Self-organized criticality as an absorbing-state phase transition

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We explore the connection between self-organized criticality and phase transitions in models with absorbing states. Sandpile models are found to exhibit criticality only when a pair of relevant parameters — dissipation ϵ and driving field h — are set to their critical values. The critical values of ϵ and h are both equal to zero. The first result is due to the absence of saturation (no bound on energy) in the sandpile model, while the second result is common to other absorbing-state transitions. The original definition of the sandpile model places it at the point ($\epsilon=0, h=0^+$): it is *critical by definition*. We argue power-law avalanche distributions are a general feature of models with infinitely many absorbing configurations, when they are subject to slow driving at the critical point. Our assertions are supported by simulations of the sandpile at $\epsilon=h=0$ and *fixed* energy density ζ (no drive, periodic boundaries), and of the slowly driven pair contact process. We formulate a field theory for the sandpile model, in which the order parameter is coupled to a *conserved* energy density, which plays the role of an effective creation rate. [S1063-651X(98)08805-9]

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I. INTRODUCTION

Avalanche behavior is common to many physical phenomena, ranging from magnetic systems (the Barkhausen effect) [1] and flux lines in high-T_c superconductors [2], to fluid flow through porous media [3], microfracturing processes [4], earthquakes [5], and lung inflation [6]. The common feature of all these systems is slow external driving, causing an intermittent, widely distributed response. Avalanches come in very different sizes, often distributed as a power law. This fact excites the interest of statistical physicists, since power laws imply the absence of a characteristic scale, a feature observed close to a critical point. In order to describe a critical point, we need only specify a set of critical exponents, whose values are determined by general symmetries and conservation laws and do not depend on microscopic details of the system.

Is there a connection between the observed power-law distribution of avalanche sizes and critical phenomena? And if so, can we understand the physics of avalanches by applying what we know about critical points and universality? A tentative answer to these questions was given by Bak, Tang, and Wiesenfeld (BTW) [7], who proposed that the power laws in avalanche statistics are due to a new kind of critical phenomenon, which they called self-organized criticality (SOC). In ordinary phase transitions, criticality is attained only by fine-tuning certain control parameters (temperature, pressure, etc.) to special values. Only close to this critical point is scale invariance observed. BTW suggested that in some dynamical systems the critical point is reached automatically, without any fine tuning, thus explaining the wide occurrence of power laws in nature. The idea was then exemplified by several dynamical models, such as the sandpile [7], and the forest-fire model [8]. The SOC hypothesis has stimulated an enormous amount of research. If this — as one of its originators has concluded — is ''how nature works'' [9], then the question of ''how SOC works'' becomes all the more urgent.

The concept of ''spontaneous'' criticality, as discussed in the SOC literature, presents, however, several ambiguities. Several authors have noted that the external driving rate is a parameter that has to be fine tuned to zero in order to observe criticality [10–13]. On the other hand, it would be amazing, without prior knowledge of the critical coupling, to define a system like the Ising model so that it is intrinsically at its critical point. SOC appears less miraculous if we suppose that there is ''generic scale invariance,'' i.e., that criticality obtains over a region of parameter space, not just a point. But we will argue that as for the Ising model, SOC typically exists at a critical *point* in the relevant parameter space. The identity of the parameters has been obscured by the manner in which the models were defined. How can a model be critical by definition, when for most statistical mechanics models, we do not even *know* the exact critical point? One way for a system to discover its own critical point is through a suitable extremal dynamics, as in invasion percolation; another is that we may know the critical parameters *a priori*, because they are fixed by a symmetry or a conservation law, and build these into the definition of the model.

Recently, a novel mean-field analysis of SOC models was presented [13], which pointed out the similarities between SOC models and models with absorbing states [14,15]. (An absorbing state is one allowing no further change or activity.) The mean-field theory provides a new insight into the origin of SOC, which, in the sandpile model, is essentially the criticality of the population of toppling sites. It turns out

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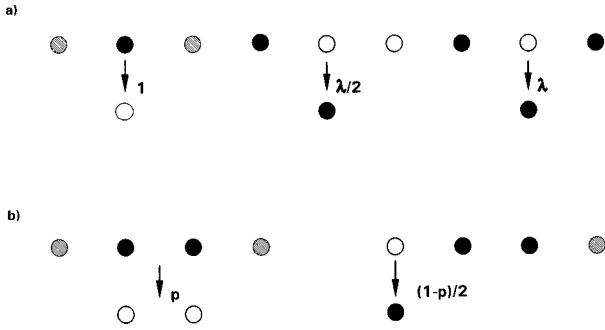‡Electronic address: zapperi@miranda.bu.edu

FIG. 1. (a) Transition rates in the one-dimensional contact process. Filled circles denote occupied sites, open circles, vacant sites; gray sites may be either occupied or vacant. (b) Transition rates in the one-dimensional pair contact process.

that SOC corresponds to the onset of nonlocality in the dynamics of the system. Nonlocality, and hence criticality, is obtained by fine tuning the control parameters, precisely as in continuous phase transitions. In this paper we focus on the similarities and differences between SOC and models with absorbing states. The latter are relatively well understood: We can identify the order parameter, and know the static and dynamic scaling behavior in the neighborhood of the critical point. The mean-field and field-theoretic analyses are well established, and one has some idea how to derive these starting from an exact master equation [16]. Applying these ideas, we arrive at a new understanding of SOC.

In Sec. II we review the models of interest: contact processes and sandpiles. The formulation of a general theory of SOC is problematic because of the nonlocal interactions implicitly present in these models. For instance, field-theoretic analysis encounters difficulties related with the singularity of the continuum limit [17]. Moreover, it is in general not possible to treat simultaneously the two time scales of avalanche propagation and external driving. Section III describes how these problems can be solved by a suitable "regularization" of the dynamics, which is local in space and time, and presents sandpile criticality as a kind of absorbing-state transition. The regularized dynamics readily lends itself to a continuum formulation, presented in Sec. IV. This motivates, in Sec. V, a study of sandpiles at the critical point, without boundaries or driving, and of the pair contact process subject to a slow drive. We present preliminary simulation results of these systems. We summarize our perspective on SOC in Sec. VI.

II. CONTACT PROCESSES AND SANDPILES

A. The contact process

One of the simplest models showing an absorbing-state transition is the contact process (CP) [18] [see Fig. 1(a)]. To each site of a d-dimensional lattice we assign a binary variable $\sigma_i = 0,1$. (Occupied sites are said to harbor a "particle.") Occupied sites ($\sigma = 1$) become empty ($\sigma = 0$) at unit rate, while empty sites become occupied with rate $w\lambda$, where w is the fraction of occupied nearest neighbor (NN) sites. The vacuum state (all sites empty), is clearly absorbing and is the only stationary state for $\lambda < \lambda_c$, while for $\lambda > \lambda_c$ there is also an active stationary state. ($\lambda_c \simeq 3.298$ in one dimen-

sion.) The order parameter is the density ρ_a of active (occupied) sites, which vanishes at the transition as

$$\rho_a \sim (\lambda - \lambda_c)^\beta. \tag{1}$$

The simplest (mean-field) description of the CP treats the σ_i as uncorrelated:

$$\frac{d\rho_a}{dt} = -(1-\lambda)\rho_a - \lambda\rho_a^2, \tag{2}$$

leading to $\lambda_c = 1$ and $\beta = 1$. As in equilibrium, we characterize the critical singularities by a set of critical exponents [19], such as β, and $\nu_\perp$, which describes the divergence of the correlation length ξ:

$$\xi \sim (\lambda - \lambda_c)^{-\nu_\perp}. \tag{3}$$

Besides the "thermal" perturbation $\Delta \equiv \lambda - \lambda_c$, a second relevant field is an external particle source h. (h is the rate of "spontaneous" creation at vacant sites.) For $\Delta = 0$, $\rho_a \sim h^{1/\delta_h}$. Other exponents are defined by considering the decay of perturbations to the stationary state [19]. Models with a single absorbing state fall generically in the universality class of directed percolation (DP), also known as Reggeon field theory [19–21].

