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Exponential instability for inverse elliptic problems with unknown boundaries

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Abstract. We prove that the inverse problem of determining unknown defects of various types in a conductor by performing electrostatic measurements at the boundary is severely ill-posed. We show that the ill-posedness does not depend on the nature of the defects to be determined and, more importantly, by the kind, choice and number of measurements performed.

1. Introduction

We discuss the instability character of the following kind of inverse boundary value problems arising in nondestructive evaluation. Let us assume that we have a homogeneous and isotropic conductor body, which is contained in a reasonably smooth and bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. We assume that the body suffers form the presence of flaws or defects which can not be directly observed since they may be on an inaccessible part of the boundary or inside the conductor. We would like to determine the shape and location of these defects by performing a certain number (possibly finite) of suitably chosen electrostatic measurements of voltage and current type on a part of the boundary of the conductor which is assumed to be known, accessible and safe. The defects may be of different nature, namely they may be either *perfectly insulating* or *perfectly conducting*. Moreover, we can classify the defects by their topological properties into four broad categories. If a defect has non-empty interior, for instance if it is a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$, we speak of a *crack*. If the crack is completely inside the conductor, we call it a *buried crack* or *crack* for short, if it reaches the boundary we call it a *surface-breaking crack* or *surface crack* for short. If the defect is the closure of a domain, we call it a *material loss*. If the material loss is compactly contained in Ω we call it a *cavity*, if it intersects the boundary we speak of a *material loss at the boundary* or *boundary material loss* for short. In any of these cases, the inverse problem may be viewed as a (part of the) boundary identification problem.

The inverse crack problem has been introduced in [12], where the first uniqueness result in two dimensions has been proved. In three dimensions uniqueness results are provided in [3]. For what concerns stability we refer to [17], in two dimensions, and to [3], in three dimensions, and their references. For other uniqueness and stability results, as well as reconstruction methods, see the recent review [7].

The uniqueness for the surface crack problem has been studied in [11] in two dimensions and in [4] in three dimensions. Stability results, in two dimensions, and other references on the problem may be found in [16].

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Uniqueness for the determination of a material loss at the boundary, in two dimensions, was proved in [6]. The same argument works for the determination of cavities, as well. Stability estimates for the determination of cavities or material losses at the boundary in two dimensions may be found, respectively, in [5] and [16], and their references, whereas in three dimensions they have been treated in [2] and [8].

It is well-known that these kinds of inverse problems are *severely ill-posed*, or *exponentially ill-posed*, that is, even if we have unique determination, the error on the defects may depend in a logarithmic way from the error on the measurements (usually due to noise).

The first example of severely ill-posedness has been constructed in [1] and deals with the problem of the determination of a boundary material loss in a planar conductor. An example similar to the one in [1] has been constructed for the problem of cavities, still in two dimensions, in [5]. These two examples are explicit in the sense that a family of solutions showing the instability character of the problem is given by explicit formulas, choosing defects whose boundaries are highly oscillating. The construction of a family satisfying the instability property looked for is not an easy task for other inverse problems. In [14] N. Mandache has proved that the inverse conductivity problem is also exponentially unstable. The procedure used in [14] does not depend on an explicit construction, it is instead constituted by a purely topological argument, which follows from the work of A. N. Kolmogorov and V. M. Tihomirov, [13]. Following the topological arguments of [13] and the procedure described in [14], a general method for determining instability has been extracted in [9]. The outline of the method has been stated in a rather abstract framework. We recall the result below. Let (X, d) be a metric space (in our case the space of the defects) and let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$. As usual we denote with H' its dual and for any $v' \in H'$ and any $v \in H$ we denote by $\langle v', v\rangle$ the duality pairing between v' and v. With $\mathcal{L}(H, H')$ we denote the space of bounded linear operators between H and H' with the usual operators norm. We observe that $\mathcal{L}(H, H')$ will represent the space of the measurements. We shall also fix $\gamma : H\backslash\{0\} \mapsto [0, +\infty]$ such that

$$\gamma(\lambda v) = \gamma(v) \quad \text{for any } v \in H\backslash\{0\} \text{ and any } \lambda \in \mathbb{R}\backslash\{0\}. \tag{1.1}$$

Let us remark that the function γ may attain both the values 0 and $+\infty$ and can be thought of as a suitable kind of Rayleigh quotient. Let F be a function from X to $\mathcal{L}(H, H')$, that is, for any $x \in X$, $F(x)$ is a linear and bounded operator between H and H'. We remark that the map F represents the map which associates to any defect in X its corresponding measurements. We also fix a reference operator $F_0 \in \mathcal{L}(H, H')$ and a reference point x_0 in X. We wish to point out that no connection is required between x_0 and F_0, in particular F_0 might be different from $F(x_0)$. For any $\varepsilon > 0$, let $X_\varepsilon = \{x \in X : \ d(x, x_0) \leq \varepsilon\}$. We recall that, given a metric space (Y, d_Y) and a positive ε, $Y_1 \subset Y$, is *ε-discrete* if for any two distinct points y_1, $y_1' \in Y_1$ we have $d_Y(y_1, y_1') \geq \varepsilon$. The following exponential instability result related to the map F is proved in [9].

Theorem 1.1 *Let us assume that the following conditions are satisfied.*

i) There exist positive constants ε_0, C_1 and α_1 such that for any ε, $0 < \varepsilon < \varepsilon_0$, we can find an ε-discrete set Z_ε contained in X_ε with at least $\exp(C_1\varepsilon^{-\alpha_1})$ elements.

ii) There exist three positive constants p, C_2 and α_2 and an orthonormal basis in H, $\{v_k\}_{k=1}^{+\infty}$, such that the following conditions hold. For any $k \in \mathbb{N}$, we have that $\gamma(v_k) < \infty$, and for any $n \in \mathbb{N}$,

$$\#\{k \in \mathbb{N} : \ \gamma(v_k) \leq n\} \leq C_2(1 + n)^p \tag{1.2}$$

where # denotes the number of elements. For any $x \in X$ and any $(k, l) \in \mathbb{N} \times \mathbb{N}$ we have

$$|\langle(F(x) - F_0)v_k, v_l\rangle| \leq C_2 \exp(-\alpha_2 \max\{\gamma(v_k), \gamma(v_l)\}). \tag{1.3}$$

Then there exists a positive constant ε_1, depending on ε_0, C_1, C_2, α_1, α_2 and p only, so that for every ε, $0<\varepsilon<\varepsilon_1$, we can find x_1 and x_2 satisfying

$$\begin{aligned} &x_1, x_2 \in X_\varepsilon; \quad d(x_1,x_2)\geq \varepsilon;\\ &\|F(x_1)-F(x_2)\|_{\mathcal{L}(H,H')}\leq 2\exp(-\varepsilon^{-\alpha_1/2(p+1)}). \end{aligned} \tag{1.4}$$

In [9], Theorem 1.1 has been applied to the inverse inclusion and inverse scattering problems, showing the exponential ill-posedness in both situations. Then, in [10], it has also been applied to boundary identification problems of parabolic type.

Here we apply the abtract method outlined in Theorem 1.1 to the determination of defects inside a conductor, in any dimension and for any of the types of defects we have described before. We show that, even if we perform all possible measurements and we assume strong a priori hypotheses on the unknown defect, still the problem is exponentially ill-posed. We observe that, in the case of cavities, we assume that the unknown cavity is star-shaped with respect to a fixed point and, in certain cases, the cavity to be determined is a priori known to be even convex, see Remark 2.1. We may therefore conclude that the logarithmic stability estimates obtained in [2, 3, 5, 8, 16, 17] are essentially optimal.

The plan of the paper is as follows. In Section 2 we carefully describe all the boundary identification problem we deal with and the instability results we prove. Then we proceed with the proofs of these results, which are in general obtained as straightforward applications of the abstract theorem. In order to apply the abstract theorem, what is essentially needed is to choose a suitable orthonormal basis and to check that all the hypotheses of the abstract theorem are satisfied. Concerning orthonormal basis, we shall employ eigenfunctions corresponding to eigenvalue problems of Stekloff type. We have collected all the information we shall need about these orthonormal basis in Section 3. Then, in Section 4, the proofs of the instability results are concluded. Using the orthonormal basis introduced in Section 3, we verify that the abstract result applies to the problems we consider and we prove their exponential instability.

2. Statement of the instability results

Before stating the main results, we need to introduce some notation about the Sobolev spaces we shall use and to describe the spaces of the unknowns. For any $N\geq 2$, any $x=(x_1,\ldots,x_N)\in\mathbb{R}^N$ and any $r>0$, we denote $B_N(x,r)=\{y\in\mathbb{R}^N:\ \|y-x\|<r\}$. We set $S^{N-1}(x,r)=\partial B_N(x,r)$. Furthermore, we set $S^{N-1}=\partial B_N(0,1)$, and $S^{N-1}_+=\{y\in S^{N-1}:\ y_N\geq 0\}$, and, analogously, $S^{N-1}_-=\{y\in S^{N-1}:\ y_N\leq 0\}$. Finally, we denote $B'_{N-1}(x,r)=\{y\in B_N(x,r):\ y_N=x_N\}$.

