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ON THE KLEIN-GORDON EQUATION ORIGINATING ON A CURVE AND APPLICATIONS TO THE TSUNAMI RUN-UP PROBLEM

By

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Abstract

Our goal is to study the linear Klein-Gordon equation in matrix form, with initial conditions originating on a curve. This equation has applications to the Cross-Sectionally Averaged Shallow Water equations, i.e. a system of nonlinear partial differential equations used for modeling tsunami waves within narrow bays, because the general Carrier-Greenspan transform can turn the Cross-Sectionally Averaged Shallow Water equations (for shorelines of constant slope) into a particular form of the matrix Klein-Gordon equation. Thus the matrix Klein-Gordon equation governs the run-up of tsunami waves along shorelines of constant slope. If the narrow bay is U-shaped, the Cross-Sectionally Averaged Shallow Water equations have a known general solution via solving the transformed matrix Klein-Gordon equation. However, the initial conditions for our Klein-Gordon equation are given on a curve. Thus our goal is to solve the matrix Klein-Gordon equation with known conditions given along a curve. Therefore we present a method to extrapolate values on a line from conditions on a curve, via the Taylor formula. Finally, to apply our solution to the Cross-Sectionally Averaged Shallow Water equations, our numerical simulations demonstrate how Gaussian and N-wave profiles affect the run-up of tsunami waves within various U-shaped bays.

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Chapter 1: Introduction and Overview

1.1 Introduction

Our project is motivated by the study of tsunami waves. Tsunami waves are a series of long waves formed via a large displacement of water. They are often caused by natural phenomena such as earthquakes, landslides, volcano eruptions, and in rare cases, asteroid impacts [BT15], [HS06], [SB06]. Tsunamis pose a major threat to coastal areas, because they are capable of generating waves that result in inundation (flooding), damages, and casualties. Although earthquakes are often an indicator for tsunami formation, using earthquake magnitude alone to predict tsunami waves can sometimes result in over warning (not fatal) or under warning (results in casualties) of a tsunami [BR09]. In other words, tsunami prediction requires measurements that do not solely rely on the readings of other natural disasters.

Alaska is especially vulnerable to tsunami waves. This is probably due to the fact that the Alaska-Aleutian arcs is one of the most active margins in the world [RLK07]. In particular, the Aleutian Arc is the region where the Pacific plate subducts beneath the North America Plate. Within the last century there have been six great earthquakes within Alaska: the M8.4 1906 Rat Islands, M8.6 1938 Shumagin Islands, M8.6 1946 Unimak Island, M8.6 1957 Andreanof Islands, M9.2 1964 Prince William Sound, and M8.0 1965 Rat Islands [BDN+11]. Since the 1946 earthquake the US installed a tsunami warning center known as the Honolulu Seismic Observatory, due to the damage done by the tsunami proceeding the earthquake [BT15]. However, it was the 1964 earthquake that caused the largest tsunami within the US [RLK07], with waves reaching as far as California [WAB+13]. The moral here is that tsunami warning systems are of a national, and potentially an international, concern.

Perhaps the most significant tsunami to affect the world within the last century is the 2004 Sumatra–Andaman tsunami, also known as the 2004 Boxing Day tsunami. The 2004 Boxing Day tsunami was a major event that resulted in the deaths of over 200,000 people. The tsunami waves originated in the Indian Ocean and they affected at least 16 countries directly, such as Indonesia, Thailand, Sri Lanka, India, Maldives, and more [CHP06], although there were waves as far as east Africa [CH06]. The major reason why the death toll was so high was due to lack of awareness and preparedness of the incoming tsunami, with some areas not having the proper warning systems. As a result of this catastrophe there have been some major advancement in tsunami warning measures. This includes installing proper warning systems along coastlines [Oka15], an increase of numerical simulations of tsunami waves [SB06], advancement in technology to monitor potential tsunami waves, and an increased alertness of tsunamis made to the public [BT15].

Since the 2004 Boxing day tsunami there have been at least 130 tsunamis, with 14 of them causing casualties [KTBS15]. In particular, the major tsunami of the last 10 years to affect Japan is the 2011 Tohoku tsunami that resulted in the deaths of 20,000 people [Oka15]. The tsunami was preceded by a M9.0 earthquake. Aneyoshi Bay experienced waves as tall as 38.9 meters, making it the largest wave recorded in Japan. In spite of the fact that Japan is the most tsunami-prepared nation in the world, Japan still underestimated the impact of the incoming tsunami waves [BR09]. One of the areas affected by the 2011 Tohoku tsunami was Hirota Bay. The bathymetry of Hirota Bay is U-shaped, meaning that the cross-section of the bay resembles the shape of a U [LST+13]. Moreover the bay had waves of uniform height, which is in part due to its bathymetry. Modeling such waves within U-shaped bays might provide key details in evacuation procedures for Japan. All of this indicates the extraordinary importance of modeling tsunami waves, which includes generation, propagation, and run-up.

In Summer 2017, the Research Experience for Undergraduates (REU) Program modeled the propagation of a tsunami wave for various U-shaped bays. The practical use of doing this was to examine the possible run-ups of tsunami waves, and how bathymetry affects run-up. The tsunami run-up problem is the process of evaluating the maximum height a tsunami wave achieves at the shore. For hypothetical models this is best achieved through modeling the propagation of waves. Because tsunami waves are characterized as long waves of small steepness [CH06], [SB06], they are viewed as shallow water waves, and therefore governed by the nonlinear Shallow Water Wave equations (SWEs). Although the 2D SWEs (2 spatial and 1 temporal variable) are most-commonly used in numerical simulations of tsunami waves [KTBS15], [SB06], within narrow bays the Cross-Sectionally Averaged (CSA) SWEs are a simpler and more effective tool for modeling propagation of waves [HW53], [Sto57], [ZPGO06]. The advantage of using the CSA SWEs is that it can be linearized for certain bays. Hence we focus on the CSA SWEs.

Given horizontal distance x and time t, the CSA SWEs are given by

$$\partial_t S + \partial_x (S\mu) = 0, \tag{1.1a}$$

$$\partial_t \mu + \mu\partial_x\mu + g\partial_x\eta = 0, \tag{1.1b}$$

where $\mu = \mu(x,t)$ is the velocity and $\eta = \eta(x,t)$ is the perturbed water displacement, g is the constant acceleration due to gravity, $H(x,t) = h(x) + \eta(x,t)$ is the total water depth, and $h = h(x)$ is the unperturbed water depth. The cross-sectional area $S = S(H)$ is dependent only on the total depth $H(x,t)$. We assume that the initial conditions $\mu_0 = \mu(x,0)$ and $\eta_0 = \eta(x,0)$ are explicitly given for all real x. All of these terms are shown in figure 1.1.

In the case of bays of constant slope, i.e. $h(x) = \alpha x$ for some $\alpha > 0$ (see figure 1.1 A), Carrier and Greenspan [CG58] discovered a special transformation known as the Carrier-Greenspan (CG) transform. It is used to turn the CSA SWEs into a system of first-order linear partial differential equations (PDEs). Later Rybkin et al [RPD14] and Raz et al [RNRP18] discovered generalized CG transforms that apply to arbitrary cross-sections. For our purposes we apply the transform shown in [RNRP18]. This generalized CG transform is shown in section 5.1.

In particular, the bathymetry of Hirota Bay motivates us to study the run-up of waves within U-shaped bays. So taking into account our bathymetry of interest and assuming our

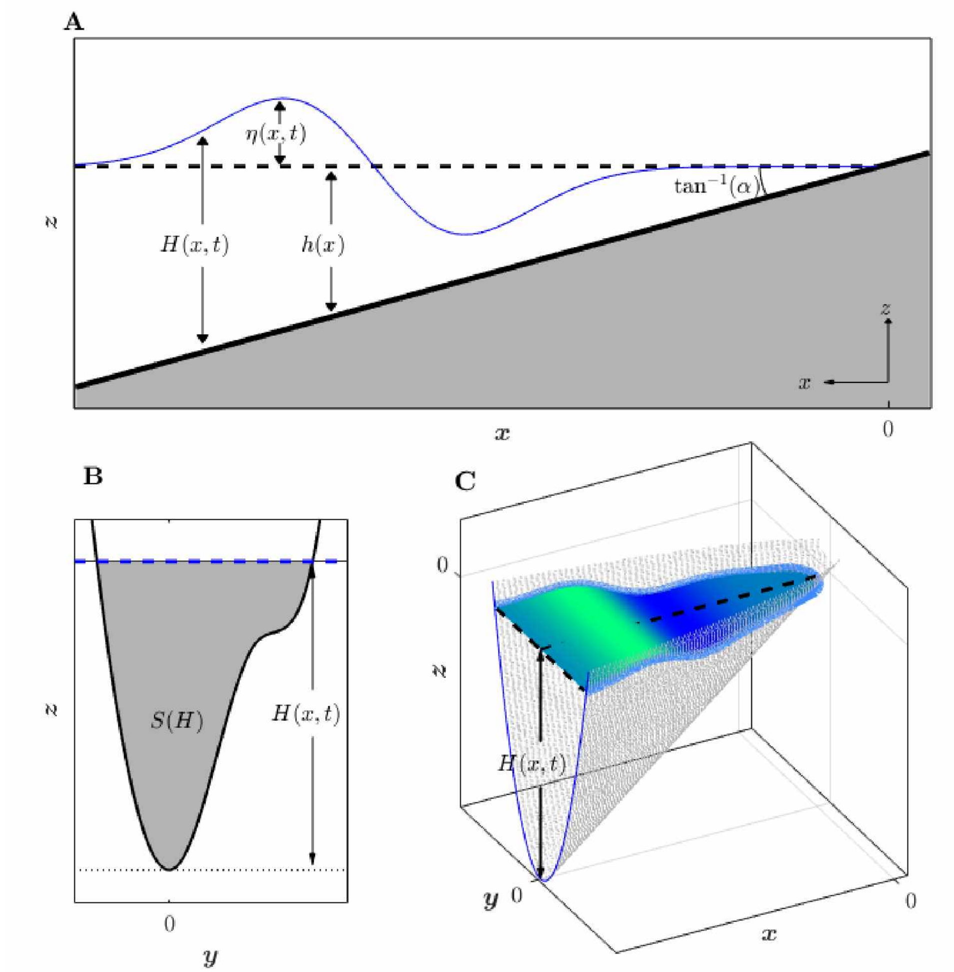

Figure 1.1: **A.** x-z cross section along the main axis of the bay, where $x = 0$ is the unperturbed shoreline. The positive x direction points to the sea, and therefore the unperturbed depth $h(x)$ (dashed) has constant slope of magnitude α. The perturbed $H(x,t)$ water level is denoted $H = h + \eta$, and equivalently $\eta = H - h$. **B.** y-z cross section of a generic asymmetric bay. **C.** three-dimensional view of a generic parabolic bay.

shorelines are of constant slope, our bay bathymetry becomes

$$Z(x,y) = \beta|y|^m - \alpha x$$

for arbitrary $\beta > 0$ and $m > 0$. For example, the equation $Z(x,y) = \beta y^2 - \alpha x$ describes the channel topography of a parabolic-shaped bay as shown in figure 1.1 C. Based on our bay bathymetry, our cross-sectional area is

$$S(H) = \frac{2m}{\beta^{1/m}(m+1)} H^{\frac{m+1}{m}},$$

and in section 5.2 we show how this is derived. The main reason why the CSA SWEs have an explicit solution for U-shaped bays is because the cross-sectional area is explicitly given, which simplifies (1.1).

