Dense-lineability of sets of Birkhoff-universal functions with rapid decay

L. Bernal-González, M.C. Calderón-Moreno and W. Luh*

Abstract

Let A be an unbounded Arakelian set in the complex plane whose complement has infinite inscribed radius, and ψ be an increasing positive function on the positive real numbers. We prove the existence of a dense linear manifold M of entire functions all of whose non-zero members are Birkhoff-universal, such that each function in M has overall growth faster than ψ and, in addition, $\exp(|z|^\alpha)f(z) \to 0$ ($z \to \infty$, $z \in A$) for all $\alpha < 1/2$ and $f \in M$. With slightly more restrictive conditions on A, we get that the last property also holds for the action Tf of certain holomorphic operators T. Our results unify, extend and complete recent work by several authors.

2000 Mathematics Subject Classification: Primary 30E10. Secondary 30D15, 47A16, 47B38.

Key words and phrases: Birkhoff-universal function, dense lineability, Arakelian set, infinite order differential operator, growth of entire functions.

*The first and second authors have been partially supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127, by MEC Grant MTM2006-13997-C02-01 and by MEC Acción Especial MTM2006-26627-E.

1 Introduction and known results

Several analysts have recently focussed their attention to find entire functions bearing a behavior which is "extremely wild" and "extremely tamed" simultaneously. To be more specific, it has been proved the existence of entire functions f that are universal in the sense of Birkhoff and are bounded (or even tend to zero as $z \to \infty$) on several families of prescribed sets, such as lines, strips, angles or even bigger ones. This decay can be very rapid. Additional properties may be incorporated, for instance, the function f can present an overall growth rate that is as fast as desired. Finally, the structure of the family of such functions is very rich, namely, it can contain, except for zero, a dense linear submanifold of the space of entire functions.

Let us fix some notation and terminology, and provide a short account of the related results up to date.

As usual, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_\infty$ stand for the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the real line, the complex plane and the extended complex plane, respectively. If $a \in \mathbb{C}$ and $r > 0$, then $B(a,r)$ ($\overline{B}(a,r)$) denotes the open (closed, respectively) ball with center a and radius r. If $A \subset \mathbb{C}$ then A^0, $\overline{A}$, ∂A will represent, respectively, the interior, the closure and the boundary of A in $\mathbb{C}$. The *inscribed radius of* A is defined by $\varrho(A) := \sup\{r > 0 : \text{there exists a ball } B \text{ of radius } r \text{ with } B \subset A\}$. If $\varepsilon > 0$, we set $B(A,\varepsilon) := \{z \in \mathbb{C} : \ d(z,A) \le \varepsilon\}$, where $d(z,A) := \inf\{|z-a| : \ a \in A\}$. A subset $A \subset \mathbb{C}$ is called an *Arakelian set* if A is closed in $\mathbb{C}$ and $\mathbb{C}_\infty \setminus A$ is connected and locally connected in $\mathbb{C}_\infty$. By Σ_1, Σ_2 we denote the family of all strips (i.e., plane regions between two parallel straight lines) and the family of all sectors $s_\beta := \{z : \ 0 \le \arg z \le \beta\}$ ($\beta \in (0, 2\pi)$), respectively.

By E we denote the space of entire functions, that is, holomorphic functions on $\mathbb{C}$. The space E is endowed with the topology of the uniform convergence in compacta, so that E becomes a completely metrizable topological vector space. If $\Phi \in E$ then its *exponential type* is defined as $\tau(\Phi) = \inf\{B \in (0,+\infty) : \text{there exists } A \in (0,+\infty) \text{ such that } |\Phi(z)| \le Ae^{B|z|} \text{ for all } z \in \mathbb{C}\}$. A function $\Phi \in E$ is said to be *of exponential type* whenever $\tau(\Phi) < +\infty$, that is, there are two constants A, $B \in (0,+\infty)$ such that $|\Phi(z)| \le Ae^{B|z|}$ ($z \in \mathbb{C}$). If, specially, $\tau(\Phi) = 0$ then we say that Φ of *subexponential type.* If $\Phi(z) := \sum_{n=0}^\infty a_n z^n$ is an entire function of exponential type then the expression

$$\Phi(D) = \sum_{n=0}^{\infty} a_n D^n$$

defines a (continuous, linear) operator $\Phi(D) : E \to E$ (see [9]). Here $D^0 = I$ (the identity), $D^1 = D$, $D^2 = D \circ D, \ldots$, where $Df := f'$ is the differentiation operator. Observe that the special cases $\Phi(z) = z$, $\Phi(z) = e^{az}$ yield the respective operator $\Phi(D) = D$, $\Phi(D) = T_a$. Here T_a denotes the translation operator with vector a, defined by $T_a f = f(\cdot + a)$. We have that an operator $T : E \to E$ commutes with the translations (i.e., $T\,T_a = T_a\,T$ for all $a \in \mathbb{C}$) if and only if $T = \Phi(D)$ for some $\Phi \in E$ with exponential type (see for instance [25]).

The maximum modulus function of a function $f \in E$ will be denoted by M_f, that is, $M_f(r) := \max\{|f(z)| : \ |z| = r\} = \max\{|f(z)| : \ |z| \le r\}$ $(r > 0)$.

In 1929, Birkhoff [11] proved the existence of a function $f \in E$ whose family of translates $\{f(a + \cdot) : \ a \in \mathbb{C}\}$ is dense in E. Such a function is called a *Birkhoff-universal function*. In a more general setting, Birkhoff-universal functions are precisely the hypercyclic vectors of E with respect to the translations operators T_a (see [26], [27] and [14]), but this generality will not be considered in the sequel. The harmonic analogue of Birkhoff's theorem –that is, E is replaced by the space H of harmonic functions on $\mathbb{R}^N$– can be found in Armitage-Gauthier's paper [4].

