UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVIII
2010

ALGEBRAIC APPROXIMATION OF SMOOTH MAPS

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To Professor Józef Siciak on his 80th birthday.

1. Introduction. This is a survey article devoted to the problem of approximation of smooth (that is, $\mathcal{C}^\infty$) maps between nonsingular real algebraic varieties by algebraic morphisms. A common ancestor of the results presented below is the classical Weierstrass approximation theorem: Any smooth real-valued function defined in an open neighborhood of a compact subset K of $\mathbb{R}^n$ can be approximated on K, in the $\mathcal{C}^\infty$ topology, by polynomial functions.

The basic objects of our investigation are *real algebraic sets*. These are subsets of $\mathbb{R}^n$, for some n, defined by systems of polynomial equations. According to Hilbert's basis theorem, each algebraic subset of $\mathbb{R}^n$ is given by a finite system of polynomial equations $p_1 = 0, \ldots, p_k = 0$, and hence by a single equation $p_1^2 + \ldots + p_k^2 = 0$. We have the usual notions of dimension, irreducibility and nonsingularity of real algebraic sets. In particular, each nonsingular real algebraic set is a smooth manifold. The converse if false. For example, the algebraic curve $C = \{(x, y) \in \mathbb{R}^2 \mid y^3 + 2x^2y - x^4 = 0\}$ is a smooth submanifold of $\mathbb{R}^2$, but 0 is a singular point of C. The algebraic subsets of $\mathbb{R}^n$ satisfy the axioms for closed sets in a topology. The topology on $\mathbb{R}^n$ determined in this way is called the *Zariski topology*, as opposed to the *Euclidean topology* induced by the usual metric. Thus each real algebraic set is endowed with the Zariski topology and the Euclidean topology. For any algebraic subset Z of $\mathbb{R}^n$ and any Zariski open subset U of Z, let $\mathcal{R}(U)$ denote the ring of regular functions on U. Recall that a function $r : U \to \mathbb{R}$ is said to be *regular* if it is of the form $r = p/q$, where p and q are polynomial functions with $q^{-1}(0) \cap U = \emptyset$. If Z is endowed with the Zariski topology, the correspondence $\mathcal{R}_Z : U \to \mathcal{R}(U)$ defines a sheaf of rings of real-valued functions on Z. This gives rise to a ringed space $(Z, \mathcal{R}_Z)$. A *real algebraic variety* is a topological space X together with a sheaf $\mathcal{R}_X$ of real-valued functions on X such that the ringed space $(X, \mathcal{R}_X)$ is isomorphic to $(Z, \mathcal{R}_Z)$ for some real algebraic set Z (such an object is often called an affine real algebraic variety, cf. [**6**]). A map $f : X \to Y$ of real algebraic varieties is said to be *regular* (or to be a *morphism*) if it is continuous and for every open subset U of Y and every function r in $\mathcal{R}_Y(U)$, the composite function $r \circ f|_{f^{-1}(U)}$ is in $\mathcal{R}_X(f^{-1}(U))$. Assuming that Y is an algebraic subset of $\mathbb{R}^p$, a map $f = (f_1, \dots, f_p) : X \to Y$ is regular if and only if each component $f_j : X \to \mathbb{R}$ is a regular function, that is, f_j belongs to $\mathcal{R}_X(X)$ for $j = 1, \dots, p$. The set of all regular maps from X into Y is denoted by $\mathcal{R}(X, Y)$. Henceforth, unless explicitly stated otherwise, all topological notions relating to real algebraic varieties will refer to the Euclidean topology.

Example 1.1.

(i) If $(X, \mathcal{R}_X)$ is a real algebraic variety, then for any Zariski open subset U of X, the ringed space $(U, \mathcal{R}_X|_U)$ is a real algebraic variety. Indeed, it can be assumed that X is an algebraic subset of $\mathbb{R}^n$. Let p be a polynomial function on $\mathbb{R}^n$ with $p^{-1}(0) = X \setminus U$. Then $(U, \mathcal{R}_X|_U)$ is isomorphic to $(Z, \mathcal{R}_Z)$, where $Z = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in X, yp(x) = 1\}$ is an algebraic subset of $\mathbb{R}^{n+1}$.

(ii) Any Zariski locally closed subset of real projective n-space $\mathbb{P}^n(\mathbb{R})$, with the usual sheaf of regular functions, can be regarded as a real algebraic variety in the sense defined above. The quickest way to see it is to observe that $\mathbb{P}^n(\mathbb{R})$ is contained in the affine complex algebraic variety $\mathbb{P}^n(\mathbb{C}) \setminus H$, where H is the hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by the equation $x_0^2 + \ldots + x_n^2 = 0$. Another, more useful, affine model of $\mathbb{P}^n(\mathbb{R})$ is constructed as follows. Let M_{n+1}denote the set of all real $(n+1) \times (n+1)$ matrices and let

$$P_n = \{A \in M_{n+1} | {}^t A = A = A^2, \operatorname{trace}(A) = 1\}.$$

Clearly, P_n is an algebraic subset of $M_{n+1} = \mathbb{R}^{(n+1)^2}$, and the map

$$\mathbb{P}^n(\mathbb{R}) \to P_n, (x_0 : \ldots : x_n) \to \left(\frac{x_i x_j}{x_0^2 + \ldots + x_n^2} \right) \in P_n$$

is an isomorphism. In a similar way, one can show that the real Grassmannians are real algebraic varieties.

(iii) Any quasiprojective complex algebraic variety W can be considered with its underlying real structure. It becomes then a real algebraic variety, denoted by $W_{\mathbb{R}}$. In particular, $\mathbb{P}^1(\mathbb{C})_{\mathbb{R}}$ is biregularly isomorphic to the unit 2-sphere $\mathbb{S}^2$.

Real algebraic varieties are much more flexible than complex varieties. This is illustrated by the following fundamental result (cf. [**41, 48**] and [**6**]).

Theorem 1.2 (Nash–Tognoli). *Any compact smooth manifold is diffeomorphic to a nonsingular real algebraic variety.*

The proof is based on a deep and difficult theorem from cobordism theory. Namely, every compact smooth manifold is cobordant to a compact nonsingular real algebraic variety. It is interesting to note that the varieties in the cobordism theorem are quite simple. They are products of $\mathbb{P}^{2n}(\mathbb{R})$ and $\mathbb{P}^q(\mathbb{R})$-bundles on $\mathbb{P}^r(\mathbb{R})$ (cf. [**39**, Lemma 1]).

Any nonsingular real algebraic variety diffeomorphic to a given smooth manifold M is called an *algebraic model* of M. The Nash–Tognoli theorem can be supplemented as follows.

THEOREM 1.3 (cf. [**11**]). *Any compact smooth manifold of positive dimension has an uncountable family of mutually birationally nonequivalent algebraic models.*

As will be seen below, the existence of various algebraic models of compact smooth manifolds has important consequences for the approximation problem.

Let X and Y be nonsingular real algebraic varieties. We regard $\mathcal{R}(X,Y)$ as a subset of the space $\mathcal{C}^\infty(X,Y)$ of all smooth maps from X into Y, endowed with the $\mathcal{C}^\infty$ compact-open topology (the weak $\mathcal{C}^\infty$ topology in the terminology used in [**29**]). The Weierstrass approximation theorem implies that $\mathcal{R}(X,\mathbb{R}^p)$ is dense in $\mathcal{C}^\infty(X,\mathbb{R}^p)$. However, in general, the set $\mathcal{R}(X,Y)$ is not dense in $\mathcal{C}^\infty(X,Y)$, and its closure $\mathcal{C}^\infty_{\mathcal{R}}(X,Y)$ is very hard to describe. We say that a smooth map from X into Y can be *approximated by regular maps* if it belongs to $\mathcal{C}^\infty_{\mathcal{R}}(X,Y)$.

THEOREM 1.4 (cf. [**16**]). *Any compact smooth manifold of positive dimension has an algebraic model Y such that $\mathcal{C}^\infty_{\mathcal{R}}(X,Y) \neq \mathcal{C}^\infty(X,Y)$ for every nonsingular real algebraic variety X of positive dimension.*

We say that a nonsingular irreducible real algebraic variety Y is a *Weierstrass variety* if $\mathcal{C}^\infty_{\mathcal{R}}(X,Y) = \mathcal{C}^\infty(X,Y)$ for some nonsingular real algebraic variety X of positive dimension.

EXAMPLE 1.5.

(i) The unit n-sphere

$$\mathbb{S}^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^n \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

is a Weierstrass variety for all $n \geq 1$. For $n \geq 2$ this is an immediate consequence of the Weierstrass theorem, while for $n = 1$ it requires an extra argument (cf. Corollaries 2.6 and 2.7).

(ii) Any nonsingular rational real algebraic variety Y is a Weierstrass variety. In fact, the set $\mathcal{R}(C,Y)$ is dense in $\mathcal{C}^\infty(C,Y)$ for every compact nonsingular real algebraic curve C (cf. [**17**]). Recall that a real algebraic variety is said to be *rational* if it is birationally equivalent to $\mathbb{P}^n(\mathbb{R})$ for some n.

(iii) A nonsingular irreducible real algebraic curve is a Weierstrass variety if and only if it is rational (cf. [**16**]).

(iv) Let V be a nonsingular irreducible complex algebraic variety defined over $\mathbb{R}$. The set $V(\mathbb{R})$ of real points of V is either empty or it is a nonsingular irreducible real algebraic variety with $\dim_{\mathbb{R}} V(\mathbb{R}) = \dim_{\mathbb{C}} V$. If $\dim_{\mathbb{C}} V \geq 1$ and the first Betti number of V is nonzero, then $V(\mathbb{R})$ is not a Weierstrass variety (cf. [**17**]). In particular, the real part of a positive-dimensional complex Abelian variety defined over $\mathbb{R}$ is never a Weierstrass variety.

(v) It is not known whether the Fermat sphere

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^4 + x_2^4 + x_3^4 = 1\}$$

is a Weierstrass variety.

It seems that the class of Weierstrass varieties is "close" to the class of nonsingular rational varieties. These two classes coincide in dimension 1. We conjecture that a compact nonsingular irreducible real algebraic surface is a Weierstrass variety if and only if it is rational. It is difficult to formulate a precise conjecture in higher dimensions. There exists a Weierstrass 3-fold that is not rational. Indeed, according to [**1**], one can find a nonsingular real algebraic 3-fold X that is not rational, but $X \times \mathbb{P}^n(\mathbb{R})$ is rational for some $n \geq 1$. In view of Example 1.5 (ii), X is a Weierstrass variety.

In some cases, the only regular maps are constant maps.

Theorem 1.6 (cf. [**19**]). *Any compact connected smooth manifold has an algebraic model Y such that for every rational real algebraic variety X, every regular map from X into Y is constant.*

The phenomenon of scarcity of regular maps is also illustrated by the following:

Example 1.7. Let V and W be nonsingular irreducible complex projective curves defined over $\mathbb{R}$, of genus $g(V)$ and $g(W)$, respectively. Assume that the sets of real points $C = V(\mathbb{R})$ and $D = W(\mathbb{R})$ are nonempty. Each regular map from C into D has a unique extension to a (complex) regular map from V into W. If $g(V) < g(W)$, then every regular map from C into D is constant by virtue of the Hurwitz–Riemann theorem [**26**, p. 140]. According to the theorem of de Franchis [**38**, p. 227], if $g(W) \geq 2$, then there exist only finitely many nonconstant regular maps from C to D.

As a special case of the previous example, we get:

Example 1.8. Let F_n be the Fermat curve of degree n,

$$F_n = \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^n + y^n = z^n\}.$$

If $k > n \geq 2$, then every regular map from F_n into F_k is constant. If $k \geq 4$, then for each $n \geq 1$, there exist only finitely many nonconstant regular maps from F_n into F_k.

If a smooth map between nonsingular real algebraic varieties can be approximated by regular maps, then it is homotopic to a regular map (cf. [**6**, Corollary 9.3.7] if the source variety is not compact). The converse is false. If $n \geq 4$, then every smooth map $F_n \to F_n$ of topological degree 1 is homotopic to a regular map (the identity map), but only finitely many of such maps can be approximated by regular maps (namely, only these which are already regular).

It is convenient to introduce a certain numerical invariant. Given a real algebraic variety Y, let $\beta(Y)$ be the supremum of all nonnegative integers n with the following property: for every n-dimensional compact connected nonsingular real algebraic variety X, every continuous map from X into Y is homotopic to a regular map.

Theorem 1.9 (cf. [**19**]). *Let Y be a compact nonsingular real algebraic variety. Then:*

(i) $\beta(Y) \leq \dim Y$*, provided that* $\dim Y \geq 1$.