Even though $S(H)$ in (1.1) has a explicit formula for U-shaped bays, there is a still a major complication in solving the CSA SWEs. When the generalized Carrier-Greenspan transform is used on (1.1) for unperturbed height $h(x) = \alpha x$, system (1.1) becomes a special form of the matrix Klein-Gordon Equation (KGE)

$$\partial_t U = A(x)\partial_x U + B(x)U, \tag{1.2}$$

where $U = U(x,t)$ is a 2×1 vector function and $A(x), B(x)$ are arbitrary 2×2 matrix-valued functions. The consequence of applying the transform is that our initial conditions to (1.2) does not necessarily originate on a straight line in (x,t)-space, compared to μ_0 and η_0 originating on the line $t = 0$. In other words, the transform simplifies our governing equations but complicates our initial conditions [Joh97]. Hence our approach to solve the CSA SWEs is to solve the (1.2) for initial conditions originating on a curve. This means solving the matrix KGE in its most general form. Hence for (1.2) we consider initial conditions that originate on a curve $\Gamma = \{(x, \tau(x)) : x \in \mathbb{R}\}$. That is, the values $U_0(x) = U(x,t)\big|_\Gamma$ are explicitly given. Our goal is to solve (1.2) provided $U(x,t)\big|_\Gamma$.

1.2 Overview

Chapter 2 begins with the derivation of the Wave Equation, a linear 1D PDE. We then present the D'Alembert formula, a special solution to the Wave equation. This prepares us to solve the Wave equation with variable speed, and therefore for how we solve the 1D Klein-Gordon Equation.

Chapter 3 addresses the matrix KGE. Because the CG transform moves conditions that are on a line to conditions on a curve, our initial conditions to the matrix KGE originate on an arbitrary curve.

In chapter 4 we solve the problem posed in chapter 3. The first section of chapter 4 presents a method of projecting initial conditions Γ onto a line. We then find a general solution to a particular form of the matrix KGE.

Chapter 5 addresses the CSA SWEs at the shore. The CG transform is presented and used to transform the CSA SWEs into a special form of the matrix KGE. We then use the solution in chapter 4 to present an analytical solution to the CSA SWEs for U-shaped bays. We follow this by simulations to tsunami run-up problems. To be consistent with [CWY03], [Kan04] and [KS06], we model tsunami run-ups with initial Gaussian and initial N-wave profiles. All of the simulations apply to the tsunami run-up of U-shaped bays.

In regards to the logical structure of the thesis, we consider the following:

1. Begin with the CSA SWEs with initial conditions originating on the line $t = 0$. (section 5.1)

2. Perform the general CG transform on CSA SWEs and initial conditions. (section 5.1)

3. Obtain a special case of the linear KGE in matrix form and initial conditions originating on a curve. (section 5.1)

4. Find the general solution to our special case of linear KGE in matrix form. (section 4.2)

5. Approximate initial conditions originating on a line from initial conditions originating on a curve. (section 4.1)

6. Perform numerical simulations for various U-shaped bays. (section 5.3)

We note that we address the KGE prior to the CSA SWEs, because the emphasis of this thesis is to solve the matrix KGE. However the reason for solving the matrix KGE is to solve the CSA SWEs.

Chapter 2: Preliminaries

The Wave equation describes the vibrations of a string. It is one of the simplest equations in shallow water theory, but its derivation shows the applications and limitations to describing the motion of water waves. Moreover if we consider particular initial conditions to the Wave equation then it has a solution known as the D'Alembert formula. Of course, the D'Alembert formula cannot accurately describe the motion of tsunami waves within narrow bays, but its derivation demonstrates the challenges in solving the matrix KGE for initial conditions along an arbitrary curve. So in sections 2.1 and 2.2 we derive the Wave equation and the D'Alembert formula respectively.

Section 2.3 explores the Wave equation with variable speed. We introduce the method of separation of variables, which is used to solve the matrix KGE in chapter 4. In section 2.4 we examine the 1D KGE. The 1D KGE is similar to the Wave equation with variable speed, with the exception that the 1D KGE contains a potential term. The 1D KGE has applications to shallow water theory. Finally, section 2.5 describes the complications of solving an initial value problem with initial conditions originating along a curve.

2.1 Wave Equation

We note that the derivation of the Wave equation is the same follows the same process in [TM04]. Given position x and time t, figure 2.1 displays the components that act on the string, where $u = u(x,t)$ is the vertical displacement, $T = T(x,t)$ is the tension, $\rho = \rho(x)$ is the density, and $\theta = \theta(x,t)$ is the angle of the string. Over a small displacement Δx, the string has mass $\rho(x)\sqrt{\Delta x^2 + \Delta u^2}$ and acceleration $\partial_t^2 u$. If we assume tension T is the only force acting on the string, then according to Newton's laws of Physics,

$$\rho(x)\sqrt{\Delta x^2 + \Delta u^2}\partial_t^2 u = T(x+\Delta x, t)\sin\theta(x+\Delta x, t) - T(x,t)\sin\theta(x,t).$$

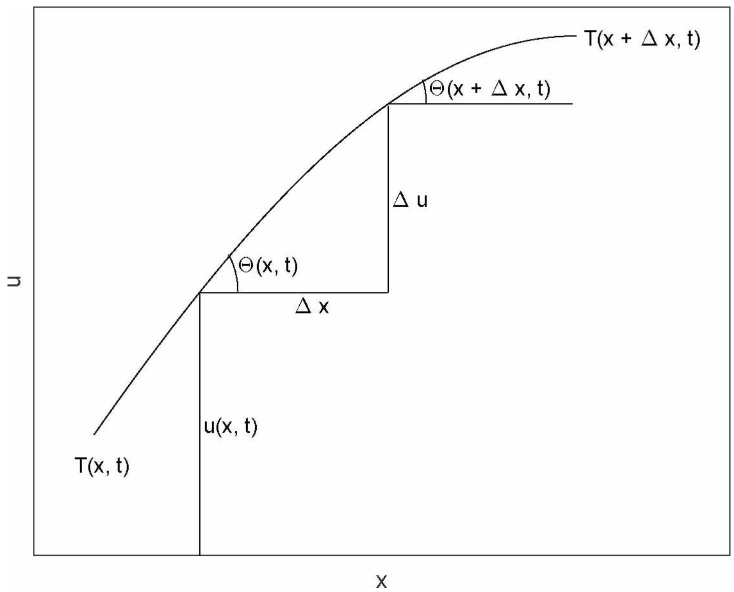

Figure 2.1: A diagram showing the vertical displacement $u(x,t)$ of a string at position x and time t. The tension to the right of the string has magnitude $T(x+\Delta x,t)$ that acts on angle $\theta(x+\Delta x,t)$ above the horizontal, while the tension to the left has magnitude $T(x,t)$ that acts on angle $\theta(x,t)$ below the horizontal.

If we divide this equation by Δx and let $\Delta x \to 0$, then it becomes

$$\rho(x)\sqrt{1+(\partial_x u)^2}\,\partial_t^2 u = \partial_x[T(x,t)\sin\theta(x,t)]. \tag{2.1}$$

Based on figure 2.1 we note that the angle of the string is dependent on the instantaneous rate of change of u with respect to x, i.e.

$$\tan\theta(x,t) = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = \partial_x u.$$

However, substituting this equation into (2.1) makes it very complicated. Instead we assume that the string is vibrating with a very low angle $\theta(x,t)$ for all x and t. So we make the estimations

$$\partial_x u = \tan\theta(x,t) \approx 0, \qquad \sin\theta(x,t) \approx \tan\theta(x,t) = \partial_x u,$$

and substitute these into (2.1) to obtain

$$\rho(x)\partial_t^2 u = \partial_x[T(x,t)\partial_x u]$$
$$= \partial_x T(x,t)\partial_x u + T(x,t)\partial_x^2 u. \tag{2.2}$$

To further simplify (2.2), we assume that the string is only moving vertically. That is, the horizontal forces of the string are zero:

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0,$$

and as $\Delta x \to 0$ we obtain the result

$$\partial_x[T(x,t)\cos\theta(x,t)] = 0. \tag{2.3}$$

Because the angle of vibration $\theta(x,t)$ is very small for all x and t then $\cos\theta(x,t) \approx 1$. So (2.3) implies that $\partial_x T(x,t)$ is negligible. Thus $T(x,t) = T(t)$ and therefore (2.2) simplifies to

$$\rho(x)\partial_t^2 u = T(t)\partial_x^2 u. \tag{2.4}$$

Finally, we assume that the applied tension does not change in t, and that the density is homogeneous. This implies $T(t) = T$ and $\rho(x) = \rho$ for arbitrary constants T and ρ. By dividing both sides of (2.4) by ρ we obtain the Wave Equation

$$\partial_t^2 u = c^2 \partial_x^2 u, \qquad x, t \in \mathbb{R}, \tag{2.5}$$

where $c = \sqrt{T/\rho}$.

We note that (2.5) is a PDE just like the CSA SWEs or the matrix KGE. Going through the effort to solve (2.5) prepares us for solving the matrix KGE along an arbitrary curve. The following section demonstrates what are the necessary conditions to solve (2.5), as well as what that solution looks like.