An interesting problem is whether (Birkhoff-)universality is compatible to apparently opposite properties, such as boundedness or even rapid decay to zero (as $z \to \infty$) on large sets. A number of results have been recently produced in this vein.

Namely, Bonilla [15, Theorem 1] discovered in 2000 that, given $\alpha > 0$, there exists a dense linear manifold $M \subset H$ such that every function $v \in M \setminus \{0\}$ is universal, and $\lim_{\substack{||x|| \to \infty \\ x \in S}} ||x||^\alpha D^\beta v(x) = 0$ for every $v \in M$, every multi-index β and every hyperplane strip S.

Calderón-Moreno [17] proved in 2002 that, given an $\alpha < \frac{1}{2}$, a continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ which is integrable on $(1, +\infty)$, a sequence $\mathcal{F} = \{h_n\}_{n=1}^{\infty}$ of non-constant entire functions, and two sequences $\{\Psi_{i,n}\}_{n=1}^{\infty}$ $(i = 1, 2)$ of entire functions of subexponential type (with $\Psi_{2,n} \neq 0$ for all n), there exists a dense linear manifold $M \subset E$ satisfying

(a) $\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{3/2} \varphi(|z|)) f(z) = 0 \quad (S \in \Sigma_1 \cup \Sigma_2,\ f \in M)$,

(b) $\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{\alpha})(\Psi_{1,n}(D) f)(z) = 0 \quad (S \in \Sigma_1 \cup \Sigma_2,\ f \in M)$,

(c) The relative growth order $\varrho_{h_n}(f) := \limsup_{r \to \infty} \dfrac{\log M_{h_n}^{-1}(M_f(r))}{\log r} = \infty$ $(f \in M \setminus \{0\},\ n \in \mathbb{N})$, and

(d) $\Psi_{2,n}(D) f$ is Birkhoff-universal $(f \in M \setminus \{0\},\ n \in \mathbb{N})$.

Independently, Costakis and Sambarino [19, Theorem 5] proved in 2004 that, given a compact set $K \subset \mathbb{C}$, there exists an entire function f whose translates $z \mapsto f(z+n)$ $(n \in \mathbb{N})$ are dense in E such that f tends to zero on every "translated" sector $K + \{z : \ \varepsilon \leq \arg z \leq 2\pi(1-\varepsilon)\}$ $(\varepsilon \in (0,1))$. And Gharibyan, Luh and Niess [24, Theorem 1.1] demonstrated that, given a sector $S := \{z = z_0 + re^{it} : \ r \geq 0,\ t \in [t_0 - \tau, t_0 + \tau]\}$ $(z_0 \in \mathbb{C},\ t_0 \in \mathbb{R},$ $0 < \tau < \pi)$, there is a dense subset $M \subset E$ such that every function $\varphi \in M$ is bounded on S and Birkhoff-universal.

Finally, Bernal and Bonilla [10, Theorem 3.3] were able to establish that, for a prescribed closed subset $F \subset \mathbb{C}$, the following conditions are equivalent:

(i) There exists a Birkhoff-universal function f that is bounded on F.

(ii) There exists an Arakelian subset A of $\mathbb{C}$ such that $F \subset A$ and $\varrho(\mathbb{C} \setminus A) = +\infty$.

In this paper, we provide a rather general statement about the existence of large linear manifolds of Birkhoff-universal functions with rapid decay on certain sets, even under the action of appropriate operators. The results established in the preceding paragraphs are then obtained as consequences. Finally, it is worth noting that our findings can be expressed in terms of the recently introduced notion of dense-lineability, see the work [6] by Aron, Gurariy and Seoane. Namely, if X is a topological vector space and $A \subset X$, then A is said to be *dense-lineable* provided that there is a dense linear manifold $M \subset X$ such that $M \setminus \{0\} \subset A$.

2 Universal functions with rapid decay

We devote this section to state our main assertions. But, before this, we need a number of preliminary results, which are incorporated in the following six lemmas. Lemma 2.1, whose proof can be found in [20], gives an "analytic" way to generate Arakelian sets. Lemma 2.2 is a special case of [10, Theorem 3.1]. Lemma 2.3 is elementary and provides us with a topological way to construct Arakelian sets from known ones. Lemma 2.4 is a special instance of [10, Lemma 2.1]. Lemma 2.5 is an important tangential approximation result due to Arakelian (see [1] or [23, pp. 153–154]). Lemma 2.6 is the Malgrange-Ehrenpreis surjectivity theorem (see [22], [28] or [9, p. 87]). Finally, Lemma 2.7 contains an elementary topological result which is frequently used in approximation theory. Since we have been unable to find an explicit reference of it, a proof will be provided.

Lemma 2.1. *Let $A \subset \mathbb{C}$. If f is a Birkhoff-universal entire function that is bounded on A, then*

$$A_0 := \{z \in \mathbb{C} : \ |f(z)| \le \sup_A |f|\}$$

is an Arakelian subset of $\mathbb{C}$.

Lemma 2.2. *If A is a subset of $\mathbb{C}$ and there exists a Birkhoff-universal function $f \in E$ such that f is bounded on A, then $\varrho(\mathbb{C} \setminus A) = +\infty$.*

Lemma 2.3. *Let A be an Arakelian subset in $\mathbb{C}$.*

(a) *If V is a bounded connected open subset of $\mathbb{C}$ with $V \setminus A \neq \emptyset$, then $A \setminus V$ is an Arakelian subset in $\mathbb{C}$.*

(b) *If $\{S_n : \ n \in \mathbb{N}\}$ is a countable family of pairwise disjoint sets with $S_n \cap A = \emptyset$ $(n \in \mathbb{N})$, such that each S_n is either a singleton or a closed ball, and the sets S_n go to ∞ (in the sense that $\inf_{z \in S_n} |z| \to \infty$ as $n \to \infty$), then*

$$F := A \cup \bigcup_{n=1}^{\infty} S_n$$

is an Arakelian subset of $\mathbb{C}$.

