrm{W}^{1,1}(0, T; \mathrm{L}^2(\Omega))$. Due to what have been proved in Lemmas 3.1 and 3.2, we only need to justify the coupling terms. From (3.56) with $m = n$ and (3.65) with $k = n$, we deduce that

$$\int_{Q_T} \theta_n \mathbf{u}_{n-1} \cdot \nabla\zeta\, \phi \, d\mathbf{x}\, dt \to \int_{Q_T} \theta \mathbf{u} \cdot \nabla\zeta\, \phi \, d\mathbf{x}\, dt$$

for any $\zeta \in \mathrm{L}^2(0, T; \mathrm{H}_0^1(\Omega))$ and any $\phi \in \mathrm{C}_0^\infty(0, T)$. To prove the convergence of the other coupling terms, lets us assume that $|\nabla \mathbf{f}| \neq 0$ and $\sigma' \neq 0$. If $|\nabla \mathbf{f}| = 0$ or $\sigma' = 0$, then the convergence of each of the corresponding terms follows as in the modified Navier–Stokes problem (see Antontsev and Oliveira [4]). The following convergence result is proved in Díaz and Galiano [8] by using (3.67) and the assumption (3.26): for any $p \geqslant 2$

$$\theta_n \to \theta \quad \text{strongly in } \mathrm{L}^p(Q_T) \text{ and a.e. in } Q_T \text{ as } n \to \infty. \tag{3.68}$$

Now, since $\mathbf{f}$ is Lipschitz-continuous (cf. (3.28)), then

$$\mathbf{f}(\theta_n) \to \mathbf{f}(\theta) \quad \text{strongly in } \mathbf{L}^p(Q_T) \text{ and a.e. in } Q_T \text{ as } n \to \infty \tag{3.69}$$

and, as a consequence,

$$\int_{Q_T} \mathbf{f}(\theta_{n-1}) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt \to \int_{Q_T} \mathbf{f}(\theta) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt$$

for any $\mathbf{v} \in \mathbf{V}$ and any $\phi \in \mathrm{C}_0^\infty(0, T)$. Analogously as for (3.69), from (3.68) and (3.21), and once that σ is Lipschitz-continuous too (cf. (3.29)),

$$q_n \to q \quad \text{strongly in } \mathrm{L}^p(Q_T) \text{ and a.e. in } Q_T \text{ as } n \to \infty. \tag{3.70}$$

On the other hand, we can write

$$\int_{Q_T} \big(|\mathbf{u}_n|^{q_{n-1}-2}\mathbf{u}_n - |\mathbf{u}|^{q-2}\mathbf{u}\big) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt = I_1 + I_2,$$

where

$$I_1 := \int_{Q_T} \big(|\mathbf{u}_n|^{q_{n-1}-2}\mathbf{u}_n - |\mathbf{u}_n|^{q-2}\mathbf{u}_n\big) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt,$$
$$I_2 := \int_{Q_T} \big(|\mathbf{u}_n|^{q-2}\mathbf{u}_n - |\mathbf{u}|^{q-2}\mathbf{u}\big) \cdot \mathbf{v}\phi \, d\mathbf{x}\, dt.$$

The convergence $I_2 \to 0$ follows by (3.54), with $\mathbf{Z} = \mathbf{u}$, and from relation (3.70) we can prove that $I_1 \to 0$, both for any $\mathbf{v} \in \mathbf{L}^{q(t)}(\Omega)$ and any $\phi \in \mathrm{C}_0^\infty(0, T)$. Thus we have proved Theorem 3.1. □

Remark 3.2. It is possible to prove the above theorem by using a slightly different approach: first, given a velocity $\mathbf{w}$, to prove the existence of a temperature θ, solution of the problem (3.60)–(3.62); next, to define a nonlinear operator $\Lambda(\mathbf{w}) = \theta$ and to prove it is continuous; after, given a temperature ω, to prove the existence of a velocity field $\mathbf{u}$, solution of the problem (3.37)–(3.40); then, to define a nonlinear operator $\Pi(\omega) = \mathbf{u}$ and to prove it is continuous; finally, to use Schauder's theorem to prove the composite operator $\Pi\Lambda$ has a fixed point. This approach was considered in Antontsev et al. [2] for a Boussinesq like stationary problem.

The following result is very useful to deal with the kinetics energy associated with the modified OB problem (1.4)–(1.8).