2.2 Wave Equation Initial Value Problem

In this section we solve the initial value problem (IVP)

$$\partial_t^2 u = c^2 \partial_x^2 u, \qquad x \in \mathbb{R}, t \geq 0 \tag{2.6a}$$

$$u(x,0) = f(x), \qquad f \in C(\mathbb{R}) \tag{2.6b}$$

$$\partial_t u(x,0) = g(x), \qquad g \in C(\mathbb{R}), \tag{2.6c}$$

where $C(\mathbb{R})$ denotes the set of continuous functions on $\mathbb{R}$. We note that (2.6a) is identical to PDE (2.5). The method to solve (2.6) is to follow the method in [BB87] and introduce the variables

$$\xi = x - ct, \qquad \nu = x + ct. \tag{2.7}$$

We call (2.7) the characteristic lines of (2.6), which have convenient properties in simplifying PDE (2.6a). The idea is to rewrite the operators ∂_x^2 and ∂_t^2 in terms of ξ and ν:

$$\partial_x = \partial_x \xi \partial_\xi + \partial_x \nu \partial_\nu = \partial_\xi + \partial_\nu, \qquad \partial_x^2 = \partial_\xi^2 + 2\partial_\xi \partial_\nu + \partial_\nu^2$$

$$\partial_t = \partial_t \xi \partial_\xi + \partial_t \nu \partial_\nu = -c\partial_\xi + c\partial_\nu, \qquad \partial_t^2 = c^2 \partial_\xi^2 - 2c^2 \partial_\xi \partial_\nu + c^2 \partial_\nu^2,$$

and when substituted into (2.6a) we obtain

$$(c^2 \partial_\xi^2 - 2c^2 \partial_\xi \partial_\nu + c^2 \partial_\nu^2) u = c^2 (\partial_\xi^2 u + 2\partial_\xi \partial_\nu u + \partial_\nu^2) u,$$

which simplifies into the linear PDE

$$\partial_\xi \partial_\nu u = 0. \tag{2.8}$$

Solving (2.8) means finding solutions to u such that differentiation with respect to ν and ξ results in the zero function. This means u must be of the form

$$u(\xi, \nu) = f_1(\xi) + f_2(\nu)$$

for arbitrary differentiable functions f_1 and f_2. In terms of x and t the previous equation becomes

$$u(x,t) = f_1(x - ct) + f_2(x + ct). \tag{2.9}$$

We note that without initial conditions (2.6b) and (2.6c) the PDE (2.6a) has infinitely many solutions (2.9). When it comes to solving the matrix KGE we would of course want to have one solution, otherwise we would not know which equation accurately describes the tsunami run-up problem. Because initial conditions for (2.6a) are known along $t = 0$ then we may pose a system of equations to find appropriate values for f_1 and f_2:

$$f(x) = u(x,0) = f_1(x) + f_2(x), \tag{2.10a}$$

$$g(x) = \partial_t u(x,0) = -cf_1'(x) + cf_2'(x). \tag{2.10b}$$

Note that

$$\int_a^x g(x^*)dx^* = -cf_1(x) + cf_2(x)$$

for arbitrary constant a, and so solving (2.10) is just like solving a system of linear equations:

$$f_1(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_a^x g(x^*)dx^*,$$
$$f_2(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_a^x g(x^*)dx^*.$$

Hence (2.6) has a convenient solution known as the D'Alembert formula [BB87], which we obtain by substituting our values for f_1 and f_2 into (2.9):

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x^*)dx^*. \tag{2.11}$$

The CSA SWEs are a system of nonlinear PDEs that govern the run-up of tsunami waves within narrow bays, and solving them is much more complicated than what we showed in this section. For now we can make progress by considering more complicated versions of the Wave equation, such as when our speed of our string is not constant.

2.3 Wave Equation for Spatially-Variable Speed

In our derivation of the Wave equation in (2.5), we assumed that our tension is constant with respect to variables x and t. As a result the speed $c = \sqrt{T/\rho}$ is constant. However we would like to relax conditions and assume that speed is not necessarily constant:

$$\partial_t^2 u = \partial_x \left(c^2(x)\partial_x u\right), \qquad x \in \mathbb{R}, t \geq 0 \tag{2.12}$$

where $c(x) = \sqrt{T(x)/\rho}$. Note this equation is easily derived from equation (2.2) by making the additional assumptions that tension T is constant with respect to time t and density ρ is constant throughout the string. Thus it is appropriate to think of (2.12) as the Wave equation for variable speed. Just like we did with Wave equation (2.5), we would like to solve the IVP

$$\partial_t^2 u = \partial_x \left(c^2(x)\partial_x u\right), \qquad x,t \in \mathbb{R}, \tag{2.13a}$$

$$u(x,0) = f(x), \qquad f \in C(\mathbb{R}) \tag{2.13b}$$

$$\partial_t u(x,0) = g(x), \qquad g \in C(\mathbb{R}). \tag{2.13c}$$

Unlike (2.6) this IVP does not have an explicit solution for arbitrary $c(x)$. In fact the method to solving this IVP is completely different from solving (2.6). Instead of introducing characteristic lines we may suggest an ansatz, i.e. a guess for the format of our solution:

$$u(x,t) = X(x)Y(t). \tag{2.14}$$

Substitute this equation into (2.13a) to get

$$X\ddot{Y} = (c^2X')'Y = (c^2(x))'X'Y + c^2(x)X''Y,$$

where $\ddot{Y} = \partial_t^2 Y$ and $X' = \partial_x X$. The previous equation implies

$$\frac{\ddot{Y}}{Y} = \frac{(c^2(x))'X' + c^2(x)X''}{X} \tag{2.15}$$

for all $X \neq 0, Y \neq 0$. Because Y is a function of t and X is a function of x then (2.15) implies that $\ddot{Y}/Y$ and $(c^2(x)X')'/X$ must equal a common constant. Note if $(c^2(x)X')'/X = \lambda$ for $\lambda > 0$ and that $c(x) = c$ for constant $c > 0$ then $X'' = (\lambda/c^2)X$, which implies that the amplitudes of the waves are increasing as x increases. So instead we assume $\ddot{Y}/Y$ equals a nonpositive constant, i.e.

$$\frac{\ddot{Y}}{Y} = -\lambda^2, \tag{2.16}$$

where $\lambda \geq 0$. We have that $\ddot{Y} = -\lambda^2 Y$, which is an ordinary differential equation (ODE) that has a well-known general solution:

$$Y(t;\lambda) = F_1(\lambda)\cos(\lambda t) + F_2(\lambda)\sin(\lambda t), \tag{2.17}$$

where F_1 and F_2 are arbitrary functions. What (2.17) tells us is that our solution Y depends on what value of λ satisfies (2.16). We run into a similar situation for X, where according to (2.15) and (2.16) we have that

$$-\lambda^2 X = (c^2(x))' X' + c^2(x) X''. \tag{2.18}$$

From solving (2.18) we obtain a solution $X(x;\lambda)$. Because (2.13a) is linear, our general solution $u(x,t)$, based on our ansatz (2.14), is given by

$$u(x,t) = \int_0^\infty X(x;\lambda) Y(t;\lambda) d\lambda. \tag{2.19}$$

To find an exact solution, apply (2.13b) and (2.13c) to (2.19) and solve a system of equations, just as we did for (2.6). Note that we have not written a general solution to (2.18) for arbitrary $c(x)$. However, in chapter 4 we provide an example in which the ansatz (2.14) allows us to find a general solution of the form (2.19) to a particular PDE.

What we showed is that our method of solving PDEs varies when we consider spatially-variable speed. This is important when it comes to the 1D Klein-Gordon equation (KGE), because it is the equation (2.12) with an additional term. In the next section we explore the 1D KGE, and show how it is related to the Wave equation.

2.4 The 1D Klein-Gordon Equation

The 1D KGE is derived from the energy equation that arises in quantum mechanics. It is easy to derive as shown in [Gre00]. We have that the total energy E of our string consists of the system at rest plus the kinetic energy:

$$E = \sqrt{(mc_0^2)^2 + p^2 c_0^2}, \tag{2.20}$$

where m is the mass, c_0 is the speed of light, and p is the momentum. In our case of modeling the perturbation of waves in (x,t)-space, the energy E of the string is given by the operator

$i\hbar\partial_t$ and the momentum p is given by $\hbar\partial_x/i$, where

$$\hbar = 1.0545718 \times 10^{-34}\text{Joules} \cdot \text{seconds}$$

is the reduced Planck constant. Hence if we square both sides of (2.20) and substitute our values for E and p we obtain

$$-\hbar^2\partial_t^2 = (mc_0^2)^2 - \hbar^2\partial_x^2 c_0^2.$$

Finally introduce the perturbed height $u(x,t)$ to obtain the 1D Klein-Gordon equation:

$$\partial_t^2 u = c_0^2\partial_x^2 u - \left(\frac{mc_0^2}{\hbar}\right)^2 u, \qquad x,t \in \mathbb{R}. \tag{2.21}$$

The term $(mc_0^2/\hbar)^2$ describes the energy of the string at rest (mc_0^2) divided by the Planck reduced constant $\hbar$, all of which is squared.

In place of the constant potential term from quantum mechanics, for applications to water waves we introduce arbitrary potential $q(x)$, and recall the variable speed of equation (2.12):

$$\partial_t^2 u = \partial_x\left(c^2(x)\partial_x u\right) + q(x)u, \qquad x,t \in \mathbb{R}. \tag{2.22}$$

Again, if we apply an ansatz to (2.22), i.e., $u(x,t) = X(x)Y(t)$ for some X and Y, then solving (2.22) results in the general solution

$$u(x,t) = \int_0^\infty X(x;\lambda)Y(t;\lambda)d\lambda,$$

where

$$Y(t;\lambda) = F_1(\lambda)\cos(\lambda t) + F_2(\lambda)\sin(\lambda t)$$

for arbitrary F_1, F_2 and $\lambda \geq 0$, and $X(x;\lambda)$ is given implicitly by the ODE

$$c^2(x)X'' + (c^2(x))'X' + \lambda^2 X = q(x).$$

As expected, if $q(x) = 0$ then solving for X is identical to solving X in (2.18). Just like the IVP (2.6), we find an exact solution to $u(x,t)$ by substituting initial conditions into our general solution and solve a system of equations to find the arbitrary functions $X(x;\lambda)$ and $Y(t;\lambda)$. In particular, knowing conditions $u(x,0)$ and $\partial_t u(x,0)$ simplify $Y(0;\lambda)$, which helps in finding an exact solution to (2.22).

2.5 Initial Conditions Along an Arbitrary Curve

Recall solving (2.6) involved solving a system of equations to find an exact solution. However, as mentioned in chapter 1, we would like to solve the matrix KGE with initial conditions which originate on an arbitrary curve Γ. Finding the general solution to the matrix KGE follows the same method we showed in solving (2.13), but substituting Γ into our general solution is problematic. So in this section we discuss the difficulties in solving the Wave equation with initial conditions along Γ. We note that solving the Wave equation with initial conditions along Γ is an original problem that we do not solve. However the IVP presented in this section is a warm-up to the problem we address in chapter 3.