Lemma 2.4. *If A is a subset of $\mathbb{C}$ with $\varrho(A) = +\infty$ and B is any closed ball, then $\varrho(A \setminus B) = +\infty$.*

If $F \subset \mathbb{C}$, then by $A(F)$ we denote the class of functions $g : F \to \mathbb{C}$ which are continuous on F and holomorphic on F^0.

Lemma 2.5. *If F is an Arakelian subset of $\mathbb{C}$, then for every $g \in A(F)$ and every continuous function $\varepsilon : [0, +\infty) \to (0, +\infty)$ with*

$$\int_1^{+\infty} t^{-3/2} \log \left(\frac{1}{\varepsilon(t)} \right) dt < +\infty,$$

there is a function $f \in E$ such that

$$|f(z) - g(z)| < \varepsilon(|z|) \qquad \text{for all } z \in F.$$

Lemma 2.6. *Let Φ be a non-zero entire function with exponential type. Then the differential operator $\Phi(D) : E \to E$ is surjective.*

Recall that a topological space X is called a T_1-*space* if each singleton $\{x\}$ is a closed set, and that X is called *perfect* if it lacks isolated points.

Lemma 2.7. *Let X be a T_1, perfect topological space whose topology is defined by an increasing family $d_1 \le d_2 \le \cdots \le d_n \le \cdots$ of pseudodistances. Assume that $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ are two countable subsets of X such that $\{x_n : n \in \mathbb{N}\}$ is dense in X and*

$$\lim_{n \to \infty} d_n(x_n, y_n) = 0.$$

Then the set $\{y_n : n \in \mathbb{N}\}$ is also dense in X.

Proof. From the hypotheses it follows that X is a metrizable space and that the expression

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x, y)}{1 + d_n(x, y)} \qquad (x, y \in X)$$

defines a distance generating the topology of X.

Each finite subset F is closed, because X is T_1. Moreover, F has empty interior. Indeed, suppose by way of contradiction that $F^0 \neq \emptyset$. Then F^0 is open, finite (hence closed) and has at least two points. Choose a minimal non-empty open subset $A \subset F^0$. Then A has at least two points a, b. Since X is T_1, there is an open set U such that $a \in U$ and $b \notin U$. Denote $B = A \cap U$. Then B is a nonempty open subset of F^0 and it is a proper subset of A, which contradicts the selection of A.

For each $n \in \mathbb{N}$ we get

$$d(x_n, y_n) \leq \sum_{k=1}^{n} \frac{1}{2^k} \frac{d_k(x_n, y_n)}{1 + d_k(x_n, y_n)} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \leq 2d_n(x_n, y_n) + \frac{1}{2^n}.$$

Thus $d(x_n, y_n) \to 0$ as $n \to \infty$. Fix $\alpha \in X$ and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ with $d(x_n, y_n) < \varepsilon/2$ for all $n > N$. If we set $F := \{x_1, x_2, \dots, x_N\}$, then F is a closed set with empty interior, so the set $\{x \in X : \ d(x, \alpha) < \varepsilon/2\} \setminus F$ is a nonempty open set. By denseness, we derive the existence of a $m > N$ such that x_m belongs to the last set, whence $d(x_m, \alpha) < \varepsilon/2$. By the triangle inequality, $d(y_m, \alpha) < \varepsilon$, which proves the denseness of $\{y_n : \ n \in \mathbb{N}\}$. $\square$

We are now ready to establish our results about dense-lineability of families of "tamed" universal functions.

Theorem 2.8. *Assume that A is an unbounded subset of $\mathbb{C}$. Then the following properties are equivalent:*

(a) *There is an Arakelian subset $A_0 \subset \mathbb{C}$ such that $A \subset A_0$ and $\varrho(\mathbb{C} \setminus A_0) = +\infty$.*

(b) *Given two functions φ, $\psi : [0, +\infty) \to (0, +\infty)$ such that φ is continuous and integrable on $[1, +\infty)$, and ψ is increasing, there exists a linear submanifold $M = M(A, \varphi, \psi) \subset E$ satisfying the following:*

   (i) *M is dense in E.*

   (ii) *Each non-zero function in M is Birkhoff-universal.*

   (iii) $\displaystyle\lim_{\substack{z \to \infty \\ z \in A}} \exp(\varphi(|z|)|z|^{3/2}) f(z) = 0$ *for all $f \in M$.*

   (iv) $\displaystyle\lim_{r \to \infty} \frac{M_f(r)}{\psi(r)} = +\infty$ *for all $f \in M \setminus \{0\}$.*

Proof. Assume that (b) holds. Then, if we choose, say, $\varphi(t) := \dfrac{1}{1+t^{4/3}}$, we obtain from (i), (ii), (iii) the existence of at least a Birkhoff-universal function f that is bounded on A. From Lemma 2.1, and with the notation of it, we get that the set A_0 is an Arakelian subset with $A \subset A_0$ such that f is bounded on A_0. By Lemma 2.2, we get $\varrho(\mathbb{C} \setminus A_0) = +\infty$, which is (a).

Suppose now that (a) is true. Since $A \subset A_0$, we have that (iii) holds for A if it does for A_0. Hence we can assume that A is an Arakelian set such that $\varrho(\mathbb{C} \setminus A) = +\infty$. Fix two functions φ, $\psi : [0, +\infty) \to (0, +\infty)$ with φ integrable on $(1, +\infty)$, φ continuous, and ψ increasing.