(ii) *If the k-th Stiefel–Whitney class of Y is nonzero for some $k \geq 1$, then $\beta(Y) \leq k$. In particular, $\beta(Y) = 0$ or $\beta(Y) = 1$, provided that Y is nonorientable as a smooth manifold.*

(iii) *If* $\dim Y = p$ *and* $H_k(Y; \mathbb{Z}/2) \neq 0$ *for some k satisfying* $0 < k < p$, *then*

$$\beta(Y) \leq \begin{cases} \max\{k, p-k\} - 1 \leq p - 2 & \text{for } k \neq p/2 \\ p/2 & \text{for } k = p/2. \end{cases}$$

Theorem 1.9 implies in particular that there is no positive-dimensional compact nonsingular real algebraic variety Y for which the set $\mathcal{R}(X,Y)$ is dense in $\mathcal{C}^\infty(X,Y)$ for every compact nonsingular real algebraic variety X. Thus the Weierstrass approximation theorem cannot be reformulated in a straightforward manner for maps with values in any compact nonsingular real algebraic variety of positive dimension.

It is hard to determine the exact value of $\beta(Y)$ even for "simple" varieties Y.

Example 1.10 (cf. [**19**]). Let Y be a compact nonsingular irreducible real algebraic variety of positive dimension.

(i) If Y is rational, then $\beta(Y) \geq 1$ and the equality holds provided that Y is nonorientable. Moreover, $\beta(\mathbb{P}^n(\mathbb{R})) = 1$ for all $n \geq 1$.

(ii) Assuming that $\dim Y = 1$, we have $\beta(Y) = 1$ if and only if Y is biregularly isomorphic to $\mathbb{S}^1$. We conjecture that, up to biregular isomorphism, the only variety Y (of any dimension) with $\beta(Y) = \dim Y$ is $\mathbb{S}^1$.

(iii) If $\dim Y = 2$, then $\beta(Y) = 0$ or $\beta(Y) = 1$, and $\beta(Y) = 1$ provided that Y is either rational or homeomorphic to $\mathbb{S}^2$.

(iv) If $\dim Y = 3$ and $H_1(Y; \mathbb{Z}/2) \neq 0$, then $\beta(Y) = 0$ or $\beta(Y) = 1$.

(v) Assuming that $\dim Y = 4$ and Y is connected and simply connected, we have $\beta(Y) \geq 3$ if and only if Y is homeomorphic to $\mathbb{S}^4$.

(vi) If Y is homeomorphic to $\mathbb{S}^n$ with $n \geq 2$, then $n-1 \leq \beta(Y) \leq n$ and no example with $\beta(Y) = n$ is known. If n is even, then $\beta(\mathbb{S}^n) = n-1$.

The following general result provides some additional information about the invariant $\beta(-)$.

Theorem 1.11 (cf. [**19**]). *A compact connected smooth manifold M has an algebraic model Y with $\beta(Y) = 0$ if and only if the fundamental group of M is nontrivial.*

As demonstrated heretofore, for nonsingular real algebraic varieties X and Y, the set $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$ only in exceptional cases. The approximation problem, motivated by the Weierstrass theorem, is that of finding a reasonable description of the closure $\mathcal{C}^\infty_{\mathcal{R}}(X, Y)$ of $\mathcal{R}(X, Y)$ in $\mathcal{C}^\infty(X, Y)$. The set $\mathcal{C}^\infty_{\mathcal{R}}(X, Y)$ tends to be very small if the target variety is sufficiently "general." Moreover, the approximation problem is hopelessly complicated without imposing some restrictions on Y. In the subsequent sections, we study $\mathcal{C}^\infty_{\mathcal{R}}(X, Y)$ assuming that X is compact and Y is rational. As a guiding principle serves the following:

Conjecture 1.12. *A smooth map from X into Y, where X is compact and Y is rational, can be approximated by regular maps if and only if it is homotopic to a regular map.*

This conjecture is known to be true in several cases. For example, when $Y = \mathbb{P}^n(\mathbb{R})$ (cf. Section 2), Y is a rational surface (cf. Section 6) or $Y = \mathbb{S}^4$ (cf. Section 7). Actually, in these three cases we have results describing $\mathcal{C}^\infty_{\mathcal{R}}(X, Y)$ in terms of certain cohomological invariants and being much more useful than Conjecture 1.12. In favorable situations these invariants can be explicitly computed. We pay special attention to maps with values in $\mathbb{S}^1 \cong \mathbb{P}^1(\mathbb{R})$ (cf. Section 3) or $\mathbb{S}^2$ (cf. Sections 4 and 5). Some results are surprising. For instance, up to biregular isomorphism, there exist exactly 18 unordered pairs $\{C, D\}$ of nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$, defined over $\mathbb{Q}$, such that the set $\mathcal{R}(C \times D, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(C \times D, \mathbb{S}^2)$ (cf. Section 5). Conjecture 1.12 remains open for $Y = \mathbb{S}^p$ with p different from 1, 2 and 4. There are, however, interesting results concerning homotopical properties of regular maps with values in $\mathbb{S}^p$ for any p (cf. Section 8).

Acknowledgements. We want to thank Professor Kamil Rusek for suggesting that we write this paper.

2. Real algebraic cycles and maps into $\mathbb{P}^n(\mathbb{R})$. Let X be a compact nonsingular real algebraic variety. Each d-dimensional algebraic subset A of X represents a homology class in $H_d(X; \mathbb{Z}/2)$. This can be seen in several different ways. For example, the pair (X, A) can be triangulated so that each $(d-1)$-simplex is the common face of an even number of d-simplices (cf. [**6**]). It follows that the sum of all d-simplices contained in A is a $\mathbb{Z}/2$-cycle. We say that the homology class in $H_d(X;\mathbb{Z}/2)$ represented by this cycle is represented by A. The subset $H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)$ of $H_d(X;\mathbb{Z}/2)$ consisting of all homology classes represented by d-dimensional algebraic subsets of X forms a subgroup. If $n = \dim X$, let $H^{n-d}_{\mathrm{alg}}(X;\mathbb{Z}/2)$ denote the inverse image of $H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)$ under the Poincaré duality isomorphism $H^{n-d}(X;\mathbb{Z}/2) \to H_d(X;\mathbb{Z}/2)$. The k-th Stiefel–Whitney class $w_k(X)$ of X is in $H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$ for all $k \geq 0$ (cf. [**21, 3**]). In particular, $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2) \neq 0$ if X is nonorientable. The groups $H_d^{\mathrm{alg}}(-;\mathbb{Z}/2)$ and $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ have the expected functorial property. If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then

$$f_*(H_d^{\mathrm{alg}}(X;\mathbb{Z}/2)) \subseteq H_d^{\mathrm{alg}}(Y;\mathbb{Z}/2) \text{ and } f^*(H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X;\mathbb{Z}/2),$$

where f_* and f^* are the homomorphisms of homology and cohomology groups induced by f (cf. [**21, 3**]).

Example 2.1. Clearly, $H^k_{\mathrm{alg}}(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2) = H^k(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2)$ for all $k \geq 0$ and $n \geq 1$.

Example 2.2 (Joost van Hamel). The algebraic surface

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + ((x^2+y^2)^2 - 1)((x^2+y^2)^2 - 2) = 0\}$$

in $\mathbb{R}^3$ is nonsingular and diffeomorphic to the standard torus $\mathbb{S}^1 \times \mathbb{S}^1$. In particular, $H^1(T;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$. We assert that $H^1_{\mathrm{alg}}(T;\mathbb{Z}/2) \cong \mathbb{Z}/2$. The assertion can be proved as follows. The algebraic curve

$$C = \{(t, z) \in \mathbb{R}^2 \mid z^2 + (t^2-1)(t^2-2) = 0\}$$

in $\mathbb{R}^2$ is nonsingular and irreducible. It has two connected components, each diffeomorphic to $\mathbb{S}^1$. Let C_+ denote the connected component of C contained in $\{(t, z) \in \mathbb{R}^2 \mid t > 0\}$. The map $e : C_+ \to T$ defined by $e(t, z) = (\sqrt{t}, 0, z)$ is a smooth embedding. Set $E = e(C_+)$ and let $[E]$ denote the homology class in $H_1(T;\mathbb{Z}/2)$ represented by the smooth curve E. If $\pi : T \to C$ is defined by $\pi(x, y, z) = (x^2+y^2, z)$, then $\pi(T) = C_+$ and the composite map $\pi \circ e : C_+ \to C$ is the inclusion map $C_+ \hookrightarrow C$. It follows that the homology class $\pi_*([E])$ in $H_1(C;\mathbb{Z}/2)$ is represented by the smooth curve C_+. In particular, $\pi_*([E])$ is not in $H_1^{\mathrm{alg}}(C;\mathbb{Z}/2)$, the curve C being irreducible. Since π is a regular map, the functoriality of $H_1^{\mathrm{alg}}(-;\mathbb{Z}/2)$ implies that the homology class $[E]$ is not in $H_1^{\mathrm{alg}}(T;\mathbb{Z}/2)$. On the other hand, the algebraic curve $D = \mathbb{S}^1 \times \{0\}$ is contained in T, and hence its homology class [D] belongs to $H_1^{\mathrm{alg}}(T;\mathbb{Z}/2)$. Clearly, $[D] \neq 0$ since D is transverse to E and the intersection $D \cap E$ consist of one point. Consequently, $H_1^{\mathrm{alg}}(T;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and $H^1_{\mathrm{alg}}(T;\mathbb{Z}/2) \cong \mathbb{Z}/2$, as asserted.

Of course, T is not biregularly isomorphic to the standard torus $\mathbb{S}^1 \times \mathbb{S}^1$ for which $H^1_{\mathrm{alg}}(\mathbb{S}^1 \times \mathbb{S}^1;\mathbb{Z}/2) = H^1(\mathbb{S}^1 \times \mathbb{S}^1;\mathbb{Z}/2)$.

The group $H^1_{\mathrm{alg}}(-;\mathbb{Z}/2)$ is of particular interest to us. We have the following general result.

THEOREM 2.3. *Let M be a compact connected smooth manifold of dimension at least 2. For a subgroup G of $H^1(M;\mathbb{Z}/2)$, the following conditions are equivalent:*

(a) *There exist an algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ satisfying*
$$\varphi^*(G) = H^1_{\mathrm{alg}}(X;\mathbb{Z}/2).$$
(b) *The first Stiefel–Whitney class $w_1(M)$ is in G.*

Obviously, (a) implies (b) since $w_1(X)$ is in $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$. The proof of the converse is too long to be included here (cf. [**10, 34**]). Theorem 2.3 gives a complete description of the subgroups of $H^1(M;\mathbb{Z}/2)$ that can be realized as $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$ for some algebraic model X of M. In particular, there is no restriction on the subgroups if M is orientable. It should be mentioned that the behavior of the group $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ for $k \geq 2$ is more complicated. Given a positive even integer k, one can find a compact connected smooth manifold M with $H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \neq H^k(X;\mathbb{Z}/2)$ for every algebraic model X of M (cf. [**2, 34, 47**]). The closure of $\mathcal{R}(-,\mathbb{P}^n(\mathbb{R}))$ in $\mathcal{C}^\infty(-,\mathbb{P}^n(\mathbb{R}))$ can be described in terms of $H^1_{\mathrm{alg}}(-;\mathbb{Z}/2)$.

THEOREM 2.4. *Let X be a compact nonsingular real algebraic variety and let n be a positive integer. For a smooth map $f : X \to \mathbb{P}^n(\mathbb{R})$, the following conditions are equivalent:*

(a) *f can be approximated by regular maps.*
(b) *f is homotopic to a regular map.*
(c) *$f^*(u_n)$ is in $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$, where u_n is the unique generator of the group $H^1(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2$.*

SKETCH OF PROOF. Obviously, (a) implies (b). The functoriality of $H^1_{\mathrm{alg}}(-;\mathbb{Z}/2)$ implies that $g^*(u_n)$ is in $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$ for every regular map $g : X \to \mathbb{P}^n(\mathbb{R})$. If f is homotopic to g, then $f^* = g^*$, and hence (c) follows from (b). In the proof that (c) implies (a), we use theory of algebraic vector bundles on real algebraic varieties, developed in [**6**, Chapter 12]. First we claim that every smooth section of an algebraic vector bundle η on X can be approximated by regular sections. Indeed, η is an algebraic vector subbundle of the trivial vector bundle $\varepsilon_X^k = X \times \mathbb{R}^k$ for some k. However, then this is just the Weierstrass approximation theorem.