A more complicated situation arises when many absorbing configurations exist. The simplest model to have been studied in detail so far is Jensen's pair contact process (PCP) [22] [see Fig. 1(b)]. In this model, a nearest-neighbor pair of particles may mutually annihilate, with probability p, or else, with probability $q \equiv 1-p$, create a new particle at a randomly chosen NN, provided it is vacant. There are infinitely many absorbing configurations, since all that is required is the absence of any NN particle pairs. In one dimension the static critical behavior at $q_c = 0.9229$ is DP-like [22,23], but the spreading or avalanche dynamics has variable exponents, depending on the particle density ϕ in the environment of the seed [24]. A special, "natural" class of absorbing configurations with particle density ϕ_{nat} are those spontaneously generated by the critical dynamics. DP spreading exponents are recovered only if the initial particle density is set to $\phi_{nat} \simeq 0.242(1)$ [24–27].

B. The sandpile model

Sandpile models are cellular automata (CA) with an integer (or in some cases continuous), variable z_i ("energy"), defined on a d-dimensional lattice. At each time step an energy grain is added to a randomly chosen site, until the energy of a site reaches a threshold z_c. When this happens the site relaxes:

$$z_i \rightarrow z_i - z_c \tag{4}$$

and energy is transferred to the nearest neighbors:

$$z_j \rightarrow z_j + y_j. \tag{5}$$

The relaxation of a site can induce NN sites to relax in turn, if they exceed the threshold because of the energy received, and so on. From the moment a site reaches threshold, until all sites have again relaxed ($z_i < z_c$, $\forall i$), the addition of
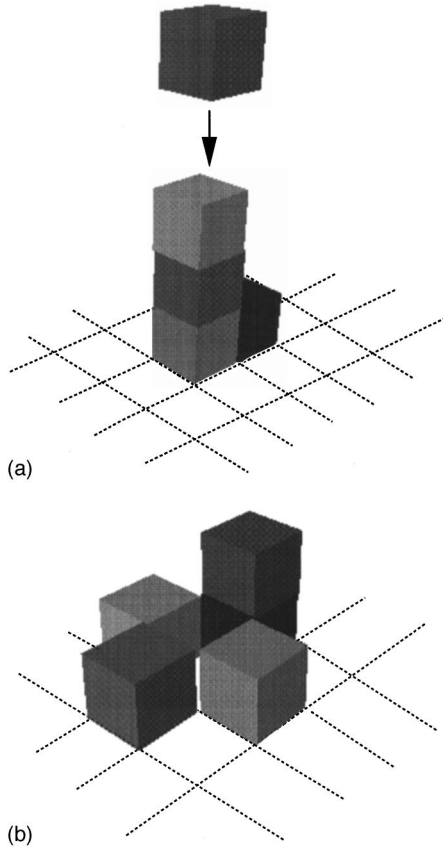
FIG. 2. The BTW sandpile model. When four grains are accumulated in one lattice site ($z_c = 4$), the site relaxes distributing the grains to the neighboring sites.

energy is suspended. The sequence of events during this interval constitutes an *avalanche*. For conservative models the transferred energy equals the energy lost by the relaxing site ($\Sigma y_j = z_c$), at least on average. Usually, dissipation occurs only at the boundary, from which energy can leave the system.

Since the energy input stops during an avalanche, we have, in effect, an infinite time scale separation between the toppling dynamics and the external source. Under these conditions the system reaches a stationary state characterized by avalanches whose sizes s are distributed as a power law [7,28–31]:

$$P(s) \sim s^{-\tau}. \qquad (6)$$

The model introduced by Bak, Tang, and Wiesenfeld (BTW) [7] is a discrete automaton in which $z_c = 2d$ and $y_j = 1$ (see Fig. 2).

An interesting variation of the original sandpile is the Manna model [32] (see Fig. 3). In this automaton the critical threshold is $z_c = 2$ independent of the dimensionality d, and if a relaxation (toppling) takes place, the energy is distributed to two randomly chosen nearest-neighbor sites (see Fig. 3). Variations in which part of the energy is kept by the relaxing site can also be considered, as well as models in which energy is transferred along a preferred direction [28].

Finally, sandpile models in which part of the energy is dissipated have been studied [33]. In continuous-energy models, some fraction of the energy removed from a relaxing
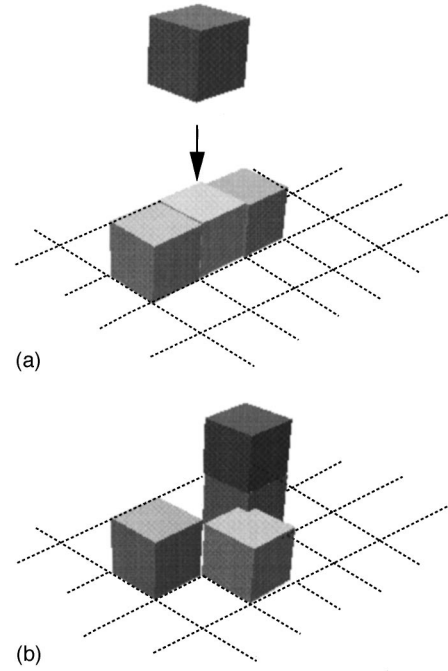

FIG. 3. The Manna sandpile model. When two grains are accumulated in one lattice site ($z_c = 2$), the site relaxes distributing the grains to two randomly chosen neighbors.

site is lost, instead of being transferred to one of the neighbors [33]. In a discrete-energy model, such as the Manna or BTW sandpiles, one can introduce a parameter ϵ representing the average energy dissipated in an elementary relaxation event. The two dissipation mechanisms lead to the same effect, namely, a characteristic length is introduced into the system and criticality is lost. The avalanche size distribution decays as

$$P(s) \sim s^{-\tau} f(s/s_c), \qquad (7)$$

where the cutoff size scales as $s_c \sim \epsilon^{-1/\sigma}$. We can also observe avalanches in the contact process, by starting the system with a single particle [34]. The activity may spread over many sites before dying out; avalanches are power-law distributed if λ is set to its critical value.