Let $\{P_n : \ n \in \mathbb{N}\}$ be a dense countable subset of E, for instance, the set of polynomials whose coefficients have rational real and imaginary parts. Our goal is to construct an appropriate sequence $\{f_n : \ n \in \mathbb{N}\}$ of universal functions such that

$$\lim_{n\to\infty} \sup_{z\in B_n} |f_n(z) - P_n(z)| = 0, \tag{1}$$

where $B_n := \overline{B}(0, n)$ $(n \in \mathbb{N})$. For if (1) were true then Lemma 2.7 applies with $X := E$ and $d_n(f, g) := \sup_{z\in B_n} |f(z) - g(z)|$ $\quad (f, g \in E, \ n \in \mathbb{N})$. Consequently, the sequence $\{f_n\}$ would be dense. Hence the set

$$M := \operatorname{span}\{f_n : \ n \in \mathbb{N}\} \tag{2}$$

would be a dense linear submanifold of E, which is (i).

Since $\mathbb{C} \setminus A \neq \emptyset$, there exists $n_0 \in \mathbb{N}$ such that $B^0_{n+1} \setminus A \neq \emptyset$ for all $n \geq n_0$. By Lemma 2.3(a), $A \setminus B^0_{n+1}$ is an Arakelian set $(n \geq n_0)$. From Lemma 2.4 we can select a sequence of pairwise disjoint closed balls $\overline{B}(c_p, p + 1)$ such that $\overline{B}(c_p, p + 1) \cap A = \emptyset$ for all $p \in \mathbb{N}$. We split the sequence of centers c_p into two sequences, namely, $a_p := c_{2p-1}$, $b_p := c_{2p}$. Let $K_p := \overline{B}(a_p, p)$ $(p \in \mathbb{N})$. Observe that

$$\lim_{p\to\infty} \min_{z\in K_p} |z| = +\infty \tag{3}$$

and

$$\lim_{p\to\infty} b_p = \infty. \tag{4}$$

Indeed, if (3) were not true, then one could choose two sequences $\{p_1 < p_2 < \cdots\} \subset \mathbb{N}$, $z_j \in K_{p_j}$ $(j \in \mathbb{N})$ and a positive finite constant M with $|z_j| \leq M$

for all $j \in \mathbb{N}$. Hence some subsequence of (z_j) would converge to a finite point, which is absurd because $|z_j - z_k| \geq 2$ for all j, k with $j \neq k$. Then (3) holds. The proof of (4) is analogous.

By extracting a subsequence if necessary, we can assume that $|b_1| < |b_2| < |b_3| < \cdots$ and

$$|z| > 2p \qquad (z \in K_p, \ p \in \mathbb{N}). \tag{5}$$

For $n \geq n_0$, denote by $J(n)$ the least $m \in \mathbb{N}$ satisfying $B_n \cap K_m = \emptyset$ and $b_m \notin B_n$. According to (3), (4) and Lemma 2.3(b), the set

$$F_n := (A \setminus B^0_{n+1}) \cup B_n \cup \{b_p : \ p \geq J(n)\} \cup \bigcup_{p=J(n)}^{\infty} K_p$$

is an Arakelian subset of $\mathbb{C}$.

Now, we consider a partition of $\mathbb{N}$ into infinitely many sequences $\{p(n,1) < p(n,2) < \cdots < p(n,k) < \cdots\}$ $(n \in \mathbb{N})$.

Let us define, by induction, two sequences of functions $\{g_n\}_{n\geq n_0}$, $\{f_n\}_{n\geq n_0}$ (after it, we can complete, if necessary, the sequences by defining $g_j = g_{n_0}$, $f_j = f_{n_0}$ for $j = 1, \ldots, n_0 - 1$). Assuming that the functions $g_j \in A(F_j)$, $f_j \in E$ have been fixed for $j < n$, we define the function $g_n : F_n \to \mathbb{C}$ $(n \geq n_0)$ as

$$g_n(z) = \begin{cases} 0 \quad \text{if } z \in A \setminus B^0_{n+1} \\ P_n(z) \quad \text{if } z \in B_n \\ 1 + |b_{p+1}|\Psi(|b_{p+1}|) + \displaystyle\sum_{j=n_0}^{n-1} M_{f_j}(|b_p|) \quad \text{if } z = b_p \ (p \geq J(n)) \\ 0 \quad \text{if } z \in K_{p(j,k)} \ (j \in \mathbb{N} \setminus \{n\}, \ k \in \mathbb{N}, \ p(j,k) \geq J(n)) \\ P_k(z - a_{p(n,k)}) \quad \text{if } z \in K_{p(n,k)} \ (k \in \mathbb{N}, \ p(n,k) \geq J(n)), \end{cases}$$

where the sum $\sum_{j=n_0}^{n-1}$ is considered as 0 for $n = n_0$, which fixes g_{n_0} without ambiguity. The functions f_n $(n \geq n_0)$ are selected inductively by using the Arakelian theorem. Specifically, we consider the function $\varepsilon_n : [0, +\infty) \to (0, +\infty)$ given by

$$\varepsilon_n(t) = \frac{1}{n(1+t)\exp(t^{1/4} + t^{3/2}\varphi(t))}.$$

It is straightforward that each of these functions satisfies

$$\int_1^\infty t^{3/2} \log \frac{1}{\varepsilon_n(t)}\, dt < +\infty.$$

By Lemma 2.5, there exists an entire function f such that

$$|f_n(z) - g_n(z)| < \varepsilon_n(|z|) \qquad (z \in F_n). \tag{6}$$

According to (6) and the definition of g_n, ε_n, we have:

$$|f_n(z)| < \exp(-|z|^{1/4} - |z|^{3/2}\varphi(|z|)) \qquad (z \in A \setminus B^0_{n+1}), \tag{7}$$

$$|f_n(z) - P_n(z)| < \frac{1}{n} \qquad (z \in B_n), \tag{8}$$

$$\left| f_n(b_p) - 1 - |b_{p+1}|\Psi(|b_{p+1}|) - \sum_{j=n_0}^{n-1} M_{f_j}(|b_p|) \right| < 1 \qquad (p \geq J(n)), \tag{9}$$

$$|f_n(z)| < \frac{1}{1+|z|} \qquad (z \in K_{p(j,k)},\ j \neq n,\ k \in \mathbb{N},\ p(j,k) \geq J(n)) \tag{10}$$

$$|f_n(z) - P_{p(n,k)}(z - a_{p(n,k)})| < \frac{1}{1+|z|} \qquad (z \in K_{p(n,k)},\ k \in \mathbb{N},\ p(n,k) \geq J(n)). \tag{11}$$