Now, let γ_n be the universal line bundle on $\mathbb{P}^n(\mathbb{R})$. The total space of γ_n is

$$E(\gamma_n) = \{(l, e) \in \mathbb{P}^n(\mathbb{R}) \times \mathbb{R}^{n+1} \mid e \in l\},$$

where $\mathbb{P}^n(\mathbb{R})$ is regarded as the set of all 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. In particular, γ_n is an algebraic subbundle of the trivial vector bundle on $\mathbb{P}^n(\mathbb{R})$ with fiber $\mathbb{R}^{n+1}$. We have $f^*(u_n) = w_1(f^*\gamma_n)$, where $w_1(-)$ stands for the first Stiefel–Whitney class and $f^*\gamma_n$ is the pullback of γ_n by f. According to (c), $w_1(f^*\gamma_n)$ is in $H^1_{\mathrm{alg}}(X; \mathbb{Z}/2)$, which implies the existence of an algebraic line bundle ξ on X and a smooth vector bundle isomorphism $\varphi : \xi \to f^*\gamma_n$ (cf. [**6**, Theorem 12.4.8]). Let $\sigma : X \to \mathrm{Hom}(\xi, f^*\gamma_n)$ be the smooth section of the vector bundle $\mathrm{Hom}(\xi, f^*\gamma_n)$ defined by $\sigma(x)(w) = \varphi(w)$ for all x in X and w in the fiber ξ_x of ξ at x. Since $f^*\gamma_n$ is a smooth subbundle of ε_X^{n+1}, we can regard σ as a smooth section of the algebraic vector bundle $\mathrm{Hom}(\xi, \varepsilon_X^{n+1})$. Hence, there exists a regular section $s : X \to \mathrm{Hom}(\xi, \varepsilon_X^{n+1})$ close to σ in the $\mathcal{C}^\infty$ topology. We can assume that the linear map $s(x) : \xi_x \to (\varepsilon_X^{n+1})_x = \{x\} \times \mathbb{R}^{n+1}$, is injective for all x in X, the variety X being compact. Define $g : X \to \mathbb{P}^n(\mathbb{R})$ by $g(x) = \rho(s(x)(\xi_x))$, where $\rho : X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the canonical projection. It is readily seen that the map g is regular (cf. [**6**, Proposition 3.4.9]), while the construction of g implies that it is close to f in the $\mathcal{C}^\infty$ topology. □

COROLLARY 2.5. *Any compact smooth manifold M has an algebraic model X such that for each positive integer n, the set $\mathcal{R}(X, \mathbb{P}^n(\mathbb{R}))$ is dense in $\mathcal{C}^\infty(X, \mathbb{P}^n(\mathbb{R}))$.*

PROOF. By Theorem 2.3, there exists an algebraic model X of M with $H^1_{\mathrm{alg}}(X; \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$, and hence it suffices to make use of Theorem 2.4. □

COROLLARY 2.6. *Let X be a compact nonsingular real algebraic curve. Then the set $\mathcal{R}(X, \mathbb{P}^n(\mathbb{R}))$ is dense in $\mathcal{C}^\infty(X, \mathbb{P}^n(\mathbb{R}))$ for all $n \geq 1$. In particular, $\mathcal{R}(X, \mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(X, \mathbb{S}^1)$.*

PROOF. It suffices to apply Theorem 2.4 since $H^1_{\mathrm{alg}}(X; \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$. □

COROLLARY 2.7. *The set $\mathcal{R}(\mathbb{S}^n, \mathbb{P}^k(\mathbb{R}))$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{P}^k(\mathbb{R}))$ for all $n \geq 1$ and $k \geq 1$. In particular, $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^1)$ for all $n \geq 1$.*

PROOF. The assertion follows from Theorem 2.4. □

3. Regular maps into $\mathbb{S}^1$. Let X be a compact nonsingular real algebraic variety. Recall that $\mathcal{C}^\infty_\mathcal{R}(X,\mathbb{S}^1)$ denotes the closure of $\mathcal{R}(X,\mathbb{S}^1)$ in $\mathcal{C}^\infty(X,\mathbb{S}^1)$. It is convenient to regard $\mathbb{S}^1$ as a multiplicative subgroup of $\mathbb{C}\setminus\{0\}$ and endow $\mathcal{C}^\infty(X,\mathbb{S}^1)$ with the induced group structure. Obviously, $\mathcal{R}(X,\mathbb{S}^1)$ and $\mathcal{C}^\infty_\mathcal{R}(X,\mathbb{S}^1)$ are subgroups of $\mathcal{C}^\infty(X,\mathbb{S}^1)$. In this section we investigate the quotient group

$$\Gamma(X) = \mathcal{C}^\infty(X,\mathbb{S}^1)/\mathcal{C}^\infty_\mathcal{R}(X,\mathbb{S}^1).$$

The set $\mathcal{R}(X,\mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(X,\mathbb{S}^1)$ if and only if $\Gamma(X)=0$.

For a smooth manifold M, let $A(M)$ denote the image of the canonical monomorphism $r: H^1(M,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}/2 \to H^1(M;\mathbb{Z}/2)$.

Theorem 3.1. *For any compact nonsingular real algebraic variety X, the group $\Gamma(X)$ is canonically isomorphic to $A(X)/(A(X)\cap H^1_{\mathrm{alg}}(X;\mathbb{Z}/2))$.*

Proof. Consider the homomorphism

$$\psi: \mathcal{C}^\infty(X,\mathbb{S}^1)\to H^1(X;\mathbb{Z}/2),\ \psi(f)=f^*(u),$$

where u is the unique generator of $H^1(\mathbb{S}^1;\mathbb{Z}/2)\cong\mathbb{Z}/2$. We claim that $\psi(\mathcal{C}^\infty(X,\mathbb{S}^1)) = A(X)$. Indeed, the following diagram of group homomorphisms

$$\begin{tikzcd}
\mathcal{C}^\infty(X,\mathbb{S}^1) \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}/2 \arrow[dl, "r"] \\
& H^1(X;\mathbb{Z}/2) &
\end{tikzcd}$$

is commutative, where φ is defined analogously to ψ, by replacing the generator u of $H^1(\mathbb{S}^1;\mathbb{Z}/2)$ with a generator of $H^1(\mathbb{S}^1,\mathbb{Z})\cong\mathbb{Z}$. Since φ is surjective, the claim follows. By Theorem 2.4, a smooth map $f: X\to\mathbb{S}^1$ is in $\mathcal{C}^\infty_\mathcal{R}(X,\mathbb{S}^1)$ if and only if $f^*(u)$ is in $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$. Therefore,

$$\psi^{-1}(A(X)\cap H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)) = \mathcal{C}^\infty_\mathcal{R}(X,\mathbb{S}^1),$$

and the homomorphism ψ induces an isomorphism

$$\Gamma(X)\to A(X)/(A(X)\cap H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)).$$

□

More information about the group $\Gamma(-)$ is contained in the following result.

Theorem 3.2. *Let M be a compact connected smooth manifold of dimension at least 2 and let*

$$\alpha(M) = \begin{cases} \operatorname{rank} H^1(M;\mathbb{Z}) - 1 & \textit{if } M \textit{ is nonorientable and } w_1(M)\in A(M), \\ \operatorname{rank} H^1(M;\mathbb{Z}) & \textit{otherwise.}\end{cases}$$

Then:

(i) *For each algebraic model X of M, one has $\Gamma(X) \cong (\mathbb{Z}/2)^s$ for some integer s satisfying $0 \leq s \leq \alpha(M)$.*

(ii) *For each integer s satisfying $0 \leq s \leq \alpha(M)$, there exists an algebraic model X of M with $\Gamma(X) \cong (\mathbb{Z}/2)^s$.*

PROOF. (i) Let X be an algebraic model of M. We have $\dim_{\mathbb{Z}/2} A(X) = \rho(M)$, where $\rho(M) = \operatorname{rank} H^1(M;\mathbb{Z})$. According to Theorem 3.1, $\Gamma(X) \cong (\mathbb{Z}/2)^s$ for some s satisfying $0 \leq s \leq \rho(M)$. Moreover, if M is nonorientable and $w_1(M)$ is in $A(M)$, then $0 \leq s \leq \rho(M) - 1$ since $w_1(X)$ belongs to $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$.

(ii) It suffices to make use of Theorems 2.3 and 3.1. □

Informally, Theorem 3.2 can be interpreted as saying that $1/2^s$ of the space $\mathcal{C}^\infty(X,\mathbb{S}^1)$ consists of the maps approximable by regular maps.

COROLLARY 3.3. *For a compact connected orientable smooth manifold M, the following conditions are equivalent:*

(a) *The set $\mathcal{R}(X,\mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(X,\mathbb{S}^1)$ for every algebraic model X of M.*

(b) *The first Betti number of M is zero or $\dim M \leq 1$.*

PROOF. The equivalence of (a) and (b) follows from Theorem 3.2 (resp. Corollary 2.6) if $\dim M \geq 2$ (resp. $\dim M = 1$). The case $\dim M = 0$ is trivial. □

THEOREM 3.4. *For a compact connected smooth surface M, the following conditions are equivalent:*

(a) *The set $\mathcal{R}(X,\mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(X,\mathbb{S}^1)$ for every algebraic model X of M.*

(b) *M is homeomorphic to the unit 2-sphere or the real projective plane or the Klein bottle.*

PROOF. The equivalence of (a) and (b) follows from Theorem 3.2 since $\alpha(M) = 0$ if and only if M is homeomorphic to one of the three surfaces named in (b). □

We next examine the group $\Gamma(X)$ for a compact connected nonsingular real algebraic surface X of topological genus g. Note that $A(X) = H^1(X;\mathbb{Z}/2)$ for X orientable and $A(X) = \{v \in H^1(X;\mathbb{Z}/2) \mid v \cup v = 0\}$ for X nonorientable, where $\cup$ stands the cup product. The Stiefel–Whitney class of $w_1(X)$ is not in $A(X)$ if and only if X is nonorientable of odd genus. Denoting by $\gamma(X)$ (resp. $\delta(X)$) the dimension of the $\mathbb{Z}/2$-vector space $H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$ (resp. $A(X) \cap H^1_{\mathrm{alg}}(X;\mathbb{Z}/2)$), in view of Theorem 3.2, we get

$$\Gamma(X) \cong \begin{cases} (\mathbb{Z}/2)^{2g-\gamma(X)} & \text{if } X \text{ is orientable,} \\ (\mathbb{Z}/2)^{g-\gamma(X)} & \text{if } X \text{ is nonorientable and } g \text{ is odd,} \\ (\mathbb{Z}/2)^{g-\delta(X)-1} & \text{if } X \text{ is nonorientable and } g \text{ is even.} \end{cases}$$

In particular, assuming that X is either orientable or nonorientable of odd genus, the set $\mathcal{R}(X, \mathbb{S}^1)$ is dense in $\mathcal{C}^\infty(X, \mathbb{S}^1)$ if and only if $H^1_{\mathrm{alg}}(X; \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$.

4. Regular maps into $\mathbb{S}^2$. We begin by introducing a new cohomological invariant, useful in the study of regular maps with values in $\mathbb{S}^{2k}$ for $k \geq 1$. In particular, it allows us to give a characterization of the smooth maps into $\mathbb{S}^2$ that are approximable by regular maps.

Let X be a compact nonsingular real algebraic variety. A *nonsingular projective complexification* of X is a pair (V, j), where V is a nonsingular complex projective variety defined over $\mathbb{R}$ and $j : X \to V$ is an injective map such that the set $V(\mathbb{R})$ of real points of V is Zariski dense in V, $j(X) = V(\mathbb{R})$ and j induces a biregular isomorphism between X and $V(\mathbb{R})$. The existence of (V, j) follows from Hironaka's resolution of singularities [**28**] (V is uniquely determined, up to isomorphism over $\mathbb{R}$, only if $\dim X \leq 1$). For each nonnegative integer k, let $H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z})$ denote the subgroup of $H^{2k}(X; \mathbb{Z})$ defined by

$$H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) = j^*(H^{2k}_{\mathrm{alg}}(V; \mathbb{Z})),$$

where $H^{2k}_{\mathrm{alg}}(V; \mathbb{Z})$ is the subgroup of $H^{2k}(V; \mathbb{Z})$ generated by the cohomology classes corresponding to irreducible algebraic subvarieties of V of (complex) codimension k. One readily shows that $H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z})$ does not depend on the choice of (V, j) (cf. [**5**]). The direct sum of the groups $H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z})$ for $k \geq 0$ is closed under cup product (since the analogous statement for the groups $H^{2k}_{\mathrm{alg}}(V; \mathbb{Z})$ holds true; cf. [**27**]).