Notations on Sobolev spaces Let $\Omega\subset\mathbb{R}^N$, $N\geq 2$, be a bounded domain and let $\partial\Omega$ be its boundary. About regularity, we assume that there exists a homeomorphism $\chi: B_N(0,1)\mapsto\Omega$ such that, for a positive constant C, we have

$$\begin{aligned} &\|\chi(\tilde{x})-\chi(\tilde{y})\|\leq C\|\tilde{x}-\tilde{y}\| && \text{for any } \tilde{x},\tilde{y}\in B_N(0,1),\\ &\|\chi^{-1}(x)-\chi^{-1}(y)\|\leq C\|x-y\| && \text{for any } x,y\in\Omega. \end{aligned} \tag{2.5}$$

Furthermore, we shall consider two internally disjoint subsets of $\partial\Omega$, Γ_A and Γ_I, so that $\Gamma_A\cup\Gamma_I=\partial\Omega$. We assume either that $\Gamma_A=\partial\Omega$ and $\Gamma_I=\emptyset$, or that Γ_A and Γ_I are not empty and are assumed to be regular enough, namely there exists a homeomorphism $\chi:\overline{B_N(0,1)}\mapsto\Omega$ satisfying (2.5), so that, if we still denote with χ its extension by continuity to $\overline{B_N(0,1)}$, then $\Gamma_A=\chi(S^{N-1}_+)$ and $\Gamma_I=\chi(S^{N-1}_-)$. We introduce the following Sobolev spaces. Let $H^1(\Omega)=\{u\in L^2(\Omega):\ \nabla u\in L^2(\Omega)\}$, where ∇u denotes the gradient of u in the sense of distributions. We recall that $H^1(\Omega)$ is a Hilbert space with scalar product $(u,v)_{H^1(\Omega)}=\int_\Omega \nabla u\cdot\nabla v+uv$. With $H^{1/2}(\Gamma_A)$ we denote the space of traces of $H^1(\Omega)$ functions

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on Γ_A, which can be endowed in a canonical way with a scalar product induced by the one of $H^1(\Omega)$ so that $H^{1/2}(\Gamma_A)$ is a Hilbert space. By $H^{-1/2}(\Gamma_A)$ we shall denote the dual space to $H^{1/2}(\Gamma_A)$. We recall that $H^{1/2}(\Gamma_A) \subset L^2(\Gamma_A) \subset H^{-1/2}(\Gamma_A)$. We shall also make use of the following spaces. Let ${}_0H^{1/2}(\Gamma_A) = \{\psi \in H^{1/2}(\Gamma_A) : \int_{\Gamma_A} \psi = 0\}$. Its dual is given by the space ${}_0H^{-1/2}(\Gamma_A) = \{\eta \in H^{-1/2}(\Gamma_A) : \langle \eta, 1 \rangle = 0\}$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. If Γ_I is not empty, we set $H^1_0(\Omega, \Gamma_I)$ and $H^1_{const}(\Omega, \Gamma_I)$ as the closed subspaces of $H^1(\Omega)$ constituted by the functions $u \in H^1(\Omega)$ so that $u = 0$ in a weak sense on Γ_I and $u = constant$ in a weak sense on Γ_I, respectively. With $H^{1/2}_0(\Gamma_A, \Omega)$ and $H^{1/2}_{const}(\Gamma_A, \Omega)$ we denote the closed subspaces of $H^{1/2}(\Gamma_A)$ constituted by the traces of $H^1_0(\Omega, \Gamma_I)$ and $H^1_{const}(\Omega, \Gamma_I)$ functions on Γ_A, respectively. For our purposes, we need to introduce on the Sobolev spaces defined above suitable scalar products, which are different but topologically equivalent to the canonical ones. We wish to remark that the definitions of these scalar products do not take into account the fact that the spaces $H^{-1/2}$ and $H^{1/2}$ are dual one to each other. For any ψ, $\varphi \in H^{1/2}(\Gamma_A)$, we set $\tilde{\psi} \in H^1(\Omega)$ as the solution to

$$\begin{cases} \Delta \tilde{\psi} = 0 & \text{in } \Omega, \\ \tilde{\psi} = \psi & \text{on } \Gamma_A, \\ \frac{\partial \tilde{\psi}}{\partial \nu} = 0 & \text{on } \Gamma_I, \end{cases} \tag{2.6}$$

and $\tilde{\varphi}$ as the solution to the same boundary value problem with ψ replaced by φ, and the scalar product we use on $H^{1/2}(\Gamma_A)$ is given by

$$(\psi, \varphi)_{H^{1/2}(\Gamma_A)} = \int_\Omega \nabla \tilde{\psi} \cdot \nabla \tilde{\varphi} + \int_{\Gamma_A} \psi \varphi. \tag{2.7}$$

We observe that ${}_0H^{1/2}(\Gamma_A)$ coincides with the subspace which is orthogonal, with respect to this scalar product, to the constant function 1. Any $\eta \in H^{-1/2}(\Gamma_A)$ can be decomposed, in a unique way, into the sum of $\hat{\eta}$, an element of ${}_0H^{-1/2}(\Gamma_A)$, and a constant function $c(\eta)$. Furthermore, to $\hat{\eta}$ we can associate $\tilde{\eta} \in H^1(\Omega)$ that solves

$$\begin{cases} \Delta \tilde{\eta} = 0 & \text{in } \Omega, \\ \frac{\partial \tilde{\eta}}{\partial \nu} = \hat{\eta} & \text{on } \Gamma_A, \\ \frac{\partial \tilde{\eta}}{\partial \nu} = 0 & \text{on } \Gamma_I. \end{cases} \tag{2.8}$$

If, in the same way, we associate to $\phi \in H^{-1/2}(\Gamma_A)$ the functions $\hat{\phi}$, $c(\phi)$ and $\tilde{\phi}$, then the scalar product on $H^{-1/2}(\Gamma_A)$ may be defined as

$$(\eta, \phi)_{H^{-1/2}(\Gamma_A)} = \int_\Omega \nabla \tilde{\eta} \cdot \nabla \tilde{\phi} + c(\eta) c(\phi). \tag{2.9}$$

We remark that, with respect to this scalar product, ${}_0H^{-1/2}(\Gamma_A)$ is the orthogonal subspace to the constant function 1. We take Γ_I not empty. If ψ belongs to $H^{1/2}_{const}(\Gamma_A, \Omega)$, then there exist (and are unique) $\hat{\psi} \in H^{1/2}_0(\Gamma_A, \Omega)$ and a constant function $c(\psi)$ so that $\psi = \hat{\psi} + c(\psi)$. Let $\tilde{\psi} \in H^1(\Omega)$ solve

$$\begin{cases} \Delta \tilde{\psi} = 0 & \text{in } \Omega, \\ \tilde{\psi} = \hat{\psi} & \text{on } \Gamma_A, \\ \tilde{\psi} = 0 & \text{on } \Gamma_I. \end{cases} \tag{2.10}$$

Then, if we associate to $\varphi \in H^{1/2}_{const}(\Gamma_A, \Omega)$ its corresponding decomposition given by $\hat{\varphi}$ and $c(\varphi)$, and its corresponding function $\tilde{\varphi}$, on $H^{1/2}_{const}(\Gamma_A, \Omega)$ we introduce the scalar product

$$(\psi, \varphi)_{H^{1/2}_{const}(\Gamma_A, \Omega)} = \int_\Omega \nabla \tilde{\psi} \cdot \nabla \tilde{\varphi} + c(\psi) c(\varphi). \tag{2.11}$$

Such a scalar product obviously induces a scalar product on $H_0^{1/2}(\Gamma_A,\Omega)$, which is the closed subspace of $H^{1/2}_{const}(\Gamma_A,\Omega)$ orthogonal to the constant function 1.

Spaces of smooth perturbations of a given set We shall consider the following examples. Let us fix integers $N\geq 2$ and $m\geq 1$ and positive constants ε and β. Let us also fix $x\in\mathbb{R}^N$ and $r>0$.

Let f be any real function defined on $\overline{B'_{N-1}(x,r)}$. We define the *graph* of f in the following way. We pose $\mathrm{graph}(f)=\{y\in\mathbb{R}^N:\ y_N=f(y_1,\ldots,y_{N-1},x_N),\ (y_1,\ldots,y_{N-1},x_N)\in\overline{B'_{N-1}(x,r)}\}$, and, assuming $f\geq x_N$, we define the *subgraph* of f as $\mathrm{subgraph}(f)=\{y\in\mathbb{R}^N:\ x_N\leq y_N\leq f(y_1,\ldots,y_{N-1},x_N),\ (y_1,\ldots,y_{N-1},x_N)\in\overline{B'_{N-1}(x,r)}\}$.