It follows from (8) that (1) holds, so the linear manifold defined by (2) is dense in E. Then (i) has been proved. In order to prove (iii), it is enough to show, by linearity and by the fact that B^0_{n+1} is bounded, that

$$\lim_{\substack{n\to\infty \\ z \in A \setminus B^0_{n+1}}} \exp(\varphi(|z|)|z|^{3/2}) f_n(z) = 0 \qquad (n \geq n_0).$$

This derives from (7).

As for the growth of the members of M, let us fix $f = \sum_{j=n_0}^{N} \lambda_j f_j \in M \setminus \{0\}$, where the λ_j are complex constants and $\lambda_N \neq 0$. Note that, thanks to (9),

$$\begin{aligned} M_f(|b_p|) &\geq |f(b_p)| \\ &\geq |\lambda_N f_N(b_p)| - \sum_{j=n_0}^{N-1} |f_j(b_p)| \\ &\geq |\lambda_N f_N(b_p)| - \sum_{j=n_0}^{N-1} M_{f_j}(|b_p)|) \\ &> |b_{p+1}|\Psi(|b_{p+1}|) \qquad (p \geq J(N)). \end{aligned}$$

To each $r \geq |b_{J(N)}|$ we can associate a unique $p = p(r) \geq J(N)$ such that $|b_p| \leq r < |b_{p+1}|$. Since Ψ is increasing and (4) holds, we get

$$\frac{M_f(r)}{\Psi(r)} \geq \frac{M_f(|b_p|)}{\Psi(|b_{p+1}|)} > |b_{p(r)+1}| \to +\infty \qquad (r \to \infty),$$

which proves (iv).

It remains to prove that each function $f = \sum_{j=n_0}^{N} \lambda_j f_j$ as before (we may assume $\lambda_N = 1$, for a nonzero scalar multiple of a universal function is still universal) is Birkhoff-universal. To this end, we consider the ball B_k (with $k \geq J(N)$) and estimate for $z \in B_k$ the following:

$$\begin{aligned} |f(z + a_{p(N,k)}) - P_k(z)| &\leq |f_N(z + a_{p(N,k)}) - P_k(z)| + \sum_{j=n_0}^{N-1} |f_j(z + a_{p(N,k)})| \\ &< \frac{N - n_0 + 1}{1 + |z + a_{p(N,k)}|} \leq \frac{N - n_0 + 1}{1 + k} \leq \frac{N - n_0 + 1}{1 + |a_{p(N,k)}| - |z|} \\ &\leq \frac{N - n_0 + 1}{1 + 2p(N,k) - k} \leq \frac{N - n_0 + 1}{1 + k}. \end{aligned} \tag{12}$$

We have used (10), (11) together with the fact that if $z \in B_k$ then $z + a_{p(N,k)} \in \overline{B}(a_{p(N,k)}, k) \subset K_{p(N,k)}$. Thus

$$\lim_{k \to \infty} \sup_{B_k} |f(\cdot + a_{p(N,k)}) - P_k| = 0.$$

A new application of Lemma 2.7 shows us the denseness of $\{f(\cdot + a_{p(N,k)}) : k \geq 1\}$. Consequently, the translates of f form a dense subset of E, and this is (ii). $\square$

Remarks 2.9. 1. In the last theorem, we can replace property (ii) by the following stronger one:

(ii') For each $f \in M \setminus \{0\}$ and each non-zero operator $T : E \to E$ commuting with the translations, the function Tf is Birkhoff-universal.

Indeed, if T is as above, then $T = \Phi(D)$ for some non-zero entire function Φ with exponential type. According to Lemma 2.6, T is surjective. Hence T has dense range. Let $f \in M \setminus \{0\}$. By (ii), the set $\{T_a f : a \in \mathbb{C}\}$ is dense in E. Since T commutes with the translations, we obtain $\{T_a(Tf) : a \in \mathbb{C}\} =$

$T(\{T_a f : \ a \in \mathbb{C}\})$, that is dense because T has dense range. Thus Tf is universal.

2. If in the formulation of Theorem 2.8, we replace the increasing function ψ by a family $\mathcal{F} = \{h_n\}_{n=1}^{\infty}$ of non-constant entire functions then we can change (iv) to

$$\text{(iv')} \lim_{r\to\infty} \frac{\log M_{h_n}^{-1}(M_f(r))}{\log r} = \infty \text{ for all } f \in M \setminus \{0\} \text{ and all } n \in \mathbb{N}.$$

Indeed, we can construct an increasing function ψ satisfying $\psi(n) = \max\{M_{h_j}(n^n) : \ j = 1,\dots,n\}$ $(n \in \mathbb{N})$. Let M be the linear manifold constructed in Theorem 2.8 for this ψ. Fix $N \in \mathbb{N}$ and $f \in M \setminus \{0\}$. Since $\lim_{r\to\infty} \frac{M_f(r)}{\psi(r)} = +\infty$, we get an $r_0 > 0$ with $M_f(r) > \psi(r)$ for $r > r_0$. If $n > \max\{N, r_0\}$ one obtains $M_f(n) > \psi(n) \ge M_{h_N}(n^n)$. Hence $\log M_{h_N}^{-1}(M_f(n)) > n \log n$, and (iv') follows. Conversely, from the Weierstrass interpolation theorem for entire functions (see [30, Chapter 15]) it follows that if an increasing function $\psi : [0,+\infty) \to (0,+\infty)$ is prescribed and (iv') holds then we can take $h_n = h$ for all n, where h is an entire function with $h(k) = k\psi(k)$ $(k \ge 1)$. Then (iv) is derived easily by using that M_f and ψ are increasing.