For every nonnegative integer i,

$$p_i(X) \in H^{4i}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) \quad \text{and} \quad \beta(w_{2i}(X) \cup w_{2i+1}(X)) \in H^{4i+2}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}),$$

where $p_i(X)$ is the ith Pontryagin class of X and

$$\beta : H^{4i+1}(X; \mathbb{Z}/2) \to H^{4i+2}(X; \mathbb{Z})$$

is the Bockstein homomorphism. Indeed, let τ_X be the tangent bundle to X. The k-th Chern class $c_k(\tau_X \otimes_{\mathbb{R}} \mathbb{C})$ of the complexification of τ_X and the k-th Chern class $c_k(V)$ of V are related by $c_k(\tau_X \otimes_{\mathbb{R}} \mathbb{C}) = j^*(c_k(V))$. Since $c_k(V)$ is in $H^{2k}_{\mathrm{alg}}(V; \mathbb{Z})$, it follows that $c_k(\tau_X \otimes_{\mathbb{R}} \mathbb{C})$ is in $H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z})$. It suffices to recall that $c_{2i}(\tau_X \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^i p_i(X)$ and $c_{2i+1}(\tau_X \otimes_{\mathbb{R}} \mathbb{C}) = \beta(w_{2i}(X) \cup w_{2i+1}(X))$ (cf. [**40**]). In particular,

$$\beta(w_1(X)) \in H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}).$$

The group $H^{2k}_{\mathbb{C}\text{-alg}}(-; \mathbb{Z})$ has the functorial property (cf. [**5**]). If $f : X \to Y$ is a regular map between compact nonsingular real algebraic varieties, then

$$f^*(H^{2k}_{\mathbb{C}\text{-alg}}(Y; \mathbb{Z})) \subseteq H^{2k}_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}).$$

EXAMPLE 4.1.

(i) If $j : \mathbb{P}^n(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{C})$ is the inclusion map, then $(\mathbb{P}^n(\mathbb{C}), j)$ is a nonsingular projective complexification of $\mathbb{P}^n(\mathbb{R})$, and hence $H^{2k}_{\mathbb{C}\text{-alg}}(\mathbb{P}^n(\mathbb{R}); \mathbb{Z}) = H^{2k}(\mathbb{P}^n(\mathbb{R}); \mathbb{Z})$ for all $k \geq 0$.

(ii) Let Σ^n_{2d} be the n-dimensional Fermat sphere of degree $2d$,

$$\Sigma^n_{2d} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^{2d} + \ldots + x_{n+1}^{2d} = 1\}.$$

We claim that $H^{2k}_{\mathbb{C}\text{-alg}}(\Sigma^n_{2d}; \mathbb{Z}) = H^{2k}(\Sigma^n_{2d}; \mathbb{Z})$ for all $d \geq 1$, $k \geq 0$ and $n \geq 1$. Since Σ^n_{2d} is diffeomorphic to $\mathbb{S}^n$, the only nontrivial case is $n = 2k$ with $k \geq 1$. Set $\Sigma = \Sigma^{2k}_{2d}$ and define

$$V = \{(u_1 : v_1 : \ldots : u_k : v_k : s : t) \in \mathbb{P}^{2k+1}(\mathbb{C}) \mid u_1^{2d} + v_1^{2d} + \ldots + u_k^{2d} + v_k^{2d} + s^{2d} = t^{2d}\}$$

$$j : \Sigma \to V,\ j(u_1, v_1, \ldots, u_k, v_k, s) = (u_1 : v_1 : \ldots : u_k : v_k : s : 1).$$

Then (V, j) is a nonsingular projective complexification of Σ. Let α be a complex number satisfying $\alpha^{2d} = -1$ and let

$$L = \{(u_1 : v_1 : \ldots : u_k : v_k : s : t) \in \mathbb{P}^{2k+1}(\mathbb{C}) \mid u_l = v_l \text{ for } 1 \leq l \leq k \text{ and } s = t\}.$$

By construction, L is a nonsingular complex subvariety of V of codimension k, $j : \Sigma \to V$ is tranverse to L, and $j^{-1}(L) = \{(0, \ldots, 0, 1)\}$. If λ is the cohomology class in $H^{2k}_{\text{alg}}(V; \mathbb{Z})$ represented by L, then $j^*(\lambda)$ is a generator of $H^{2k}(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$ (cf. [**21**, Proposition 2.15]). The claim follows since $j^*(\lambda)$ is in $H^{2k}_{\mathbb{C}\text{-alg}}(\Sigma; \mathbb{Z})$. In particular, $H^{2k}_{\mathbb{C}\text{-alg}}(\mathbb{S}^n; \mathbb{Z}) = H^{2k}(\mathbb{S}^n; \mathbb{Z})$ for all $k \geq 0$ and $n \geq 1$.

(iii) A compact nonsingular irreducible real algebraic curve C is said to be *dividing* if the set $V \setminus j(C)$ is disconnected, where (V, j) is a nonsingular projective complexification of C (V is uniquely determined up to isomorphism over $\mathbb{R}$ since $\dim C = 1$). If C is dividing and connected, then $H^2_{\mathbb{C}\text{-alg}}(C \times D; \mathbb{Z}) = 0$ for every compact nonsingular real algebraic curve D. Indeed, $j^*(H^q(V; \mathbb{Z})) = 0$ for all $q \geq 1$. Moreover, if (W, l) is a nonsingular projective complexification of D, then $(V \times W, j \times l)$ is that of $C \times D$. The assertion follows since $(j \times l)^*(H^2(V \times W; \mathbb{Z})) = 0$ by virtue of Künneth's theorem. In particular, $H^2_{\mathbb{C}\text{-alg}}(\mathbb{S}^1 \times D; \mathbb{Z}) = 0$ since $\mathbb{S}^1$ is a dividing curve.

EXAMPLE 4.2. For each nonsingular complex projective variety W, we have $H^{2k}_{\mathrm{alg}}(W;\mathbb{Z}) \subseteq H^{2k}_{\mathbb{C}\text{-alg}}(W_{\mathbb{R}};\mathbb{Z})$ for all $k \geq 0$. This assertion can be proved as follows. Suppose that W is a subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $\sigma : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ be complex conjugation and let $\overline{W} = \sigma(W)$. The map $\tau : W \times \overline{W} \to W \times \overline{W}$, $\tau(x, y) = (\sigma(y), \sigma(x))$ is an antiholomorphic involution. Thus $W \times \overline{W}$ can be regarded as a complex projective variety defined over $\mathbb{R}$, whose set of real points consists of the points fixed by τ. If $j : W_{\mathbb{R}} \to W \times \overline{W}$ is defined by $j(x) = (x, \sigma(x))$, then $(W \times \overline{W}, j)$ is a nonsingular projective complexification of $W_{\mathbb{R}}$. The assertion follows since the composition of j with the canonical projection $\pi : W \times \overline{W} \to W$ is the identity map of $W_{\mathbb{R}}$, and $\pi^*(H^{2k}_{\mathrm{alg}}(W;\mathbb{Z})) \subseteq H^{2k}_{\mathrm{alg}}(W \times \overline{W};\mathbb{Z})$. In particular, $H^2_{\mathbb{C}\text{-alg}}(W_{\mathbb{R}};\mathbb{Z}) = H^2(W_{\mathbb{R}};\mathbb{Z})$ if $\dim_{\mathbb{C}} W = 1$. Furthermore, $H^{2k}_{\mathbb{C}\text{-alg}}(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}};\mathbb{Z}) = H^{2k}(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}};\mathbb{Z})$ for all $k \geq 0$ and $n \geq 1$.

The next two results can be viewed as a partial counterpart of Theorem 2.3 for the group $H^2_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$.

THEOREM 4.3. *Any compact smooth manifold M has an algebraic model X with $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = H^2(X;\mathbb{Z})$.*

For the proof we refer to [**13**]. Theorem 4.3 cannot be generalized on the groups $H^{2k}_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ for $k \geq 2$. There exists a compact connected smooth manifold M such that $H^4_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) \neq H^4(X;\mathbb{Z})$ for every algebraic model X of M (cf. [**35**]).

THEOREM 4.4. *Let M be a compact connected smooth surface.*

(i) *If M is nonorientable of odd genus, then $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = H^2(X;\mathbb{Z})$ for every algebraic model X of M.*
(ii) *If M is nonorientable of even genus, then there exist two algebraic models X_0 and X_1 of M satisfying $H^2_{\mathbb{C}\text{-alg}}(X_0;\mathbb{Z}) = 0$ and $H^2_{\mathbb{C}\text{-alg}}(X_1;\mathbb{Z}) = H^2(X_1;\mathbb{Z})$.*
(iii) *If M is orientable, then for each integer b, there exists an algebraic model X_b of M with $H^2_{\mathbb{C}\text{-alg}}(X_b;\mathbb{Z}) = bH^2(X_b;\mathbb{Z})$.*

Depending on whether the surface M is orientable or not, $H^2(M;\mathbb{Z}) = \mathbb{Z}$ or $H^2(M;\mathbb{Z}) = \mathbb{Z}/2$. Thus Theorem 4.4 gives a complete description of the subgroups of $H^2(M;\mathbb{Z})$ that can be realized as $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ for some algebraic model X of M. If M is nonorientable of odd genus, then $\beta(w_1(M))$ generates $H^2(M;\mathbb{Z})$ and hence (i) follows. The proofs of (ii) and (iii) are significantly harder (cf. [**9**] and [**14**], respectively).

The group $H^2_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ is of interest to us since it can be used to describe the closure of $\mathcal{R}(-,\mathbb{S}^2)$ in $\mathcal{C}^\infty(-,\mathbb{S}^2)$.

THEOREM 4.5. *Let X be a compact nonsingular real algebraic variety. For a smooth map $f : X \to \mathbb{S}^2$, the following conditions are equivalent:*

(a) *f can be approximated by regular maps.*
(b) *f is homotopic to a regular map.*
(c) *$f^*(s_2)$ is in $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$, where s_2 is a generator of $H^2(\mathbb{S}^2;\mathbb{Z}) \cong \mathbb{Z}$.*

SKETCH OF PROOF. Recall that $\mathbb{S}^2$ is biregularly isomorphic to $\mathbb{P}^1(\mathbb{C})_{\mathbb{R}}$. The analogous, more general, theorem can be formulated for maps with values in $\mathbb{P}^n(\mathbb{C})_{\mathbb{R}}$, where s_2 in condition (c) is a generator of $H^2(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}};\mathbb{Z})$. The proof is then parallel to that of Theorem 2.4. One works with algebraic $\mathbb{C}$-line bundles (on real algebraic varieties) instead of real algebraic line bundles, substitutes $H^2_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ for $H^1_{\text{alg}}(-;\mathbb{Z}/2)$, and makes use of the functoriality of $H^2_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ and the equality $H^2_{\mathbb{C}\text{-alg}}(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}};\mathbb{Z}) = H^2(\mathbb{P}^n(\mathbb{C})_{\mathbb{R}};\mathbb{Z})$. More details can be found in [**5, 9, 14**]. □

Several interesting results concerning regular maps into $\mathbb{S}^2$ can now be derived.

THEOREM 4.6. *Any compact smooth manifold M has an algebraic model X such that the set $\mathcal{R}(X;\mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(X;\mathbb{S}^2)$.*

PROOF. It suffices to combine Theorems 4.3 and 4.5. □

THEOREM 4.7. *For a compact connected smooth surface M, the following conditions are equivalent:*

(a) *The set $\mathcal{R}(X;\mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(X;\mathbb{S}^2)$ for every algebraic model X of M.*
(b) *M is nonorientable of odd genus.*

PROOF. It suffices to make use of Theorems 4.4 and 4.5. □

Let X be a compact nonsingular real algebraic surface. If X is connected and orientable, then

$$H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = b(X)H^2(X;\mathbb{Z})$$

for some uniquely determined nonnegative integer $b(X)$. According to Theorem 4.5 and Hopf's theorem,

$$\mathcal{C}^\infty_{\mathcal{R}}(X,\mathbb{S}^2) = \{f \in C^\infty(X,\mathbb{S}^2) \mid \deg(f) \in b(X)\mathbb{Z}\},$$

where $\deg(f)$ denotes the topological degree of f (computed with respect to some fixed orientations on X and on $\mathbb{S}^2$). Thus the description of $\mathcal{C}^\infty_{\mathcal{R}}(X,\mathbb{S}^2)$ is reduced to the computation of the numerical invariant $b(X)$. In particular, the set $\mathcal{R}(X,\mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(X,\mathbb{S}^2)$ (resp. $\mathcal{R}(X,\mathbb{S}^2)$ contains only null homotopic maps) if and only if $b(X) = 1$ (resp. $b(X) = 0$). The invariant $b(-)$ can take any nonnegative integer as its value. By Theorem 4.4 (iii), if M is a compact connected orientable smooth surface, then for each nonnegative integer b, there exists an algebraic model X_b of M with $b(X_b) = b$. The computation of $b(X)$ for a given surface X is a very subtle problem (cf. Section 5).