As we explored in finding the D'Alembert formula, we went from a general solution (2.9) to an exact solution (2.11) by plugging in initial conditions and solving a system of equations. In the case where $\Gamma = \mathbb{R} \times \{0\}$, we have that substituting our conditions along Γ into the general solution (2.9) is what results in the D'Alembert formula. However, even if the curve Γ is described as a function of x, we still run into complications to finding an exact solution, which is what we explore now.

Let us examine the following IVP:

$$\partial_t U = A\partial_x U, \qquad x,t \in \mathbb{R}, \tag{2.23a}$$

$$U(x,\tau(x)) = F(x), \tag{2.23b}$$

where $U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ for smooth u, v, $A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$ for arbitrary constant speed c, and

$$F(x) = \begin{pmatrix} f(x) \\ \int_0^x \left[g(s)/c + c\tau'(s)(f'(s) - \tau'(s)g(s))\right] ds + v(0,\tau(0)) \end{pmatrix}$$

for arbitrary $f, \tau \in C^{(1)}(\mathbb{R})$ and arbitrary $g \in C(\mathbb{R})$. The purpose of solving a vector-valued IVP is to show how to approach solving systems of PDEs. Moreover (2.23a) is a specific form of the matrix KGE (1.2), and finding a general solution to (1.2) is our problem of interest.

According to (2.23a) we have that $\partial_t u = c\partial_x v$ and $\partial_t v = c\partial_x u$. Differentiate each equation to yield $\partial_t^2 u = c\partial_t\partial_x v$ and $\partial_x\partial_t v = c\partial_x^2 u$. Since v is smooth then $\partial_t\partial_x v = \partial_x\partial_t v$, and hence we have the relation $\partial_t^2 u = c^2\partial_x^2 u$. Thus (2.23a) simplifies into the Wave equation. Meanwhile, initial conditions F were chosen so that $u(x,\tau(x)) = f(x)$ and $\partial_t u(x,\tau(x)) = g(x)$. Hence the IVP we are actually trying to solve is

$$\partial_t^2 u = c^2\partial_x^2 u, \qquad x,t \in \mathbb{R}, \tag{2.24a}$$

$$u(x,\tau(x)) = f(x), \tag{2.24b}$$

$$\partial_t u(x,\tau(x)) = g(x). \tag{2.24c}$$

For this IVP we have that initial conditions originate on the curve $\Gamma = \{(x,\tau(x)) : x \in \mathbb{R}\}$. Because the PDE (2.24a) is identical to the PDE in the IVP (2.6) then the general solution

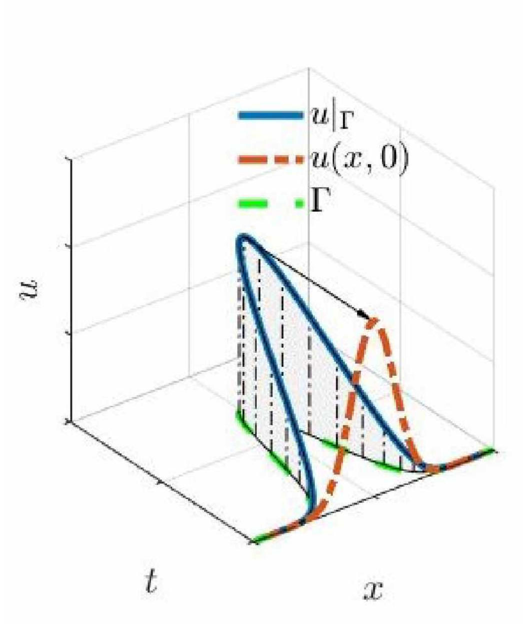

Figure 2.2: A figure illustrating initial conditions (solid curve) for (2.24) along a Gaussian curve ($t = A\exp(a(x-b)^2)$). Instead of substituting values $u|_\Gamma$ into our general solution (2.9), we would like to solve for the conditions along $\mathbb{R}\times\{0\}$ (dashed curve along $t=0$).

to (2.24a) is given by (2.9). Now let us substitute our initial conditions into (2.9):

$$f(x) = u(x,\tau(x)) = f_1(x - c\tau(x)) + f_2(x + c\tau(x)), \tag{2.25a}$$

$$g(x) = \partial_t u(x,\tau(x)) = -cf_1'(x - c\tau(x)) + cf_2'(x + c\tau(x)). \tag{2.25b}$$

The purpose of the system (2.25) is to solve for the functions f_1 and f_2. If τ is a constant function then solving (2.25) is similar to solving f_1 and f_2 for (2.6), because we solve a system of linear equations. For non-constant τ our problem is more complicated, because we cannot integrate (2.25b) to find f_1 and f_2.

Instead of finding an exact solution using $\{(x,\tau(x)) : x \in \mathbb{R}\}$, we would like to find conditions along $\mathbb{R}\times\{0\}$. Figure 2.2 illustrates what initial conditions (solid blue) along an arbitrary curve Γ (dashed green) may look like, versus how initial conditions look along $\mathbb{R}\times\{0\}$ (dashed red) in (x,t)-space. In chapter 4 we explore how to find conditions along $\mathbb{R}\times\{0\}$ using conditions along Γ.

The purpose of this chapter was to set the foundation for shallow water theory and solving IVPs. The Wave equation is not typically used for modeling water waves in general, but it is a valuable learning tool. It is possible to solve (2.13) using numerical methods, but an explicit solution like the D'Alembert formula is more convenient when it comes to studying properties of water waves. We must consider more complicated PDEs than the Wave equation, in particular the matrix KGE. It is time to present our problem of interest.

Chapter 3: Statement of the problem

Our statement of the problem is to solve the following IVP:

$$\partial_t U = LU, \qquad x, t \in \mathbb{R}, \tag{3.1a}$$

$$U_0(x) = U(x,t)\big|_\Gamma, \tag{3.1b}$$

where $U = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$ for functions u and v, $L = A\partial_x + B$ for arbitrary 2×2 matrix functions $A = A(x)$ and $B = B(x)$, and $\Gamma = \{(x, \tau(x)) : x \in \mathbb{R}\}$ for arbitrary curve τ in (x,t)-space.

The idea for solving (3.1) is to first simplify (3.1a) into two PDEs: one involving only u and the other involving only v. This is similar to rewriting (2.23a) into the Wave equation. To reduce (3.1a) into a one-system equation, we rely heavily on the entries of matrices A and B. If A and B are simple enough then (3.1a) may reduce to familiar PDEs that have known general solutions.

We have seen in finding an explicit solution to (2.6) that initial conditions are critical in finding an exact solution to a PDE. After we found a general solution to (2.6), we applied initial conditions (2.6b) and (2.6c) to our general solution to solve a system of equations, and obtain the exact solution (2.11). Even though our initial conditions for (3.1) are given along an arbitrary curve Γ instead of along $\mathbb{R} \times \{0\}$ as in (2.6), we would like to solve (3.1a) knowing conditions along $\mathbb{R} \times \{0\}$. We accomplish this goal by finding a projection such that we extrapolate each value $U(x,0)$ from each known value $U(x,t)\big|_\Gamma$.

In the next chapter we find a formula for projecting our values along Γ to values along $\mathbb{R} \times \{0\}$ in (x,t)-space. We follow this by finding a general solution to (3.1a) for particular matrices A and B. Our projection and general solution are enough to find an exact solution to the CSA SWEs for U-shaped bays.

Chapter 4: Solution to the problem

Although our initial conditions originate on a curve Γ, the idea in this chapter is to extrapolate values along $\mathbb{R} \times \{0\}$ using values along Γ. This is done by applying Taylor's formula. The one caveat of using the Taylor formula is that we must impose conditions on Γ to make the formula work. Section 4.1 covers what our projection looks like.

We would like our solution to (3.1a) as explicit as possible. This means imposing constraints on (3.1a), i.e. consider specific matrices A and B. We value this approach over numerical methods to solving (3.1a), because our explicit solution reduces error in using computer software to model the profile of waves within narrow U-shaped bays. So in section 4.2 we derive a general solution to a specific form of (3.1a).

4.1 Projection of Initial Conditions

The approach to solving (3.1a) along Γ is as follows. Consider a projection of U_0 onto some $\widetilde{U}_0$ such that, given any $\varepsilon > 0$ we have that $\|U(x,0) - \widetilde{U}_0(x)\| < \varepsilon$ for any $x \in \mathbb{R}$, where $\|\cdot\|$ is the maximum norm on $\mathbb{R}^2$:

$$\|(a,b)\| := \max\{|a|, |b|\}.$$

We call the map $U_0 \to \widetilde{U}_0$ the projection of U_0 onto $\mathbb{R}$. Since section 4.2 presents a general solution that easily obtains an exact solution via substitution of $\widetilde{U}_0$, our immediate goal is to find the map $(x, \tau(x)) \mapsto (x, 0)$. Such a map is found by applying Taylor's formula to $U(x,0)$:

$$U(x,0) = \sum_{n=0}^{n} \left(\frac{(-t)^k \partial_t^k U(x,t)}{k!} \right) \Bigg|_{\Gamma} + E_n(x), \tag{4.1}$$

where $E_n(x)$ is our error term:

$$E_n(x) = \frac{(-\tau(x))^{n+1}}{(n+1)!}\partial_t^{n+1}\begin{pmatrix} u(x,\xi_u(x)) \\ v(x,\xi_v(x)) \end{pmatrix},$$

for some $\xi_u(x), \xi_v(x)$ between 0 and $\tau(x)$. Thus we set

$$\widetilde{U}_0^{(n)}(x) = \sum_{k=0}^{n} \frac{(-\tau(x))^k}{k!} \left(\partial_t^k U(x,t)\right)\big|_\Gamma, \tag{4.2}$$

and call $\widetilde{U}_0^{(n)}$ the n^{th} order approximation of $U(x,0)$ from data U_0. Since we rely on Taylor's formula in (4.2), $U\big|_\Gamma$ and τ must satisfy the necessary conditions for Taylor's theorem. So for the remainder of this chapter we require that $U|_\Gamma$ and τ are smooth on $\mathbb{R}$.

First Note that if our data U_0 is given along the curve $\mathbb{R} \times \{0\}$ then our approximation is of zero order, i.e.

$$\widetilde{U}_0^{(0)}(x) = U(x,t)\big|_\Gamma = U_0(x).$$

However we expect that this is not likely the case for U_0. So we focus on simplifying (4.2).