3. For any unbounded set $A \subset \mathbb{C}$ and any function $g : A \to \mathbb{C}$ one has that $\lim_{\substack{z\to\infty \\ z\in A}} g(z)$ exists and equals w_0 if and only if $\lim_{\substack{z\to\infty \\ z\in B}} g(z)$ exists and equals w_0 for all unbounded sets $B \subset \mathbb{C}$ with $B \setminus A$ bounded. If we choose $\varphi(t) := (1+t)^{-3/2}$ and $A := \mathbb{C} \setminus \{z = x+iy : \ y^2 < x\}$, then for each compact set $K \subset \mathbb{C}$ and each $\varepsilon \in (0,1)$ the set $B := K + \{z : \ \varepsilon \le \arg z \le 2\pi(1-\varepsilon)\}$ satisfies that $B \setminus A$ is bounded. Since the sequence (a_n) in the proof of Theorem 2.8 can be selected here within $\mathbb{N}$, we obtain Costakis-Sambarino's result [19, Section 5] given in Section 1. And an adequate rotation of the last set A can be used to cover Gharibyan-Luh-Niess' theorem [24, Theorem 1.1] also given in the Introduction.

Theorem 2.10. *Let A be an unbounded subset of $\mathbb{C}$. Suppose that $\mu \in (0,+\infty)$ is such that $B(A,\mu) \subset A_0$ for some Arakelian set with $\varrho(\mathbb{C} \setminus A_0) = +\infty$. Assume that $\psi : [0,+\infty) \to (0,+\infty)$ is an increasing function.*

Then there exists a linear submanifold $M = M(A,\mu,\psi) \subset E$ satisfying the following:

(i) *M is dense in E.*

(ii) *For every $f \in M \setminus \{0\}$, f is Birkhoff-universal.*

(iii) $\displaystyle\lim_{\substack{z\to\infty \\ z\in A}} \exp(|z|^\alpha)(\Phi(D)f)(z) = 0$ *for all $f \in M$, all $\alpha < 1/2$ and all $\Phi \in E$ with $\tau(\Phi) < \mu$.*

(iv) $\displaystyle\lim_{r\to\infty} \frac{M_f(r)}{\psi(r)} = +\infty$ *for all $f \in M \setminus \{0\}$.*

Proof. According to the statement of Theorem 2.8 and its proof, for the function $\varphi(t) := \dfrac{1}{(2+t)\log^2(2+t)}$ $(t \geq 0)$ –which is continuous on $[0, +\infty)$ and integrable on $(1, +\infty)$– there are functions f_n $(n \in \mathbb{N})$ such that the linear manifold $M := \operatorname{span}\{f_n : \ n \in \mathbb{N}\}$ satisfies (i), (ii), (iv) and

$$\lim_{\substack{z\to\infty \\ z\in A_0}} \exp(|z|^{3/2}\varphi(|z|))f(z) = 0 \text{ for all } f \in M. \tag{13}$$

Then our unique task is to show that, given $f \in E$, then (iii) holds provided that (13) is true for f.

By hypothesis, $\overline{B}(z, \mu) \subset A_0$ for all $z \in A$. Fix a real number $\alpha < 1/2$ as well as a function $\Phi \in E$ with $\tau(\Phi) < \mu$. Choose any $\beta \in (\alpha, \frac{1}{2})$ and any $\nu \in (\tau(\Phi), \mu)$. Let $\Phi(z) = \sum_{n=0}^\infty c_n z^n$ be the Taylor expansion of Φ. According to [13, p. 11], we have the following expression for $\tau(\Phi)$:

$$\tau(\Phi) = \limsup_{n\to\infty} (n!|c_n|)^{1/n}.$$

Then there is a constant $C \in (0, +\infty)$ such that

$$|c_n| \leq C\,\frac{\nu^n}{n!} \qquad (n \in \mathbb{N}_0). \tag{14}$$

On the other hand, we have that

$$\lim_{t\to+\infty} \frac{(t+\mu)^\alpha}{t^{3/2}\varphi(t)} = 0.$$

In particular, there is $R_0 > 0$ such that

$$(|\xi| + \mu)^\alpha < |\xi|^{3/2}\varphi(|\xi|) \qquad (|\xi| > R_0). \tag{15}$$

Let $\varepsilon > 0$. By (13), there exists $R_1 > R_0$ satisfying

$$\exp(|\xi|^{3/2}\varphi(|\xi|))|f(\xi)| < \frac{\varepsilon(\mu-\nu)}{C\mu} \qquad (|\xi| > R_1). \tag{16}$$

If $z \in A$, then the counterclockwise oriented circle $\gamma(z)$ with center z and radius μ is contained in A_0. By using (14), (15), (16) and the Cauchy integral formula for derivatives, we obtain the following estimates, that are valid for $|z| > R_0 + \mu$:

$$|\exp(|z|^\beta)(\Phi(D)f)(z)| = \left|\exp(|z|^\beta)\sum_{n=0}^{\infty} c_n f^{(n)}(z)\right|$$

$$= \left|\exp(|z|^\beta)\sum_{n=0}^{\infty}\frac{n!c_n}{2\pi i}\oint_{\gamma(z)}\frac{f(\xi)}{(\xi-z)^{n+1}}\,d\xi\right|$$

$$\leq \exp(|z|^\beta)\sum_{n=0}^{\infty}\frac{n!|c_n|}{2\pi}\,2\pi\mu\cdot\frac{1}{\mu^{n+1}}\cdot\sup\{|f(\xi)| :\ \xi \in \overline{B}(z,\mu)\}$$

$$\leq \exp(|z|^\beta)\sum_{n=0}^{\infty} C\left(\frac{\nu}{\mu}\right)^n \sup\{|f(\xi)| :\ \xi \in \overline{B}(z,\mu)\}$$

$$\leq \frac{C\mu}{\mu-\nu}\cdot\sup\{\exp((|\xi|+\mu)^\alpha)|f(\xi)| :\ \xi \in \overline{B}(z,\mu)\}$$

$$\leq \frac{C\mu}{\mu-\nu}\cdot\sup\{\exp(|\xi|^{3/2}\varphi(|\xi|))|f(\xi)| :\ \xi \in \overline{B}(z,\mu)\} < \varepsilon.$$

