

## A NOTE ON RELATION BETWEEN AZUKAWA ISOMETRIES AT ONE POINT AND GLOBAL BIHOLOMORPHISMS

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**Abstract.** We prove that under certain assumptions holomorphic functions which are Azukawa isometries at one point are in fact biholomorphisms.

**1. Introduction.** The holomorphic contractibility of Carathéodory–Reiffen and Kobayashi–Royden pseudometrics have put much interest in the relation of global biholomorphicity and Carathéodory or Kobayashi isometricity at one point. While from the mentioned property it immediately follows that a biholomorphism must be a Carathéodory and Kobayashi isometry, the opposite statement is obviously not true in the general case. In 1984 Jean-Pierre Vigué proved the following result.

**THEOREM** (see [9]). *Let  $\Omega$  be a bounded convex domain in  $\mathbb{C}^n$  and let  $M$  be a complex manifold on which a Carathéodory–Reiffen pseudodistance is a distance. Suppose  $F : \Omega \rightarrow M$  is a holomorphic mapping which is a Carathéodory–Reiffen isometry at a point  $p \in \Omega$ . Then  $F$  is a biholomorphism.*

A few years later Ian Graham proved analogous theorem for a Kobayashi–Royden isometry.

**THEOREM** (see [2]). *Suppose  $M$  is a taut complex manifold of dimension  $n$ . Suppose  $\Omega$  is a strictly convex bounded domain in  $\mathbb{C}^n$ . Suppose  $F : \Omega \rightarrow M$  is a holomorphic mapping which is a Kobayashi–Royden isometry at a point  $p \in \Omega$ . Then  $F$  is a biholomorphism.*

In this paper we switch our interest to another holomorphically contractible pseudometric: the Azukawa pseudometric  $A_G$ . We obtain the following main result.

THEOREM 1. Let  $G_1, G_2 \subset \mathbb{C}^n$  be domains. Let  $a \in G_1$  and let  $F : G_1 \rightarrow G_2$  be such that:

- (1)  $F \in \mathcal{O}(G_1, G_2)$ ;
- (2)  $G_1$  is taut;
- (3)  $G_2$  is bounded;
- (4) for any  $z \in G_1$  we have  $g_{G_1}(a, z) = l_{G_1}^*(a, z)$ ;
- (5) for any  $X \in \mathbb{C}^n$  we have  $A_{G_2}(F(a); F'(a)X) = A_{G_1}(a; X)$ .

Then  $F$  is a biholomorphism.

Proof of this result is given in the last section of this paper.

The following example shows a material difference between our result and theorems of Vigue and Graham.

EXAMPLE 2. Observe that if  $G$  is a taut balanced pseudoconvex domain, then assumption (4) in Theorem 1 with  $G_1 = G$  is satisfied (see [3, Proposition 4.2.7]). We can also use Theorem 1 with  $G_2 = G$  (provided  $G_1$  is such that (5) is satisfied). However, taut balanced pseudoconvex domains do not need to be convex, thus in many cases we may apply neither Vigue's nor Graham's theorems with  $G$  in the role of  $\Omega$ . A good example of that situation is the set  $G := \{(z, w) \in \mathbb{C}^n : |z| < 1, |w| < 1, |zw| < \alpha\}$ , for a fixed  $0 < \alpha < 1$ .

**2. Preliminaries.** In this section we recall the definitions and basic properties of the Green function and Azukawa pseudometric. For the detailed proofs, more interesting facts about these objects and their connections to Kobayashi and Carathéodory pseudodistances and pseudometrics see for instance [1, 3–5], and [6].

Let  $G$  be a domain in  $\mathbb{C}^n$ . To simplify the definitions, for  $a \in G$  let  $\exp \mathcal{L}_a$  denote the family of functions  $u : G \rightarrow [0, 1)$  such that  $\log u$  is plurisubharmonic on  $G$  and there exists a positive constant  $M$  such that  $u(z) \leq M\|z - a\|$ ,  $z \in G$ .