At first glance, the BTW sandpile looks quite unlike the CP. One difference is that the avalanche dynamics in the sandpile is *nonlocal* and *deterministic*. The sandpile model is inherently nonlocal because of the implicit time scale separation. A site can receive energy only if the system is quiescent, i.e., no active sites are present on the lattice. This implies that transition rates depend upon the entire set of lattice variables present in the system, giving rise to a strongly nonlocal dynamical rule. Given the configuration prior to the avalanche, and the location of the newly added particle, deterministic toppling rules govern the evolution to the next stable configuration, and this evolution can affect sites anywhere in the system. To have any hope of applying the methods used for the CP, we have to assume that the deterministic sandpile dynamics can be realized as a limiting case of models with local, stochastic dynamics, *belonging to the same universality class as the sandpile*. (We refer to this as ''regularizing'' the sandpile rules.) The latter hypothesis

would need to be verified, but seems plausible if the rules respect the same symmetries and conservation laws as those of the original model. (We provide an example in Sec. V.) Similar considerations apply to "extremal dynamics," which requires the action of an omniscient agent to choose the next event. Such a dynamics can presumably emerge as a limiting case of local rules in which each unit only has information about a finite number of neighbors.

As originally defined, the sandpile seems to involve *no* parameters. There is only the toppling rule, which, after some time, miraculously yields a critical state. But in devising a regularized dynamics, we are forced to include a non-vanishing driving rate by introducing the probability h per unit time that a site will receive a grain of energy [13]. (We may fix the relaxation rate for active sites at unity.) Energy is distributed homogeneously and the total energy flux is given by $J_{in} = hL^d$. The parameter h sets the driving time scale or equivalently the typical waiting time between different avalanches as $\tau_d \sim 1/h$. As $h \to 0$, we recover the slow driving limit; i.e., during an avalanche the system does not receive energy. This formulation of the dynamics has the advantage of being local in space and time. The state of a single site depends only on the state of the site itself and its nearest-neighbor sites at the previous time step, through a transition probability that is given by the reaction and driving rates.

III. TOWARDS A LOCAL THEORY OF SOC

After reformulating the sandpile rules as local and stochastic, we can proceed along the path followed for nonequilibrium phase transitions. From the master equation we can derive mean-field equations that give a qualitative picture of the phenomenon, exploiting several analogies with models with absorbing states. The mean-field analysis of regularized sandpiles shows that the order parameter is the density ρ_a of active sites (i.e., whose height $z \geq z_c$), and that ρ_a is coupled to the densities of "critical" ($z = z_c - 1$) and "stable" ($z < z_c - 1$) sites [13]. In mean-field theory, the dependence of the order parameter ρ_a on the parameters h and ϵ can be obtained on the back of an envelope. Since energy is conserved in the stationary state, the incoming energy flux J_{in} must be balanced by the dissipated energy $J_{out} = \epsilon \rho_a L^d$. From $J_{in} = J_{out}$, we obtain

$$\rho_a = \frac{h}{\epsilon}. \tag{8}$$

There is no stationary state for $h > \epsilon$; see Fig. 4. The model is critical just in the double limit $h, \epsilon \to 0, h/\epsilon \to 0$, since the zero-field susceptibility $\chi \equiv d\rho_a/dh$ diverges, implying a long-ranged (critical) response function. Critical behavior emerges in the limit of vanishing driving field, corresponding to locality breaking in the sandpile dynamics. The driving and dissipation rates are the *control parameters* of the model; the stationary order parameter naturally vanishes at the critical point. When $h = 0$, any configuration with $\rho_a = 0$ is absorbing. Thus there are an infinite number of absorbing configurations for a sandpile, just as for the pair contact process. (In close analogy with Ref. [13], in the field
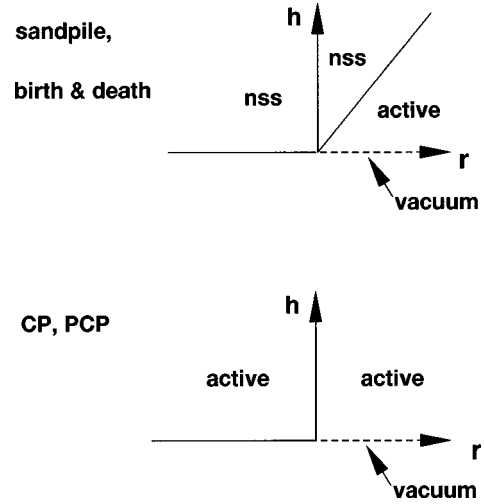

FIG. 4. Phase diagrams of the sandpile and birth-and-death processes, and of the contact process and the PCP. The thermal parameter r corresponds to ϵ in the sandpile, $1 - \lambda$ in the birth-and-death process, and to $\lambda_c - \lambda$ in the contact process. "nss" denotes a region where no stationary state is possible.

theory of the PCP the order parameter — the density of nearest-neighbor *pairs* — is coupled to a non-order-parameter field [23].)

In absorbing-state transitions, it is very useful to consider the spread of active sites from an isolated seed. Following the scaling framework developed by Grassberger and de la Torre [19], we expect that the probability that a small perturbation imposed on an absorbing configuration activates s sites scales as

$$P(s, \epsilon) = s^{-\tau} \mathcal{G}(s/s_c(\epsilon)), \tag{9}$$

where $s_c \sim \epsilon^{-1/\sigma}$ is the cutoff in the avalanche size. The perturbation decays in the stationary subcritical state as

$$\rho_a(t) \sim t^\eta \mathcal{F}(t/t_c(\epsilon)). \tag{10}$$

Here t_c denotes the characteristic time, which scales as $t_c \sim \epsilon^{-\nu_\parallel}$. In this way we have translated the avalanche description into the formalism commonly employed to study models with absorbing states [34].

It is natural to regard sandpile models as having two parameters, ϵ and h, with the original models poised, by definition, at the point $(0, 0^+)$. It should be evident that ϵ in the sandpile model is a "temperaturelike" variable, playing the same role as $\lambda_c - \lambda \equiv -\Delta$ in the CP. For $\epsilon > 0$ we cannot have sustained avalanches; they decay exponentially in this *subcritical* regime. To have self-sustained avalanches, or an active stationary state with $h \equiv 0$ in the sandpile, we would need $\epsilon < 0$, that is to say, the possibility of creating additional energy quanta when a site topples. But this immediately raises a new problem: the energy will never be lost (except at the boundaries), so in the thermodynamic limit we shall have a runaway "chain reaction" instead of a stationary state for $\epsilon < 0$. The impossibility of a stationary state for $\epsilon < 0$ is analogous to the absence of a well-defined free energy in the Gaussian model below T_c. Neither model has the saturation effect needed for stability in the "low-temperature" phase. In the Gaussian model the stabilizing

$u\phi^4$ term is missing from the Hamiltonian, while in the sandpile there is nothing to stop energy from accumulating. Indeed, to do so would mean to lose energy from the system, destroying the conservation law even for $\epsilon = 0$. Criticality would then *require* that ϵ take some negative but *a priori unknown* value. *Thus the possibility of not having to tune the system is predicated on the absence of saturation*, or, equivalently, on having strict energy conservation when $\epsilon = 0$.