This had to be shown. □

Remarks 2.11. 1. Under the hypotheses of the last theorem, we have that, in particular, the decay given by (iii) holds for every $\Phi \in E$ with subexponential type.

2. Concerning Theorem 2.10, the set $A := \mathbb{C} \setminus [\{z = x+iy :\ x \geq 0,\ 0 \leq y \leq \frac{1}{x+1}\} \cup \bigcup_{n=2}^{\infty} \overline{B}(2^n, n)$ is an Arakelian set with $\varrho(\mathbb{C}\setminus A) = +\infty$ such that, for any $\mu > 0$, $\overline{B}(A,\mu)$ is not contained in any Arakelian set A_0 satisfying $\varrho(\mathbb{C} \setminus A_0) = +\infty$.

3. Under the less restrictive hypothesis on A in Theorem 2.8, we get a linear manifold M satisfying (i), (ii), (iv) and $\lim_{\substack{z\to\infty \\ z\in A}} \exp(|z|^\alpha) f(z) = 0$ for all $\alpha < \frac{1}{2}$ and all $f \in M$.

If in the proof of Theorem 2.8 we replace the functions ε_n used there by the functions

$$\varepsilon_n(t) := \frac{1}{n(1+t)\exp(t^{1/4} + t^{3/2}\varphi(t) + \frac{t^{3/2}}{(2+t)\log^2(2+t)})} \qquad (n \ge 1,\ t \ge 0)$$

then we can combine the formulations and the proofs of both Theorems 2.8, 2.10 to yield the following result.

Theorem 2.12. *Assume that A is an unbounded subset of $\mathbb{C}$ and that there exists $\mu > 0$ satisfying $B(A,\mu) \subset A_0$ for some Arakelian set A_0 with $\varrho(\mathbb{C} \setminus A_0) = +\infty$. Suppose that $\varphi,\ \psi : [0,+\infty) \to (0,+\infty)$ are functions such that φ is continuous, ψ is increasing and φ is integrable on $(1,+\infty)$.*

Then there exists a linear submanifold $M = M(\varphi,\psi,\mu) \subset E$ satisfying the following properties:

(i) *M is dense in E.*

(ii) *Each nonzero function in M is Birkhoff-universal.*

(iii) $\lim_{\substack{z\to\infty \\ z\in A}} \exp(\varphi(|z|)|z|^{3/2}) f(z) = 0$ *for all* $f \in M$ *and*
$\lim_{\substack{z\to\infty \\ z\in A}} \exp(|z|^\alpha)(\Phi(D)f)(z) = 0$ *for all* $f \in M$*, all* $\alpha < \frac{1}{2}$ *and all* $\Phi \in E$ *with* $\tau(\Phi) < \mu$.

(iv) $\lim_{r\to\infty} \dfrac{M_f(r)}{\psi(r)} = +\infty$ *for all* $f \in M \setminus \{0\}$.

3 Concluding remarks

1. If we choose $A := \mathbb{C} \setminus \{z = x + iy :\ x > 1,\ -x^{1/2} < y < -x^{1/3}\}$ (with any $\mu > 0$) in Theorem 2.12 then every set $B \in \Sigma_1 \cup \Sigma_2$ (see notation in Section 1) satisfies that $B \setminus A$ is bounded. From this and Remarks 2.9.1, 2.9.2, first

part of 2.9.3 and 2.11.1, the results by Calderón-Moreno [17] given in the Introduction follow. But observe that our linear manifold *does not depend upon* α (the second limit in (iii) holds for all $\alpha < \frac{1}{2}$).

2. In view of Theorems 2.10 and 2.12, it is natural to pose the following question, whose answer is unknown to us: If the condition concerning μ is satisfied by A *for all* $\mu > 0$, does it hold that

$$\lim_{\substack{z\to\infty \\ z\in A}} \exp(|z|^\alpha)(Tf)(z) = 0$$

for all $\alpha < \frac{1}{2}$, all $f \in M$ and all operators $T : E \to E$ *commuting with the translations*?

3. In the harmonic setting, Bonilla's result [15, Theorem 1] stated in the Introduction can be extended, replacing the strips S by a fixed set $A \subset \mathbb{R}^N$ for which there are $\mu > 0$ and an Arakelian set $A_0 \subset \mathbb{R}^N$ with $B(A, \mu) \subset A_0$ and $\varrho(\mathbb{C}\setminus A_0) = +\infty$ (with similar meanings as in $\mathbb{C}$ for "Arakelian set", "$B(A, \mu)$" and "$\varrho(\cdot)$"). To see this, just use (as in [15]) Cauchy's estimates for harmonic functions [7, p. 3] (not needed if one only wants to get $\lim_{\substack{x\to\infty \\ x\in A}} ||x||^\alpha v(x) = 0$, that is, the case $\beta = (0, \ldots, 0)$) as well as the harmonic analogue of Arakelian's approximation theorem given in [5, Theorem 1.1].