DEFINITION 3. Define

$$g_G(a, z) := \sup\{u(z) : u \in \exp \mathcal{L}_a\}, \quad a \in G, \quad z \in G;$$

$$A_G(a; X) := \sup\{\limsup_{0 \neq \lambda \rightarrow 0} \frac{u(a + \lambda X)}{|\lambda|} : u \in \exp \mathcal{L}_a\}, \quad a \in G, \quad X \in \mathbb{C}^n.$$

The function  $g_G$  is called a pluricomplex Green function with a pole at point  $a$  and  $A_G$  is called an Azukawa pseudometric.

Both  $g_G$  and  $A_G$  are holomorphically contractible, i.e. for a domain  $D \subset \mathbb{C}^m$  and a holomorphic function  $F : G \rightarrow D$  we have

$$g_D(F(a), F(z)) \leq g_G(a, z), \quad a, z \in G,$$

and

$$A_D(F(a); F'(a)X) \leq A_G(a; X), \quad a \in G, \quad X \in \mathbb{C}^n.$$

Obviously, if  $F$  is a biholomorphism, we get equalities: this property is called biholomorphic invariance. Moreover

$$g_{\mathbb{D}}(\lambda', \lambda'') = m(\lambda', \lambda'') := |(\lambda' - \lambda'')/(1 - \lambda' \overline{\lambda''})|, \quad \lambda', \lambda'' \in \mathbb{D},$$

and

$$A_{\mathbb{D}}(\lambda; X) = |X| \gamma(\lambda) := |X|/(1 - |\lambda|^2), \quad \lambda \in \mathbb{D}, \quad X \in \mathbb{C},$$

where by  $\mathbb{D}$  we denote a unit disc in  $\mathbb{C}$ .

One can show that  $g_G(a, \cdot) \in \exp \mathcal{L}_a$ ,  $a \in G$ , (see [5]). Consequently we obtain an equivalent and much more useful definition of Azukawa pseudometric

$$A_G(a; X) = \limsup_{0 \neq \lambda \rightarrow 0} \frac{g_G(a, a + \lambda X)}{|\lambda|}, \quad a \in G, \quad X \in \mathbb{C}^n.$$

Now let us introduce the following notation. By  $l_G$  we denote the Lempert function, by  $k_G$  the Kobayashi pseudodistance and by  $K_G$  the Kobayashi–Royden pseudometric. We also use the convention: for a function  $f$  let  $f^*$  denote  $\tanh f$ .

**DEFINITION 4.** We say that a domain  $G \subset \mathbb{C}^n$  is *k-hyperbolic* (resp. *K-hyperbolic*), if for any  $z, w \in G$ ,  $z \neq w$ , we have  $k_G(z, w) > 0$  (resp. for any  $z \in G$ ,  $X \in \mathbb{C}^n$ ,  $X \neq 0$ , we have  $K_G(z; X) > 0$ ).

For a domain in  $\mathbb{C}^n$  *k-hyperbolicity* is equivalent to *K-hyperbolicity* (see for instance [3, Theorem 7.2.2]). In the proof of Theorem 1 we will use the following property.

**LEMMA 5** (see [3, Corollary 7.2.4]). *Any taut domain in  $\mathbb{C}^n$  is k-hyperbolic.*

**3. Proof of the main theorem.** Before we proceed to the proof, we need one easy but interesting lemma.

**LEMMA 6.** *Let  $G \subset \mathbb{C}^n$  be a domain such that  $0 \in G$  and let  $\varphi \in \mathcal{O}(\mathbb{D}, G)$  be such that  $\varphi(0) = 0$ . Then the following conditions are equivalent:*

- (i) *there exists a  $\lambda' \in \mathbb{D} \setminus \{0\}$  such that  $g_G(0, \varphi(\lambda')) = |\lambda'|$ ;*
- (ii)  *$g_G(0, \varphi(\lambda)) = |\lambda|$  for all  $\lambda \in \mathbb{D}$ ;*
- (iii)  *$A_G(0; \varphi'(0)) = 1$ .*

**PROOF.** The proof is similar to the proof of an analogous theorem for complex Carathéodory and Carathéodory–Reiffen geodesics (see Proposition 8.1.3 in [3]).  $\square$

**PROOF OF THEOREM 1.** Using biholomorphic invariance of  $g_G$  and  $A_G$  we may assume  $a = 0$  and  $F(a) = 0$ . Observe that  $A_{G_1}(0; X) > 0$  for  $X \neq 0$ .

