Rooted Trees Searching for Cocyclic Hadamard Matrices over D_{4t}

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Abstract. A new reduction on the size of the search space for cocyclic Hadamard matrices over dihedral groups D_{4t} is described, in terms of the so called *central distribution*. This new search space adopt the form of a forest consisting of two rooted trees (the vertices representing subsets of coboundaries) which contains all cocyclic Hadamard matrices satisfying the constraining condition. Experimental calculations indicate that the ratio between the number of constrained cocyclic Hadamard matrices and the size of the constrained search space is greater than the usual ratio.

Keywords: Hadamard matrix, cocyclic matrix, dihedral groups.

1 Introduction

Since Hadamard matrices (that is, $\{1, -1\}$-square matrices whose rows are pairwise orthogonal) were introduced at the end of the XIXth century, the interest in their construction has grown substantially, because of their multiple applications (see [7] for instance).

For this reason, many attempts and efforts have been devoted to the design of good construction procedures, the latest involving heuristic techniques (see [5], [1], [4] for instance). Even alternative theoretical descriptions characterizing Hadamard matrices have been proposed (for instance, Ito's works involving Hadamard graphs [11] in the middle eighties, and Hadamard groups [6,12] more recently). But no matter what one may think, Hadamard matrices keep on being elusive anyway.

The point is that though it may be easily checked that the size of a Hadamard matrix is to be 2 or a multiple of 4, there is no certainty whether such a Hadamard matrix exists for every size $4t$. This is the *Hadamard conjecture*, which remains unsolved for more than a century.

In fact, the design of a procedure which outputs a Hadamard matrix of the desired size has shown to be as important as solving the Hadamard conjecture itself.

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In the early 90s, a surprising link between homological algebra and Hadamard matrices [9] led to the study of cocyclic Hadamard matrices [10]. The main advantages of the cocyclic framework are that:

- On one hand, the Hadamard test for cocyclic matrices [10] runs in $O(t^2)$ time, better than the $O(t^3)$ algorithm for usual (not necessarily cocyclic) Hadamard matrices.
- On the other hand, the search space reduces drastically, though it still is often of exponential size (see [4,2] for details).

Among the groups for which cocyclic Hadamard matrices have been found, it seems that dihedral groups D_{4t} are more likely to give a more density of cocyclic Hadamard matrices, even for every order multiple of 4 (see [8,3,2] for instance).

Unfortunately, the task of explicitly construct cocyclic Hadamard matrices over D_{4t} is considerably difficult, since the search space inherits a exponential size. New ideas dealing with this problem are welcome.

The purpose of this paper is to describe a new reduction on the size of the search space for cocyclic Hadamard matrices over dihedral groups D_{4t}. The key idea is exploiting the notions of *i-paths* and *intersections* introduced in [2], in order to design a forest consisting of two rooted trees (the vertices representing subsets of coboundaries) which contains all cocyclic Hadamard matrices satisfying the so called *central distribution*. We will explain these notions in the following section.

We organize the paper as follows. Section 2 is devoted to preliminaries on cocyclic matrices. Section 3 describes the new search space for cocyclic Hadamard matrices over D_{4t}. We include some final remarks and comments.

2 Preliminaries on Cocyclic Matrices

Consider a multiplicative group $G=\{g_1=1, g_2, \dots, g_{4t}\}$, not necessarily abelian. A cocyclic matrix M_f over G consists in a binary matrix $M_f = (f(g_i, g_j))$ coming from a 2-cocycle f over G, that is, a map $f : G \times G \to \{1, -1\}$ such that

$$f(g_i, g_j)f(g_ig_j, g_k) = f(g_j, g_k)f(g_i, g_jg_k), \qquad \forall\, g_i, g_j, g_k \in G.$$

We will only use normalized cocycles f (and hence normalized cocyclic matrices M_f), so that $f(1, g_j) = f(g_i, 1) = 1$ for all $g_i, g_j \in G$ (and correspondingly $M_f = (f(g_i, g_j))$ consists of a first row and column all of 1s).