We denote $X_{m\beta\varepsilon}(B'_{N-1}(x,r))=\{\mathrm{graph}(f):\ f\in C_0^m(B'_{N-1}(x,r)),\ \|f\|_{C^m(B'_{N-1}(x,r))}\leq\beta$ and $x_N\leq f\leq x_N+\varepsilon\}$ and with $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ we indicate the space obtained by taking the subgraphs of all the functions belonging to the same class as before. We consider the spaces $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ as metric spaces with the Hausdorff distance. To any strictly positive function g defined on $S^{N-1}(x,r)=\partial B_N(x,r)$, we denote its *radial graph* as $\mathrm{graph}_{rad}(g)=\{y\in\mathbb{R}^N:\ y=x+g(\omega)\cdot\left(\frac{\omega-x}{r}\right),\ \omega\in S^{N-1}(x,r)\}$ and its *radial subgraph* as $\mathrm{subgraph}_{rad}(g)=\{y\in\mathbb{R}^N:\ y=x+\rho\cdot\left(\frac{\omega-x}{r}\right),\ 0\leq\rho\leq g(\omega),\ \omega\in S^{N-1}(x,r)\}$. Then, with the notation $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ we denote the space given by $\{\mathrm{graph}_{rad}(g):\ g\in C^m(S^{N-1}(x,r)),\ \|g\|_{C^m(S^{N-1}(x,r))}\leq\beta$ and $r\leq g\leq r+\varepsilon\}$ and with $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ we denote the space of radial subgraphs of all the functions belonging to the same class used before. Also the spaces $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are metric spaces endowed with the Hausdorff distance. It is an easy remark the fact that $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ are contained in the closed ball, with respect to the Hausdorff distance between closed sets, of radius ε centred at $\overline{B'_{N-1}(x,r)}$ and $S^{N-1}(x,r)$, respectively. Analogously, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are contained in the closed ball, again with respect to the Hausdorff distance, of radius ε and centre $\overline{B'_{N-1}(x,r)}$ and $\overline{B_N(x,r)}$, respectively. Maybe more interesting and significant is the following remark.

Remark 2.1 For any m, β and ε, the elements of $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ are all compact subsets which are star-shaped with respect to the point $x\in\mathbb{R}^N$. Furthermore, for any integers $N\geq 2$ and $m\geq 3$, and for any positive β and r, there exists $\tilde{\varepsilon}$, depending on N, m, β and r only, such that if $0<\varepsilon<\tilde{\varepsilon}$, then any element of $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$ is a compact convex set.

The determination of convex or star-shaped sets is usually considered to be more stable than the determination of other kinds of sets. Nevertheless our results show that even with a convexity or star-shapedness assumption the instability is still of exponential type. We would like to study properties of ε-discrete sets of $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$ and $X_{m\beta\varepsilon}(S^{N-1}(x,r))$, $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$. We have the following proposition.

Proposition 2.2 *Let us fix integers $N\geq 2$ and $m\geq 1$ and positive constants β and r. We also fix $x\in\mathbb{R}^N$. Fixed $\varepsilon>0$, let X_ε be equal to one of the following four metric spaces: $X_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $Y_{m\beta\varepsilon}(B'_{N-1}(x,r))$, $X_{m\beta\varepsilon}(S^{N-1}(x,r))$ or $Y_{m\beta\varepsilon}(S^{N-1}(x,r))$. Then, there exists a positive constant ε_0, depending on N, m, β and r only, so that for any ε, $0<\varepsilon<\varepsilon_0$, we can find Z_ε satisfying the following properties. We have that the set Z_ε is contained in X_ε; Z_ε is ε-discrete, with respect to the Hausdorff distance; and, finally, Z_ε has at least $\exp(2^{-N}\varepsilon_0^{(N-1)/m}\varepsilon^{-(N-1)/m})$ elements.*

Proof. The proof can be obtained, with slight modifications, along the lines of the proof of Lemma 2 in [14]. □

2.1. Inverse crack problem

Let $\Omega = B_N(0,1)$, $N \geq 2$, be the region occupied by a homogeneous conducting body. Let us assume that inside the conductor there is a *crack* σ, that is a closed set inside Ω so that $\Omega\backslash\sigma$ is connected and, locally, σ can be represented by the graph of a smooth function. We can consider two different types of cracks, *perfectly insulating* and *perfectly conducting*, and we can prescribe on the (exterior) boundary of Ω either the voltage or the current density. Thus, the electrostatic potential u in Ω satisfies either

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\sigma; \end{cases} \tag{2.12}$$

if σ is perfectly insulating, or, when σ is assumed to be perfectly conducting,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ u = constant & \text{on } \sigma. \end{cases} \tag{2.13}$$

We remark that, in (2.12), on $\partial\sigma$ means on either sides of σ. On the boundary the potential satisfies either

$$u = \psi \text{ on } \partial\Omega; \quad \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, 1\rangle = 0; \tag{2.14}$$

where $\psi \in H^{1/2}(\partial\Omega)$ is the prescribed voltage at the boundary, or, if we prescribe the current density on the boundary to be $\eta \in {}_0H^{-1/2}(\partial\Omega)$,

$$\frac{\partial u}{\partial \nu} = \eta \text{ on } \partial\Omega; \quad \int_{\partial\Omega} u = 0; \tag{2.15}$$

we wish to remark that normalization conditions have been added to the boundary conditions. We have that all the direct problems (2.12)-(2.14), (2.12)-(2.15), (2.13)-(2.14) and (2.13)-(2.15) admit a unique (weak) solution. The inverse crack problem consists of recovering the shape and location of the unknown crack σ by performing electrostatic measurements at the boundary. In this subsection we shall state the instability character of such an inverse problem, in all the possible cases, that is when we consider either insulating or conducting cracks, and when either we prescribe voltages and measure corresponding currents or we prescribe currents and measure corresponding voltages. We wish to remark that, for what concerns uniqueness and stability results, these have been obtained with a finite number of boundary measurements, usually with two suitably chosen measurements. Our instability example shows the optimality of the stability estimates previously obtained and that the stability can not be improved by taking different or more measurements. The framework of our example is as follows. Let $N \geq 2$ and m, positive integers, and β, a positive constant, be fixed. Let $X = X_{m\beta(1/4)}(B'_{N-1}(0,1/2))$ with the Hausdorff distance. To any $\sigma \in X$, we can associate the following four operators. Let $\Lambda_1(\sigma) : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ be given by

$$\langle \Lambda_1(\sigma)\psi, \varphi\rangle = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\tilde{\varphi},$$

where ψ, $\varphi \in H^{1/2}(\partial\Omega)$, u solves (2.12)-(2.14) and $\tilde{\varphi}$ is any $H^1(\Omega\backslash\sigma)$ function whose trace on $\partial\Omega$ coincides with φ. Let $\mathcal{N}_1(\sigma) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ be given by

$$\mathcal{N}_1(\sigma)\eta = u|_{\partial\Omega},$$

where $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and u solves (2.12)-(2.15). Let $\Lambda_2(\sigma) : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ be given by

$$\langle \Lambda_2(\sigma)\psi, \varphi\rangle = \langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla\tilde{\varphi},$$

where ψ, $\varphi \in H^{1/2}(\partial\Omega)$, u solves (2.13)-(2.14) and $\tilde{\varphi}$ is any $H^1_{const}(\Omega,\sigma)$ function whose trace on $\partial\Omega$ coincides with φ. Let $\mathcal{N}_2(\sigma) : {}_0H^{-1/2}(\partial\Omega) \mapsto {}_0H^{1/2}(\partial\Omega)$ be given by

$$\mathcal{N}_2(\sigma)\eta = u|_{\partial\Omega},$$

where $\eta \in {}_0H^{-1/2}(\partial\Omega)$ and u solves (2.13)-(2.15). Let us remark that for any $\sigma \in X$, each $\Lambda_i(\sigma)$ and $\mathcal{N}_i(\sigma)$, $i = 1, 2$, is a bounded linear operator between a Hilbert space and its dual, it is self-adjoint and its norm is always assumed to be the canonical one of bounded operators between these two Hilbert spaces. The operator Λ_i, $i = 1, 2$, is usually referred to as the Dirichlet-to Neumann map, whereas $\mathcal{N}_i$, $i = 1, 2$, is usually called the Neumann-to-Dirichlet map. Keeping in mind the notation and this remark, we are able to state our instability result.

Proposition 2.3 *Let us fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. Let (X, d) be the metric space where $X = X_{m\beta(1/4)}(B'_{N-1}(0, 1/2))$ and d is the Hausdorff distance. Let us fix $T \in \{\Lambda_1, \mathcal{N}_1, \Lambda_2, \mathcal{N}_2\}$. Then there exists a positive ε_1, depending on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two cracks σ_1, σ_2 belonging to X satisfying*

$$\begin{aligned} &d(\sigma_j, \overline{B'_{N-1}(0, 1/2)}) \leq \varepsilon, \text{ for any } j = 1, 2; \quad d(\sigma_1, \sigma_2) \geq \varepsilon; \\ &\|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{aligned} \tag{2.16}$$

For the proof of this proposition we refer to Subsection 4.1.

2.2. Inverse cavity problem

The inverse cavity problem can be treated if we substitute, in the previous subsection, the set of cracks inside Ω, $X = X_{m\beta(1/4)}(B'_{N-1}(0, 1/2))$, with the set of *cavities* inside Ω given by $X = Y_{m\beta(1/4)}(S^{N-1}(0, 1/2))$. With almost no modification in the proof, a result completely analogous to the one described in Proposition 2.3 can be obtained. So also the inverse cavity problem shows an exponential instability character.