4. Some restriction on α is necessary in Theorems 2.10, 2.12. Indeed, if $\alpha > 1/2$ and A is the sector $S_{1/\alpha}$ then an application of Phragmén-Lindeloff's theorem (see for instance [29]) yields that f is constant (so not universal) whenever $\lim_{\substack{z\to\infty \\ z\in S_{1/\alpha}}} \exp(|z|^\alpha)f(z) = 0$. We do not know whether $\alpha = 1/2$ is possible.

5. Existence of Birkhoff-universal entire functions enjoying some kind of controlled overall growth (but not necessary bounded on prescribed sets) was obtained in the analytic case by Duios-Ruis [21], Chan and Shapiro [18], Arakelian and Hakobian [2], and in the harmonic case by Armitage [3].

6. A stronger kind of universality is the so-called *frequent hypercyclicity*, a concept coined by Bayart and Grivaux in which some vector "visits many times" a prescribed open set under the action of an operator (see [8] and [16] for the precise definition and properties). In the special case of translation operators on E, Blasco, Bonilla and Grosse-Erdmann [12] have recently

proved the existence of frequent Birkhoff-universal entire functions whose growth satisfies certain restrictions.

References

[1] N.U. Arakelian, Approximation complexe et proprietés des fonctions analytiques, *Actes Congrès Intern. Math.* **2** (1970), 595–600.

[2] N. Arakelian and A. Hakobian, Entire functions with infinite sets of deficient functions, in: Entire functions in modern analysis (Tel-Aviv, 1997), 11–19, *Israel Math. Conf. Proc.* 15, Bar-Ilan Univ., Ramat Gan, 2001.

[3] D.H. Armitage, Permissible growth rates for Birkhoff type universal harmonic functions, *J. Math. Anal. Appl.* **136** (2005), 230–243.

[4] D.H. Armitage and P.M. Gauthier, Recent developments in harmonic approximation, with applications, *Results in Math.* **29** (1996), 1–15.

[5] D.H. Armitage and M. Goldstein, Better than uniform approximation on closed sets by harmonic functions with singularities, *Proc. London Math. Soc.* **60** (1990), 319–343.

[6] R.M. Aron, V.I. Gurariy and J.B. Seoane, Lineability and spaceability of sets of functions on $\mathbf{R}$, *Proc. Amer. Math. Soc.* **133** (2005), 795–803.

[7] S. Axler, P. Bourdon and W. Ramsey, *Harmonic Function Theory*, Springer, New York, 1992.

[8] F. Bayart and S. Grivaux, Frequently hypercyclic operators, *Trans. Amer. Math. Soc.* **358** (2006), 5083–5117.

[9] C.A. Berenstein and R. Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, New York, 1995.

[10] L. Bernal-González and A. Bonilla, Universality of entire functions bounded on closed sets, *J. Math. Anal. Appl.* **315** (2006), 302–316.

[11] G.D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189** (1929), 473–475.

[12] O. Blasco, A. Bonilla and K.G. Grosse-Erdmann, Rate of growth of frequently hypercyclic functions, *Preprint.*

[13] R.P. Boas, *Entire functions*, Academic Press, 1954.

[14] J. Bonet, F. Martínez-Giménez and A. Peris, Linear chaos on Fréchet spaces, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), 1649–1655.

[15] A. Bonilla, "Counterexamples" to harmonic Liouville's theorem and harmonic functions with zero nontangential limits, *Colloq. Math.* **83** (2000), 155–160.

[16] A. Bonilla and K.G. Grosse-Erdmann, Frequently hypercyclic operators and vectors, *Ergod. Th. Dynam. Sys.* **20** (2006), 1–23.

[17] M.C. Calderón-Moreno, Universal functions with small derivatives and extremely fast growth, *Analysis* **22** (2002), 57–66.

[18] K.C. Chan and J.H. Shapiro, The cyclic behavior of translation operators on Hilbert spaces of entire functions, *Indiana Math. J.* **40** (1991), 1421–1449.

[19] G. Costakis and M. Sambarino, Genericity of wild holomorphic functions and common hypercyclic vectors, *Adv. Math.* **182** (2004), 278–306.

[20] A.A. Danyelyan and G. Schmieder, Topological properties of level sets of entire functions, *Results Math.* **33** (1998), 266–273.

[21] S.M. Duios Ruis, On the existence of universal functions, *Soviet Math. Dokl.* **27** (1983), 9–13.

[22] L. Ehrenpreis, Mean periodic functions I, *Amer. J. Math.* **77** (1955), 293–328.

[23] D. Gaier, *Lectures on Complex Approximation*, Birkhäuser, Basel-London-Stuttgart, 1987.

[24] T.L. Gharibyan, W. Luh and M. Niess, Birkhoff functions that are bounded on prescribed sets, *Arch. Math.* **86** (2006), 261–267.

[25] G. Godefroy and J.H. Shapiro, Operators with dense invariant, cyclic vector manifolds, *J. Funct. Anal.* **98** (1991), 229–269.

[26] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* (*N.S.*) **36** (1999), 345–381.

[27] K.G. Grosse-Erdmann, Recent developments in hypercyclicity, *Rev. R. Acad. Cien. Serie A. Mat.* **97** (2003), 273–286.

[28] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier* (*Grenoble*) **6** (1955/1956), 271–355.

[29] A.I. Markushevich, *Theory of Functions of a Complex Variable*, Prentice-Hall, Englewood Cliffs, N.J, 1967.

[30] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York-St. Louis-San Francisco, 1987.

L. BERNAL-GONZÁLEZ and M.C. CALDERÓN-MORENO
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE MATEMÁTICAS, APDO. 1160
AVENIDA REINA MERCEDES
41080 SEVILLA, SPAIN
E-mails: `lbernal@us.es, mccm@us.es`

WOLFGANG LUH
FACHBEREICH MATHEMATIK
UNIVERSITÄT TRIER
D-54286 TRIER
GERMANY
E-mail: `luh@uni-trier.de`