Another example is furnished by removing saturation (i.e., the restriction to at most one particle per site) from the CP, resulting in an exactly soluble birth-and-death branching process. In this model, particles disappear at unit rate, and produce offspring at rate λ, at neighboring sites, whether they are occupied or not. This corresponds to setting $b=0$ in the mean-field equation

$$\frac{d\rho}{dt} = (\lambda - 1)\rho - b\rho^2. \qquad (11)$$

The density grows without limit for $\lambda > \lambda_c = 1$. It is important to note that $\lambda_c = 1$ not only in mean-field theory, but in fact for the actual birth-and-death process. Avalanches follow power laws, with the survival probability $P(t) \sim t^{-1}$, for example. Restoring saturation ($b > 0$) permits the existence of an active stationary state, but at the cost of shifting λ_c to some larger but *a priori unknown* value. (The birth-and-death process is free of higher-order terms that would renormalize the critical value of the thermal parameter from that given by mean-field theory.) As shown in Fig. 4, neither model has a stationary state for negative values of the thermal parameter r; this is the main difference from the phase diagram of the CP. The sandpile, however, presents a further subtlety: while there is no upper bound on *energy*, the order parameter is subject to saturation, since ρ_a cannot exceed unity.

The critical point of the birth-and-death process is at $\lambda = 1$ because this point corresponds to a balance, on average, between births and deaths. Similarly, the sandpile is critical at $\epsilon = 0$ because a toppling site sends a particle to each of g neighbors, and each of these neighbors is critical with probability $1/g$, so the gain and loss terms for the number of active sites balance on average. Thus the sandpile and the birth-and-death process have the same phase diagram. This does not mean, of course, that the two models share the same avalanche dynamics — that of the birth-and-death process is rather trivial. An important aspect of the sandpile is that the condition needed for critical avalanches — that a fraction $1/g$ of the nearest neighbors of an active site be critical — is established by the transient dynamics of the model. The leftovers from preceding avalanches provide the environment in which activity is critical. Memory appears to be the crucial feature of SOC models, and is due to the presence of a non-trivial threshold for activity; for sandpiles this means that $z_c \geq 2$. (For completeness, we note that a sandpile with $z_c = 1$ corresponds to a simple random walk, with well-known scaling properties. One may think of it as the analog of the birth-and-death process, in the family of models obeying strict conservation of particle number for $\epsilon = 0$.) The mean-field analysis [13] shows that having (on average) a fraction $1/g$ of the nearest neighbors of an active site critical is necessary for having a stationary state, in which energy input is balanced by dissipation. That is, the only stationary state for a sandpile at $(0,0^+)$ is a critical state.

What happens when we impose an activity threshold on the CP? One realization of such a threshold corresponds to the PCP. We study the *driven* PCP in Sec. V. Here we consider the field-theoretic description of the PCP [23]. As noted above, the order parameter ρ is coupled to a non-order-parameter field n representing the density of isolated particles. The equations take the form

$$\frac{\partial \rho}{\partial t} = D_\rho \nabla^2 \rho - a\rho - b\rho^2 - wn\rho + \cdots + \eta_\rho \qquad (12)$$

and

$$\frac{\partial n}{\partial t} = D_n \nabla^2 \rho + r\rho - u\rho^2 - \bar{w}n\rho + \cdots + \eta_n, \qquad (13)$$

where the noise terms satisfy

$$\langle \eta_i(x,t)\eta_j(x',t') \rangle = \Gamma_{i,j}\rho(x,t)\delta(x-x')\delta(t-t'). \qquad (14)$$

The field $n(x,t)$ is frozen in regions where $\rho = 0$. [If $w = 0$, Eq. (12) is the minimal field theory for the CP [35].] Now, because of the simple form of the n equation, we can formally eliminate this field to obtain

$$\frac{\partial \rho}{\partial t} = D_\rho \nabla^2 \rho - a\rho - b\rho^2 + \eta_\rho - wr\rho(x,t)$$

$$\times \int_0^t dt' \rho(x,t')e^{-\bar{w}\int_{t'}^t ds\rho(x,s)}, \qquad (15)$$

which exhibits a long-memory effect [23]. The nonlocal term turns out to be irrelevant to the stationary properties of the active phase: it is exponentially small if the density of active sites is different from zero. The situation can be different for spreading from a seed, in which case the active sites form only an infinitesimal fraction of the lattice.

IV. FIELD THEORY OF SANDPILES

A field theory of sandpiles should parallel that for the PCP in many respects. As noted before, a crucial point of the sandpile dynamics is the coupling of the density field ρ_a with the background of critical sites ρ_c. Each region devoid of active sites is frozen until such a site is generated. The activity spreads and in general alters the configuration before it moves away or disappears. The active sites leave a trace of their dynamical history in the frozen configurations of critical and stable sites they produce. If new active sites are created in the same region at some later time, they will feel the effect of the active sites present earlier in the region. This creates a long-range interaction in time and space among active sites. The range of this interaction depends on the characteristic time scale of the driving, because the fluctuations induced by h destroy the memory effect. Close to the infinite time scale separation, the characteristic driving time scale diverges and the range of the nonlocal interaction extends to the entire system. This picture is valid also for the PCP, since in both systems the response function diverges as ρ_a approaches zero and the nonlocal term becomes more and

more important as $\rho_a \to 0$. In sandpile models the density of active sites is proportional to the external field h and nonlocality is recovered in the limit $h \to 0$.

We turn now to a detailed continuum description of the BTW model. Let $\rho_i(\mathbf{x}, t)$ be the density of sites with height i at $\mathbf{x}$. We note that each site is subject to an input of energy due to three sources: (1) The external field, h; (2) toppling of active sites at any of the four NN's: $(4 - \epsilon)\rho_a$ where $\rho_a = \Sigma_{i \geqslant 4} \rho_i$ and ϵ is the average energy dissipated; (3) a diffusion-like contribution: $(1 - \epsilon/4)\nabla^2 \rho_a$. The diffusive term arises because a gradient in ρ_a leads to a particle flux: the excess in the mean number of particles arriving at $\mathbf{x}$ from the left, over those arriving from the right, is $j_x(\mathbf{x}, t) = -(1 - \epsilon/4)\partial_x \rho_a$. The net inflow of particles at $\mathbf{x}$ is therefore $-\nabla \cdot \mathbf{j} = (1 - \epsilon/4)\nabla^2 \rho_a$. Applying these observations to the mean-field equations derived in Ref. [13], we can write down the following set of continuum equations:

$$\frac{\partial \rho_i}{\partial t} = \rho_{i+4} + (\rho_{i-1} - \rho_i)\{(4 - \epsilon)[\rho_a + \tfrac{1}{4}\nabla^2 \rho_a] + h\} + \eta_{i+4}^T$$
$$- \eta_i + \eta_{i-1}, \quad 0 \leqslant i \leqslant 3 \tag{16}$$

and

$$\frac{\partial \rho_i}{\partial t} = -\rho_i + \rho_{i+4} + (\rho_{i-1} - \rho_i)\{(4 - \epsilon)[\rho_a + \tfrac{1}{4}\nabla^2 \rho_a] + h\}$$
$$- \eta_i^T + \eta_{i+4}^T - \eta_i + \eta_{i-1}, \quad i \geqslant 4. \tag{17}$$