To find the n^{th} order approximation we do so recursively. This means we start with finding the 1^{st} order projection explicitly in terms of (3.1b). This means solving for $(\partial_t U)\big|_\Gamma$ in terms of known values. According to (3.1a),

$$(\partial_t U)\big|_\Gamma = (LU)\big|_\Gamma. \tag{4.3}$$

Since $U\big|_\Gamma = U(x,\tau(x))$ then

$$\frac{d}{dx}(U\big|_\Gamma) = (\partial_x U)\big|_\Gamma + (\partial_t U)\big|_\Gamma \tau'(x). \tag{4.4}$$

It follows from (4.4) that

$$L\left(U\big|_{\Gamma}\right) = (LU)\big|_{\Gamma} + \tau' A(\partial_t U)\big|_{\Gamma} = (I + \tau' A)\left(\partial_t U\right)\big|_{\Gamma}, \tag{4.5}$$

where I is the 2×2 identity matrix. Let

$$D = I + \tau' A, \tag{4.6}$$

where I is the 2×2 identity matrix. From (4.5) deduce that

$$(\partial_t U)\big|_{\Gamma} = D^{-1} L\left(U\big|_{\Gamma}\right), \tag{4.7}$$

and hence our 1^{st} order projection is given by

$$\widetilde{U}_0^{(1)} = U_0 - \tau D^{-1} L U_0.$$

What (4.7) shows is that our partial derivative $\partial_t U$ along Γ can be written as an equation in terms of A, B, τ', and $\left(U\big|_{\Gamma}\right)$. Moreover (4.7) reveals a pattern. For $n \geq 1$ define

$$U_n = D^{-1} L U_{n-1}, \tag{4.8}$$

where U_0 is our initial data. Observe that the operator ∂_t commutes with $D^{-1}\Delta$. Thus

$$\begin{aligned}(\partial_t^2 U)\big|_{\Gamma} &= \partial_t(\partial_t U)\big|_{\Gamma} = (\partial_t D^{-1} L U)\big|_{\Gamma} \\ &= D^{-1} L(\partial_t U)\big|_{\Gamma} = D^{-1} L\left(D^{-1} L\left(U\big|_{\Gamma}\right)\right) \\ &= D^{-1} L U_1 = U_2.\end{aligned}$$

Continue iteratively to conclude the following.

Theorem 4.1. *Suppose we are given the initial-value problem*

$$\partial_t U = LU, \qquad x, t \in \mathbb{R}, \tag{4.9a}$$

$$U(x,t)|_\Gamma = U_0(x), \tag{4.9b}$$

where $U(x,t)$ is smooth on $\Gamma = \{(x, \tau(x)) : x \in \mathbb{R}\}$, and $\Delta = A\partial_x + B$ for arbitrary smooth 2×2 matrices $A = A(x)$ and $B = B(x)$. If τ is smooth on $\mathbb{R}$ and $D = I + \tau' A$ is invertible on $\mathbb{R}$, then for $n \geq 1$,

$$\left(\partial_t^n U\right)\big|_\Gamma = U_n, \tag{4.10}$$

where $U_n = D^{-1}\Delta U_{n-1}$.

By Theorem 4.1 we can rewrite our n^{th} order projection of U_0 as

$$\widetilde{U}_0^{(n)}(x) = \sum_{k=0}^{n} \frac{(-\tau(x))^k}{k!} U_k(x). \tag{4.11}$$

Recall that for any $\varepsilon > 0$ we would like that $\|U(x,0) - \widetilde{U}_0(x)\| < \varepsilon$ for any $x \in \mathbb{R}$. In other words, we want our n^{th} order projection $\widetilde{U}_0^{(n)}$ to converge absolutely to $\widetilde{U}_0$. The way this is achieved is that the error term $E_n(x)$ converges to 0 absolutely, i.e.

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \|E_n(x)\| = 0.$$

We note that for each x, there must be a one-to-one correspondence between $U(x, \tau(x))$ and $U(x, 0)$ for our projection to work. This means for each value of x, there is exactly one value of τ. Otherwise τ is not a function of x. When it comes to our numerical simulations we consider the cases when τ is a Gaussian and an N-wave curve, both of which satisfy the necessary requirements for our projection to work. So our next focus is finding a suitable solution to (3.1a).

4.2 Solutions to the Klein-Gordon Equation with Specific A and B

The solution we present in this section involves reducing the matrix KGE into a 1-system PDE. The proof is straightforward, and it has practical applications for modeling the CSA SWEs. The solution is as follows:

Proposition 4.2. *Suppose we are given the linear PDE*

$$\partial_t U = LU, \qquad 0 \leq x, t \in \mathbb{R}, \tag{4.12}$$

where $L = A\partial_x + B$ *and*

$$U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, A(x) = \begin{pmatrix} 0 & a \\ bx & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \tag{4.13}$$

for $0 < |b| \leq 1$, $0 < ab$, *and* $1 \leq c/b$. *If* $\|U(0,t)\| < \infty, \lim_{x\to\infty} U(x,t) = 0$, *and* U *is smooth then the general solution to (4.12) is given by*

$$u(x,t) = x^{-\frac{c}{2b}} \int_0^\infty K(k,t) J_{c/b}(2k\sqrt{x})dk, \tag{4.14a}$$

$$v(x,t) = -\frac{1}{a} x^{\frac{1}{2}(1-c/b)} \int_0^\infty \frac{K_t(k,t)}{k} J_{c/b-1}(2k\sqrt{x})dk, \tag{4.14b}$$

where

$$K(k,t) = C_1(k)\sin(\sqrt{ab}kt) + C_2(k)\cos(\sqrt{ab}kt)$$

for arbitrary $C_1(k)$ *and* $C_2(k)$, *and* $J_\alpha(z)$ *is the Bessel function of the first kind of order* α.

Proof. We first note (4.12) is the system of equations

$$\partial_t u = a\partial_x v \tag{4.15a}$$

$$\partial_t v = bx\partial_x u + cu. \tag{4.15b}$$

Differentiate (4.15) with respect to t to get

$$\partial_t^2 u = a\partial_t\partial_x v, \tag{4.16a}$$

$$\partial_t^2 v = bx\partial_t\partial_x u + c\partial_t u. \tag{4.16b}$$

Now differentiate (4.15) with respect to x

$$\partial_x\partial_t u = a\partial_x^2 v, \tag{4.17a}$$

$$\partial_x\partial_t v = b\partial_x u + bx\partial_x^2 u + c\partial_x u. \tag{4.17b}$$

Since U is smooth then $\partial_x\partial_t U = \partial_t\partial_x U$. Thus we substitute equations (4.17) into equations (4.16) to yield

$$\partial_t^2 u = abx\partial_x^2 u + a(b+c)u_x, \tag{4.18a}$$

$$\partial_t^2 v = abx\partial_x^2 v + acv_x. \tag{4.18b}$$

Note by solving (4.18a) we may easily derive (4.18b). To find u, consider the ansatz

$$u(x,t) = X(x)Y(t) \tag{4.19}$$

to rewrite (4.18a) as

$$X\ddot{Y} = abxX''Y + a(b+c)X'Y. \tag{4.20}$$

After separating the functions of x from the functions of t, we get

$$\ddot{Y} = -\lambda^2 Y \tag{4.21a}$$

$$abxX'' + a(b+c)X' = -\lambda^2 X, \tag{4.21b}$$

where $\lambda \geq 0$. Equation (4.21a) is a second-order linear ODE with solution

$$Y(t) = a_1(\lambda)\sin(\lambda t) + a_2(\lambda)\cos(\lambda t), \tag{4.22}$$

where $a_1(\lambda)$ and $a_2(\lambda)$ are arbitrary functions. Equation (4.21b) is equivalent to

$$x^2X'' + \left(1 + \frac{c}{b}\right)xX' + \left(\frac{\lambda}{\sqrt{ab}}\right)^2 xX = 0. \tag{4.23}$$

Let $(1 + c/b) = (2p + 1), r = 1/2$, and $\beta = 0$. Then (4.23) appears as the modified Bessel equation given in [Bow58]:

$$x^2X'' + (2p+1)xX' + \left(\left(\frac{\lambda}{\sqrt{ab}}\right)^2 x^{2r} + \beta^2\right)X = 0. \tag{4.24}$$

The solution to (4.24) is given by

$$X(x) = x^{-p}\left[b_1(\lambda)J_{n/r}\left(\frac{\lambda}{r\sqrt{ab}}x^r\right) + b_2(\lambda)Y_{n/r}\left(\frac{\lambda}{r\sqrt{ab}}x^r\right)\right], \tag{4.25}$$

where $n = \sqrt{p^2 - \beta^2}$, $b_1(\lambda)$ and $b_2(\lambda)$ are arbitrary functions, and $Y_\alpha(z)$ is the Bessel function of the second kind of order α. Note n simplifies to $|p|$. Moreover, because $Y_{n/r}(x)$ is unbounded at $x = 0$, and $|U(x = 0, t)| < \infty$, then we require $b_2(\lambda) = 0$. Thus equation (4.25) becomes

$$X(x) = b_1(\lambda)x^{-\frac{c}{2b}}J_{c/b}\left(2\lambda\sqrt{\frac{x}{ab}}\right). \tag{4.26}$$

Combine equations (4.22) and (4.26) into (4.19) to get

$$u(x, t; \lambda) = x^{-\frac{c}{2b}}\{C_1(\lambda)\sin(\lambda t) + C_2(\lambda)\cos(\lambda t)\}J_{c/b}\left(2\lambda\sqrt{\frac{x}{ab}}\right), \tag{4.27}$$

where $C_1(\lambda)$ and $C_2(\lambda)$ are arbitrary functions. Imposing $\lambda(k) = \sqrt{ab}\,k$, our solution given by (4.27) becomes

$$u(x,t;k) = x^{-\frac{c}{2b}} K(k,t) J_{c/b}(2k\sqrt{x}). \tag{4.28}$$

A similar argument for (4.18b) yields

$$v(x,t;k) = x^{\frac{1}{2}(1-c/b)} F(k,t) J_{c/b-1}(2k\sqrt{x}), \tag{4.29}$$

where $F(k,t) = D_1(k)\cos(\sqrt{ab}kt) + D_2(k)\sin(\sqrt{ab}kt)$ for arbitrary functions $D_1(k)$ and $D_2(k)$.