EXAMPLE 4.8. It would be interesting to complete the computation of $b(F_k \times F_k)$, where F_k is the Fermat curve of degree k (cf. Example 1.8 for the definition of F_k). By Example 4.1 (iii), $b(F_2 \times F_2) = 0$ since F_2 is isomorphic to $\mathbb{S}^1$. If $k \geq 3$, then according to [**14**, Example 1.14], $b(F_k \times F_k) = 1$ for k odd and $1 \leq b(F_k \times F_k) \leq 2$ for k even. A more delicate argument in [**20**, Example 17] allows to prove the equality $b(F_4 \times F_4) = 2$. We conjecture that $b(F_k \times F_k) = 2$ for k even and greater than 4.

THEOREM 4.9. *Let C be a compact nonsingular real algebraic curve. If C is dividing and connected, then for every compact nonsingular real algebraic curve D, the set $\mathcal{R}(C \times D, \mathbb{S}^2)$ contains only null homotopic maps.*

PROOF. The assertion follows from Example 4.1 (iii), Theorem 4.5 and Hopf's theorem. □

The assumption in Theorem 4.9 that the curve C be dividing is necessary. If C is connected and nondividing, then one can find a compact connected nonsingular real algebraic curve D such that $b(C \times D) = 2$ (cf. [**15**]; it remains an open problem if there exists a curve D with $b(C \times D) = 1$). There are also connected nondividing curves C for which the set $\mathcal{R}(C \times C, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(C \times C, \mathbb{S}^2)$ (cf. Section 5). Such curves are exceptional. For "most" nondividing curves C, the set $\mathcal{R}(C \times C, \mathbb{S}^2)$ contains only null homotopic maps. The meaning of "most" can be made precise by introducing moduli spaces of real algebraic curves (cf. [**20**] and, for cubic curves, Section 5).

THEOREM 4.10. *For any nonsingular complex projective curve W, the set $\mathcal{R}(W_\mathbb{R}, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(W_\mathbb{R}, \mathbb{S}^2)$.*

PROOF. The assertion follows from Example 4.2 and Theorem 4.5. □

Curiously, the set $\mathcal{R}(W_\mathbb{R} \times W_\mathbb{R}, \mathbb{S}^2)$ is never dense in $\mathcal{R}(W_\mathbb{R} \times W_\mathbb{R}, \mathbb{S}^2)$, unless W has genus 0 (cf. [**18**]).

The approximation problem has a complete solution for maps from the Fermat sphere Σ^n_{2d} into $\mathbb{S}^2$ (cf. Example 4.1 (ii) for the definition of Σ^n_{2d}).

THEOREM 4.11. *The set $\mathcal{R}(\Sigma^n_{2d}, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(\Sigma^n_{2d}, \mathbb{S}^2)$ for all $d \geq 1$ and $n \geq 1$. In particular, $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^2)$ for all $n \geq 1$.*

PROOF. It suffices to make use of Example 4.1 (ii) and Theorem 4.5. □

We next investigate maps from nonsingular algebraic surfaces in $\mathbb{P}^3(\mathbb{R})$ into $\mathbb{S}^2$. Such surfaces are subject to certain topological restrictions since each compact smooth surface in $\mathbb{P}^3(\mathbb{R})$ has at most one nonorientable connected component, which is necessarily of odd genus.

THEOREM 4.12. *Let $V \subset \mathbb{P}^3(\mathbb{C})$ be a nonsingular surface defined by a homogenous polynomial of degree at least 4, with all coefficients real and algebraically independent over $\mathbb{Q}$. The set $X = V(\mathbb{R})$ of real points of V is either empty or it is a nonsingular irreducible real algebraic surface in $\mathbb{P}^3(\mathbb{R})$ with the following properties:*

(i) *If X is orientable, then the set $\mathcal{C}^\infty_{\mathcal{R}}(X, \mathbb{S}^2)$ consists of the null homotopic smooth maps.*

(ii) *If X is nonorientable and X_0 is its unique nonorientable connected component, then the set $\mathcal{C}^\infty_{\mathcal{R}}(X, \mathbb{S}^2)$ consists of the smooth maps $f : X \to \mathbb{S}^2$ such that the restriction $f|_{X\setminus X_0} : X \setminus X_0 \to \mathbb{S}^2$ is null homotopic (the last condition is understood to be vacuous whenever $X = X_0$).*

In particular, assuming $X \neq \emptyset$, the set $\mathcal{R}(X; \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(X; \mathbb{S}^2)$ if and only if X is nonorientable and connected.

SKETCH OF PROOF. By the Lefschetz theorem, V is connected and hence irreducible. Consequently, if X is nonempty, then it is a nonsingular irreducible surface in $\mathbb{P}^3(\mathbb{R})$.

Let $\rho : \mathrm{Pic}(\mathbb{P}^3(\mathbb{C})) \to \mathrm{Pic}(V)$ be the homomorphism of the Picard groups induced by the inclusion map $V \hookrightarrow \mathbb{P}^3(\mathbb{C})$. If $e : X \hookrightarrow \mathbb{P}^3(\mathbb{R})$, $j : X \hookrightarrow V$ and $l : \mathbb{P}^3(\mathbb{R}) \hookrightarrow \mathbb{P}^3(\mathbb{C})$ are the inclusion maps, then the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Pic}(\mathbb{P}^3(\mathbb{C})) \arrow[r, "\rho"] \arrow[d, "c_1"'] & \mathrm{Pic}(V) \arrow[d, "c_1"'] \\
H^2(\mathbb{P}^3(\mathbb{C}); \mathbb{Z}) \arrow[d, "l^*"'] & H^2(V; \mathbb{Z}) \arrow[d, "j^*"'] \\
H^2(\mathbb{P}^3(\mathbb{R}); \mathbb{Z}) \arrow[r, "e^*"] & H^2(X; \mathbb{Z}),
\end{tikzcd}$$

where c_1 is the first Chern class homomorphism. It is well known that ρ is an isomorphism (cf. [**24**]). Note that $(\mathbb{P}^3(\mathbb{C}), l)$ and (V, j) are nonsingular projective complexifications of $\mathbb{P}^3(\mathbb{R})$ and X, respectively. Therefore,

$$\mathrm{Im}\ l^* = H^2_{\mathbb{C}\text{-alg}}(\mathbb{P}^3(\mathbb{R}); \mathbb{Z}) = H^2(\mathbb{P}^3(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}/2 \ \text{ and } \ \mathrm{Im}\ j^* = H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}).$$

Since $c_1(\mathrm{Pic}(-)) = H^2_{\mathrm{alg}}(-; \mathbb{Z})$, it follows that

$$H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) = \mathrm{Im}\ e^*.$$

Thus either $H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) = 0$ or $H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) \cong \mathbb{Z}/2$.

If X is orientable, then $H^2(X;\mathbb{Z})$ is a nontrivial free abelian group, and hence $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = 0$. If X is nonorientable, then the smooth surface X_0 has odd genus, and hence $\beta(w_1(X)) \neq 0$. Since $\beta(w_1(X))$ belongs to $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$, it follows that $H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = \mathbb{Z}\beta(w_1(X))$.

Thus both (i) and (ii) follow from Theorem 4.5 and Hopf's theorem. The last assertion is obvious. □

Let Ω_k be the vector space of real homogenous polynomials of degree k in 4 variables. The subset of Ω_k consisting of the polynomials whose coefficients are algebraically dependent over $\mathbb{Q}$ is a countable union of proper algebraic subsets. Thus Theorem 4.12 holds for most surfaces V of degree at least 4. Some assumption on the coefficients of the polynomial defining V is necessary. Indeed, consider the Fermat sphere Σ^2_{2d} as a surface in $\mathbb{P}^3(\mathbb{R})$. The Zariski closure of Σ^2_{2d} in $\mathbb{P}^3(\mathbb{C})$ is nonsingular. Moreover, Σ^2_{2d} is orientable and, according to Theorem 4.12, the set $\mathcal{R}(\Sigma^2_{2d}, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(\Sigma^2_{2d}, \mathbb{S}^2)$.

Theorem 4.13. *Let X be a nonsingular algebraic subset of $\mathbb{P}^n(\mathbb{R})$ with $n+2 \leq 2\dim X$. Assume that the Zariski closure of X in $\mathbb{P}^n(\mathbb{C})$ is nonsingular. Then X is irreducible and*

$$\mathcal{C}^\infty_{\mathcal{R}}(X, \mathbb{S}^2) = \{f \in \mathcal{C}^\infty(X, \mathbb{S}^2) \mid f^*(s_2) \in e^*(H^2(\mathbb{P}^n(\mathbb{R});\mathbb{Z}))\},$$

where s_2 is a generator of $H^2(\mathbb{S}^2;\mathbb{Z})$ and $e : X \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map.

Sketch of proof. Let V be the Zariski closure of X in $\mathbb{P}^n(\mathbb{C})$ and let $\rho : \mathrm{Pic}(\mathbb{P}^n(\mathbb{C})) \to \mathrm{Pic}(V)$ be the homomorphism induced by the inclusion map $V \hookrightarrow \mathbb{P}^n(\mathbb{C})$. According to the Barth–Larsen theorem (cf. [**36**]), V is irreducible and ρ is an isomorphism. In particular, X is irreducible. Moreover, as in the proof of Theorem 4.12, we get

$$H^2_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) = \mathrm{Im}\ e^* = e^*(H^2(\mathbb{P}^n(\mathbb{R});\mathbb{Z})).$$

Thus it suffices to apply Theorem 4.5. □

Theorem 4.13 is of interest when $\dim X \geq 3$. If $\dim X = 2$, then $n = 2$ and $X = \mathbb{P}^2(\mathbb{R})$.

5. Regular maps from the product of real cubic curves into $\mathbb{S}^2$. In this section we study the set of regular maps $\mathcal{R}(C \times D, \mathbb{S}^2)$, where C and D are nonsingular real cubic curves in $\mathbb{P}^2(\mathbb{R})$. For proofs we refer to [**14**].

We first recall the classification of nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$. For any real number α in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, set

$$\tau_\alpha = \begin{cases} \frac{1}{2}(1 + \alpha\sqrt{-1}) & \text{if } \alpha > 0 \\ \alpha\sqrt{-1} & \text{if } \alpha < 0. \end{cases}$$

The numbers

$$g_2(\tau_\alpha) = 60 \sum_{\omega \in \Lambda'_\alpha} \omega^{-4}, \quad g_3(\tau_\alpha) = 140 \sum_{\omega \in \Lambda'_\alpha} \omega^{-6},$$

where $\Lambda_\alpha = \mathbb{Z} + \mathbb{Z}\tau_\alpha$ is a lattice in $\mathbb{C}$ and $\Lambda'_\alpha = \Lambda_\alpha \setminus \{0\}$, are real and

$$D_\alpha = \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \,|\, y^2 z = 4x^3 - g_2(\tau_\alpha)xz^2 - g_3(\tau_\alpha)z^3\}$$

is a nonsingular cubic curve in $\mathbb{P}^2(\mathbb{R})$ (cf. [**32**]). Each nonsingular cubic curve in $\mathbb{P}^2(\mathbb{R})$ is biregularly isomorphic to exactly one cubic D_α. Thus $\mathbb{R}^*$ can be regarded as a moduli space of nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$. For $\alpha > 0$ (resp. $\alpha < 0$) the cubic D_α is connected (resp. has 2 connected components). The Zariski closure E_α of D_α in $\mathbb{P}^2(\mathbb{C})$ is a complex elliptic curve. For each $\alpha \in \mathbb{R}^*$ there is precisely one $\beta \in \mathbb{R}^*$, $\beta \neq \alpha$, such that the complex elliptic curves E_α and E_β are isomorphic: $\beta = -\alpha$ if $\alpha^2 = 1$ and $\beta = 1/\alpha$ otherwise. The corresponding real cubics D_α and D_β are said to be *associated*.

THEOREM 5.1. *Let C and D be nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$. Then $C \times D$ can be oriented in such a way that for each map f in $\mathcal{R}(C \times D, \mathbb{S}^2)$, the topological degree $\deg(f|_A)$ of the restriction of f to a connected component A of $C \times D$ is independent of the choice of A. Moreover,*

$$\mathrm{Deg}_{\mathcal{R}}(C, D) = \{m \in \mathbb{Z} \,|\, m = \deg(g|_A),\, g \in \mathcal{R}(C \times D, \mathbb{S}^2)\}$$

is a subgroup of $\mathbb{Z}$.

Let $b(C, D)$ be the nonnegative integer defined by $\mathrm{Deg}_{\mathcal{R}}(C, D) = b(C, D)\mathbb{Z}$. For $C \times D$ connected, $b(C, D)$ coincides with the invariant $b(C \times D)$ introduced in Section 4.