(For $i = 0$, of course, ρ_{-1} and η_{-1} are identically zero.) The terms η_i represent noise arising due to fluctuations in the number of events of a given kind; η_i is the contribution associated with the reaction $i \to i+1$, and η_i^T with toppling: $i \to i-4$ for $i \geqslant 4$. Since the number of events is Poisson distributed (approaching a Gaussian in the continuum limit), the variance equals the mean, and the noise variance is proportional to the mean rate of the corresponding process. Thus we have

$$\langle \eta_i(\mathbf{x}, t) \eta_j(\mathbf{x}', t') \rangle = \Gamma \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \rho_i(\mathbf{x}, t)$$
$$\times [(4 - \epsilon)\rho_a(\mathbf{x}, t) + h] \tag{18}$$

and

$$\langle \eta_i^T(\mathbf{x}, t) \eta_j^T(\mathbf{x}', t') \rangle = \Gamma^T \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \rho_i(\mathbf{x}, t). \tag{19}$$

The noise terms $\propto \nabla^2 \rho_a$ have been dropped, as they are expected to be irrelevant.

This set of equations satisfies probability conservation: $\Sigma_s \rho_s$ is a constant, equal to unity by normalization. Let $\zeta(\mathbf{x}, t) \equiv \Sigma_s s \rho_s(\mathbf{x}, t)$ be the local energy density. From Eqs. (16) and (17) we have

$$\frac{\partial \zeta}{\partial t} = \left(1 - \frac{\epsilon}{4}\right)\nabla^2 \rho_a + h - \epsilon \rho_a + \eta_\zeta, \tag{20}$$

where

$$\langle \eta_\zeta(\mathbf{x}, t) \eta_\zeta(\mathbf{x}', t') \rangle = \Gamma^\zeta \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')[h + \epsilon \rho_a(\mathbf{x}, t)]. \tag{21}$$

(Again we have neglected diffusive noise, and have used the fact that in the absence of a source and of dissipation, the total energy does not fluctuate.) For $h = \epsilon = 0$ we have simply

$$\frac{\partial \zeta}{\partial t} = \nabla^2 \rho_a, \tag{22}$$

so that $E = \int d^2 x \zeta(\mathbf{x}, t)$ is conserved.

The generalization of Eqs. (16) and (17) to other dimensions, or to other sandpile models (e.g., Manna's), is straightforward, but the analysis of this complicated set of equations is problematic. One might try to cut off the hierarchy by simply declaring $\rho_i \equiv 0$ for i greater than some i_c. The choice of cutoff, however, is not obvious, and one would have to add suitable correction terms to ensure that energy is conserved when one active site topples onto another. (Altering the sandpile rules to *forbid* such transfers — in effect, constraining all sites to have $z \leqslant z_c$ — raises an interesting possibility, but one that we shall not pursue further here.) As a step toward simplification of the continuum equations, we sum up Eq. (17) for $i \geqslant 4$ to obtain

$$\frac{\partial \rho_a}{\partial t} = -\rho_a + \rho_a^* + \rho_3\{(4 - \epsilon)[\rho_a + \tfrac{1}{4}\nabla^2 \rho_a] + h\}$$
$$- \eta_a^T + \eta_a^* + \eta_3, \tag{23}$$

where $\rho_a^* \equiv \Sigma_{i \geqslant 8} \rho_i$, $\eta_a^T \equiv \Sigma_{i \geqslant 4} \eta_i^T$, and $\eta_a^* \equiv \Sigma_{i \geqslant 8} \eta_i^T$. Since the density of sites with heights $\geqslant 8$ should be negligible, we might ignore the terms with asterisks. Then the active-site density is coupled only to ρ_3, identified in Ref. [13] as the density of *critical* sites, ρ_c. In that work, sites with heights < 3 are considered in a unified manner, as the density of *stable* sites, ρ_s. Equation (16) shows, however, that the evolution of ρ_c is coupled specifically to ρ_2, not simply to $\rho_s \equiv \rho_0 + \rho_1 + \rho_2$. In the mean-field theory [13], the quantity $u = \rho_2/\rho_s$ is therefore introduced. In a spatially homogeneous stationary state, the value of u can be deduced from energy conservation. But in the present context, $u = u(\mathbf{x}, t)$ is another dynamical variable. Thus our attempt to reduce Eqs. (16) and (17) to a description in terms of three basic categories meets with difficulties.

Rather than pursuing a systematic derivation of a reduced set of equations from Eqs. (16) and (17), we shall use what we have learned so far, together with the observation that in constructing a field theory, a detailed accounting is unimportant, so long as one respects the symmetries and conservation laws of the original model. In the present instance, it is essential to ensure conservation of energy when $\epsilon = h = 0$. In fact, Eq. (20) represents this explicitly, and shows how the energy density ζ is coupled to ρ_a. We therefore retain Eq. (20) as one of our basic equations.

We obtain the other equation by replacing ρ_3 in Eq. (23) with $f(\zeta)(1 - \rho_a)$: only nonactive sites can contribute to the gain term for ρ_a, and they do so at a rate that depends on the local energy density. [$f(\zeta)$ plays a role analogous to that of u in the mean-field theory.] For small ζ, far from criticality, one expects the height distribution to be Poissonian, so that $f(\zeta) \propto \zeta^3$; for large values of ζ, f will approach a limiting value. The case of most immediate interest is a system near the critical stationary state, with $\zeta \simeq \zeta_c$ and $f \simeq \rho_c$,

where ζ_c and ρ_c represent the average values of energy and the density of critical sites, respectively, at the critical point. From the MF solution we have $\rho_c = 1/4$ for the BTW model in the limit $h \rightarrow 0$, i.e., the density of critical sites is still rather small, and f is an increasing function: $f(\zeta) = \rho_c + A(\zeta - \zeta_c) + \cdots$, with $A > 0$. Presumably, only the linear term need be retained in the vicinity of the critical point. The resulting field theory for the regularized BTW sandpile is given by

$$\frac{\partial \rho_a}{\partial t} = -\rho_a + [\rho_c + A(\zeta - \zeta_c)](1 - \rho_a)$$
$$\times \{(4 - \epsilon)[\rho_a + \tfrac{1}{4}\nabla^2 \rho_a] + h\} + \eta_a, \qquad (24)$$

where

$$\langle \eta_a(\mathbf{x}, t) \eta_a(\mathbf{x}', t') \rangle = \Gamma \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \zeta(\mathbf{x}, t)$$
$$\times [(4 - \epsilon)\rho_a(\mathbf{x}, t) + h], \qquad (25)$$

together with Eqs. (20) and (21). As in the PCP, our field theory for the sandpile supports an infinite number of absorbing configurations: any $\zeta(\mathbf{x}, t)$ consistent with $\rho_a \equiv 0$ (when $h = 0$). In both theories, the non-order-parameter field enters the equation for the order parameter in the role of an *effective creation rate*. The crucial difference between the PCP and the sandpile is that in the latter case, this auxiliary field is conserved at the critical point.