Now observe,

$$\begin{aligned} \frac{d}{dx}\left(x^{\frac{1}{2}(1-c/b)} J_{c/b-1}\left(2k\sqrt{x}\right)\right) &= -\frac{k}{\sqrt{x}} x^{\frac{1}{2}(1-c/b)} J_{c/b}\big(2k\sqrt{x}\big) \\ &= -kx^{-\frac{c}{2b}} J_{c/b}\big(2k\sqrt{x}\big). \end{aligned}$$

Thus $a\partial_x v$ is given by

$$a\partial_x v = -akx^{-\frac{c}{2b}} F(k,t) J_{c/b}\left(2k\sqrt{x}\right). \tag{4.30}$$

Since $\partial_t u = a\partial_x v$, then we may differentiate (4.28) with respect to t and compare it to (4.30), which shows the relation

$$x^{-\frac{c}{2b}} K_t(k,t) J_{c/b}(2k\sqrt{x}) = -akx^{-\frac{c}{2b}} F(k,t) J_{c/b}\left(2k\sqrt{x}\right). \tag{4.31}$$

This implies $F = -K_t/(ak)$. We may rewrite (4.29) as

$$v(x,t;k) = -\frac{1}{a} x^{\frac{1}{2}(1-c/b)} \frac{K_t(k,t)}{k} J_{c/b-1}(2k\sqrt{x}). \tag{4.32}$$

Finally, integrate (4.28) and (4.32) for all $k \geq 0$ to obtain the solution to (4.12):

$$
\begin{aligned}
u(x,t) &= x^{-\frac{c}{2b}} \int_0^\infty K(k,t) J_{c/b}(2k\sqrt{x})dk, \\
v(x,t) &= -\frac{1}{a} x^{\frac{1}{2}(1-c/b)} \int_0^\infty \frac{K_t(k,t)}{k} J_{c/b-1}(2k\sqrt{x})dk.
\end{aligned}
$$

$\square$

In this chapter we went through deliberate efforts to find values along $\mathbb{R} \times \{0\}$ using values along Γ. We note that the projection places several constraints on Γ, and that the n^{th} order projection only approximates values $\widetilde{U}_0$. Moreover, we note that an exact solution emerges from (4.14), provided we know our values on $\mathbb{R} \times \{0\}$. This is because at $t = 0$ the solution (4.14) simplifies into a system of equations, at which point we may explicitly solve for the functions $C_1(k)$ and $C_2(k)$. In other words, solving (3.1) would be much simpler if our initial conditions originate on $\mathbb{R} \times \{0\}$. That is why in the next chapter we examine the practical reason for considering initial conditions along Γ.

Although solution (4.14) involves many constraints on matrices A and B, the following chapter makes practical use of (4.14). As mentioned earlier, the matrix KGE has applications to shallow water theory, but (4.14) explicitly shows us how the matrix KGE actually models water waves. Hence we examine the CSA SWEs, and how it is actually a particular form of the matrix KGE.

Chapter 5: Applications to the shallow water wave equation

The CSA SWEs are a system of PDEs used to model the profiles of waves within narrow bays. Its solution describes the perturbation and velocity of waves, although we are particularly interested in modeling just the perturbation. If we consider waves hitting the shoreline of a beach of constant slope, the Carrier-Greenspan (CG) transform turns the CSA SWEs into a system of linear PDEs, namely a particular form of the matrix KGE. This first section presents the CSA SWEs and the transformation used to linearize it.

If we consider a power-shaped cross section then the CSA SWEs have an explicit solution. This is because the CG transform turns the CSA SWEs into the particular form of the matrix KGE presented in proposition 4.2. This confirms the link between the matrix KGE and the tsunami run-up problem.

5.1 Cross-Sectionally Averaged Shallow Water Equations

We consider the Cross-sectionally averaged Shallow Water equations (CSA SWEs) described in [Sto57]:

$$\partial_{t^*} S^* + \partial_{x^*}(S^* \mu^*) = 0, \tag{5.1a}$$

$$\partial_{t^*} \mu^* + \mu^* \partial_{x^*} \mu^* + g \partial_{x^*} \eta^* = 0, \tag{5.1b}$$

where $S^*(H^*(x,t))$ is the cross-sectional area under the total water height $H^*(x,t)$, g is the acceleration of the water due to gravity, $\mu^* = \mu^*(x^*, t^*)$ is the velocity, and $\eta^* = \eta^*(x^*, t^*)$ is the perturbed water height. Given unperturbed water height $h = h(x)$ we have the relation $H^* = h^* + \eta^*$. Because we are interested in the run-up of tsunami waves along a shoreline of constant slope then the unperturbed height h^* is described by $h^*(x^*) = \alpha x^*$ for some slope of magnitude $\alpha > 0$.

The system in (5.1) can be reduced to a dimensionless form. The purpose of doing this is two-fold: the dimensionless form of the CSA SWEs is a simpler system of PDEs to solve and the dimensionless form requires less data in order to graph its solution. The idea is that we introduce an arbitrary scalar l and transform our terms into dimensionless values:

$$x = \frac{x^*}{l}, \qquad t = t^*\sqrt{\frac{g\alpha}{l}}, \qquad \mu = \frac{\mu^*}{\sqrt{g\alpha l}}, \qquad \eta = \frac{\eta^*}{\alpha l}, \qquad h = \frac{h^*}{\alpha l}, \qquad S = \frac{S^*}{A_0}, \tag{5.2}$$

where A_0 is an arbitrary area scalar. After condition (5.2) is applied to (5.1) then we have a dimensionless form of the CSA SWEs. Therefore the problem presented in this chapter is as follows:

$$\partial_t S + \partial_x(S\mu) = 0, \qquad x \in \mathbb{R}, t \geq 0 \tag{5.3a}$$

$$\partial_t \mu + \mu\partial_x\mu + \partial_x\eta = 0, \tag{5.3b}$$

$$\mu(x, 0) = \mu_0(x), \qquad \mu_0 \in C^\infty(\mathbb{R}) \tag{5.3c}$$

$$\eta(x, 0) = \eta_0(x), \qquad \eta_0 \in C^\infty(\mathbb{R}). \tag{5.3d}$$

It is practical to assume that μ_0 and η_0 are continuously differentiable, because they are the dimensionless form of physical data. Moreover we obtain our initial conditions at $t = 0$ because it is reasonable to collect data when the wave is initially formed.

We can transform (5.3a)-(5.3b) into a linear system using the generalized CG transform, which is derived in [RPD14] and later generalized in [RNRP18]. The CG transform takes on the following form:

$$\varphi = \mu, \qquad \psi = \eta + \frac{\mu^2}{2}, \tag{5.4a}$$

$$s = x + \eta, \qquad \tau = t - \mu. \tag{5.4b}$$

Based on (5.4), system (5.3a)-(5.3b) becomes

$$\partial_\tau \varphi + \partial_s \psi = 0 \tag{5.5}$$

$$\partial_\tau \psi + \varphi + \frac{S}{S'} \partial_s \varphi = 0, \tag{5.6}$$

where $S' = dS(H)/dH$. Equations (5.5)-(5.6) can be rewritten as a particular form of the matrix KGE

$$\Phi_\tau = L\Phi, \tag{5.7}$$

where $L = A\partial_s + B$ for

$$\Phi(s,\tau) = \begin{pmatrix} \varphi(s,\tau) \\ \psi(s,\tau) \end{pmatrix}, A(s) = \begin{pmatrix} 0 & -1 \\ -\frac{S}{S'} & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{5.8}$$

We expect S to respect physical considerations, and therefore S is smooth. Since S is the cross-sectional area of the bay, S should be positive and strictly increasing as s increases. This implies $S|_{H=0} = 0$ and $S'(s) > 0$ for all $s > 0$. Therefore our matrix A is continuous for all $s > 0$.

In (x,t) space, we are given initial conditions u_0 and η_0 along the curve $\mathbb{R} \times \{0\}$. In (s,τ) space, the conditions u_0 and η_0 are transformed onto the curve

$$\Gamma = \{(s|_{t=0}, \tau|_{t=0}) : s > 0\}.$$

Since

$$s|_{t=0} = x + \eta_0, \qquad \tau|_{t=0} = -\mu_0,$$

then Γ is explicitly given by

$$\Gamma = \{ (\, s, -\mu_0(\, x(s)\,)\,) : s > 0 \}\,, \tag{5.9}$$

where $x(s)$ is implicitly given via the relation at $s|_{t=0}$:

$$s = x(s) + \eta_0(\, x(s)\,). \tag{5.10}$$

It is vital that $x(s)$ is a bijective function. That way, we obtain one value of s for each value of x. We note that the CSA SWEs describe the perturbation of shallow waves, and thus we can rely on η being very small compared to h. So we impose the condition that $\eta_0'(x) > -1$ for all x. Thus (5.10) is strictly increasing for all values of x, and is therefore bijective. Along Γ, the initial conditions for Φ become

$$\Phi(s,\tau)|_{\Gamma} = \begin{pmatrix} \varphi(s,\tau)|_{t=0} \\ \psi(s,\tau)|_{t=0} \end{pmatrix} = \begin{pmatrix} \mu_0(\, x(s)\,) \\ \eta_0(\, x(s)\,) + \mu_0^2(\, x(s)\,)/2 \end{pmatrix} = \Phi_0(s). \tag{5.11}$$

By proposition 4.1 we have a method of approximating $\Phi(s, 0)$ using Φ_0:

$$\widetilde{\Phi}_0^{(n)}(s) = \sum_{k=0}^{n} \frac{\left(\, \mu_0(\, x(s)\,)\,\right)^k}{k!} \Phi_k(s), \tag{5.12}$$

where $\Phi_n = D^{-1} L \Phi_{n-1}$. Based on formula (5.12), we require $D = I - u_0' A$ to be nonsingular for $s > 0$, i.e. the determinant of D is nonzero:

$$1 - \frac{S}{S'} (\mu_0'(s))^2 \neq 0.$$

Since $\mu_0(s)$ is smooth for $s > 0$ then either $|\mu_0'(s)| < \sqrt{S'/S}$ or $|\mu_0'(s)| > \sqrt{S'/S}$. For physical considerations, we have that $\mu_0'(s) \to 0$ as $s \to \infty$. Thus we impose the condition

$$|\mu_0'| < \sqrt{\frac{S'}{S}}. \tag{5.13}$$

We note that we do not have a general solution to (5.7) for arbitrary $S(H)$. However, if $S(H)/S'(H) = aH$ for some $a < 0$ then we may apply proposition 4.2 to (5.7). Fortunately this is precisely the case for U-shaped bays. In this next section we explore how U-shaped bays affect $S(H)$, and how to obtain an exact solution using initial conditions along $\mathbb{R} \times \{0\}$ in (s, τ)-space.