Theorem 3.2. *Assume that the conditions of Theorem* 3.1 *are fulfilled. Then any couple of weak solutions of the modified OB problem* (1.4)–(1.8) *in the sense of Definition* 3.1 *satisfies to*

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)}+\nu\left\|\nabla\mathbf{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)}+\alpha\int_\Omega\left|\mathbf{u}(t)\right|^{q(t)}d\mathbf{x}\leqslant\int_\Omega\mathbf{u}(t)\cdot\mathbf{f}\big(\theta(t)\big)\,d\mathbf{x}\tag{3.71}$$

for a.a. $t\in[0,T]$*, where, according to* (3.21)*,* $q=\sigma\circ\theta$*.*

Proof. The proof is straightforward. We take the limit inf, as $n\to\infty$, of the equation from which one derives (3.44), with ω replaced by θ. Then from (3.52), (3.53), (3.56) and a classical property of weak limits, we obtain (3.71). □

With respect to the uniqueness of weak solutions for the modified problem (1.4)–(1.8), we know that the answer to this question is closely related with the same issue for the Navier–Stokes problem obtained from (1.4)–(1.8) by assuming an isothermal process. Uniqueness of weak solutions, in the large, is proved only for $N=2$. For $N=3$ this result is proved only for a short interval of time. In consequence, we are only able to establish a uniqueness result for the problem (1.4)–(1.8) in the 2-D case.

Theorem 3.3. *Let* $N=2$ *and assume that* $\varphi^{-1}\in \mathrm{C}^{0,1}[0,\infty)$ *and the conditions of Theorem* 3.1 *are fulfilled. Then a weak solution of the OB modified problem* (1.4)–(1.8) *in the sense of Definition* 3.1 *is unique.*

Proof. Let $(\mathbf{u}_1,\theta_1)$ and $(\mathbf{u}_2,\theta_2)$ be two weak solutions of the OB modified problem (1.4)–(1.8). We firstly notice that assumption $\varphi^{-1}\in\mathrm{C}^{0,1}[0,\infty)$ and Theorem 3.1 imply that $\nabla\varphi(\theta)\in\mathbf{L}^2(Q_T)$. This and the fact that $\theta_t\in\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$ is used in Díaz and Galiano [8] to prove that $\theta_1=\theta_2$. Arguing as in Antontsev and Oliveira [4] and using (2.19), we can prove that $\mathbf{u}_1=\mathbf{u}_2$. □

4. Asymptotic stability

In this section we shall study the behavior in time of the velocity component in the weak solutions of the modified OB problem (1.4)–(1.8). The properties established here are concerned with the vanishing of the velocity component in a finite time and, when this is not possible, to see how it decays for large t. From the Fluid Mechanics viewpoint, these properties are related with stopping the fluid flow at some time, possibly infinity. It is well known, in Fluid Mechanics, that it is possible to stop a fluid only by the thermo-mechanics process of phase changing. Therefore this work to be consistent, we have to consider a small range of temperature, say $\theta\in[m,M]$, where $0\leqslant m<M<\infty$, to avoid any phase changing process. Here, we shall assume that T is sufficiently large or even let $T=\infty$. With this in mind, we shall consider in this section the time domain $(0,\infty)$. It is worth to recall that, to the best of our knowledge, the late studies on the asymptotic behavior of the weak solutions for the classical OB problem provide only power time-decays (see the references cited in Section 1).

Connected with the behavior of the velocity $\mathbf{u}$ is the forcing term. This in turn, in OB type problems, corresponds to the buoyancy force and therefore comes as a function of the temperature. So, when we assume the buoyancy force is zero, tacitly we are saying that the temperature is zero. Therefore we start by studying the conditions under which the temperature vanishes in a finite time. If we assume the thermal conductivity function satisfies to

$$\varphi'(s)\geqslant Cs^{m-1}\quad\text{with } 0<m<1 \text{ and } C=\mathit{constant}>0\tag{4.72}$$

and additionally

$$\varphi_*=0\quad\text{a.e. on } \Gamma_T,\tag{4.73}$$

then we can prove the existence of a finite time, say $t^\#$, such that for any weak solution $(\mathbf{u},\theta)$ of the modified OB problem (1.4)–(1.8), $\theta=0$ for all $t\geqslant t^\#$ and a.e. in Ω. The proof is carried out by using a suitable energy method with θ^p, for a suitable p, as a test function in Eq. (1.6) and with the assumptions (4.72)–(4.73) (see Galiano [10, Theorem 3.1]). In consequence the assumptions made in the following theorem on the buoyancy force can be reasonably satisfied.