Indeed, using (4), Proposition 3.18 from [7], and Lemma 5 we obtain

$$\begin{aligned} A_{G_1}(0; X) &= \limsup_{0 \neq \lambda \rightarrow 0} \frac{g_{G_1}(0, \lambda X)}{|\lambda|} = \limsup_{0 \neq \lambda \rightarrow 0} \frac{l_{G_1}^*(0, \lambda X)}{|\lambda|} \\ &= \limsup_{0 \neq \lambda \rightarrow 0} \frac{l_{G_1}(0, \lambda X)}{|\lambda|} = K_{G_1}(0; X) > 0. \end{aligned}$$

Now, from (5) we get  $A_{G_2}(0; F'(0)X) > 0$ ,  $X \neq 0$ . Thus,  $F'(0)$  is an isomorphism and so  $F$  is injective in a neighborhood  $U$  of zero. From the tautness of  $G_1$  we get the equality between the Euclidean topology of  $G_1$  and its Kobayashi topology (see [3, Proposition 3.3.4]). Using (5) we may assume  $U \supset B_{g_{G_1}}(r_0) := \{z \in G_1 : g_{G_1}(0, z) < r_0\}$  for some  $r_0 \in (0, 1)$ .

We show that  $g_{G_2}(0, F(z)) = g_{G_1}(0, z)$  for  $z \in G_1$ . Fix a  $z_0 \in G_1 \setminus \{0\}$ . Since  $G_1$  is taut, there exists an extremal disc  $\varphi \in \mathcal{O}(\mathbb{D}, G_1)$  for the pair  $(0, z_0)$ , i.e.  $\varphi(0) = 0$ ,  $\varphi(\lambda_0) = z_0$ , and  $l_{G_1}^*(0, \varphi(\lambda_0)) = |\lambda_0|$  for some  $\lambda_0 \in (0, 1)$ . Thus  $g_{G_1}(0, \varphi(\lambda_0)) = |\lambda_0|$  and from Lemma 6 we obtain  $A_{G_1}(0; \varphi'(0)) = 1$ . From (5) we get  $A_{G_2}((F \circ \varphi)(0); (F \circ \varphi)'(0)) = 1$ . Thus once again from Lemma 6 we get  $g_{G_2}((F \circ \varphi)(0), (F \circ \varphi)(\lambda)) = |\lambda|$  for any  $\lambda \in \mathbb{D}$ . Hence

$$g_{G_2}(0, F(z_0)) = |\lambda_0| = g_{G_1}(0, z_0).$$

Consequently,  $F^{-1}(B_{g_{G_2}}(r)) \subset B_{g_{G_1}}(r)$  and  $F(B_{g_{G_1}}(r)) \subset B_{g_{G_2}}(r)$ ,  $r \in (0, 1)$ . Therefore  $F$  is a proper holomorphic map  $G_1 \rightarrow G_2$ . In particular,  $F$  is surjective. Consequently,  $F(B_{g_{G_1}}(r)) = B_{g_{G_2}}(r)$  and  $F^{-1}(B_{g_{G_2}}(r)) = B_{g_{G_1}}(r)$ ,  $r \in (0, 1)^1$ .

Define  $V := \{z \in G_1 : J_F(z) = 0\}$ . Then (see [8, Chapter 15.1])

- $F(V)$  is an analytic subset of  $G_2$ ;
- there exists an  $m \in \mathbb{N}$  such that  $\sharp F^{-1}(w) = m$  for  $w \notin F(V)$  and  $\sharp F^{-1}(w) < m$  for  $w \in F(V)$ ;
- $F : G_1 \setminus F^{-1}(F(V)) \rightarrow G_2 \setminus F(V)$  is a holomorphic covering map.

Since  $F : B_{g_{G_1}}(r_0) \rightarrow B_{g_{G_2}}(r_0)$  is biholomorphic, we conclude that  $m = 1$ . Thus  $F$  is a biholomorphism.  $\square$

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<sup>1</sup>Observe that we could not get the equalities without surjectivity of  $F$ .

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