A basis $\mathcal{B}$ for 2-cocycles over G consists of some representative 2-cocycles (coming from inflation and transgression) and some elementary 2-coboundaries ∂_i, so that every cocyclic matrix admits a unique representation as a Hadamard (pointwise) product $M = M_{\partial_{i_1}} \dots M_{\partial_{i_w}} \cdot R$, in terms of some coboundary matrices $M_{\partial_{i_j}}$ and a matrix R formed from representative cocycles.

Recall that every *elementary coboundary* ∂_d is constructed from the characteristic set map $\delta_d : G \to \{\pm 1\}$ associated to an element $g_d \in G$, so that

$$\partial_d(g_i, g_j) = \delta_d(g_i)\delta_d(g_j)\delta_d(g_ig_j) \qquad \text{for} \qquad \delta_d(g_i) = \begin{cases} -1 & g_d = g_i \\ 1 & g_d \neq g_i \end{cases} \qquad (1)$$

Although the elementary coboundaries generate the set of all coboundaries, they might not be linearly independent (see [3] for details). Moreover, since the elementary coboundary ∂_{g_1} related to the identity element in G is not normalized, we may assume that $\partial_{g_1} \notin \mathcal{B}$.

The cocyclic Hadamard test asserts that a cocyclic matrix is Hadamard if and only if the summation of each row (but the first) is zero [10]. In what follows, the rows whose summation is zero are termed *Hadamard rows.*

This way, a cocyclic matrix M_f is Hadamard if and only if every row $(M_f)_i$ is a Hadamard row, $2 \leq i \leq 4t$.

In [2] the Hadamard character of a cocyclic matrix is described in an equivalent way, in terms of *generalized coboundary matrices*, *i-walks* and *intersections*. We reproduce now these notions.

The *generalized coboundary matrix* $\bar{M}_{\partial_j}$ related to a elementary coboundary ∂_j consists in negating the j^{th}-row of the matrix M_{∂_j}. Note that negating a row of a matrix does not change its Hadamard character. As it is pointed out in [2], every generalized coboundary matrix $\bar{M}_{\partial_j}$ contains exactly two negative entries in each row $s \neq 1$, which are located at positions (s,i) and (s,e), for $g_e = g_s^{-1} g_i$. We will work with generalized coboundary matrices from now on.

A set $\{\bar{M}_{\partial_{i_j}} : 1 \leq j \leq w\}$ of generalized coboundary matrices defines an *i-walk* if these matrices may be ordered in a sequence $(\bar{M}_{l_1}, \ldots, \bar{M}_{l_w})$ so that consecutive matrices share exactly one negative entry at the i^{th}-row. Such a walk is called an *i-path* if the initial and final matrices do not share a common -1, and an *i-cycle* otherwise. As it is pointed out in [2], every set of generalized coboundary matrices may be uniquely partitioned into disjoint maximal i-walks.

From the definition above, it is clear that every maximal i-path contributes two negative occurrences at the i^{th}-row. This way, a characterization of Hadamard rows (consequently, of Hadamard matrices) may be easily described in terms of i-paths.

Proposition 1. *[2] The i^{th} row of a cocyclic matrix $M = M_{\partial_{i_1}} \ldots M_{\partial_{i_w}} \cdot R$ is Hadamard if and only if*

$$2c_i - 2I_i = 2t - r_i \tag{2}$$

where c_i denotes the number of maximal i-paths in $\{\bar{M}_{\partial_{i_1}}, \ldots, \bar{M}_{\partial_{i_w}}\}$, r_i counts the number of -1s in the i^{th}-row of R and I_i indicates the number of positions in which R and $\bar{M}_{\partial_{i_1}} \ldots \bar{M}_{\partial_{i_w}}$ share a common -1 in their i^{th}-row.

From now on, we will refer to the positions in which R and $\bar{M}_{\partial_{i_1}} \ldots \bar{M}_{\partial_{i_w}}$ share a common -1 in a given row simply as *intersections*, for brevity.

We will now focus on the case of dihedral groups.