In a simple mean-field treatment (spatially homogeneous, no noise), we have

$$\frac{d\rho_a}{dt} = -\left[\frac{\epsilon}{4} - a(\bar{\zeta} - 1)\left(1 - \frac{\epsilon}{4}\right)\right]\rho_a - [1 + a(\bar{\zeta} - 1)]$$
$$\times \left(1 - \frac{\epsilon}{4}\right)\rho_a^2 + [1 + a(\bar{\zeta} - 1)]\frac{h}{4}, \qquad (26)$$

and

$$\frac{d\zeta}{dt} = -\epsilon \rho_a + h, \qquad (27)$$

where we define $4A(\zeta - \zeta_c) = a(\bar{\zeta} - 1)$ by introducing $\bar{\zeta} \equiv \zeta/\zeta_c$. For ϵ and h small, and $h/\epsilon \ll 1$, the mean-field equations have the stable stationary solution $\rho_a = h/\epsilon$, $\bar{\zeta} = 1 - h/a\epsilon$. In the case $h = 0^+$ — the slowly driven limit $h, \epsilon \rightarrow 0$, with $h/\epsilon \rightarrow 0$ — the stationary value of ζ approaches the critical height ζ_c. It is easy to recognize then that ϵ plays the role of a control parameter, analogous to λ in the CP, with the critical point at $\epsilon = 0$.

A different situation is faced when we impose $\epsilon = h = 0$ from the outset, rather than via the slowly driving limit. We have from Eq. (27) that ζ is strictly conserved in this case. The average energy density is thus an external parameter that can be freely fixed in the initial condition. In this case, in fact, ζ is the only control parameter. In the following section we present simulations of just such a situation.

The full analysis of the field theory will be deferred to a future publication. Here we simply observe that for $\epsilon = h = 0$,

$$\zeta(\mathbf{x}, t) = \zeta(\mathbf{x}, t = 0) + \int_0^t dt' \nabla^2 \rho_a(\mathbf{x}, t'). \qquad (28)$$

The evolution of the active-site density contains long-memory terms. If $\zeta(\mathbf{x}, t = 0) = \zeta_0(\mathbf{x}) \approx \zeta_c$, then to leading order

$$\frac{\partial \rho_a}{\partial t} = \frac{1}{4}\nabla^2 \rho_a + a(\bar{\zeta}_0 - 1)\rho_a - \rho_a^2$$
$$+ \frac{a}{\zeta_c} \rho_a \int_0^t dt' \nabla^2 \rho_a(\mathbf{x}, t') + \eta_a, \qquad (29)$$

Unlike the PCP, in which the memory terms decay $\propto \exp(-C \int dt \rho_a)$, here the memory decays more slowly, via the diffusive relaxation of ρ_a. It is worth noting that even in active regions, fluctuations in the height field ζ cannot relax directly; they only do so by inducing similar fluctuations in ρ_a. Relaxation of the latter then redistributes energy along with active sites.

In summary, all of the models discussed so far — sandpiles, the CP, PCP, and the birth-and-death process — have a critical point in a space of two relevant parameters, one temperaturelike (r), the other fieldlike (h). Criticality requires $h = 0$. Models such as the sandpile and PCP have a nontrivial threshold for activity and therefore exhibit multiple absorbing configurations. When such models are run at ($r = 0, h = 0^+$), then out of a range of possible values for one or more non-order-parameter densities (the critical-site density in the sandpile, the density of isolated particles in the PCP), the dynamics selects a unique value. From this vantage, the fact that certain models are critical by definition is of secondary importance. The essential feature is the behavior of a critical system under slow drive. We can study a critical, but non-SOC sandpile by setting $\epsilon = h = 0$; conversely, we can observe avalanches on all scales in the PCP if we set $p = p_c$ and $h = 0^+$.

V. SIMULATION RESULTS

The preceding discussion motivates several new kinds of simulations of the sandpile and the PCP. We report some preliminary results in this section.

A. Sandpiles at $\epsilon = h = 0$

In a regularized theory of sandpiles, we need to introduce a dissipation rate $\epsilon \gg h$ to realize a stationary state. In the original model, a stationary state is achieved by imposing open boundary conditions. While this may be appropriate for modeling processes in which stress may only be released at the boundaries of the system, it is an inconvenience theoretically: it is easier to study criticality in uniform systems; once bulk behavior is understood, the effects of various kinds of boundaries can be analyzed. We therefore study a sandpile with periodic boundaries. We performed simulations of the stochastic BTW sandpile at ($\epsilon = 0, h = 0$). With $\epsilon = h = 0$, the mean-height $\zeta \equiv N/L^d$ is strictly conserved; it is an additional parameter at our disposal.

Initial configurations are generated by distributing at random a fixed number N of particles among L^d lattice sites.
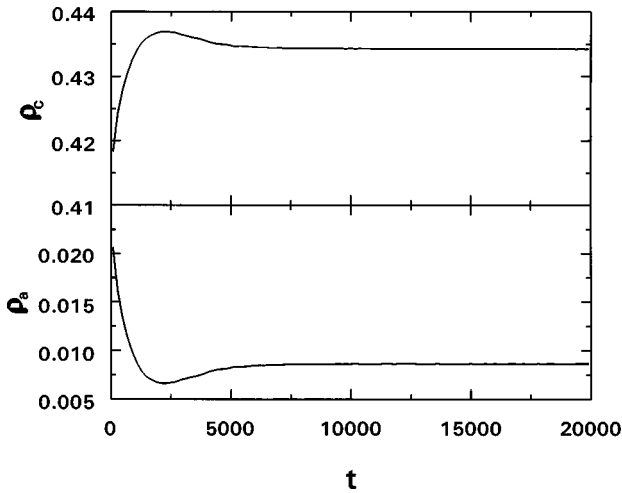
FIG. 5. Evolution of the active-site density ρ_a, and of the density ρ_c of critical sites, in the two-dimensional stochastic sandpile at $\epsilon = h = 0$. System size $L = 160$; mean height $\zeta = \zeta_c = 2.125$.

(Since the initial configuration is on average spatially homogeneous, all average properties such as densities and correlation functions are translation invariant.) Once all N particles have been placed (but not before), the dynamics begins: each active site (i.e., having $z \geq z_c = 2d$) topples at unit rate. In practice, we maintain a list of the current set of N_a active sites, choose one at random as the next to topple, and update the list following the redistribution of energy to the $2d$ neighbors. At each toppling event, time is incremented by $1/N_a$ — the mean waiting time to the next event. We compute average properties over a set of N_{samp} independent trials, each using a distinct initial configuration. ($N_{samp} = 10^3 - 10^5$ depending on the lattice size and the distance from criticality.)

Sustained activity depends upon two factors. First, there must be at least one active site in the initial configuration. This condition is trivially satisfied on large lattices, as the probability of having *no* active sites becomes exponentially small. [For large L, the initial height at a given site is essentially a Poisson random variable, $P_n \simeq \zeta^n e^{-\zeta}/n!$, so the probability of having no active sites $\sim (1 - P_{2d})^{L^d}$.] The second requirement is that there should be on average at least one critical site among the nearest neighbors of an active site. One expects the latter condition to depend sensitively on ζ, raising the possibility of a phase transition as we vary this parameter.