In the planar case, an explicit example developed in [5] shows the exponential instability character of the inverse cavity problem and, consequently, that the stability estimates therein contained are essentially optimal. Our results here confirm this fact and extend it to the higher dimensional case, thus providing the essential optimality of stability estimates of logarithmic type.

2.3. Inverse surface crack problem

Let $\Omega = B_N(0, 1)\backslash\{x \in B_N(0, 1) : \; x_{N-1} \geq 0 \text{ and } x_N = 0\}$, $N \geq 2$, and let $\Gamma_I = \{x \in \overline{B_N(0, 1)} : x_{N-1} \geq 0 \text{ and } x_N = 0\}$. Inside Ω, we consider the geodesic distance between two points as the infimum of the lengths of smooth paths contained in Ω connecting the two points. If we consider the boundary of Ω with respect to this distance, we notice that this boundary contains two overlapping copies of Γ_I, one obtained by approaching Γ_I with points x in Ω such that $x_N > 0$ and the other obtained by approaching it with points $x \in \Omega$ so that $x_N < 0$. The set Γ_A is obtained from Γ_I by taking the closure, in the topology of $\partial\Omega$ induced by the geodesic distance defined above, of $\partial\Omega\backslash\Gamma_I$. We remark that Γ_A coincides, from a set point of view, with S^{N-1}, but each point belonging to the intersection of Γ_I and S^{N-1} should be counted with multiplicity two, as for points of Γ_I. With $H^{1/2}(\Gamma_A)$ we denote the space of traces of $H^1(\Omega)$ functions on Γ_A and with $H^{-1/2}(\Gamma_A)$ we shall denote its dual. On these two spaces, we consider scalar products which are defined exactly as we have done before for regular domains, in (2.7) and (2.9), respectively. We notice that $H^{1/2}(\Gamma_A) \subset L^2(S^{N-1}) \subset H^{-1/2}(\Gamma_A)$. Finally, we notice that the spaces ${}_0H^{1/2}(\Gamma_A)$ and ${}_0H^{-1/2}(\Gamma_A)$ are the orthogonal subspaces, respectively in $H^{1/2}(\Gamma_A)$ and $H^{-1/2}(\Gamma_A)$, to the constant function 1 and are dual one to each other. We observe that the spaces $H^1_0(\Omega, \Gamma_I)$

and $H^1_{const}(\Omega,\Gamma_I)$ are given by the spaces of $H^1(B_N(0,1))$ functions which are, respectively, identically zero or constant in a weak sense on Γ_I. The spaces of traces on Γ_A of functions belonging to $H^1_0(\Omega,\Gamma_I)$ and $H^1_{const}(\Omega,\Gamma_I)$, respectively, are again denoted by $H^{1/2}_0(\Gamma_A,\Omega)$ and $H^{1/2}_{const}(\Gamma_A,\Omega)$. On these two last spaces, a scalar product is defined in the same fashion as we have done for regular domains in (2.11). If $N=2$, let $\sigma_0=\{x\in\overline{B_2(0,1)}:\ x_1\geq -1/2 \text{ and } x_2=0\}$. If $N\geq 3$, let $f\in C^\infty_0(B'_{N-2}(0,1/4))$ so that $-1/4\leq f\leq 0$. Let $\sigma_0=\Gamma_I\cup\{y\in B'_{N-1}(0,1):\ f((y_1,\ldots,y_{N-2},0))\leq y_{N-1}\leq 0,\ (y_1,\ldots,y_{N-2},0)\in B'_{N-2}(0,1/4)\}$. By definition if $N=2$, and by a suitable choice of f if $N\geq 3$, we can always assume that $B'_{N-1}(\tilde{x}_0,1/16)$ is contained in σ_0, where $\tilde{x}_0=(0,\ldots,0,-1/8,0)$. Then we fix a positive integer m and a positive constant β and we define X as the set

$$X=\{\sigma=(\sigma_0\backslash B'_{N-1}(\tilde{x}_0,1/16))\cup\sigma':\ \sigma'\in X_{m\beta(1/4)}(B'_{N-1}(\tilde{x}_0,1/16))\}. \tag{2.17}$$

We remark that each $\sigma\in X$ is a connected closed set inside $\overline{B_N(0,1)}$ so that $\Gamma_I\subset\sigma$ and $\sigma\backslash\Gamma_I\subset B_N(0,4/5)$. If we assume that $B_N(0,1)$ is occupied by a homogeneous conductor, we can think any $\sigma\in X$ as a *surface crack* inside $B_N(0,1)$. We can distinguish between two different kinds of surface cracks, namely *insulating* and *conducting*. Let us assume that $\sigma\in X$ is an insulating surface crack and that we prescribe on Γ_A the current density to be equal to $\eta\in {}_0H^{-1/2}(\Gamma_A)$. Then the electrostatic potential u in $B_N(0,1)$ satisfies

$$\begin{cases}\Delta u=0 & \text{in } B_N(0,1)\backslash\sigma,\\ \frac{\partial u}{\partial\nu}=0 & \text{on either sides of } \sigma,\\ \frac{\partial u}{\partial\nu}=\eta & \text{on } \Gamma_A,\\ \int_{\Gamma_A}u=0, & \end{cases} \tag{2.18}$$

where we have also added a normalization condition. We have that u is a weak solution to (2.18) if and only if $u\in H^1(B_N(0,1)\backslash\sigma)$, $\int_{\Gamma_A}u=0$, and

$$\int_{B_N(0,1)\backslash\sigma}\nabla u\cdot\nabla w=\langle\eta,w|_{\Gamma_A}\rangle,\quad \text{for any } w\in H^1(B_N(0,1)\backslash\sigma).$$

Clearly such a function u exists and is unique. We have that $u|_{\Gamma_A}$ belongs to ${}_0H^{1/2}(\Gamma_A)$ and that the operator $\mathcal{N}_3(\sigma):{}_0H^{-1/2}(\Gamma_A)\mapsto{}_0H^{1/2}(\Gamma_A)$ so that, for any $\eta\in{}_0H^{-1/2}(\Gamma_A)$, $\mathcal{N}_3(\sigma)\eta=u|_{\Gamma_A}$, u solution to (2.18), is linear, bounded and self-adjoint. When, otherwise, $\sigma\in X$ is a conducting surface crack in $B_N(0,1)$ and we prescribe the voltage on Γ_A to be $\psi\in H^{1/2}_{const}(\Gamma_A,\Omega)$, then the potential u in $B_N(0,1)$ solves

$$\begin{cases}\Delta u=0 & \text{in } B_N(0,1)\backslash\sigma,\\ u=c(\psi) & \text{on } \sigma,\\ u=\psi & \text{on } \Gamma_A,\end{cases} \tag{2.19}$$

where $c(\psi)$ is a constant so that $\hat{\psi}=\psi-c(\psi)\in H^{1/2}_0(\Gamma_A,\Omega)$. Let $\tilde{\psi}$ be any $H^1_0(B_N(0,1)\backslash\sigma,\sigma)$ function so that $\tilde{\psi}|_{\Gamma_A}=\hat{\psi}$. Then u solves in a weak sense (2.19) if and only if $u-c(\psi)-\tilde{\psi}\in H^1_0(B_N(0,1)\backslash\sigma)$ and

$$\int_{B_N(0,1)\backslash\sigma}\nabla u\cdot\nabla w=0,\quad \text{for any } w\in H^1_0(B_N(0,1)\backslash\sigma).$$

By standard elliptic equations methods we infer that u, solution to (2.19), exists and it is unique. To such a solution we can associate $\frac{\partial u}{\partial\nu}|_{\Gamma_A}\in(H^{1/2}_{const}(\Gamma_A,\Omega))'$ as follows. For any

$\varphi \in H^{1/2}_{const}(\Gamma_A, \Omega)$, let the constant $c(\varphi)$ be so that $\hat{\varphi} = \varphi - c(\varphi) \in H^{1/2}_0(\Gamma_A, \Omega)$, and let $\tilde{\varphi}$ be any $H^1_0(B_N(0,1)\backslash\sigma, \sigma)$ function so that $\tilde{\varphi}|_{\Gamma_A} = \hat{\varphi}$. Then,

$$\langle \frac{\partial u}{\partial \nu}|_{\Gamma_A}, \varphi \rangle = \int_{B_N(0,1)\backslash\sigma} \nabla u \cdot \nabla \tilde{\varphi}.$$

The operator $\Lambda_3(\sigma) : H^{1/2}_{const}(\Gamma_A, \Omega) \mapsto (H^{1/2}_{const}(\Gamma_A, \Omega))'$ so that, for any $\psi \in H^{1/2}_{const}(\Gamma_A, \Omega)$, $\Lambda_3(\sigma)\psi = \frac{\partial u}{\partial \nu}|_{\Gamma_A}$, u solution to (2.19), is linear, bounded and self-adjoint. The inverse surface crack problem consists of the determination of the unknown surface crack from suitable information on the operator $\mathcal{N}_3$ or Λ_3, respectively. The operators $\mathcal{N}_3$ and Λ_3 correspond to electrostatic boundary measurements. Many papers have treated this problem when a finite number of measurements is performed, that is when either $\mathcal{N}_3(\eta)$ is measured for a finite number of different η or $\Lambda_3(\psi)$ is measured for a finite number of different ψ.