To simplify the exposition in this section, let us recall the notation $q:=\sigma\circ\theta$ introduced in (3.21), and let us set now

$$q^-(t):=\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega}q(t,\mathbf{x}),\qquad q^+(t):=\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega}q(t,\mathbf{x}).\tag{4.74}$$

Given $\sigma\in\mathcal{P}(\mathbb{R})$ satisfying to (2.11) and $\theta\in L^\infty(Q_T)$, we already know that $q\in\mathcal{P}(Q_T)$ and also satisfies to (2.11). In addition, one can readily see that $q^-(t),q^+(t)\in\mathcal{P}(\Omega)$ and still satisfy to (2.11).

Theorem 4.1 *(Extinction in time). Let $(\mathbf{u},\theta)$ be a weak solution of the modified OB problem* (1.4)–(1.8) *in the sense of Definition* 3.1 *such that* (4.72)–(4.73) *hold,*

$$0 \leqslant m \leqslant \theta(\mathbf{x},t) \leqslant M < \infty \quad \forall(\mathbf{x},t) \in Q_T$$

and

$$1 < q^-(t) \leqslant q(\mathbf{x},t) \leqslant q^+(t) < 2 \quad \forall(\mathbf{x},t) \in Q_T, \tag{4.75}$$

where $q^-(t)$ and $q^+(t)$ are defined in (4.74).

(1) *Assume that $\mathbf{f}(\theta(t)) = \mathbf{0}$ for a.a. $t \geqslant t^\#$ and a.e. in Ω, where $t^\#$ is a fixed positive time. If $\mathbf{u}_\# := \mathbf{u}(t^\#) \in \mathbf{H}$, then there exists $t^* > t^\#$ such that $\mathbf{u} = \mathbf{0}$ for all $t \geqslant t^*$ and a.e. in Ω.*
(2) *Assume that $\mathbf{f}(\theta) \neq \mathbf{0}$, but*

$$\big\|\mathbf{f}\big(\theta(t)\big)\big\|_{\mathbf{V}'} \leqslant \epsilon\left(1-\frac{t}{t_\theta}\right)_+^{\frac{1}{2(\mu-1)}} \quad \textit{for a.a. } t \geqslant 0, \tag{4.76}$$

where $g_+ = \max(0,g)$, t_θ is a fixed positive time and μ is given by (4.86) *below. If $\mathbf{u}_0 \in \mathbf{H}$, then there exists a constant $\epsilon_0 > 0$ such that $\mathbf{u} = \mathbf{0}$ for all $t \geqslant t_\theta$ and a.e. in Ω, provided $0 < \epsilon \leqslant \epsilon_0$.*

Proof. *First assertion.* If $\mathbf{f}(\theta(t)) = \mathbf{0}$ for a.a. $t \geqslant t^\#$ and a.e. in Ω, we obtain from (3.71)

$$\frac{d}{dt}E(t) + CE_{2,q(\cdot,t)}(t) \leqslant 0 \quad \text{for a.a. } t \geqslant t^\#, \tag{4.77}$$

where

$$E(t) := \frac{1}{2}\big\|\mathbf{u}(t)\big\|^2_{\mathbf{L}^2(\Omega)}, \qquad E_{2,q(\cdot,t)}(t) := \big\|\nabla\mathbf{u}(t)\big\|^2_{\mathbf{L}^2(\Omega)} + A_{q(\cdot,t)}\big(\mathbf{u}(t)\big), \tag{4.78}$$

where $A_{q(\cdot,t)}(\mathbf{u}(t))$ denotes the semimodular of $\mathbf{u}(t)$ (see (2.9)). With no loss of generality, we may assume that

$$E(t) + A_{q(\cdot,t)}\big(\mathbf{u}(t)\big) \leqslant 1 \quad \text{for a.a. } t \geqslant t^\#. \tag{4.79}$$

By the generalized Hölder's inequality (2.13), properties (2.12) and (2.14), and assumption (4.79), we can prove that

$$\big\|\mathbf{u}(t)\big\|_{\mathbf{L}^{q^-(t)}(\Omega)} \leqslant C(t)A_{q(\cdot,t)}\big(\mathbf{u}(t)\big)^{\frac{1}{q^+(t)}} \quad \text{for a.a. } t \geqslant t^\#, \tag{4.80}$$

where $C(t) = C(\Omega, q^-(t), q^+(t))$. Now we shall use Gagliardo–Nirenberg–Sobolev inequality in the Sobolev spaces of functions defined in Ω and depending on t as a parameter. So, taking $p = q = 2$ and $r = q^-(t)$ in (2.15), we obtain

$$\big\|\mathbf{u}(t)\big\|_{\mathbf{L}^2(\Omega)} \leqslant C(t)\big\|\nabla\mathbf{u}(t)\big\|^{\gamma(t)}_{\mathbf{L}^2(\Omega)}\big\|\mathbf{u}(t)\big\|^{1-\gamma(t)}_{\mathbf{L}^{q^-(t)}(\Omega)} \quad \text{for a.a. } t \geqslant t^\#, \tag{4.81}$$