In one dimension, not surprisingly, we observe a rather simple behavior. For $N < L$, all trials die out rapidly, so that the only stationary state is the vacuum. For $N \geq L$, on the other hand, virtually all trials survive indefinitely. (We verified this up to $L = 1000$. In some instances the system evolves to a configuration of the form ...11112011111... in which the active site must forever circulate.) Thus we see a first-order transition at $\zeta = 1$; the stationary active-site $\bar{\rho}_a$ density jumps from 0 to about 0.15.

In two dimensions the nondriven sandpile exhibits a critical point. Figure 5 shows that the active-site density in surviving trials exhibits a nonmonotonic approach to its stationary value. By performing studies of this kind, always being careful to check that the system has reached a stationary
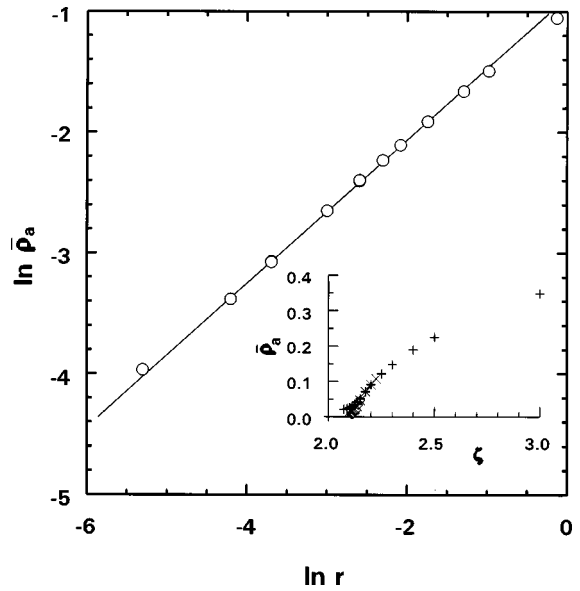

FIG. 6. Stationary active-site density $\bar{\rho}_a$ in the two-dimensional stochastic sandpile at $\epsilon = h = 0$, as a function of $r \equiv \zeta - \zeta_c$. The inset shows $\bar{\rho}_a$ vs mean height ζ on linear scales. $+$: $L = 40$; $\times$: $L = 80$; $\diamond$: $L = 160$.

state, we determined $\bar{\rho}_a(\zeta, L)$ for a range of ζ values and for $L = 20$, 40, 80, and 160. In Fig. 6 we see that $\bar{\rho}_a$ appears to increase continuously from zero at a critical value of ζ. To fix ζ_c we study the dependence of $\bar{\rho}_a$ on L, as it should follow a nontrivial power law ($\bar{\rho}_a \sim L^{-\beta/\nu_\perp}$ in the usual notation), only at the critical point. Figure 7 shows $\bar{\rho}_a$ falling owing a power law for $\zeta = 2.125$, but clearly not for 2.124 or 2.126, allowing us to conclude that $\zeta_c = 2.1250(5)$ for the two-dimensional sandpile. Indeed, this value for the mean height is in perfect agreement with the exact result $\zeta = 2.1248, \ldots$ derived by Priezzhev for the *driven* sandpile
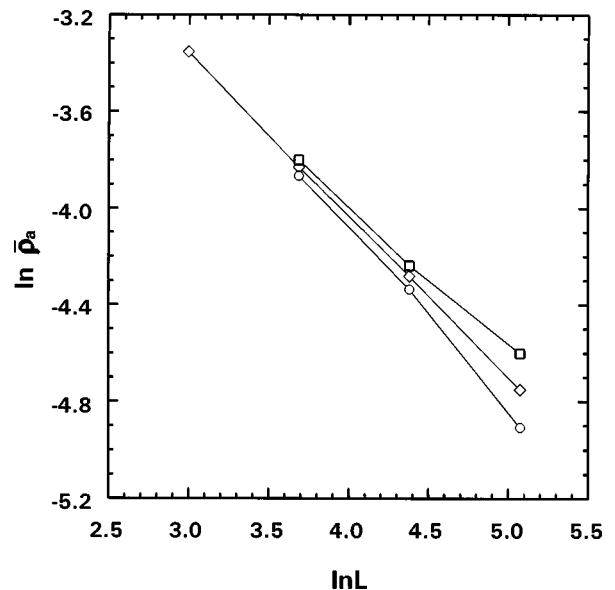

FIG. 7. Stationary active-site density $\bar{\rho}_a$ in the two-dimensional stochastic sandpile at $\epsilon = h = 0$, as a function of system size L. $\bigcirc$: $\zeta = 2.124$; $\diamond$: $\zeta = 2.125$; $\square$: $\zeta = 2.126$.

[36]. We also verify that at $\zeta = \zeta_c$, each active site has, on average, one critical nearest neighbor. The overall density of critical sites is $\bar{\rho}_c = 0.434$, again in agreement with driven sandpile simulations [37]. (At the critical point, about 10% of critical sites have heights in excess of 4.)

Having located the critical point, we can examine the critical scaling of various quantities. Figure 6 shows a clear power-law dependence of the active-site density on the distance from the critical point: $\bar{\rho}_a \sim (\zeta - \zeta_c)^\beta$ with $\beta = 0.59(1)$. The dependence of $\bar{\rho}_a(\zeta_c, L)$ on system size yields $\beta / \nu_\perp = 0.67(1)$. (Figures in parentheses denote two standard deviations in a least-squares linear fit.) We also monitored $P(t)$, the fraction of surviving trials at time t. (Approximately half of the trials appear to survive indefinitely at ζ_c.) Associated with the (approximately exponential) approach of $P(t)$ to its limit is a relaxation time, τ. We find that τ has a power-law dependence on L at the critical point: $\tau \sim L^{-\nu_\parallel / \nu_\perp}$, with $\nu_\parallel / \nu_\perp = 1.86(8)$. For comparison, we note the values for DP in $2+1$ dimensions: $\beta \simeq 0.58$, $\beta / \nu_\perp \simeq 0.80$, and $\nu_\parallel / \nu_\perp \simeq 1.76$. The similarity in β values is curious, but the differences in the other ratios indicate that the sandpile is not in the DP universality class. (This is as expected, given the differences between the sandpile and the CP discussed in Sec. III.) Studies of correlation functions, that will allow determination of $\nu_\parallel$ and $\nu_\perp$ separately, will be reported in a future publication.

In the simulations just described, we have fixed ζ, one of the variables that the dynamics selects in *driven* sandpiles with dissipation at the open boundaries. We observe criticality just at the value ζ_c observed in the driven case, and other variables such as the critical site density assume the same value in the two cases. In effect, we are able to study sandpiles in either of two "ensembles," one with fixed energy, the other with this variable adjusted by the system dynamics. Open boundaries, which served, in earlier sandpile simulations, as an outlet for accumulated energy, are now seen not to be essential for criticality. Finally, we note that our observation of criticality — at the same ζ_c as in the BTW model — in a stochastic sandpile with fully local rules, supports the expectation voiced in Sec. II, that we can study SOC using a regularized dynamics.