Proposition 2.4 *We fix integers $N \geq 2$ and $m \geq 1$ and a positive constant β. Let X be the set of closed sets described in* (2.17) *and let (X, d) be a metric space with the Hausdorff distance. Let σ_0 be defined as before. Let us fix $T \in \{\mathcal{N}_3, \Lambda_3\}$. Then we can find $\varepsilon_1 > 0$, that depends on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two surface cracks σ_1, σ_2 belonging to X so that*

$$\begin{aligned} &d(\sigma_j, \sigma_0) \leq \varepsilon, \text{ for any } j = 1, 2; \quad d(\sigma_1, \sigma_2) \geq \varepsilon; \\ &\|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{aligned} \tag{2.20}$$

We prove this result in Subsection 4.2.

2.4. Inverse boundary material loss problem

Let $\Omega = \{x \in B_N(0,1) : \ x_N > 0\}$, and let $\Gamma_A = \{x \in \partial B_N(0,1) : \ x_N \geq 0\} = S^{N-1}_+$ and $\Gamma_I = \{x \in \overline{B_N(0,1)} : \ x_N = 0\} = \overline{B'_{N-1}(0,1)}$. Fixed a positive integer m and a positive constant β, let

$$X = \{\sigma = \Gamma_I \cup \sigma' : \ \sigma' \in Y_{m\beta(1/4)}(B'_{N-1}(0, 1/2))\}. \tag{2.21}$$

Then every $\sigma \in X$ is a closed subset contained in $\overline{\Omega}$ so that $(\sigma\backslash\Gamma_I) \subset B_N(0, 4/5)$. We assume that Ω is the region occupied by a homogeneous conductor and $\sigma \in X$ is a *boundary material loss*, which might be due to a corrosion phenomenon, for instance. We assume that Γ_A is an accessible part of the boundary of our conductor, whereas $\Gamma_\sigma = \partial(\Omega\backslash\sigma)\backslash\Gamma_A$, that is the other part of the boundary where the material loss occurs, is not. Also in this case we distinguish two kinds of boundary material losses, insulating and conducting. In the first case, no current passes through Γ_σ, the part of boundary of $\Omega\backslash\sigma$ which is contained in σ. In the second case, the voltage is constant on σ. More precisely, we have that if σ is an insulating boundary material loss and if we prescribe the current density on Γ_A to be equal to $\eta \in {}_0H^{-1/2}(\Gamma_A)$, then the electrostatic potential u inside $\Omega\backslash\sigma$ is the unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_\sigma, \\ \frac{\partial u}{\partial \nu} = \eta & \text{on } \Gamma_A, \\ \int_{\Gamma_A} u = 0, & \end{cases} \tag{2.22}$$

where the last line is a normalization condition. Otherwise, if σ is conducting, then the electrostatic potential u in Ω is given by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma, \\ u = c(\psi) & \text{on } \sigma, \\ u = \psi & \text{on } \Gamma_A, \end{cases} \tag{2.23}$$

where $\psi \in H^{1/2}_{const}(\Gamma_A, \Omega)$ is the prescribed voltage on Γ_A and $c(\psi)$ is a constant so that $\psi - c(\psi)$ belongs to $H^{1/2}_0(\Gamma_A, \Omega)$. In the insulating case, for every $\sigma \in X$, we define $\mathcal{N}_4(\sigma) : {}_0H^{-1/2}(\Gamma_A) \mapsto {}_0H^{1/2}(\Gamma_A)$ so that for any $\eta \in {}_0H^{-1/2}(\Gamma_A)$, then $\mathcal{N}_4(\sigma)\eta = u|_{\Gamma_A}$, u being the unique solution to (2.22). We have that $\mathcal{N}_4(\sigma)$ is a linear, bounded and self-adjoint operator. In the conducting case, if $\sigma \in X$, let us define $\Lambda_4(\sigma) : H^{1/2}_{const}(\Gamma_A, \Omega) \mapsto (H^{1/2}_{const}(\Gamma_A, \Omega))'$ as follows. For any ψ, $\varphi \in H^{1/2}_{const}(\Gamma_A, \Omega)$,

$$\langle \Lambda_4(\sigma)\psi, \varphi \rangle = \langle \frac{\partial u}{\partial \nu}|_{\Gamma_A}, \varphi \rangle = \int_{\Omega\setminus\sigma} \nabla u \cdot \nabla \tilde{\varphi},$$

where u solves (2.23) and $\tilde{\varphi}$ is any $H^1_{const}(\Omega, \sigma)$ so that $\tilde{\varphi}|_{\Gamma_A} = \varphi$. Also $\Lambda_4(\sigma)$ is a linear, bounded and self-adjoint operator, for any $\sigma \in X$. The inverse problem consists of the determination of the shape and the location of the unknown boundary material loss σ from electrostatic measurements performed on the accessible part of the boundary, that is Γ_A. The case of a single electrostatic measurement is particularly interesting and uniqueness and optimal stability estimates have been obtained for this situation. In two dimensions, the severely ill-posedness of this problem has been shown through an explicit example in [1]. In the next proposition, proven in Subsection 4.2, we confirm that logarithmic stability is essentially optimal in any dimension, no matter how many and which measurements we perform.

Proposition 2.5 *Let $N \geq 2$ and $m \geq 1$ be integers and β be a positive constant. Let X be defined as in* (2.21)*, endowed with the Hausdorff distance d. Let us fix $T \in \{\mathcal{N}_4, \Lambda_4\}$. Then there exists a constant $\varepsilon_1 > 0$, that depends on N, m and β only, so that for any ε, $0 < \varepsilon < \varepsilon_1$, there exist two boundary material losses σ_1, σ_2 belonging to X so that*

$$\begin{array}{l} d(\sigma_j, \overline{B'_{N-1}(0,1)}) \leq \varepsilon, \text{ for any } j = 1, 2; \quad d(\sigma_1, \sigma_2) \geq \varepsilon; \\ \|T(\sigma_1) - T(\sigma_2)\| \leq 2\exp(-\varepsilon^{-(N-1)/(2mN)}). \end{array} \tag{2.24}$$

3. Stekloff eigenvalue problems

In this section we collect some results which will be repeatedly used later, when we shall apply the abstract theorem. Most of the results described in this section are obtained by standard methods, thus, for the sake of brevity, we do not enter into any detail and we limit ourselves to fix the notation and to state the results which will be needed later, referring to the literature when necessary. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let Γ_A and Γ_I be two internally disjoint subsets of $\partial\Omega$, so that $\Gamma_A \cup \Gamma_I = \partial\Omega$. About the regularity and the properties of Ω, Γ_A and Γ_I, we shall consider the same assumptions used at the beginning of Section 2. The following eigenvalue problems of Stekloff type will be discussed; first

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_I, \end{cases} \tag{3.25}$$

and then, assuming Γ_I not empty,

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \mu v & \text{on } \Gamma_A, \\ v = 0 & \text{on } \Gamma_I. \end{cases} \tag{3.26}$$

We state the following propositions concerning the eigenvalues and eigenfunctions of (3.25) and (3.26) respectively.

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Proposition 3.1 *Under the assumptions on Ω, Γ_A and Γ_I previously made, we have that the eigenvalues of* (3.25)*, counted with their multiplicity, are given by an increasing sequence*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$$

so that $\lim_{k\to\infty} \lambda_k = \infty$*. For any $n \in \mathbb{N}$, we set $N_1(n) = \#\{k \in \mathbb{N} : \lambda_k \leq n\}$. Then the asymptotic behaviour of the eigenvalues is as follows. There exists a constant C_1 depending on Ω, Γ_A and Γ_I only so that*

$$N_1(n) \leq C_1 n^{N-1}, \quad \text{for any } n \in \mathbb{N}. \tag{3.27}$$

Moreover, there exists a corresponding sequence of eigenfunctions, $\{u_k\}_{k\in\mathbb{N}}$, that is $u_k \in H^1(\Omega)\backslash\{0\}$ and the couple (λ_k, u_k) solves (3.25) *for any $k \in \mathbb{N}$, so that the following three conditions holds*

$$\{u_k|_{\Gamma_A}\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } L^2(\Gamma_A);$$

$$\left\{\frac{u_k}{\sqrt{1+\lambda_k}}|_{\Gamma_A}\right\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } H^{1/2}(\Gamma_A);$$

$$\{1|_{\Gamma_A}\} \cup \{\sqrt{\lambda_k} u_k|_{\Gamma_A}\}_{k\geq 2} \text{ is an orthonormal basis of } H^{-1/2}(\Gamma_A);$$

where we have considered the spaces $H^{1/2}(\Gamma_A)$ and $H^{-1/2}(\Gamma_A)$ with the scalar products defined in (2.7) *and* (2.9) *respectively. We remark that u_1 is a constant function not identically equal to zero.*