Assuming that $C \times D$ is oriented as in Theorem 5.1, a smooth map $h : C \times D \to \mathbb{S}^2$ can be approximated by regular maps if and only if $\deg(h|_A)$ is a multiple of $b(C, D)$ and does not depend on A. In particular, $\mathcal{R}(C \times D, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(C \times D, \mathbb{S}^2)$ (resp. $\mathcal{R}(C \times D, \mathbb{S}^2)$ contains only null homotopic maps) if and only if $C \times D$ is connected and $b(C, D) = 1$ (resp. $b(C, D) = 0$).

For (α, β) in $\mathbb{R}^* \times \mathbb{R}^*$, set $b(\alpha, \beta) = b(D_\alpha, D_\beta)$. The explicit formulas for $b(\alpha, \beta)$ are given in [**14**]. Here we only characterize the pairs (α, β) for which $b(\alpha, \beta) = 0$ and give the values for $b(\alpha, \alpha)$.

THEOREM 5.2. *For α and β in $\mathbb{R}^*$, the following conditions are equivalent:*

(a) *Every map in $\mathcal{R}(D_\alpha \times D_\beta, \mathbb{S}^2)$ is null homotopic.*
(b) *$b(\alpha, \beta) = 0$.*
(c) *$\alpha\beta$ is in $\mathbb{R} \setminus \mathbb{Q}$.*

It follows that the set of pairs (α, β) in $\mathbb{R}^* \times \mathbb{R}^*$ for which the set $\mathcal{R}(D_\alpha \times D_\beta, \mathbb{S}^2)$ contains a homotopically nontrivial map is very small. It is contained in the union of a countable family of hyperbolas in $\mathbb{R}^2$.

Corollary 5.3. *For α in $\mathbb{R}^*$, the following conditions are equivalent:*
(a) *There exists a homotopically nontrivial map in $\mathcal{R}(D_\alpha \times D_\alpha, \mathbb{S}^2)$.*
(b) *α^2 is in $\mathbb{Q}$.*
(c) *The complex elliptic curve E_α has complex multiplication.*

Since $b(\alpha, \alpha) = 0$ if and only if α^2 is in $\mathbb{R}^* \setminus \mathbb{Q}$, we now examine the values of $b(\alpha, \alpha)$ for α^2 in $\mathbb{Q}$.

Theorem 5.4. (i) *If $\alpha = (p/q)\sqrt{d}$, where p, q, $d \in \mathbb{Z}^+$, $(p, q) = 1$ and d is square free, then*

$$b(\alpha,\alpha) = \begin{cases} q^2/(p,q) & \text{if } d \equiv 3 \pmod 4 \text{ and } pq \equiv 1 \pmod 2 \\ 2q^2/(p,q) & \text{if } d \equiv 1 \pmod 4 \text{ and } pq \equiv 1 \pmod 2 \\ 4q^2/(p,q) & \text{if } pqd \equiv 0 \pmod 2. \end{cases}$$

(ii) *If $\alpha < 0$ and $\alpha^2 = p/q$, where p, $q \in \mathbb{Z}^+$ and $(p, q) = 1$, then*

$$b(\alpha, \alpha) = q.$$

The following density result is of particular interest.

Theorem 5.5. *Let $C_1, \ldots, C_n$ be nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$ and let $X = C_1 \times \cdots \times C_n$, $n \geq 2$. Then the following conditions are equivalent:*
(a) *$\mathcal{R}(X, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(X, \mathbb{S}^2)$.*
(b) *The C_i are connected and $b(C_i, C_j) = 1$ for all $i \neq j$.*
(c) *Each C_i is biregularly isomorphic to D_{α_i} with $\alpha_i = (p_i/q_i)\sqrt{d}$, p_i, q_i, $d \in \mathbb{Z}^+$, $(p_i, q_i) = 1$, d square free, $d \equiv 3 \pmod 4$, where for all $i \neq j$, one has $p_i p_j q_i q_j \equiv 1 \pmod 2$ and $p_i p_j d$ is divisible by $q_i q_j$.*

In particular, (a) *holds only for a countable collections of n-tuples of cubics.*

Corollary 5.6. *For a nonsingular real cubic curve C in $\mathbb{P}^2(\mathbb{R})$, the set $\mathcal{R}(C^n, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(C^n, \mathbb{S}^2)$, $n \geq 2$, if and only if C is biregularly isomorphic to D_α with $\alpha = \sqrt{4k-1}$ for some $k \in \mathbb{Z}^+$.*

The next corollary plays a crucial role in the proof of Theorem 4.4 (iii).

Corollary 5.7. *Let b a positive integer and let $\alpha = \sqrt{(4b-1)/b}$. Then D_α is connected and $b(D_\alpha, D_\alpha) = b$.*

We now examine consequences of the results above for cubic curves in $\mathbb{P}^2(\mathbb{R})$ defined over $\mathbb{Q}$. It is well known that there exist (up to isomorphism) exactly 13 complex elliptic curves, defined over $\mathbb{Q}$, with complex multiplication. These are the curves E_α with α in

$$\Gamma = \{-\sqrt{7}, -\sqrt{3}, -\sqrt{2}, -2, -1, \sqrt{3}, 3\sqrt{3}, \sqrt{7}, \sqrt{11}, \sqrt{19}, \sqrt{43}, \sqrt{67}, \sqrt{163}\}$$

(cf. [**32**, p. 233]). It follows, making use of Corollary 5.3, that there are precisely 26 nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$, defined over $\mathbb{Q}$, with $b(C, C) \neq 0$.

Since the j-invariant of E_α for $\alpha \in \Gamma$ is explicitly known (cf. [**32**, p. 233]), we can produce explicit equations of these 26 real cubics. Let

$$\Omega_1 = \{3, 7, 11, 19, 27, 43, 67, 163\},\ \Omega_2 = \{8, 12, 16, 28\},$$
$$\Omega = \Omega_1 \cup \Omega_2 \cup \{4\}.$$

For $k \in \Omega$, define nonsingular cubic curves A_k and A_k^* in $\mathbb{P}^2(\mathbb{R})$ by the following affine equations:

$$\begin{aligned}
&A_3 : y^2 = x^3 - 1, && A_3^* : y^2 = x^3 + 1 && \text{if } k = 3\\
&A_4 : y^2 = x^3 - x, && A_4^* : y^2 = x^3 + x && \text{if } k = 4\\
&A_k : y^2 = 4x^3 - a_k x - a_k, && A_k^* : y^2 = 4x^3 - a_k x + a_k && \text{if } k > 4,
\end{aligned}$$

where $a_k = 27j_k/(j_k - 1728)$ and

k	7	11	19	27	43	67	163
$-j_k$	$(3{\cdot}5)^3$	2^{15}	$(2^5{\cdot}3)^3$	$2^{15}{\cdot}3{\cdot}5^3$	$(2^6{\cdot}3{\cdot}5)^3$	$(2^5{\cdot}3{\cdot}5{\cdot}11)^3$	$(2^6{\cdot}3{\cdot}5{\cdot}23{\cdot}29)^3$

k	8	12	16	28
j_k	$(2 \cdot 3 \cdot 11)^3$	$2^4 \cdot 3^3 \cdot 5^3$	$(2^2 \cdot 5)^3$	$(3 \cdot 5 \cdot 17)^3$

The curves A_k, A_k^* for $k \in \Omega_1$ (resp. $k \in \Omega_2$) and A_4^* (resp. A_4) are connected (resp. disconnected). Observe that j_k is the j-invariant of A_k and A_k^*, and that no two of these real 26 cubics are isomorphic. In particular, each pair (A_k, A_k^*) is a pair of associated cubics. It can be shown that

A_k (resp. A_k^*) is isomorphic to $D_{\sqrt{k}}$ (resp. $D_{1/\sqrt{k}}$) for $k \in \Omega_1$,
A_k (resp. A_k^*) is isomorphic to $D_{-\sqrt{k}/2}$ (resp. $D_{-2/\sqrt{k}}$) for $k \in \Omega_2$,
A_4 (resp. A_4^*) is isomorphic to D_{-1} (resp. D_1).

Taking all these into account, one obtains several interesting results.

THEOREM 5.8. *There exist* (*up to isomorphism*) *exactly* 18 *unordered pairs* $\{C, D\}$ *of nonsingular cubic curves in* $\mathbb{P}^2(\mathbb{R})$, *defined over* $\mathbb{Q}$, *such that* $\mathcal{R}(C \times D, \mathbb{S}^2)$ *is dense in* $\mathcal{C}^\infty(C \times D, \mathbb{S}^2)$. *These unordered pairs are* $\{A_k, A_k\}$, $\{A_k, A_k^*\}$ *for* $k \in \Omega_1$, $\{A_3, A_{27}\}$ *and* $\{A_3^*, A_{27}\}$.

More generally, there exist (up to isomorphism) exactly 28 unordered pairs $\{C, D\}$ of connected nonsingular cubic curves in $\mathbb{P}^2(\mathbb{R})$, defined over $\mathbb{Q}$, with $b(C, D)$ odd.

EXAMPLE 5.9. Consider the following compact connected nonsingular curves in $\mathbb{R}^2$:

$$\begin{aligned}
C_1 &: x^2 + 3y^4 + 6y^2 = 1\\
C_2 &: x^2 + y^4 = 1\\
C_3 &: x^2 + 3y^4 - 6y^2 = 1.
\end{aligned}$$

One can show that C_1, C_2, C_3 are isomorphic to A_3, A_4^*, A_3^*, respectively. It follows that $b(C_k \times C_k) = k$ for $k = 1, 2, 3$. In particular, $\mathcal{R}(C_1 \times C_1, \mathbb{S}^2)$ is dense in $\mathcal{C}^\infty(C_1 \times C_1, \mathbb{S}^2)$, but a map f in $\mathcal{C}^\infty(C_k \times C_k, \mathbb{S}^2)$ can be approximated by regular maps if and only if $\deg(f)$ is in $k\mathbb{Z}$ for $k = 2, 3$.

Theorem 5.10. *There exist (up to isomorphism) precisely* 26 *nonsingular cubic curves C in $\mathbb{P}^2(\mathbb{R})$, defined over $\mathbb{Q}$, with $b(C, C) \neq 0$. These cubics are A_k, A_k^* for $k \in \Omega$, and one has*

$$b(C, C) = \begin{cases} 1 & \text{for } C \cong A_k,\ k \in \Omega, \\ k & \text{for } C \cong A_k^*,\ k \in \Omega_1, \\ k/4 & \text{for } C \cong A_k^*,\ k \in \Omega_2, \\ 2 & \text{for } C \cong A_4^*. \end{cases}$$

Arithmetical digression. The following famous statement was considered by Gauss and proved in 1966 independently by Baker and Stark: There exist exactly 9 imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$, for which the ring of algebraic integers is factorial (equivalently, there exist exactly 13 binary integral positive primitive quadratic forms with class number 1), cf. [**42**]. Strangely enough, Theorem 5.10 is equivalent to the Gauss problem.

Observe that $\{-\Delta \mid \Delta \in \Omega\}$ is the set of discriminants of binary integral positive primitive quadratic forms with class number 1, and that the set

$$\{b \in \mathbb{Z}^+ \mid b = \Delta/(\Delta, 4),\ \Delta \in \Omega\}$$

coincides with the set

$$\{b \in \mathbb{Z}^+ \mid b = b(C, C),\ C \subset \mathbb{P}^2(\mathbb{R}) \text{ is a nonsingular cubic defined over } \mathbb{Q}\}.$$

6. Regular maps into real rational surfaces. Recent papers [**4, 30**], extending earlier results [**22, 23**], give a surprizing classification of real rational surfaces.

Theorem 6.1. *Two compact nonsingular real rational surfaces are biregularly isomorphic if and only if they are homeomorphic.*

Compact nonsingular real rational surfaces are classified, up to biregular isomorphism, as follows. There are exactly two orientable rational surfaces, $\mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^1$. For each positive integer g, there exists exactly one nonorientable rational surface of topological genus g, namely the surface obtained by blowing up $\mathbb{S}^2$ at any g distinct points.

The first proof of Theorem 6.1 given in [**4**] was simplified in [**30**]. The key result of [**30**] is the following:

THEOREM 6.2. *Let Y be a compact nonsingular real rational surface. For any two systems $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ of k points of Y, there exists a biregular isomorphism $\varphi\colon Y \to Y$ with $\varphi(a_i) = b_i$ for $i = 1, \ldots, k$.*

The structure of $\mathcal{C}^\infty_{\mathcal{R}}(-, Y)$, where Y is a real rational surface, is fully controlled by the invariants $H^1_{\mathrm{alg}}(-; \mathbb{Z}/2)$ and $H^2_{\mathbb{C}\text{-alg}}(-; \mathbb{Z})$.