where $C(t) = C(N, q^-(t))$ is the constant resulting from applying (2.15) and, according to (2.16),

$$\gamma(t) = 1 - \frac{2q^-(t)}{(2-q^-(t))N + 2q^-(t)}. \tag{4.82}$$

Plugging (4.80) into (4.81), we get after some algebraic manipulations

$$E(t) \leqslant C(t)E_{2,q(\cdot,t)}(t)^{\mu(t)} \quad \text{for a.a. } t \geqslant t^\#, \tag{4.83}$$

where $C(t) = C(\Omega, N, q^-(t))$, and from (4.81) and (4.82),

$$\mu(t) = 1 + \frac{2(2-q^+(t))}{q^+(t)[(2-q^-(t))N + 2q^-(t)]}. \tag{4.84}$$

Then (4.77) and (4.83) lead us to the homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C(t)E(t)^{\frac{1}{\mu(t)}} \leqslant 0 \quad \text{for a.a. } t \geqslant t^\#, \tag{4.85}$$

where now $C(t) = C(\nu, \alpha, \Omega, q^-(t), q^+(t))$. Now, let us analyze the exponent of nonlinearity $\mu(t)$ given by (4.84). Recall that $\mu(t)$ is written in terms of the interpolation exponent $\gamma(t)$. According to Lemma 2.1 with $p = q = 2$ and $r = q^-(t)$, and also (4.82), the admissible range of $\gamma(t)$ shows us that $0 \leqslant \gamma(t) \leqslant 1$ if and only if $0 \leqslant q^-(t) \leqslant 2$ for $N = 1$ or $N \geqslant 3$, and $0 \leqslant \gamma(t) < \frac{2}{2+q^-(t)}$ if and only if $0 < q^-(t) \leqslant 2$ for $N = 2$. In consequence, we see that $\mu(t) > 1$ if and only if $1 <$

$q^-(t), q^+(t) < 2$. The other possible values of $q^-(t)$ and $q^+(t)$, *i.e.* $0 \leqslant q^-(t), q^+(t) \leqslant 1$ go out of our initial assumption that $q^-(t), q^+(t) > 1$. For this range $1 < q^-(t), q^+(t) < 2$, we can prove, from (4.84), that $1 < \mu(t) < \frac{3}{2}$. Proceeding analogously, we can prove that $C(t)$ is also bounded, for the same range $1 < q^-(t), q^+(t) < 2$, in some domain independent of t. Then we can define

$$C := \inf_{t \geqslant 0} C(t) \quad \text{and} \quad \mu := \inf_{t \geqslant 0} \mu(t). \tag{4.86}$$

Notice that the previous analysis is valid for all $t \geqslant 0$ and, consequently, $\mu > 1$. Then, gathering these information in (4.85) and from assumption (4.79), we obtain

$$\frac{d}{dt}E(t) + C(t)E(t)^{\frac{1}{\mu}} \leqslant 0 \quad \text{for a.a. } t \geqslant t^{\#}.$$

Proceeding as in Antontsev and Oliveira [4], we prove the first assertion with

$$t^* = t^{\#} + \frac{E(t^{\#})^{\frac{\mu-1}{\mu}}}{C},$$

where μ and C are given by (4.86).

Second assertion. If $\mathbf{f}(\theta) \neq \mathbf{0}$, we use first Schwarz's and Cauchy's inequalities on the right-hand term of (3.71), to obtain

$$\frac{d}{dt}E(t) + C_1 E_{2,q(\cdot,t)}(t) \leqslant C_2 \big\|\mathbf{f}\big(\theta(t)\big)\big\|^2_{\mathbf{V}'} \quad \text{for a.a. } t \geqslant 0,$$

where $C_1 = C(\nu, \alpha)$ and $C_2 = C(\nu)$. Using (4.76) and (4.83), we obtain the following non-homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C_3(t)E(t)^{\frac{1}{\mu(t)}} \leqslant C_2\epsilon^2\left(1 - \frac{t}{t_\theta}\right)_+^{\frac{1}{\mu-1}} \quad \text{for a.a. } t \geqslant 0,$$

where $C_3(t) = C(\nu, \alpha, \Omega, q^-(t), q^+(t))$ and $\mu(t)$ is defined in (4.84). Defining the absolute constants C_3 and μ as in (4.86), we achieve to

$$\frac{d}{dt}E(t) + C_3 E(t)^{\frac{1}{\mu}} \leqslant C_2\epsilon^2\left(1 - \frac{t}{t_\theta}\right)_+^{\frac{1}{\mu-1}} \quad \text{for a.a. } t \geqslant 0.$$