Proposition 3.2 *Under the assumptions on Ω, Γ_A and Γ_I previously made, and assuming that Γ_I is not empty, then the eigenvalues of* (3.26)*, counted with their multiplicity, constitute an increasing sequence*

$$0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_k \leq \ldots$$

so that $\lim_{k\to\infty} \mu_k = \infty$*. For any $n \in \mathbb{N}$, we set as before $N_2(n) = \#\{k \in \mathbb{N} : \mu_k \leq n\}$. Then the eigenvalues satisfy the following asymptotic behaviour. There exists a constant C_2 depending on Ω, Γ_A and Γ_I only so that*

$$N_2(n) \leq C_2 n^{N-1}, \quad \text{for any } n \in \mathbb{N}. \tag{3.28}$$

Furthermore, we can find a sequence $\{v_k\}_{k\in\mathbb{N}}$ of corresponding eigenfunctions, that is $v_k \in H^1(\Omega)\backslash\{0\}$ and the couple (μ_k, v_k) is a solution to (3.26) *for any $k \in \mathbb{N}$, so that*

$$\{v_k|_{\Gamma_A}\}_{k\in\mathbb{N}} \text{ is an orthonormal system of } L^2(\Gamma_A);$$

$$\{1|_{\Gamma_A}\} \cup \left\{\frac{v_k}{\sqrt{\mu_k}}|_{\Gamma_A}\right\}_{k\in\mathbb{N}} \text{ is an orthonormal basis of } H^{1/2}_{const}(\Gamma_A, \Omega);$$

where we have considered the space $H^{1/2}_{const}(\Gamma_A, \Omega)$ with the scalar product defined in (2.11)*.*

Beyond the asymptotic behaviour of the eigenvalues, we are interested in the asymptotic behaviour of the eigenfunctions, in particular in a kind of exponential decay, in terms of the eigenvalues, of the eigenfunctions away from Γ_A. In the next examples, we present some particular cases in which such kind of decay holds.

Example 3.3 Let $\Omega = B_N(0,1)$ and $\partial\Omega = S^{N-1}$, and let $\Gamma_A = \partial\Omega$ and $\Gamma_I = \emptyset$. In this case the problem (3.25) is a classical Stekloff eigenvalue problem and it is well-known that the orthonormal basis of $L^2(S^{N-1})$ constituted by the traces of eigenfunctions, as described in Proposition 3.1, coincides with

$$\{f_{jp} :\ j \geq 0 \text{ and } 1 \leq p \leq p_j\} \tag{3.29}$$

where each f_{jp} is a *spherical harmonic* of degree j, j being a nonnegative integer. We have that the function

$$u_{jp}(x) = \|x\|^j f_{jp}(x/\|x\|) \tag{3.30}$$

is harmonic in $\mathbb{R}^N$ and solves the eigenvalue problem (3.25) with eigenvalue $\lambda = j$. So, the sequence $\{u_{jp} :\ j \geq 0 \text{ and } 1 \leq p \leq p_j\}$ coincides with the sequence of eigenfunctions we have described in Proposition 3.1. The integers p_j are the dimensions of the spaces of spherical harmonics of degree j and we have that, see for instance [15, page 4],

$$p_j = \begin{cases} 1 & \text{if } j = 0, \\ \frac{(2j+N-2)(j+N-3)!}{j!(N-2)!} & \text{if } j \geq 1, \end{cases}$$

so that

$$p_j \leq 2(j+1)^{N-2}, \quad j \geq 0,$$

and

$$N_1(n) \leq \sum_{j=0}^{n} p_j \leq \sum_{j=0}^{n} 2(j+1)^{N-2} \leq 2(n+1)^{N-1}, \quad \text{for any } n \in \mathbb{N}.$$

Furthermore, for any r_0, $0 < r_0 < 1$, there exist two positive constants, $C(r_0, N)$ and $\alpha(r_0)$, so that for any u_{jp} as in (3.30) it holds

$$\|u_{jp}\|_{H^1(B_N(0,r_0))} \leq C(r_0, N)\exp(-\alpha(r_0)j). \tag{3.31}$$

Example 3.4 Let $\Omega = \{x \in B_N(0,1) :\ x_N > 0\}$, and let $\Gamma_A = \{x \in \partial B_N(0,1) :\ x_N \geq 0\} = \overline{S^{N-1}_+}$ and $\Gamma_I = \{x \in \overline{B_N(0,1)} :\ x_N = 0\} = \overline{B'_{N-1}(0,1)}$. First of all, we notice that the hypoteses of Proposition 3.1 and Proposition 3.2 are satisfied, so the conclusions of Proposition 3.1 and of Proposition 3.2 hold for the eigenvalues and eigenfunctions related to problem (3.25) and problem (3.26) with these data, respectively. The following exponential decay property can be obtained, as well. We have that if $u \in H^1(\Omega)\backslash\{0\}$ solves (3.25) for a constant λ, then, by a reflection argument, it follows that there exist j, a nonnegative integer, and f, a spherical harmonic function on S^{N-1} of degree j, so that $u(x) = \|x\|^j f(x/\|x\|)$ for any $x \in \Omega$ and $\lambda = j$. Thus, if we assume that $\|f\|_{L^2(\Gamma_A)} = 1$, we can conclude that for any r_0, $0 < r_0 < 1$,

$$\|u\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C(r_0, N)\exp(-\alpha(r_0)\lambda), \tag{3.32}$$

where the constants $C(r_0, N)$ and $\alpha(r_0)$ coincide with the ones obtained in Example 3.3. Again by a reflection argument, we have that if $v \in H^1(\Omega)\backslash\{0\}$ and a constant μ solve (3.26) then there exist j, a positive integer, and f, a spherical harmonic function on S^{N-1} of degree j, so that $v(x) = \|x\|^j f(x/\|x\|)$ for any $x \in \Omega$ and $\mu = j$. Thus, if we assume as before that $\|f\|_{L^2(\Gamma_A)} = 1$, we immediately infer that for any r_0, $0 < r_0 < 1$,

$$\|v\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C(r_0, N)\exp(-\alpha(r_0)\mu), \tag{3.33}$$

with the same constants $C(r_0, N)$ and $\alpha(r_0)$ as before.

Example 3.5 Let $\Omega = B_N(0,1)\backslash\{x \in B_N(0,1) :\ x_{N-1} \geq 0 \text{ and } x_N = 0\}$, and let $\Gamma_I = \{x \in \overline{B_N(0,1)} :\ x_{N-1} \geq 0 \text{ and } x_N = 0\}$, as in Subsection 2.3. We set Γ_A as in Subsection 2.3, as well. Also the notations concerning Sobolev spaces on Γ_A are the ones introduced in Subsection 2.3. The eigenvalue problem (3.25) with these data can be rewritten as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on (either sides of) } \Gamma_I, \end{cases} \tag{3.34}$$

that is, $u \in H^1(\Omega)$ solves (3.34) if

$$\int_\Omega \nabla u \cdot \nabla w = \int_{\Gamma_A} \lambda u w, \quad \text{for any } w \in H^1(\Omega).$$

Then, all the conclusions of Proposition 3.1 still hold true for the eigenvalue problem (3.34), also with the possibility to replace the space $L^2(\Gamma_A)$ with the space $L^2(S^{N-1})$. The exponential decay of the eigenfunctions is still valid. By separation of variables, we have that if $u \in H^1(\Omega)\backslash\{0\}$ solves (3.34) with a constant λ, then there exists a function $g \in L^2(S^{N-1})$ so that $u(x) = \|x\|^\lambda g(x/\|x\|)$ for any $x \in \Omega$. Assuming that $\|g\|_{L^2(S^{N-1})} = 1$, we obtain that for any r_0, $0 < r_0 < 1$,

$$\|u\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C_1(r_0, N)\exp(-\alpha(r_0)\lambda), \tag{3.35}$$

where $C_1(r_0, N)$ is a positive constant not depending on λ and $\alpha(r_0)$ coincides with the one defined in Example 3.3. For what concerns the eigenvalue problem (3.26) with these data, that is,

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \mu v & \text{on } \Gamma_A, \\ v = 0 & \text{on } \Gamma_I, \end{cases} \tag{3.36}$$

we have that v solves (3.36), in a weak sense, for a constant μ, if $v \in H^1_0(\Omega, \Gamma_I)$ and

$$\int_\Omega \nabla v \cdot \nabla w = \int_{\Gamma_A} \mu v w, \quad \text{for any } w \in H^1_0(\Omega, \Gamma_I).$$

Then, all the results of Proposition 3.2 are still valid for the eigenvalue problem (3.36), and we can again replace the space $L^2(\Gamma_A)$ with the space $L^2(S^{N-1})$. If $v \in H^1_0(\Omega, \Gamma_I)\backslash\{0\}$ and μ are a solution to (3.36), then, by separation of variables, we can find a function $g \in L^2(S^{N-1})$ so that $v(x) = \|x\|^\mu g(x/\|x\|)$ for any $x \in \Omega$. If we further suppose $\|g\|_{L^2(S^{N-1})} = 1$, we have that for any r_0, $0 < r_0 < 1$,

$$\|v\|_{H^1(B_N(0,r_0)\cap\Omega)} \leq C_2(r_0, N)\exp(-\alpha(r_0)\mu), \tag{3.37}$$

where $C_2(r_0, N)$ is a positive constant not depending on μ and $\alpha(r_0)$ is the same as before.

4. Proofs of the main results

In this section we apply the abstract theorem to the inverse problems described in Section 2 and we conclude the proofs of our instability results.