5.2 U-shaped Bays

For U-shaped bays, the bay bathymetry is determined by

$$Z(x, y) = \beta|y|^m - h(x) = \beta|y|^m - x$$

for arbitrary $\beta > 0$ and $m > 0$ (see figure 1.1 B). In other words, $Z(x, y)$ describes the shape of the bay beneath the water surface. Therefore we have that $Z \leq \eta$, or rather $\beta|y|^m \leq H$. This implies that y takes on values between $-(H/\beta)^{1/m}$ and $(H/\beta)^{1/m}$.

Across the y-axis the total water depth is given by $\eta - Z$. This is because η marks the peak and Z marks the bottom of the water at position x, y and time t. Since $Z(x, y) = \beta|y|^m - x$ and $H(x, t) = x + \eta(x, t)$ then $\eta - Z = H - \beta|y|^m$. Hence

$$S(H) = \int_{-(H/\beta)^{1/m}}^{(H/\beta)^{1/m}} (\eta - Z)dy = \int_{-(H/\beta)^{1/m}}^{(H/\beta)^{1/m}} (H - \beta|y|^m)dy = \frac{2m}{\beta^{1/m}(m+1)} H^{\frac{m+1}{m}}. \tag{5.14}$$

Most importantly, $S(H)$ is characterized by the relation $S(H) \propto H^{(m+1)/m}$. Thus $S/S' = \frac{m}{m+1}s$ and (5.7) changes into

$$\Phi_\tau = L\Phi, \tag{5.15}$$

where $L = A\partial_s + B$ and

$$A(s) = \begin{pmatrix} 0 & -1 \\ -\frac{m}{m+1}s & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{5.16}$$

The general solution to (5.15) is found in proposition 4.2:

$$\varphi(s,\tau) = s^{-\frac{1}{2}(1+1/m)} \int_0^\infty K(k,\tau) J_{1+1/m}(2\sqrt{s}k)dk, \tag{5.17a}$$

$$\psi(s,\tau) = s^{-\frac{1}{2m}} \int_0^\infty \frac{K_\tau(k,\tau)}{k} J_{1/m}(2\sqrt{s}k)dk, \tag{5.17b}$$

where $c = \sqrt{\frac{m}{m+1}}$,

$$K(k,\tau) = C_1(k)\sin(ck\tau) + C_2(k)\cos(ck\tau)$$

for arbitrary functions $C_1(k)$ and $C_2(k)$, and $J_\gamma(z)$ is the Bessel function of the first kind of order γ. Because of the asymptotic behavior of the Bessel function, $\varphi(s,\tau)$ and $\psi(s,\tau)$ remain bounded at $s = 0$.

Given an initial condition for Φ along $\tau = 0$, $C_1(k)$ and $C_2(k)$ can be explicitly solved by applying the inverse Hankel transform to equations (5.17b)-(5.17a). Hence,

$$C_1(k) = \frac{2k}{c} \int_0^\infty s^{\frac{1}{2m}} \psi(s,0) J_{1/m}(2k\sqrt{s})ds \tag{5.18a}$$

$$C_2(k) = 2k \int_0^\infty s^{\frac{1}{2}(1+1/m)} \varphi(s,0) J_{1+1/m}(2k\sqrt{s})ds, \tag{5.18b}$$

and we now have an analytical solution to (5.15).

While we want an initial condition for Φ along $\tau = 0$, instead we are given

$$\Phi(s,\tau)|_\Gamma = \Phi_0.$$

To solve for Φ in (5.15) given Φ_0, we apply (5.12) to Φ_0 to find $\Phi(s,0)$.

What we have shown so far is the relation between the CSA SWEs and the matrix

KGE. The key ingredient in this link is the CG transform. A word of warning is that the CG transform shows the limits of how the matrix KGE can model the tsunami run-up problem, because the CG transform only works on shorelines of constant slope. Moreover the CG transform also transforms our initial conditions μ_0 and η_0 along $\mathbb{R} \times \{0\}$ in (x, t)-space to conditions on a curve in (s, τ)-space. If we are not careful about how μ_0 and η_0 are chosen then there may not be a one-to-one correspondence between $(\mu(x,t), \eta(x,t))$ and $(\varphi(s,\tau), \psi(s,\tau))$. That is why we imposed further conditions on μ_0 and η_0 beyond the conditions required in chapter 4. Our next procedure is to test our solutions (5.17b)-(5.17a) for various u_0 and η_0.

5.3 Numerical simulations

Because (5.15) has an explicit solution, our numerical simulations model the tsunami run-up on different U-shaped bays. However, determining appropriate initial conditions for (5.15) is rather difficult. There is no known way of measuring initial perturbation and initial velocity of an incoming wave. This means that exact conditions for μ_0 and η_0 are unknown. Instead we derive μ_0 in terms of η_0, and then provide a reasonable guess for η_0, as given in previous research papers. This means examining a Gaussian, followed by an N-wave, for our initial perturbation. Our numerical simulations examine the run-up of solitary waves, followed by the perturbation at the shoreline.

In regards to waves running toward the shore, we rely on the initial velocity given by [DP11]:

$$\mu_i(x) = -\frac{2}{c}\left(\sqrt{x + \eta_i(x)} - \sqrt{x}\right) \approx -\frac{\eta_i(x)}{c\sqrt{x}}, \tag{5.19}$$

where $c = \sqrt{\frac{m}{m+1}}$ and $i = 1, 2$. For the tsunami run-up problem our initial perturbation occurs above the unperturbed water surface, i.e. $\eta_i > 0$. Thus our initial velocity (5.19) is negative, because our positive x-axis points away from the shoreline. As for η_i, we take into consideration [CWY03], [Kan04] and [KS06] and examine a Gaussian and an N-wave profile.

Thus we use

$$\eta_1(x) = A_1 e^{-\kappa(x-x_1)^2}, \tag{5.20a}$$

$$\eta_2(x) = A_1 e^{-\kappa(x-x_1)^2} - A_2 e^{-\kappa(x-x_2)^2}, \tag{5.20b}$$

where $A_1 = 0.001$, $A_2 = 0.00025$, $\kappa = 4$, $x_1 = 2$ and $x_2 = 1$.

We apply these initial conditions (5.19) and (5.20) for V-shaped bays ($m = 1$), parabolic bays ($m = 2$), U-shaped bays ($m = 4$), and planar beaches ($m = \infty$). Because our numerical simulations require computer software to model wave propagation, we let $m = 1000$ represent $m = \infty$ and note that $m = 100$ results in a very planar beach.

In regards to our application of solution (5.17), we note that there are numerical approximations we must apply. We begin with a 3^{rd} order Taylor series approximation (5.12) applied to our initial conditions (5.20) and (5.19), noting that the 2^{nd} order and 3^{rd} order approximation are nearly identical. Then we approximate $C_1(k)$ and $C_2(k)$ in (5.18) via integration from $s = 0$ to $s = 4$, noting that our initial conditions (5.20) and (5.19) are small in magnitude. Next the solution φ and ψ in (5.17) are approximated via integration from $k = 0$ to $k = 20$, because $C_1(k)$ and $C_2(k)$ are very close to 0 for $k \geq 15$. We consider solutions φ and ψ for $0.01 \leq s \leq 4$ and $0 \leq \tau \leq 10$, being careful to plug in $s = 0$ to (5.17) directly. Finally we determine our solutions μ and η in (x,t)-space via the generalized CG transform (5.4).

We note x, t, μ, η are presented in dimensionless form. What figure 5.1 shows is the perturbed height for time $t = 1, 2, ..., 8$ and for $m = 1, 2, 4, \infty$, given initial Gaussian perturbation (5.20a). What we notice is that for narrower bays (i.e. lower m) the leading wave hits the shoreline at a later time, but it begins with a higher perturbed height. This is because as m increases the cross-sectional area increases, which results in faster wave propagation. Note at times $t = 5, ..., 8$ all four waves are traveling in the opposite direction. Again, we see that a narrower bay results in a higher wave reflected back. We may also note that

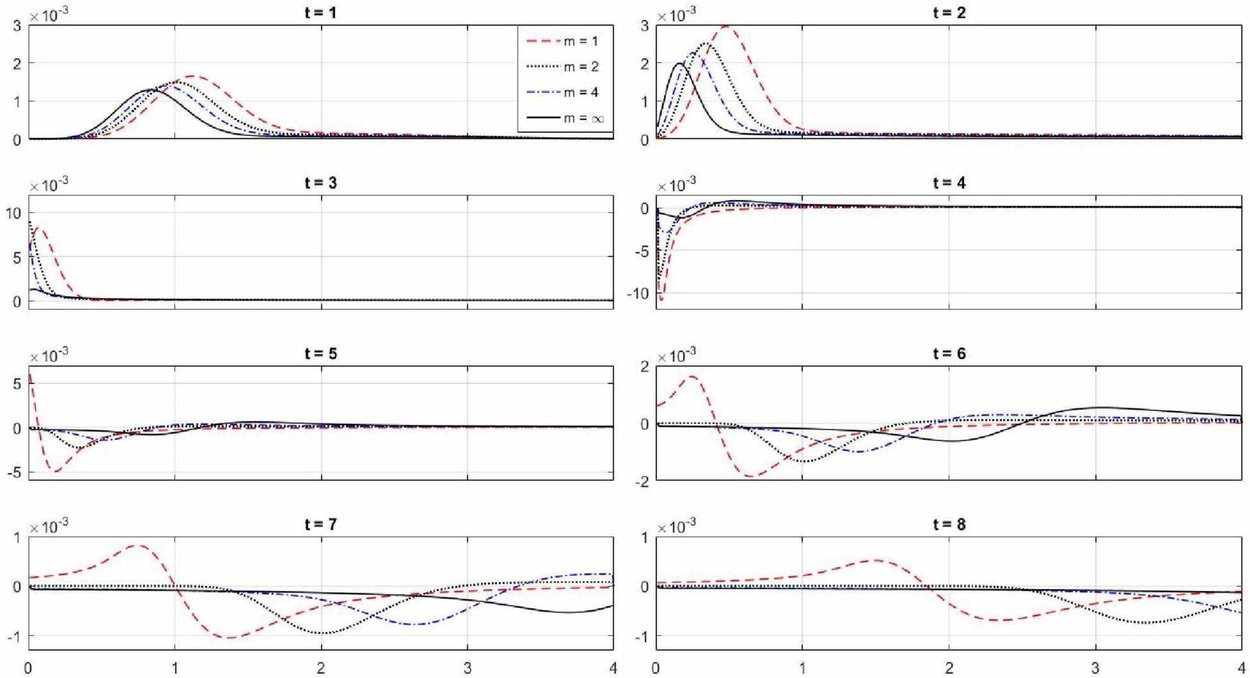

Figure 5.1: The perturbed water height, η, at $t = 1, 2, ..., 8$ for the incident Gaussian wave in a U-shaped bay where $m = 1, 2, 4$, and ∞. Note that the amplitudes of the waves are all decreasing with respect to time.

for V-shaped bays ($m = 1$), the wave reflected back is a leading depression N-wave, while $m = 2, 4, \infty$ reflect leading elevation N-waves. However, as time progresses, the perturbation of the reflected wave approaches zero. By time $t = 8$ the plane-shaped bay shows almost no change in perturbation.