Now the proof follows just in the same way as the correspondingly one given in [4]. Thus, taking

$$\epsilon_0 = \sqrt{\frac{C_1}{C_2}(1-\bar{k})\left(\frac{\mu-1}{\mu}\bar{k}\right)^{\frac{1}{\mu-1}}}, \quad \text{for some } \bar{k} \in (0,1)$$

we prove second assertion. □

Remark 4.1. These results can be extended to the limit case of $q \equiv 1$. In fact, if $q = 1$, then $q^+, q^- = 1$ and consequently $\mu = 1 + 2/(N+2)$. Theorem is proved easily because here q is constant. If $q = 2$ these results are no longer valid. For instance, taking $q^+ = q^- \equiv 2$ in (4.84), we obtain from (4.85) a linear differential inequality which provide us only an exponential decay. From (3.21), the cases $q = 1$ and $q = 2$ correspond in the modified OB problem (1.4)–(1.8) to assume $\sigma = 1$ and $\sigma = 2$.

The following theorem shows us that, for $q(\cdot, t) \geqslant q^-(t) > 2$ for a.a. $t \geqslant 0$, the velocity component in the weak solutions of the modified OB problem (1.4)–(1.8) have a power time-decay rate, not only for buoyancy forces extinguishing in a finite time, but also for non-zero buoyancy forces with suitable power time-decay rates.

Theorem 4.2 *(Power decay). Let* $(\mathbf{u}, \theta)$ *be a weak solution of the modified OB problem* (1.4)–(1.8) *in the sense of Definition* 3.1 *such that* (4.72)–(4.73) *hold and*

$$q(\mathbf{x}, t) \geqslant q^-(t) > 2 \quad \forall (\mathbf{x}, t) \in Q_T, \tag{4.87}$$

where $q^-(t)$ *is defined in* (4.74).

(1) *Assume that* $\mathbf{f}(\theta(t)) = \mathbf{0}$ *for a.a.* $t \geqslant t^{\#}$ *and a.e. in* Ω*, where* $t^{\#}$ *is a fixed positive time. If* $\mathbf{u}_{\#} \in \mathbf{H}$*, then there exist positive constants* C_1*,* C_2 *and* μ $(\mu < 1)$ *such that*

$$\big\|\mathbf{u}(t)\big\|_{\mathbf{L}^2(\Omega)} \leqslant (C_1 t + C_2)^{-\frac{\mu}{1-\mu}} \quad \text{for a.a. } t \geqslant t^{\#}. \tag{4.88}$$

(2) *Assume that* $\mathbf{f}(\theta) \neq \mathbf{0}$, *but exist positive constants* K_1, K_2, K_3 *and* μ ($\mu < 1$) *such that*

$$\left\|\mathbf{f}\big(\theta(t)\big)\right\|_{\mathbf{L}^2(\Omega)} \leqslant \frac{K_1}{(K_2 t + K_3)^{\frac{2-\mu}{1-\mu}}} \quad \text{for a.a. } t \geqslant 0. \tag{4.89}$$

If $\mathbf{u}_0 \in \mathbf{H}$, *then*

$$\left\|\mathbf{u}(t)\right\|_{\mathbf{L}^2(\Omega)} \leqslant (K_2 t + K_3)^{-\frac{\mu}{2(1-\mu)}} \quad \text{for a.a. } t \geqslant 0. \tag{4.90}$$

Proof. Here we use the same notations introduced in the proof of Theorem 4.1 and, with no loss of generality, we assume (4.79) as well. Firstly we observe that by using generalized Hölder's and Sobolev's inequalities, respectively (2.13) and (2.17), we obtain

$$\left\|\mathbf{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \leqslant C(t)\left\|\mathbf{u}(t)\right\|_{\mathbf{L}^{q(\cdot,t)}(\Omega)}\left\|\nabla\mathbf{u}(t)\right\|_{\mathbf{L}^2(\Omega)} \quad \text{for a.a. } t \geqslant 0, \tag{4.91}$$

where $C(t) = C(N, \Omega, q^-(t), q^+(t))$, where $q^-(t)$ and $q^+(t)$ are defined in (4.74). Using property (2.12) and assumption (4.79) in the generalized norm of (4.91), we obtain, after some algebraic manipulations,

$$E(t) \leqslant C(t) E_{2,q(\cdot,t)}(t)^{\mu(t)} \quad \text{for a.a. } t \geqslant 0, \tag{4.92}$$

where $E_{2,q(\cdot,t)}(t)$ is defined in (4.78) and

$$\mu(t) := \frac{2 + q^+(t)}{2q^+(t)}.$$

Notice that the assumption $2 < q^-(t) \leqslant q^+(t) < \infty$ for all $t \geqslant 0$ implies that $1/2 < \mu(t) < 1$ for all $t \geqslant 0$.