4.1. Inverse crack problem

The proof of Proposition 2.3 follows directly from the abstract theorem stated in Theorem 1.1. We just need to check that all the hypotheses of Theorem 1.1 are satisfied. Therefore, the proof is divided into two steps, each corresponding to one of the hypotheses of Theorem 1.1.

Proof of Proposition 2.3 - First step. First, let $x_0 \in X$ be $\overline{B'_{N-1}(0,1/2)}$. Then, by Proposition 2.2, X satisfies assumption *i)* of Theorem 1.1, with constants ε_0 and C_1 depending on N, m and β only, and constant $\alpha_1 = (N-1)/m$. We recall also that $\sigma \subset B_N(0,4/5)$ for any $\sigma \in X$. □For what concerns the second step, we turn our attention to assumption *ii)* of Theorem 1.1. Each case, corresponding to operators Λ_i and $\mathcal{N}_i$, $i=1,2$, should be treated separately. We limit ourselves to two cases, namely the cases corresponding to $\mathcal{N}_1$ and Λ_2, in order to show the main points of the proof, and we leave the details concerning the other two cases to the reader.

Insulating crack & Neumann-to-Dirichlet case

Proof of Proposition 2.3 - Second step. First, we notice that u is a solution to (2.12)-(2.15) if and only if $u \in H^1(\Omega\backslash\sigma)$, $\int_{\partial\Omega} u = 0$, and

$$\int_{\Omega\backslash\sigma} \nabla u \cdot \nabla w = \langle \eta, w|_{\partial\Omega}\rangle, \quad \text{for any } w \in H^1(\Omega\backslash\sigma).$$

We observe that for any $\sigma \in X$, $\mathcal{N}_1(\sigma)$ is a bounded and linear operator between ${}_0H^{-1/2}(\partial\Omega)$ and its dual. Hence we take H to be ${}_0H^{-1/2}(\partial\Omega)$ and $F : X \mapsto \mathcal{L}(H,H')$ to be defined as $F(\sigma) = \mathcal{N}_1(\sigma)$ for any $\sigma \in X$. With F_0 we denote in an analogous way the Neumann-to-Dirichlet map related to (2.12)-(2.15) when $\sigma = \emptyset$, that is the Neumann-to-Dirichlet map associated to the body where no crack is present. For any $\eta \in {}_0H^{-1/2}(\partial\Omega)\backslash\{0\}$, we define

$$\gamma(\eta) = \frac{\|\eta\|^2_{L^2(\partial\Omega)}}{\|\eta\|^2_{H^{-1/2}(\partial\Omega)}}. \tag{4.38}$$

Referring to Proposition 3.1, Example 3.3 and (3.29), $\{v_k\}_{k\in\mathbb{N}}$, the orthonormal basis of H we shall employ, is given by

$$\left\{\sqrt{j} f_{jp} :\ j \geq 1 \text{ and } 1 \leq p \leq p_j\right\} \tag{4.39}$$

with the natural order. We have that $\gamma(\sqrt{j} f_{jp}) = j$, for any j and p. Again by our remarks in Example 3.3, we deduce that $\#\{k \in \mathbb{N} :\ \gamma(v_k) \leq n\} \leq 2(1+n)^{N-1}$, for any $n \in \mathbb{N}$, For what concerns (1.3), we argue in this way. We need a kind of self-adjointness of $F(\sigma) - F_0$ for every $\sigma \in X$. We have that

$$\langle (F(\sigma) - F_0)\eta, \phi\rangle = \langle (F(\sigma) - F_0)\phi, \eta\rangle$$

for any η, $\phi \in {}_0H^{-1/2}(\partial\Omega)$, where $\langle\cdot,\cdot\rangle$ is again the duality pairing between H' and H. In fact, if u solves (2.12)-(2.15), u_0 solves the same boundary value problem with σ replaced by the empty set, v and v_0 solves the same boundary value problems with η replaced by ϕ, then

$$\langle (F(\sigma) - F_0)\eta, \phi\rangle = \int_{\Omega\backslash\sigma} \nabla v \cdot \nabla u - \int_{\Omega} \nabla v_0 \cdot \nabla u_0.$$

By the self-adjointness of the operator $F(\sigma) - F_0$, for any $\sigma \in X$, in order to prove (1.3) we have to show that there exist positive constants C_2 and α_2, which depend on N, m and β only, so that, for any j and p,

$$\left\|(F(\sigma) - F_0)\sqrt{j} f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_2 \exp(-\alpha_2 j). \tag{4.40}$$

We can find a constant C_3, depending on N only, so that, for any $\sigma \in X$,

$$\left\|(F(\sigma)-F_0)\sqrt{j}f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_3\|v_{jp}\|_{H^1(\Omega\backslash\overline{B_N(0,4/5)})},$$

where v_{jp} satisfies

$$\begin{cases} \Delta v_{jp} = 0 & \text{in } \Omega\backslash\sigma, \\ \frac{\partial v_{jp}}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \frac{\partial v_{jp}}{\partial\nu} = -j^{-1/2}\frac{\partial u_{jp}}{\partial\nu} & \text{on } \partial\sigma, \\ \int_{\partial\Omega} v_{jp} = 0, \end{cases} \tag{4.41}$$

with u_{jp} given by formula (3.30). Since $\int_{\partial\Omega} v_{jp} = 0$, a Poincaré type inequality implies that there exists a constant C_4, depending on N only, so that, for any $\sigma \in X$, we have

$$\left\|(F(\sigma)-F_0)\sqrt{j}f_{jp}\right\|_{H^{1/2}(\partial\Omega)} \leq C_4\left(\int_{\Omega\backslash\overline{B_N(0,4/5)}}\|\nabla v_{jp}\|^2\right)^{1/2}.$$

We can estimate the right hand side of the last equation as follows. We fix a cut-off function χ so that $\chi \in C_0^\infty(B_N(0,5/6))$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B_N(0,4/5)$. Without loss of generality, we can assume that for every $x \in \mathbb{R}^N$, $\|\nabla\chi(x)\| \leq C_5$, C_5 being a constant depending on N only. Let us observe that (4.41) means that for every $w \in H^1(\Omega\backslash\sigma)$ we have

$$\int_{\Omega\backslash\sigma}\nabla v_{jp}\cdot\nabla w = -\int_{\Omega\backslash\sigma} j^{-1/2}\nabla u_{jp}\cdot\nabla(\chi w).$$

Then, by taking $w = v_{jp}$, we infer that

$$\int_{\Omega\backslash\sigma}\|\nabla v_{jp}\|^2 = -\int_{\Omega\backslash\sigma} j^{-1/2}\nabla u_{jp}\cdot\nabla(\chi v_{jp}).$$

Straightforward computations allow us to prove that there exists a constant C_6, depending on N only, so that

$$\left(\int_{\Omega\backslash\sigma}\|\nabla v_{jp}\|^2\right)^{1/2} \leq C_6\left(\int_{B_N(0,5/6)}\|\nabla u_{jp}\|^2\right)^{1/2}.$$

Then we can conclude using (3.31). □

Conducting crack & Dirichlet-to-Neumann case
Proof of Proposition 2.3 - Second step. We begin with a description of the weak formulation of the boundary value problem (2.13)-(2.14). With $H^1_{const}(\Omega,\sigma)$ we denote the subspace of $H^1(\Omega)$ functions which are constant on σ. For any $c \in \mathbb{R}$, we set $H^1_c(\Omega,\sigma)$ as the subset of $H^1(\Omega)$ functions which are equal to the constant c on σ. For any $c \in \mathbb{R}$, we have that there exists and it is unique a solution to the following boundary value problem

$$\begin{cases} \Delta u_c = 0 & \text{in } \Omega\backslash\sigma, \\ u_c = c & \text{on } \sigma, \\ u_c = \psi & \text{on } \partial\Omega, \end{cases} \tag{4.42}$$

that is a function $u_c \in H^1_c(\Omega,\sigma)$ so that $u_c|_{\partial\Omega} = \psi$ and that

$$\int_{\Omega\backslash\sigma}\nabla u_c\cdot\nabla w = 0, \quad \text{for any } w \in H^1_0(\Omega)\cap H^1_0(\Omega,\sigma).$$

Given u_c, solution to (4.42), we can define $\frac{\partial u_c}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ as follows

$$\langle \frac{\partial u_c}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_{\Omega\backslash\sigma} \nabla u_c \cdot \nabla\tilde{\varphi},$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $\tilde{\varphi}$ is any $H^1_0(\Omega,\sigma)$ function so that $\tilde{\varphi}|_{\partial\Omega} = \varphi$. We claim that there exists a unique $c \in \mathbb{R}$ so that $\langle \frac{\partial u_c}{\partial \nu}|_{\partial\Omega}, 1\rangle = 0$, that is existence and uniqueness of a solution to (2.13)-(2.14). We have that u solves (2.13)-(2.14) if and only if $u \in H^1_{const}(\Omega,\sigma)$ so that $u|_{\partial\Omega} = \psi$ and that

$$\int_{\Omega\backslash\sigma} \nabla u \cdot \nabla w = 0, \quad \text{for any } w \in H^1_0(\Omega) \cap H^1_{const}(\Omega,\sigma).$$