THEOREM 6.3. *Let X be a compact nonsingular real algebraic variety and let Y be a compact nonsingular real rational surface. For a smooth map $f\colon X \to Y$, the following conditions are equivalent:*
(a) *f can be approximated by regular maps.*
(b) *f is homotopic to a regular map.*
(c) *Either Y is homeomorphic to $\mathbb{S}^2$ and $f^*(H^2(Y; \mathbb{Z})) \subseteq H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z})$, or Y is not homeomorphic to $\mathbb{S}^2$ and $f^*(H^1(Y; \mathbb{Z}/2)) \subseteq H^1_{\mathrm{alg}}(X; \mathbb{Z})$.*

The proof is given in [**33**]. Some steps in it can be simplified since Theorem 6.1 is now available.

THEOREM 6.4. *Let X and Y be compact nonsingular real rational surfaces. If X is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and Y is homeomorphic to $\mathbb{S}^2$, then the set $\mathcal{R}(X, Y)$ contains only null homotopic maps. In all other cases, the set $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$.*

PROOF. It follows from the classification of real rational surfaces that $H^1_{\mathrm{alg}}(X; \mathbb{Z}/2) = H^1(X; \mathbb{Z}/2)$. Moreover, $H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) = 0$ if X is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, and $H^2_{\mathbb{C}\text{-alg}}(X; \mathbb{Z}) = H^2(X; \mathbb{Z})$ otherwise. Thus is suffices to make use of Theorem 6.3. □

7. Regular maps into $\mathbb{S}^4$. Behind the results presented in Sections 3, 4 and 5 is the existence of biregular isomorphisms $\mathbb{S}^1 \cong \mathbb{P}^1(\mathbb{R})$ and $\mathbb{S}^2 \cong \mathbb{P}^1(\mathbb{C})_{\mathbb{R}}$. In this section we make use of the quaternions $\mathbb{H}$ and the quaternion projective line $\mathbb{P}^1(\mathbb{H})$. Identifying $\mathbb{H}$ with $\mathbb{R}^4$, we regard $\mathbb{P}^1(\mathbb{H})$ as a real algebraic variety. The map

$$\mathbb{P}^1(\mathbb{H}) \to \mathbb{S}^4,\ (u : v) \to \left(\frac{2u\overline{v}}{|u|^2 + |v|^2}, \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2} \right)$$

is a biregular isomorphism, which is used to identify $\mathbb{P}^1(\mathbb{H})$ with $\mathbb{S}^4$.

A topological $\mathbb{H}$-vector bundle on a real algebraic variety X is said to admit an *algebraic structure* if it is topologically isomorphic to an algebraic $\mathbb{H}$-vector subbundle of the trivial $\mathbb{H}$-vector bundle $X \times \mathbb{H}^n$ for same n. Clearly, the universal $\mathbb{H}$-line bundle γ on $\mathbb{P}^1(\mathbb{H}) = \mathbb{S}^4$ admits an algebraic structure.

Theorem 7.1. *Let X be a compact nonsingular real algebraic variety. For a smooth map $f\colon X \to \mathbb{S}^4$, the following conditions are equivalent:*

(a) *f can be approximated by regular maps.*

(b) *f is homotopic to a regular map.*

(c) *The $\mathbb{H}$-line bundle $f^*\gamma$ on X admits an algebraic structure.*

The proof of this theorem follows in general lines that of Theorem 2.4. (cf. [**7**] for details).

Corollary 7.2. *The set $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^4)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^4)$ for all $n \geq 1$.*

Proof. Every $\mathbb{H}$-vector bundle on $\mathbb{S}^n$ admits an algebraic structure (cf. [**46**, Theorem 11.1.], where this fact is expressed in terms of projective modules), and hence it suffices to apply Theorem 7.1. □

When a topological $\mathbb{H}$-vector bundle ξ is regarded as a $\mathbb{C}$-vector bundle, we indicate this by writing $\xi_{\mathbb{C}}$. Denote by $H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$ the subset of $H^4(X;\mathbb{Z})$ consisting of all cohomology classes of the form $c_2(\xi_{\mathbb{C}})$, where ξ is an $\mathbb{H}$-vector bundle on X admitting an algebraic structure. Since $c_1(\xi_{\mathbb{C}}) = 0$, it follows that $H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$ is a subgroup of $H^4(X;\mathbb{Z})$. If $f\colon X \to Y$ is a regular map between real algebraic varieties, then

$$f^*(H^4_{\mathbb{H}\text{-alg}}(Y;\mathbb{Z})) \subseteq H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z}).$$

Example 7.3. We have $H^4_{\mathbb{H}\text{-alg}}(\mathbb{S}^4;\mathbb{Z}) = H^4(\mathbb{S}^4;\mathbb{Z})$ since $s_4 = c_2(\gamma_{\mathbb{C}})$ is a generator of $H^4(\mathbb{S}^4;\mathbb{Z})$.

Theorem 7.4. *Let X be a compact nonsingular real algebraic 4-fold. For a smooth map $f\colon X \to \mathbb{S}^4$, the following conditions are equivalent:*

(a) *f can be approximated by regular maps.*

(b) *f is homotopic to a regular map.*

(c) *$f^*(s_4)$ is in $H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$.*

Sketch of proof. The equivalence of (a) and (b) follows from Theorem 7.1. In view of Example 7.3 and the functoriality of $H^4_{\mathbb{H}\text{-alg}}(-;\mathbb{Z})$, condition (b) implies (c).

Suppose that (c) holds and $f^*(s_4) = c_2(\xi_{\mathbb{C}})$, where ξ is an $\mathbb{H}$-vector bundle on X admitting an algebraic structure. Since $\dim X = 4$, it can be assumed that ξ is an $\mathbb{H}$-line bundle, and hence ξ is isomorphic to the pullback $g^*\gamma$ for some smooth map $g\colon X \to \mathbb{S}^4$. According to Theorem 7.1, we may assume that g is a regular map. Since

$$g^*(s_4) = g^*(c_2(\gamma_{\mathbb{C}})) = c_2((g^*\gamma)_{\mathbb{C}}) = c_2(\xi_{\mathbb{C}}) = f^*(s_4),$$

the maps f and g are homotopic by virtue of Hopf's theorem. Thus (c) implies (b). □

Theorem 7.4 with X connected and orientable is equivalent to [**9**, Theorem 5.1, Proposition 5.3]. For such a 4-fold X, we have

$$H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z}) = b_4(X)H^4(X;\mathbb{Z}),$$

where $b_4(X)$ is a uniquely determined nonnegative integer. According to Theorem 7.4 and Hopf's theorem,

$$\mathcal{C}^\infty_{\mathcal{R}}(X,\mathbb{S}^4) = \{f \in \mathcal{C}^\infty(X,\mathbb{S}^4) \,|\, \deg(f) \in b_4(X)\mathbb{Z}\},$$

where $\deg(f)$ denotes the topological degree of f. In particular, the set $\mathcal{R}(X,\mathbb{S}^4)$ is dense in $\mathcal{C}^\infty(X,\mathbb{S}^4)$ (resp. $\mathcal{R}(X,\mathbb{S}^4)$ contains only null homotopic maps) if and only if $b_4(X)=1$ (resp. $b_4(X)=0$).

EXAMPLE 7.5. Let p and q be positive integers. If p is even, then the map $\psi\colon \mathbb{S}^p \times \mathbb{S}^q \to \mathbb{S}^{p+q}$ defined by

$$\psi(x_0,\dots,x_p,y_0,\dots,y_q) = (y_0x_0,\dots,y_0x_p,y_1,\dots,y_q)$$

is of topological degree 2. In particular, $b_4(\mathbb{S}^2\times\mathbb{S}^2)=1$ or $b_4(\mathbb{S}^2\times\mathbb{S}^2)=2$. It is not known if there is a regular map form $\mathbb{S}^2\times\mathbb{S}^2$ into $\mathbb{S}^4$ of odd topological degree. The existence of such a map would, of course, imply $b_4(\mathbb{S}^2\times\mathbb{S}^2)=1$.

The group $H^4_{\mathbb{H}\text{-alg}}(-;\mathbb{Z})$ is very hard to compute. Therefore it is useful to compare it to the group $H^4_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ introduced in Section 4. Recall that the first Pontryagin class $p_1(-)$ is in $H^4_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$.

PROPOSITION 7.6. *If X is a compact nonsingular real algebraic variety, then*

$$2H^4_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) \subseteq H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z}).$$

In particular, $2p_1(X)$ is in $H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$.

SKETCH OF PROOF. Let (V,j) be a nonsingular projective complexification of X and let U be an affine Zariski open subset of V, defined over $\mathbb{R}$, containing $V(\mathbb{R})$. Let $e\colon X\to U$ be defined by $e(x)=j(x)$ for all x in X.

Each cohomology class w in $H^4_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ is of the form $w=j^*(v)$ for some v in $H^4_{\mathrm{alg}}(V;\mathbb{Z})$. Clearly, $w=e^*(u)$, where u in $H^4(U;\mathbb{Z})$ is the restriction of v. According to Grothendieck's formula [**27**, Example 15.3.6], there exists an algebraic (complex) vector bundle ζ on U with $c_1(\zeta)=0$ and $c_2(\zeta)=u$. The pullback $\eta=e^*\zeta$ is an algebraic $\mathbb{C}$-vector bundle on X with $c_1(\eta)=0$ and $c_2(\eta)=w$. The quaternionification $\xi=\eta\otimes_{\mathbb{C}}\mathbb{H}$ of η is an $\mathbb{H}$-vector bundle on X admitting an algebraic structure. The topological $\mathbb{C}$-vector bundles $\xi_{\mathbb{C}}$ and $\eta\oplus\overline{\eta}$, where $\overline{\eta}$ is the conjugate of η, are isomorphic. The cohomology class

$$c_2(\xi_{\mathbb{C}}) = c_2(\eta\oplus\overline{\eta}) = -c_1(\eta)\cup c_1(\eta) + 2c_2(\eta) = 2w$$

is in $H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$. Thus $2H^4_{\mathbb{C}\text{-alg}}(X;\mathbb{Z}) \subseteq H^4_{\mathbb{H}\text{-alg}}(X;\mathbb{Z})$, as asserted. $\square$

EXAMPLE 7.7. According to Example 4.1 (ii), Theorem 7.4 and Proposition 7.6, for the Fermat 4-sphere of degree $2k$, we have $b_4(\Sigma^4_{2k}) = 1$ or $b_4(\Sigma^4_{2k}) = 2$. If k is odd, then the map

$$\Sigma^4_{2k} \to \mathbb{S}^4,\ (x_1, \ldots, x_5) \to \left(x_1^k, \ldots, x_5^k\right)$$

is regular of topological degree 1, and hence $b(\Sigma^4_{2k}) = 1$. Thus the set $\mathcal{R}(\Sigma^4_{4l+2}, \mathbb{S}^4)$ is dense in $\mathcal{C}^\infty(\Sigma^4_{4l+2}, \mathbb{S}^4)$ for all $l \geq 0$. No example with $b_4(\Sigma^4_{4l}) = 2$ and $l \geq 1$ is known.

THEOREM 7.8. *Let X be a compact nonsingular real algebraic 4-fold and let $f\colon X \to \mathbb{S}^4$ be a smooth map.*

(i) *If $f^*(s_4)$ is in $2H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$, then f can be approximated by regular maps.*
(ii) *If $f^*(s_4)$ is a multiple of $2p_1(X)$, then f can be approximated by regular maps.*

PROOF. It suffices to apply Theorem 7.4 and Proposition 7.6. □

It should be noted that the assumption in Theorem 7.8 (ii) is purely topological.

A simple but important numerical invariant of compact oriented smooth manifolds M is the signature $\sigma(M)$ (cf. [**40**]). The following result comes from [**9**].

THEOREM 7.9. *Let X be a compact nonsingular real algebraic 4-fold and let $f\colon X \to \mathbb{S}^4$ be a smooth map. If X is connected and oriented, and* $\deg(f)$ *is a multiple of $6\sigma(X)$, then f can be approximated by regular maps.*

PROOF. According to the signature theorem [**40**, p. 224], $3\sigma(X)$ is the Pontryagin number of X. If $\deg(f)$ is a multiple of $6\sigma(X)$, then $f^*(s^4)$ is a multiple of $2p_1(X)$. If suffices to apply Theorem 7.8 (ii). □

EXAMPLE 7.10. Let X be a nonsingular real algebraic variety homeomorphic to $\mathbb{P}^2(\mathbb{C})$. Then every smooth map from X into $\mathbb{S}^4$ of topological degree divisible by 6 can be approximated by regular maps. Indeed, the assertion follows from Theorem 7.9 since $\sigma(X) = 1$.