First assertion. If $\mathbf{f}(\theta(t)) = \mathbf{0}$ for a.a. $t \geqslant t^\#$ and a.e. in Ω, then (4.77) and (4.92) lead us to the homogeneous ordinary differential inequality

$$\frac{d}{dt}E(t) + C(t)E(t)^{\frac{1}{\mu(t)}} \leqslant 0 \quad \text{for a.a. } t \geqslant t^\#, \tag{4.93}$$

where $C(t) = C(N, \Omega, q^-(t), q^+(t), \nu, \alpha)$. Now we define

$$C := \inf_{t \geqslant 0} C(t) \quad \text{and} \quad \mu := \sup_{t \geqslant 0} \mu(t). \tag{4.94}$$

Then, gathering these information in (4.93) and from assumption (4.79), we obtain

$$\frac{d}{dt}E(t) + CE(t)^{\frac{1}{\mu}} \leqslant 0 \quad \text{for a.a. } t \geqslant t^\#. \tag{4.95}$$

Integrating (4.95) between $t = 0$ and $t > 0$ we obtain (4.88) with $C_1 = (1-\mu)/\mu C$ and $C_2 = (1-\mu)/\mu(\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)}/2)^{(\mu-1)/\mu}$, where the constants C and μ are defined in (4.94).

Second assertion. Now, we assume that $\mathbf{f}(\theta) \neq \mathbf{0}$ and satisfies to (4.89). Using Hölder's inequality on the right-hand side of (3.71) and then the same reasoning used to obtain (4.95), lead us to

$$\frac{d}{dt}E(t) + C_1 E(t)^{\frac{1}{\mu}} \leqslant \sqrt{2E(t)}\left\|\mathbf{f}\big(\theta(t)\big)\right\|_{\mathbf{L}^2(\Omega)} \quad \text{for a.a. } t \geqslant 0, \tag{4.96}$$

where $C_1 = C$, and C and μ are the constants defined in (4.94). Now we introduce the new function $\mathcal{E}(t) := \sqrt{2E(t)}$ and (4.96) comes

$$\frac{d}{dt}\mathcal{E}(t) + C_2\mathcal{E}(t)^{\frac{2-\mu}{\mu}} \leqslant \left\|\mathbf{f}\big(\theta(t)\big)\right\|_{\mathbf{L}^2(\Omega)} \quad \text{for a.a. } t \geqslant 0, \tag{4.97}$$

where $C_2 = 2^{1/\mu}C_1$. Solving the homogeneous ordinary differential equation associated to (4.97), we obtain

$$\mathcal{E}(t) = (C_3 t + C_4)^{-\frac{\mu}{2(1-\mu)}} \tag{4.98}$$

where $C_3 = 2(1-\mu)/\mu C_2$ and $C_4 = \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^{-\frac{2(1-\mu)}{\mu}}$. Let us consider the function

$$\mathcal{F}(t) = \big((C_3 - C)t + C_4\big)^{-\frac{\mu}{2(1-\mu)}} \tag{4.99}$$

for some positive constant $C < C_3$. One can readily sees the function $\mathcal{F}(t)$ defined in (4.99) is an upper bound for the function $\mathcal{E}(t)$ defined in (4.98). On the other hand, the function $\mathcal{F}(t)$ satisfies to

$$\frac{d}{dt}\mathcal{F}(t) + C_2\mathcal{F}(t)^{\frac{2-\mu}{\mu}} = \frac{C_5}{((C_3 - C)t + C_4)^{\frac{2-\mu}{1-\mu}}},$$

where $C_5 = \mu/(2(1-\mu))C$ and is therefore a solution of (4.97) if condition (4.89) is satisfied with $K_1 = C_5$, $K_2 = C_3 - C$ and $K_3 = C_4$. In consequence, we obtain (4.90). □

Remark 4.2. Notice that the validity of the generalized Sobolev's inequality (2.17), used in (4.91), implies that $q^-(t) \geqslant 2$ for all $t \geqslant 0$. Therefore assumption (4.87) can be relaxed to $q(\mathbf{x}, t) \geqslant q^-(t) \geqslant 2$ for all $(\mathbf{x}, t) \in Q_T$. In this case, we need to avoid that $q^+(t) = 2$ for all $t \geqslant 0$, by assuming, for instance, that $q^+(t) > q^-(t)$ for all $t \geqslant 0$. In the limit case $q^-(t) = q^+(t) = 2$ for all $t \geqslant 0$, (4.93) becomes a linear differential inequality and, again, we can derive an exponential decay.