Next in Figure 5.2 we examine how perturbed water height at the shoreline (i.e. $s = 0$) behaves for $m = 1, 2, 4, \infty$, given the initial Gaussian profile (5.20a). As expected, a narrower bay results in a higher wave hitting the shoreline, but it does so at a slower time. One item to note is that the V-shape bay manages to have a second wave hit the shore at approximately $t = 4.8$.

Figure 5.3 shows the perturbed water height for bays $m = 1, 2, 4, \infty$, given initial N-wave perturbation (5.20b). Similar to the Gaussian waves, a narrower bay results in a higher wave but a slower time in hitting the shore. Also the perturbation approaches zero as time increases. We note that figure 5.1 and 5.3 have very similar wave propagation.

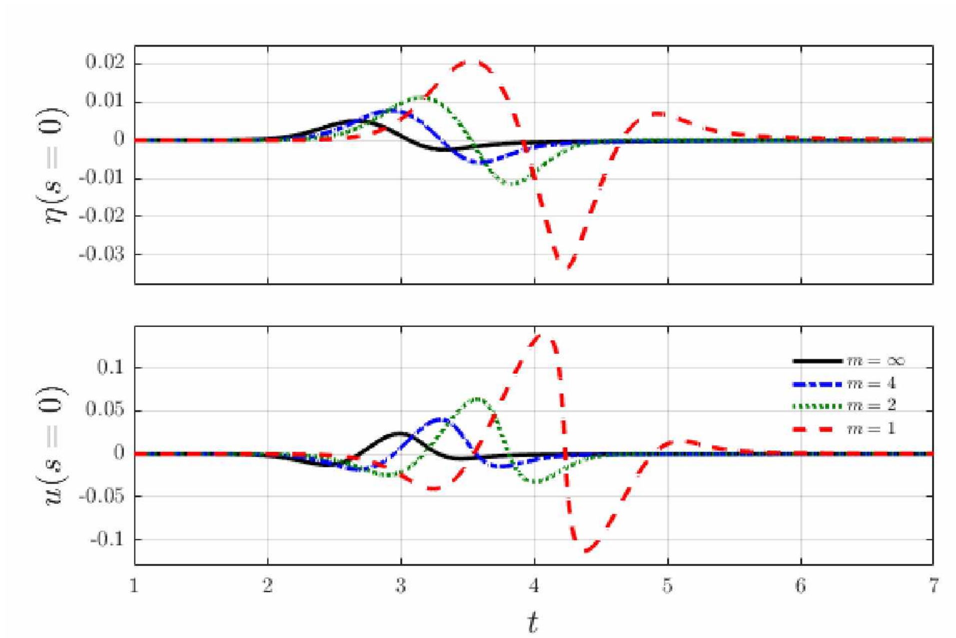

Figure 5.2: The perturbed water height and velocity, η and u, at the shoreline for the incident Gaussian wave in a U-shaped bay where $m = 1, 2, 4$, and ∞.

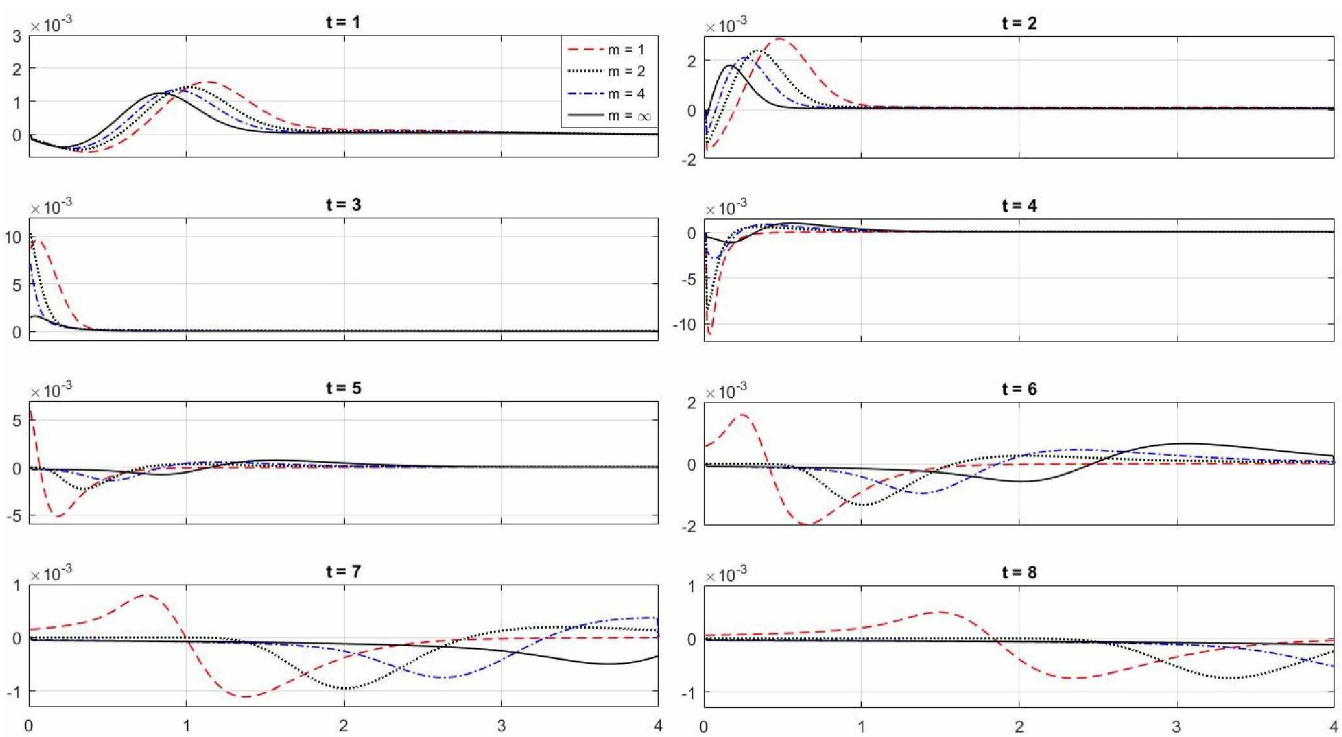

Figure 5.3: The perturbed water height, η, at $t = 1, 2, ..., 8$ for the incident leading depression N-wave shown in a U-shaped bay where $m = 1, 2, 4$, and ∞. Similar to Figure 5.1, the amplitudes for all waves decrease over time.

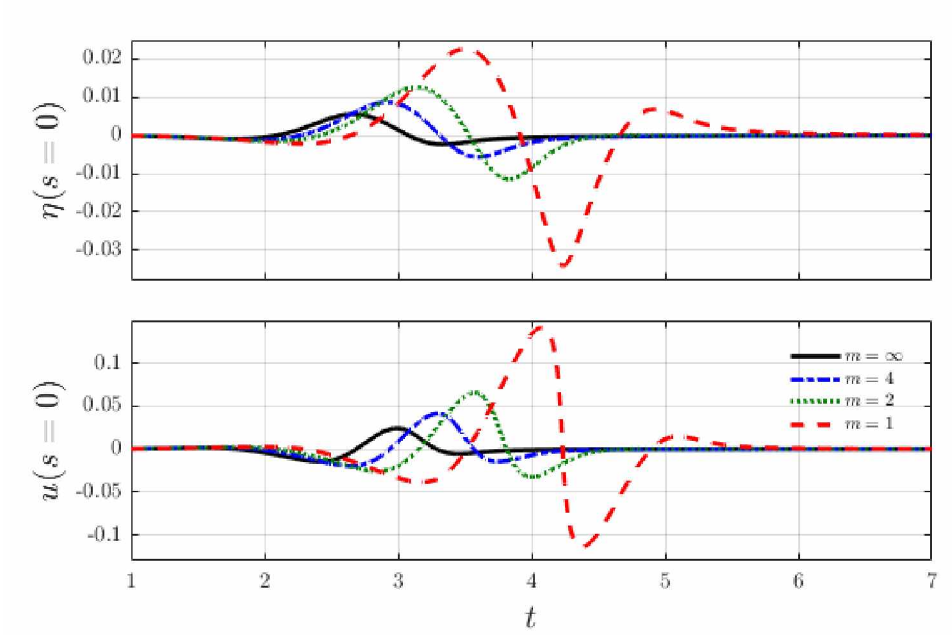

Figure 5.4: The perturbed water height and velocity, η and u, at the shoreline for the incident leading depression N-wave in a U-shaped bay where $m = 1, 2, 4$, and ∞.

Figure 5.4 is nearly identical to figure 5.2, suggesting that a Gaussian and N-wave profile hit the shoreline the same way. However, note that the waves result in a slightly higher run-up compared to the run-up in figure 5.2. This is because the leading depression of the N-wave is pushed onto the shoreline and contributes to the maximum amplitude of the water at $s = 0$.

What our numerical simulations demonstrate is that the bay bathymetry affects the run-up of waves in a significant way. The idea is that as the value of m gets smaller our run-up becomes higher. This is a reasonable conclusion, because smaller m means sharper bathymetry, and conversely higher m means smoother floors.

Chapter 6: Conclusions

What we have shown in this project is that the matrix Klein-Gordon equation can generate a solution to the cross-sectionally averaged Shallow Water equations. Although the Carrier-Greenspan turns the nonlinear CSA SWEs into the linear KGE, it also transforms our initial conditions originating on a line to conditions along a curve. The Taylor formula is used to extrapolate initial conditions to a line. In fact the Taylor formula, along with our general solution to the CSA SWEs for U-shaped bays, demonstrate that the benefit of linearizing the CSA SWEs far outweighs the cost of sending initial conditions to conditions on a curve.

Our solution to the CSA SWEs for U-shaped bays has significance in modeling tsunami waves. The solution is limited to waves along shorelines of constant slope, but having an analytical solution is a valuable tool for modeling. This is why we would like to determine analytical solutions to the CSA SWEs for arbitrary-shaped bays. However, finding an analytical solution to the linear SWEs is a difficult problem, even though it is a restricted form of the linear KGE in matrix form. This is why our statement to the problem has significant applications to the tsunami run-up problem.

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