3 Cocyclic Matrices over D_{4t}

Denote by D_{4t} the dihedral group $\mathbb{Z}_{2t} \times_\chi \mathbb{Z}_2$ of order $4t$, $t \geq 1$, given by the presentation

$$< a, b | a^{2t} = b^2 = (ab)^2 = 1 >$$

and ordering

$$\{1=(0,0), a=(1,0), \ldots, a^{2t-1}=(2t-1,0), b=(0,1), \ldots, a^{2t-1}b=(2t-1,1)\}$$

In [6] a representative 2-cocycle f of $[f] \in H^2(D_{4t}, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$ is written interchangeably as a triple (A, B, K), where A and B are the inflation variables and K is the transgression variable. All variables take values ± 1. Explicitly,

$$f(a^i, a^j b^k) = \begin{cases} A^{ij}, & i+j < 2t, \\ A^{ij}K, & i+j \geq 2t, \end{cases} \qquad f(a^i b, a^j b^k) = \begin{cases} A^{ij}B^k, & i \geq j, \\ A^{ij}B^k K, & i < j, \end{cases}$$

Let β_1, β_2 and γ denote the representative 2-cocycles related to $(A, B, K) = (-1,1,1), (1,-1,1), (1,1,-1)$ respectively.

A basis for 2-coboundaries is described in [2], and consists of the elementary coboundaries $\{\partial_a, \ldots, \partial_{a^{2t-3}b}\}$. This way, a basis for 2-cocycles over D_{4t} is given by $\mathcal{B} = \{\partial_a, \ldots, \partial_{a^{2t-3}b}, \beta_1, \beta_2, \gamma\}$.

Computational results in [6,2] suggest that the case $(A, B, K) = (1, -1, -1)$ contains a large density of cocyclic Hadamard matrices.

Furthermore, as it is pointed out in Theorem 2 of [2], cocyclic matrices over D_{4t} using $R = \beta_2\gamma$ are Hadamard matrices if and only if rows from 2 to t are Hadamard, so that the cocyclic test runs four times faster than usual.

From now on, we assume that $R = M_{\beta_2} \cdot M_\gamma = \begin{pmatrix} A & A \\ B & -B \end{pmatrix}$ for

$$A = \begin{pmatrix} 1 & 1 \cdots & 1 \\ 1 & \iddots & -1 \\ \vdots & \iddots \iddots & \vdots \\ 1 & -1 \cdots & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \cdots & -1 \\ \vdots & \ddots \ddots & \vdots \\ 1 & \ddots & -1 \\ 1 & 1 \cdots & 1 \end{pmatrix} \tag{3}$$

Now we would like to know how the 2-coboundaries in $\mathcal{B}$ have to be combined to form i-paths, $2 \leq i \leq t$. This information is given in Proposition 7 of [2].

Proposition 2. *[2] For $1 \leq i \leq 2t$, a maximal i-walk consists of a maximal subset in*

$$(M_{\partial_1}, \ldots, M_{\partial_{2t}}) \quad or \quad (M_{\partial_{2t+1}}, \ldots, M_{\partial_{4t}})$$

formed from matrices $(\ldots, M_j, M_k, \ldots)$ which are cyclically separated in $i-1$ positions (that is $j \pm (i-1) \equiv k \mod 2t$).

Notice that since $r_i = 2(i-1)$ for $2 \leq i \leq t$, the cocyclic Hadamard test reduces to check whether $c_i - I_i = t - i + 1$, for $2 \leq i \leq t$. Thus c_i uniquely determines I_i and reciprocally, $2 \leq i \leq t$.

In fact, the way in which intersections may be introduced at the i^{th}-row is uniquely determined. More explicitly

Lemma 1. *The following table gives a complete distribution of the coboundaries in $\mathcal{B}$ which may create an intersection at a given row. For clarity in the reading, we note the generalized coboundary $\bar{M}_{\partial_i}$ simply by i:*

row	$coboundaries$
2	$2t, 2t+1$
3	$2, 2t-1, 2t, 2t+1, 2t+2$
$4 \le k \le t$	$2, \ldots, k-1, 2t-k+2, \ldots, 2t+k-1, 4t-k+2, \ldots, 4t-2$

Proof
It may be seen by inspection, taking into account the distribution of the negative occurrences in R and the form of the generalized coboundary matrices. □