In Theorem 4.1 we have seen that for $1 < q^-(t) \leqslant q(\cdot, t) \leqslant q^+(t) < 2$ for all $t \geqslant 0$ and $\mathbf{f}(\theta(t)) = \mathbf{0}$ for a.a. $t \geqslant t^\#$, or for $\mathbf{f}(\theta)$ satisfying to (4.76), it was possible to establish that the velocity component in the weak solutions of the modified OB problem (1.4)–(1.8) extinct in a finite time. If $q(\cdot, t) \geqslant q^-(t) \geqslant 2$, with $q^+(t) > 2$, for a.a. $t \geqslant 0$ and $\mathbf{f}(\theta(t)) = \mathbf{0}$ for a.a. $t \geqslant t^\#$ or for $\mathbf{f}(\theta)$ satisfying to (4.89), it was possible to prove, in Theorem 4.2, that the velocity component decay at a power time rate. Now we shall study the case when $\mathbf{f}(\theta)$ is non-zero and merely belongs to a suitable function space. Using Cauchy's inequality with a suitable ε in (3.71), we obtain

$$\frac{d}{dt}E(t) + C_1 E_{2,q(\cdot,t)}(t) \leqslant C_2 \int_\Omega \left|\mathbf{f}(\theta(t))\right|^2 d\mathbf{x} \quad \text{for a.a. } t \geqslant 0, \tag{4.100}$$

where $E(t)$ and $E_{2,q(\cdot,t)}(t)$ are defined in (4.78) and $C_1 = C(\nu, \alpha)$ and $C_2 = C(\nu)$. We assume that

$$\int_\Omega \left|\mathbf{f}(\theta(t))\right|^2 d\mathbf{x} \leqslant C_\mathbf{f} \quad \text{for a.a. } t \geqslant 0, \tag{4.101}$$

where $C_\mathbf{f}$ is a positive constant. If (4.75) holds, we use (4.83), and in the case of (4.87) holds, we use (4.92). In any case, we obtain from (4.100) and (4.101)

$$\frac{d}{dt}E(t) + C_1(t)E(t)^{\xi(t)} \leqslant C_2 \quad \text{for a.a. } t \geqslant 0, \tag{4.102}$$

where, for a.a. $t \geqslant 0$,

$$\xi(t) := \frac{q^+(t)[(2 - q^-(t))N + 2q^-(t)]}{q^+(t)(2 - q^-(t))N + 4q^-(t)} \quad \text{if (4.75) holds} \tag{4.103}$$

or

$$\xi(t) := \frac{2q^+(t)}{q^+(t) + 2} \quad \text{if (4.87) holds,} \tag{4.104}$$

and $C_1(t) = C(\xi(t), \nu, \alpha)$, $C_2 = C(C_\mathbf{f}, \nu)$. Notice that in the limit case of $q(\cdot, t) \equiv 2$ both expressions of ξ in (4.103)–(4.104) converge to the same value $\xi = 1$. To overcome the dependence of ξ and C_1 on the parameter t, let us define

$$C_1 := \inf_{t \geqslant 0} C_1(t) \quad \text{and} \quad \xi := \sup_{t \geqslant 0} \xi(t). \tag{4.105}$$

According to assumption (4.79), we obtain from (4.102) and (4.105),

$$\frac{d}{dt}E(t) + C_1 E(t)^\xi \leqslant C_2 \quad \text{for a.a. } t \geqslant 0.$$

Let us now set

$$\frac{d}{dt}E(t) + C_1 E(t)^\xi = C_2 \quad \Leftrightarrow \quad \frac{d}{dt}E(t) = C_2 - C_1 E(t)^\xi := \Lambda(t). \tag{4.106}$$

If $\Lambda(t) < 0$ or $\Lambda(t) > 0$ at some time t (possibly different), then $E(t)$ is decreasing or increasing, respectively, at that time. In consequence, the asymptotically stable equilibrium of (4.106) is reached when

$$\Lambda(t) = 0 \quad \Leftrightarrow \quad E(t) = \left(\frac{C_2}{C_1}\right)^{\frac{1}{\xi}} := \frac{\mathcal{E}_*}{2} \equiv E_*. \tag{4.107}$$

A simple analysis of (4.106) and (4.107) shows us that if there exists a positive time t_0 such that $E(t_0) < E_*$, then $0 \leqslant E(t) < E_*$ for all time $t > t_0$ and, consequently, $E(t) \nearrow E_*$ as $t \to \infty$. In this case, we are done and we obtain

$$\left\|\mathbf{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \leqslant \mathcal{E}_* \quad \text{for all } t > t_0.$$

The reciprocal case stated in the following theorem can be proved directly from Antontsev and Oliveira [4].