THEOREM 7.11. *Let M be a compact connected oriented smooth manifold of dimension* 4. *Then the following conditions are equivalent:*

(a) *M is the boundary of a compact orientable smooth manifold.*
(b) $\sigma(M) = 0$
(c) *There exists an algebraic model X of M such that every regular map from X into $\mathbb{S}^4$ is null homotopic.*

Of course, the equivalence of (a) and (b) is well known (cf. [**40**]). According to Theorem 7.9, condition (c) implies (b). For the proof that (a) implies (c) we refer to [**9**].

8. Homotopy classes of regular maps into spheres. Recall that $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^p)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^p)$ for all $n \geq 1$ and $p = 1, 2$ or 4 (cf. Sections 3, 4 and 7). If p is a positive integer different from $1, 2$ and 4, no example is known of a compact nonsingular real algebraic variety X with $\dim X \geq p$ and $\mathcal{R}(X, \mathbb{S}^p)$ dense in $\mathcal{C}^\infty(X, \mathbb{S}^p)$. There are, however, several results concerning homotopy classes represented by regular maps from X into $\mathbb{S}^p$.

Fixing a generator s_p of $H^p(\mathbb{S}^p; \mathbb{Z})$, to each continuous map $f : X \to \mathbb{S}^p$ we assign the cohomology class $f^*(s_p)$ in $H^p(X; \mathbb{Z})$. The correspondence $f \to f^*(s_p)$ gives rise to a map

$$\rho_p : \pi^p(X) \to H^p(X; \mathbb{Z})$$

defined on the set of homotopy classes of continuous maps form X into $\mathbb{S}^p$.

If $\dim X \leq 2p - 2$, then $\pi^p(X)$ forms an Abelian group, called the pth cohomotopy group of X, and ρ_p is a group homomorphism (cf. [**31**]). Both the kernel and the cokernel of ρ_p are finite groups (cf. [**45**, p. 289, Proposition 2']). In particular, $\pi^p(X)$ is a finitely generated Abelian group with

$$\operatorname{rank} \pi^p(X) = \operatorname{rank} H^p(X; \mathbb{Z}).$$

According to Hopf's theorem, ρ_p is a group isomorphism if $\dim X = p$. Of interest to us is the subset $\pi^p_{\mathrm{alg}}(X)$ of $\pi^p(X)$ consisting of the homotopy classes represented by regular maps from X into $\mathbb{S}^p$. It is not known whether $\pi^p_{\mathrm{alg}}(X)$ is a subgroup of $\pi^p(X)$ (it is so if $\dim X = p$ with $p = 2$ or 4, or with p odd; cf. Theorems 4.5, 7.4 and Theorem 8.1). We let $\overline{\pi}^p_{\mathrm{alg}}(X)$ denote the subgroup of $\pi^p(X)$ generated by $\pi^p_{\mathrm{alg}}(X)$. Clearly, $\overline{\pi}^p_{\mathrm{alg}}(X) = 0$ if and only if every regular map from X into $\mathbb{S}^p$ is null homotopic.

THEOREM 8.1. *If p is odd and $\dim X \leq 2p - 2$, then $2\pi^p(X) \subseteq \pi^p_{\mathrm{alg}}(X)$. In particular,*

$$\operatorname{rank} \overline{\pi}^p_{\mathrm{alg}}(X) = \operatorname{rank} \pi^p(X).$$

Moreover, if X is connected with $\dim X = p$ odd, then $\pi^p_{\mathrm{alg}}(X) = \pi^p(X)$ or $\pi^p_{\mathrm{alg}}(X) = 2\pi^p(X)$.

For the proof we refer to [**8**] or [**6**]. No example is known with $\pi^p_{\mathrm{alg}}(X) = 2\pi^p(X)$ in the last assertion of Theorem 8.1.

We encounter a radically different behavior of the group $\overline{\pi}^p_{\mathrm{alg}}(X)$ for p even. If $\dim X \leq 4k - 2$, then the homomorphism $\rho_{2k} : \pi^{2k}(X) \to H^{2k}(X; \mathbb{Z})$ satisfies

$$\rho_{2k}(\overline{\pi}^{2k}_{\mathrm{alg}}(X)) \subseteq H^{2k}_{\mathbb{C}\text{-}\mathrm{alg}}(X; \mathbb{Z}).$$

This claim follows immediately from the functoriality of $H^{2k}_{\mathbb{C}\text{-}\mathrm{alg}}(-; \mathbb{Z})$ and Example 4.1(ii). In particular,

$$\operatorname{rank} \overline{\pi}^{2k}_{\mathrm{alg}}(X) \leq \operatorname{rank} H^{2k}_{\mathbb{C}\text{-}\mathrm{alg}}(X; \mathbb{Z}).$$

Since rank $H^{2k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ can be "small" as compared to rank $H^{2k}(X;\mathbb{Z})$, we obtain that rank $\overline{\pi}^{2k}_{\mathrm{alg}}(X)$ can be "small" as compared to rank $\pi^{2k}(X)$. This informal remark is illustrated by Theorems 8.2, 8.3, 8.5 and 8.6 below.

THEOREM 8.2. *Let Y be a compact nonsingular real algebraic variety of dimension $2k-l$, with l odd. Then $\overline{\pi}^{2k}_{\mathrm{alg}}(Y\times\mathbb{S}^l)=0$, that is, every regular map from $Y\times\mathbb{S}^l$ into $\mathbb{S}^{2k}$ is null homotopic.*

PROOF. Since $\rho_{2k}:\pi^{2k}(Y\times\mathbb{S}^l)\to H^{2k}(Y\times\mathbb{S}^l;\mathbb{Z})$ is an isomorphism, it suffices to prove that $H^{2k}_{\mathbb{C}\text{-alg}}(Y\times\mathbb{S}^l;\mathbb{Z})=0$. To this end we define

$$W=\{(z_1:\ldots:z_{l+2})\in\mathbb{P}^{l+1}(\mathbb{C})\mid z_1^2+\ldots+z_{l+1}^2=z_{l+2}^2\},$$

$$e:\mathbb{S}^l\to W,\ e(x_1,\ldots,x_{l+1})=(x_1:\ldots,x_{l+1}:1).$$

Obviously, (W,e) is a nonsingular projective complexification of $\mathbb{S}^l$. Since l is odd, we have $H^l(W;\mathbb{Z})=0$ (cf. [**25**, pp. 43, 153]). Consequently, $e^*(H^q(W;\mathbb{Z}))=0$ in $H^q(\mathbb{S}^l;\mathbb{Z})$ for all $q\geq 1$. If (V,j) is a nonsingular projective complexification of Y, then $(V\times W,j\times e)$ is that of $Y\times\mathbb{S}^l$. By Künneth's theorem, $(j\times e)^*(H^{2k}(V\times W;\mathbb{Z}))=0$, and hence $H^{2k}_{\mathbb{C}\text{-alg}}(Y\times\mathbb{S}^l;\mathbb{Z})=0$, as required. □

Theorem 8.2 comes from [**9**]. For Y orientable, a proof based on K-theory is given in [**8**] and [**6**]. This theorem inspired [**43**], where a different proof can be found.

THEOREM 8.3. *Let $d_1,\ldots,d_r$ be positive integers, with $r\geq 2$, and let $d=d_1+\ldots+d_r$. Then the following conditions are equivalent:*
(a) *Every regular map from $\mathbb{S}^{d_1}\times\ldots\times\mathbb{S}^{d_r}$ into $\mathbb{S}^d$ is null homotopic.*
(b) *The integer d is even and at least two of the integers $d_1,\ldots,d_r$ are odd.*

PROOF. Making use of Theorem 8.1 and Example 7.5, we conclude that (a) implies (b) (Theorem 8.1 is not required if $r=2$). According to Theorem 8.2, condition (b) implies (a). □

A different proof of Theorem 8.3, based on K-theory, is presented in [**8**] and [**6**].

For any smooth manifold M and any nonnegative integer k, let $P^{4k}(M)$ denote the subgroup of $H^{4k}(M;\mathbb{Z})$ generated by all cup products $p_{i_1}(M)\cup\ldots\cup p_{i_r}(M)$, with $i_1+\ldots+i_r=k$, of the Pontryagin classes of M. In general, $P^{4k}(M)$ is a "small" subgroup of $H^{4k}(M;\mathbb{Z})$. If X is a compact nonsingular real algebraic variety, then $P^{4k}(X)\subseteq H^{4k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ (cf. Section 4).

THEOREM 8.4. *Any compact connected orientable smooth manifold has an algebraic model X such that the groups $H^{4k+2}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})$ and $H^{4k}_{\mathbb{C}\text{-alg}}(X;\mathbb{Z})/P^{4k}(X)$ are finite for all $k\geq 0$.*

For the proof we refer to [**35**, Theorem 1.4].

THEOREM 8.5. *Any compact connected orientable smooth manifold M has an algebraic model X such that*

(i) $\overline{\pi}^{4k+2}_{\mathrm{alg}}(X)$ *is a finite group for* $k \geq 0$ *with* $\dim M \leq 8k+2$,

(ii) $\operatorname{rank} \overline{\pi}^{4k}_{\mathrm{alg}}(X) \leq \operatorname{rank} P^{4k}(M)$ *for* $k \geq 1$ *with* $\dim M \leq 8k-2$.

PROOF. If X is an algebraic model of M as in Theorem 8.4, then both (i) and (ii) are satisfied. $\square$

THEOREM 8.6. *Let M be a compact connected oriented smooth manifold of dimension $2k$. If k is even, assume that the disjoint union of two copies of M is an oriented boundary. Then there exists an algebraic model X of M such that every regular map from X into $\mathbb{S}^{2k}$ is null homotopic.*

PROOF. Recall that a compact oriented smooth manifold is said to be an oriented boundary if it is the boundary, with induced orientation, of a compact oriented smooth manifold with boundary. If k is even, then $P^{2k}(M) = 0$ (cf. [**40**, Lemma 17.3]). Hence, according to Theorem 8.5, there exists an algebraic model X of M such that the group $\overline{\pi}^{2k}_{\mathrm{alg}}(X)$ is finite. Consequently, $\overline{\pi}^{2k}_{\mathrm{alg}}(X) = 0$, the group $\pi^{2k}(X)$ being infinite cyclic. $\square$

Theorem 8.6 comes from [**35**]. A special case was earlier proved in [**9**].

Smooth manifolds also have algebraic models with properties opposed to those exhibited in Theorems 8.5 and 8.6.

THEOREM 8.7. *Any compact smooth manifold M has an algebraic model X such that $\overline{\pi}^{p}_{\mathrm{alg}}(X) = \pi^p(X)$ for all positive integers p satisfying* $\dim M \leq 2p-2$. *Moreover, $\pi^{p}_{\mathrm{alg}}(X) = \pi^p(X)$ for $p = \dim M$.*

For the proof we refer to [**8, 35**].

The question whether every continuous map from $\mathbb{S}^n$ into $\mathbb{S}^p$ is homotopic to a regular map, for all pairs (n,p) with $n \geq p \geq 1$, remains open and seems to be a hard problem. The answer is known to be affirmative for some pairs (n,p) (cf. [**6, 8, 44, 50**]). For example, it is so for $n = p$. Actually, for each integer d, one can find an explicit formula for a regular map $\mathbb{S}^n \to \mathbb{S}^n$ of topological degree d (cf. [**8, 50**] and [**6**]).

Working with real algebraic sets it is also natural to consider polynomial maps between them. For algebraic subsets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^p$, a map $f = (f_1, \ldots, f_p) : X \to Y$ is said to be a polynomial map if each component $f_j : X \to \mathbb{R}$ is the restriction of a polynomial function from $\mathbb{R}^n$ into $\mathbb{R}$. The behavior of polynomial maps is quite different from the behavior of regular maps. For example, each polynomial map from $\mathbb{S}^n$ into $\mathbb{S}^p$ is constant, provided that $n \geq 2^m > p$ for some integer m (cf. [**50**] or [**6**]). On the other hand, as we already know, $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^p)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^n, \mathbb{S}^p)$ for $p = 1, 2$ or 4. More on polynomial maps between spheres can be found in [**6, 49, 50**].

The following result should be compared with Theorem 8.3.

THEOREM 8.8. *Let $d_1, \dots, d_r$ be positive integers, with $r \geq 2$, and let $d = d_1 + \ldots + d_r$. Then the following conditions are equivalent:*

(a) *Every polynomial map from $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_r}$ into $\mathbb{S}^d$ is null homotopic.*

(b) *At least two of the integers $d_1, \dots, d_r$ are odd.*

The proof can be found in [**12**] or [**6**]. Special cases of Theorems 8.2, 8.3 and 8.8, only for polynomial maps, are proved in [**37**].

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Received December 30, 2010

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