B. Driven pair contact process

The one-dimensional PCP has a continuous absorbing-state transition at q_c; below this value of the creation probability, the system falls into one of an exponentially large (with L) number of absorbing configurations, each devoid of NN pairs. In contrast with previous studies, here we study a *driven* PCP. Starting from an empty lattice, we add particles at randomly chosen vacant sites, until a NN pair is formed. We then suspend the addition of particles, and permit the system dynamics, as described in Sec. II, to operate, until the system again falls into an absorbing configuration. We simulate a system of size $L = 1000$ with periodic boundary conditions and study the avalanche distributions for different values of q, with both parallel and sequential updating.

We collect statistics on the size and duration of the avalanches for various values of q. As illustrated in Fig. 8, the avalanche-size distribution $P(s)$ is a power law for some
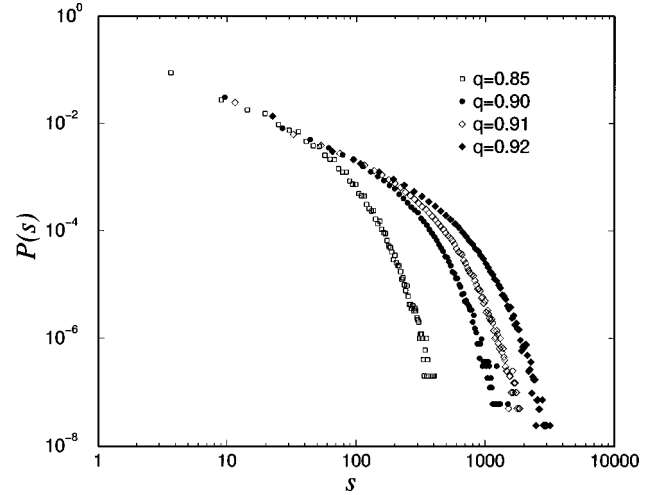

FIG. 8. Avalanche-size distribution $P(s)$ in the slowly driven, one-dimensional PCP, for various values of the creation probability q. The system size is $L = 1000$ and 10^6 avalanches are recorded for each curve.

range of s, but suffers an exponential cutoff at s_c, which grows as $q \to q_c$ as

$$s_c \sim (q_c - q)^{-1/\sigma}. \tag{30}$$

(Note that due to parallel updating, the critical creation rate $q_c \simeq 0.95$ rather than 0.9229 as found in sequentially updated simulations.) We see that the slope of the power-law distribution is consistent with DP (i.e., $\tau = 1.08$). Sequentially updated simulations (not shown) yield $\tau = 1.12$ and $\sigma = 0.45$, while the DP value is 0.39. In addition, we observe that at the critical point, the isolated-particle density approaches its natural value, $\phi_{nat} \simeq 0.2$ (parallel updating) (see Fig. 9). (Similarly, in the sequentially updated case we observe $\phi \to \phi_{nat} \simeq 0.242$.) A detailed comparison of avalanche scaling under parallel and sequential dynamics will be presented elsewhere [38].

In the slowly driven PCP, the system dynamics "self-organizes" the isolated-particle density ϕ to its natural
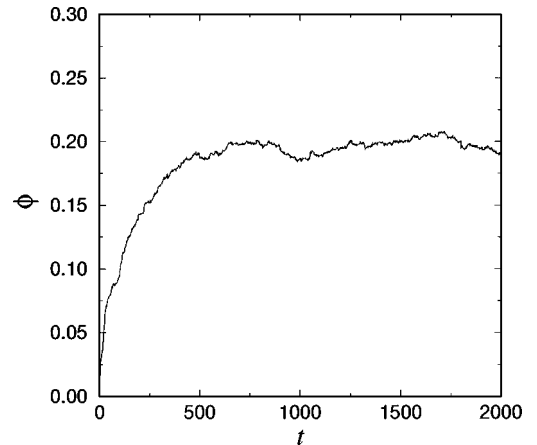

FIG. 9. The density ϕ of isolated particles in the slowly driven, one-dimensional PCP at q_c, approaches the natural value.

value, the same as in the nondriven system. This is similar to what happens in the sandpile, where the driven system selects the same critical mean height, ζ_c that we found in simulations without driving. There is, however, one rather striking difference between the models. In the PCP, activity can spread at q_c for *any* ϕ in the range [0,1/2]. In the sandpile, by contrast, activity cannot spread at all if the critical-site density is too low. Each toppling destroys an active site, and at least one of the neighbors must take its place if activity is to persist. In the PCP, each particle creation generates at least one new pair as well, so the activity has a possibility of surviving even in an *empty* lattice. This suggests that one investigate a modified PCP, in which a pair creates at particle at a (vacant) second neighbor, rather than at a NN; in this case new pairs will only be formed if ϕ is sufficiently large. Other potentially interesting models are a saturation-free version of the PCP, and the PCP in two dimensions, where only two distinct universality classes are predicted, namely, DP and dynamical percolation [39]. We defer investigation of these models to future work.

VI. SUMMARY AND PERSPECTIVE

In this paper we have argued that SOC can be understood as an aspect of multiple absorbing-state models under a slow drive. We pointed out the similarities in the phase diagrams of the two classes of models (for the sandpile and the birth-and-death process, they are identical), and in terms of avalanches and of bulk critical behavior, without boundary dissipation. We demonstrated that the sandpile exhibits an absorbing-state transition as we vary the mean height, and that the PCP, heretofore studied only as an absorbing-state transition, exhibits a power-law avalanche distribution under a slow drive. We also suggested several new models to investigate, and derived a field theory of sandpiles.

Beyond these and other avenues for quantitative investigation, we propose a viewpoint of SOC itself. What ''goes critical'' in sandpiles is ρ_a, the density of active sites. The evolution of ρ_a is intertwined with other fields, which are frozen when $\rho_a = 0$. These fields describe an energy density

that is strictly conserved at the critical point. In order for avalanches to be critical, two conditions are needed. First, the parameters h and ϵ must be set to their critical values, i.e., to zero. This is accomplished by the definition of the model, rather than by tuning parameters, but seems very similar in principle to criticality in CP-like models. The second condition is that the environmental density is such as to support avalanches on all scales. Particle conservation plays an essential role in this aspect, with the threshold for toppling providing a certain independence between ρ_a and the overall particle density. From this vantage, SOC is an absorbing-state transition riding atop a substrate that preserves a record of the previous activity. SOC typifies the behavior under slow drive, at the critical point of a model with an infinite number of absorbing configurations.

Finally, we offer a comment on the significance of sandpiles as models or paradigms of physical processes. The intention of the remark that the sandpile sits, by definition, at the critical point in a two-dimensional parameter space, is not to trivialize it, but rather to provide insight and access to new conceptual and computational tools. One may argue whether there is any point introducing ϵ and h as parameters for the sandpile; we merely posit that their discussion seems natural if one wishes to draw an analogy between sandpiles and other models with critical absorbing-state transitions. The question ''why is nature filled with systems that tune themselves to a critical point?'' may be replaced with: ''why do so many systems share the typical features of conservative, saturation-free dynamics, a threshold for activity, and widely separated time scales for external driving, on one hand, and above-threshold dynamics on the other.'' The question of how this facet of nature works remains a deep one.

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