Theorem 4.3 *(Exponential decay). Assume* $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{f}(\theta) \neq \mathbf{0}$ *satisfies to* (4.101). *Let* $(\mathbf{u}, \theta)$ *be a weak solution of the modified OB problem* (1.4)–(1.8) *in the sense of Definition* 3.1. *In addition, assume that exists a positive time* t_0 *such that* $\|\mathbf{u}(t_0)\|^2_{\mathbf{L}^2(\Omega)} > \mathcal{E}_*$. *Then there exists a positive constant C such that*

$$\left\|\mathbf{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} \leqslant \left(\left\|\mathbf{u}(t_0)\right\|^2_{\mathbf{L}^2(\Omega)} - \mathcal{E}_*\right)e^{-C(t-t_0)} + \mathcal{E}_* \quad \textit{for all } t > t_0,$$

where $\mathcal{E}_*$ *is given in* (4.107).

The conclusions about the time properties we proved in this section are based on the analysis of nonlinear ordinary differential inequalities. When dealing with these we always reduce them, by means of suitable assumptions, to the nonlinear ordinary differential inequalities with constant exponents of nonlinearity. The study of such nonlinear ordinary differential inequalities with variable exponents of nonlinearity is still an open problem.

5. Conclusions

Throughout this paper we have seen that the modification we made to the Oberbeck–Boussinesq model in (1.4)–(1.6) allowed us to obtain time properties for the velocity component of the weak solutions which are not known for the classical model (1.1)–(1.3). Using the same techniques these properties can be derived for modified problems obtained from more generalized Oberbeck–Boussinesq models. For instance, we could have considered the modified problem obtained from the Oberbeck–Boussinesq problem studied in Díaz and Galiano [8], with the velocity field and the temperature coupled by the following system of equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}(\theta) - \alpha|\mathbf{u}|^{\sigma(\theta)-2}\mathbf{u} - \nabla p + \operatorname{div}\big(\nu(\theta)\mathbf{D}\big);$$
$$\frac{\partial \mathcal{C}(\theta)}{\partial t} + \mathbf{u} \cdot \nabla \mathcal{C}(\theta) = \kappa \Delta \varphi(\theta).$$

Here $\mathbf{D}$ is the symmetric part of $\nabla \mathbf{u}$, the kinematics viscosity ν depends on the temperature θ, and $\mathcal{C}$ and φ are functions of the specific heat and thermal conductivity, respectively. To obtain the properties of the previous section, besides the assumptions on σ, we need to assume the temperature-depending viscosity is bounded:

$$0 < \textit{constant} = \nu^- \leqslant \nu(\theta) \leqslant \nu^+ = \textit{constant} < \infty \quad \text{for all } \theta \in [m, M].$$

The results still remain valid if we modify the Oberbeck–Boussinesq problem by an anisotropic thermo-absorption term:

$$\left(\alpha_1|u_1|^{\sigma_1(\theta)-2}u_1, \ldots, \alpha_N|u_N|^{\sigma_N(\theta)-2}u_N\right);$$

requiring analogous assumptions for the σ_i functions as for the σ function in the isotropic thermo-absorption case. Interesting is that, in the anisotropic case, we may allow that one α_i, but only one, can be zero. See Antontsev and Oliveira [3] where it was considered the isothermal case. Concerning the same properties for the temperature θ, we know that when Eq. (1.6) is linear, we obtain an exponential decay in time. Different properties can be derived for the temperature of the problem resulting from the modification of the Oberbeck–Boussinesq problem considered in Díaz and Galiano [8]. As we have seen at the very beginning of Section 4, the proofs of the finite speed of propagations and of the waiting time properties established in Galiano [10], for the temperature component of weak solutions to the same problem considered in Díaz and Galiano [8], can be easily adapted for the correspondingly modified problem. With respect with similar properties in space, so far we are only able to prove the velocity component of the weak solutions have compact support in Ω for 2D stationary modified problems (see Antontsev et al. [2]). The results of this paper can be generalized for more complete models as those that assume a varying density ρ. In this case, the properties are obtained by considering a bounded density such as in Antontsev et al. [5]. The extension of these results to exothermic non-Newtonian models is also possible. Here the main interest is for dilatant fluids, because for pseudo-plastic fluids the structure of the stress tensor alone is responsible for stopping the fluid in a finite time (see Oliveira [19]).

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