If we take $\tilde{\psi}$ to be any $H^1_0(\Omega,\sigma)$ function so that $\tilde{\psi}|_{\partial\Omega} = \psi$, we have that u solves (2.13)-(2.14) if and only if $\tilde{u} = u - \tilde{\psi}$ belongs to $H^1_0(\Omega) \cap H^1_{const}(\Omega,\sigma)$ and satisfies

$$\int_{\Omega\backslash\sigma} \nabla\tilde{u} \cdot \nabla w = -\int_{\Omega\backslash\sigma} \nabla\tilde{\psi} \cdot \nabla w, \quad \text{for any } w \in H^1_0(\Omega) \cap H^1_{const}(\Omega,\sigma).$$

Standard elliptic theory provides us with existence and uniqueness of such a solution. By the property $\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, 1\rangle = 0$, we can infer that $\frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ can be also defined as

$$\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, \varphi\rangle = \int_{\Omega\backslash\sigma} \nabla u \cdot \nabla\tilde{\varphi},$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $\tilde{\varphi}$ is any $H^1_{const}(\Omega,\sigma)$ function so that $\tilde{\varphi}|_{\partial\Omega} = \varphi$. Now we can denote with H the space $H^{1/2}(\partial\Omega)$. Concerning the function γ our choice is the following. For any $\psi \in H^{1/2}(\partial\Omega)\backslash\{0\}$, let

$$\gamma(\psi) = \frac{\|\psi\|^2_{H^{1/2}(\partial\Omega)}}{\|\psi\|^2_{L^2(\partial\Omega)}}. \tag{4.43}$$

Then it remains to choose an orthonormal basis of H, $\{v_k\}_{k\in\mathbb{N}}$, so that $\gamma(v_k)$ is finite for any k and (1.2) and (1.3) are satisfied. Recalling Example 3.3, in particular (3.29), we consider the set

$$\left\{ \frac{f_{jp}}{\sqrt{1+j}} \,:\, j \geq 0 \text{ and } 1 \leq p \leq p_j \right\} \tag{4.44}$$

with the natural order. This set, by Proposition 3.1, is an orthonormal basis of H and it is the one we choose. We also recall that f_{jp} is a spherical harmonic of degree j so that $\|f_{jp}\|_{L^2(\partial\Omega)} = 1$, hence $\gamma(f_{jp}/\sqrt{1+j}) = 1+j$, for any j and p. Fixed $n \in \mathbb{N}$, $\#\{k \in \mathbb{N} : \gamma(v_k) \leq n\}$ is clearly bounded from above by $2(1+n)^{N-1}$, see Example 3.3. The map $F : X \mapsto \mathcal{L}(H,H')$ is given by $F(\sigma) = \Lambda_2(\sigma)$, for any $\sigma \in X$, and F_0 denotes the Dirichlet-to-Neumann map corresponding to $\sigma = \emptyset$. We recall that the operator $F(\sigma)$ is self-adjoint for any $\sigma \in X$, as well as F_0 is. We proceed to verify (1.3) in this case. First, there exists a constant C_7, depending on N only, so that, for any $\sigma \in X$,

$$\left\| (F(\sigma) - F_0)\frac{f_{jp}}{\sqrt{1+j}} \right\|_{H^{-1/2}(\partial\Omega)} \leq C_7 \left(\int_{\Omega\backslash\overline{B_N(0,4/5)}} \|\nabla v_{jp}\|^2 \right)^{1/2} \tag{4.45}$$

where $v_{jp} = u_{jp}(\sigma) - \frac{u_{jp}}{\sqrt{1+j}}$, $u_{jp}(\sigma)$ being the solution to (2.13)-(2.14) with ψ replaced by $\frac{f_{jp}}{\sqrt{1+j}}$ and u_{jp} being as in (3.30). Hence, v_{jp} satisfies

$$\begin{cases} \Delta v_{jp} = 0 & \text{in } \Omega\backslash\sigma, \\ v_{jp} = 0 & \text{on } \partial\Omega, \\ v_{jp} = c - \frac{u_{jp}}{\sqrt{1+j}} & \text{on } \partial\sigma, \\ \langle \frac{\partial v_{jp}}{\partial \nu}|_{\partial\Omega}, 1\rangle = 0, \end{cases} \tag{4.46}$$

where $c = u_{jp}(\sigma)|_\sigma$. We notice that, if χ is the cut-off function previously defined in this subsection, then $w_{jp} = (v_{jp} - c + \chi\frac{u_{jp}}{\sqrt{1+j}}) \in H_0^1(\Omega, \sigma)$ and $v_{jp}|_{\partial\Omega} = -c$. So,

$$\int_{\Omega\setminus\sigma} \nabla v_{jp} \cdot \nabla w_{jp} = \langle \frac{\partial v_{jp}}{\partial \nu}|_{\partial\Omega}, -c\rangle = 0,$$

that is

$$\int_{\Omega\setminus\sigma} \nabla v_{jp} \cdot \nabla v_{jp} = \int_{\Omega\setminus\sigma} \nabla v_{jp} \cdot \nabla(\chi\frac{u_{jp}}{\sqrt{1+j}}),$$

from which we easily deduce that

$$\left(\int_{\Omega\setminus\sigma} \|\nabla v_{jp}\|^2\right)^{1/2} \leq C_8 \|u_{jp}\|_{H^1(B_N(0,5/6))}, \tag{4.47}$$

where C_8 depends on N only. So (1.3) is obtained by combining (4.45), (4.47) and (3.31) and the self-adjointness of the operator $F(\sigma) - F_0$. □

4.2. Inverse cavity problem, inverse surface crack problem and inverse boundary material loss problem

As we have already observed, the inverse problem of cavities can be treated in a way which is completely analogous to the treatment of the inverse crack problem.

Proof of Proposition 2.4. It can be obtained along the lines of the proof of Proposition 2.3, with obvious modifications. In particular, the reference point in X is given by σ_0, the orthonormal basis used are those described in Example 3.5, whereas the reference operator is the one related to the domain Ω, Ω as in Example 3.5. □

Proof of Proposition 2.5. Also the arguments for the proof of Proposition 2.5 are simple modifications of what we have used to prove Proposition 2.3, clearly making use of the orthonormal basis described in Example 3.4. □

References

[1] Alessandrini G 1997 Examples of instability in inverse boundary-value problems *Inverse Problems* **13** 887–897.

[2] Alessandrini G, Beretta E, Rosset E and Vessella S 2000 Optimal stability for inverse elliptic boundary value problems with unknown boundaries *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** 755–806.

[3] Alessandrini G and Di Benedetto E 1997 Determining 2-dimensional cracks in 3-dimensional bodies: uniqueness and stability *Indiana Univ. Math. J.* **46** 1–82.

[4] Alessandrini G and Di Cristo M 2000 Unique determination of surface breaking cracks in three-dimensional bodies *J. Inv. Ill-Posed Problems* **8** 469–482.

[5] Alessandrini G and Rondi L 2001 Optimal stability for the inverse problem of multiple cavities *J. Diff. Eq.* **176** 356–386.

[6] Andrieux S, Ben Abda A and Jaoua M 1993 Identifiabilité de frontière inaccessible par des mesures de surface *C. R. Acad. Sci. Paris Sér. I* **316** 429–434.

[7] Bryan K and Vogelius M S 2004 A review of selected works on crack identification in *Geometric Methods in Inverse Problems and PDE Control* ed C B Croke *et al* (New York: Springer) pp 25–46.

[8] Cheng J, Hon Y C and Yamamoto M 2001 Conditional stability estimation for an inverse boundary problem with non-smooth boundary in $\mathbb{R}^3$ *Trans. Amer. Math. Soc.* **353** 4123–4138.

[9] Di Cristo M and Rondi L 2003 Examples of exponential instability for inverse inclusion and scattering problems *Inverse Problems* **19** 685–701.

[10] Di Cristo M, Rondi L and Vessella S 2006 Stability properties of an inverse parabolic problem with unknown boundaries *Ann. Mat. Pura Appl.* **185** 223–255.

[11] Elcrat A R, Isakov V and Neculoiu O 1995 On finding a surface crack from boundary measurements *Inverse Problems* **11** 343–351.

[12] Friedman A and Vogelius M 1989 Determining cracks by boundary measurements *Indiana Univ. Math. J.* **38** 527–556.

[13] Kolmogorov A N and Tihomirov V M 1959 ϵ-entropy and ϵ-capacity of sets in functional spaces *Uspehi Mat. Nauk.* **14** 3–86 (Russian). English translation in 1961 *Amer. Math. Soc. Transl.* **17** 277–364.

[14] Mandache N 2001 Exponential instability in an inverse problem for the Schrödinger equation *Inverse Problems* **17** 1435–1444.

[15] C. Müller 1966 *Spherical Harmonics* (Berlin: Springer).

[16] Rondi L 1999 *Uniqueness and Optimal Stability for the Determination of Multiple Defects by Electrostatic Measurements* Ph.D. thesis (Trieste: SISSA) (downloadable from `http://www.sissa.it/library/`).

[17] Rondi L 2005 Optimal stability of reconstruction of plane Lipschitz cracks *SIAM J. Math. Anal.* **36** 1282–1292.