Lemma 2. *In particular, there are some coboundaries which do not produce any intersection at all, at rows* $2 \le k \le t$*, which we term free intersection coboundaries. More concretely,*

t	$coboundaries$
2	$2, 3, 6$
$t > 2$	$t, t+1, 3t, 3t+1$

Proof
It suffices to take the set difference between $\mathcal{B}$ and the set of coboundaries used in the lemma above. □

Lemma 3. *Furthermore, the following table distributes the coboundaries which produce a intersection at every row, so that coboundaries which produce the same negative occurrence at a row are displayed vertically in the same column.*

row	$coboundaries$									
2					$2t$	$2t+1$				
3				$\boxed{2t-1}$	$2t$ $\boxed{2}$	$2t+1$	$\boxed{2t+2}$			
$4 \le k \le t$	$\boxed{2t-k+2}$	$2t-k+3$ 2	$\ldots$ $\ldots$	$2t-1$ $k-2$	$2t$ $\boxed{k-1}$	$2t+1$ $\boxed{4t-k+2}$	$2t+2$ $4t-k+1$	$\ldots$ $\ldots$	$2t+k-3$ $4t-2$	$2t+k-2$	$\boxed{2t+k-1}$

Proof
It may be seen by inspection. □

Remark 1. Notice that:

- The set of coboundaries which may produce an intersection at the i^{th}-row is included in the analog set corresponding to the $(i+1)^{th}$-row.
- The boxed coboundaries do not produce any intersection at the precedent rows.

Now one could ask whether cocyclic Hadamard matrices exist for any formal distribution of pairs (c_i, I_i) satisfying the relations $c_i - I_i = t - i + 1$, for $2 \le i \le t$. Actually, this is not the case.

Proposition 3. *Not all of the formal sequences* $[(c_2, I_2), \ldots, (c_t, I_t)]$ *satisfying* $c_i - I_i = t - i + 1$ *give rise to cocyclic Hadamard matrices over* D_{4t}*, for* $t \ge 3$.

Proof

Proposition 10 of [2] bounds the number w of coboundaries in $\mathcal{B}$ to multiply with R so that a cocyclic Hadamard matrix is formed, so that $t-1 \leq w \leq 3t+2$.

In particular, for $t \geq 6$, we know that $5 \leq w$. Consequently, the case $I_2 = \ldots = I_t = 0$ is not feasible, since from Lemma 2 we know that only up to 4 coboundaries may be combined so that no intersection is generated at any row.

This proves the Lemma for $t \geq 6$. We now study the remaining cases.

Taking into account that $0 \leq I_i \leq r_i = 2i-2$, we may have a look in the way in which cocyclic Hadamard matrices are distributed regarding the number of intersections I_i, for those groups D_{4t} for which the whole set of cocyclic Hadamard matrices have been computed until now. These are precisely $t = 2, 3, 4, 5$.

For $t=2$, we formally have 3 solutions for the equation $c_2 - I_2 = 1$,

c_2	1	2	3
I_2	0	1	2

Each of these solutions gives rise to some cocyclic Hadamard matrices M_f,

I_2	0	1	2
$\lvert M_f \rvert$	4	10	2

For $t=3$, we formally have 15 solutions for the system $\left\{\begin{array}{l} c_2 - I_2 = 2 \\ c_3 - I_3 = 1 \end{array}\right.$ coming from the combination of any solution of each of the equations

c_2	2	3	4
I_2	0	1	2

c_3	1	2	3	4	5
I_3	0	1	2	3	4

Only 9 of the 15 hypothetical solutions are real solutions (there are no combinations of coboundaries meeting the other 6 "theoretical" solutions), distributed in the following way:

(I_2, I_3)	$(0,1)$	$(0,2)$	$(0,3)$	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$\lvert M_f \rvert$	6	10	2	8	20	8	2	10	6

For $t=4$, we formally have 105 solutions for the system $\left\{\begin{array}{l} c_2 - I_2 = 3 \\ c_3 - I_3 = 2 \\ c_4 - I_4 = 1 \end{array}\right.$ coming from the combination of any solution of each of the equations

c_2	3	4	5
I_2	0	1	2

c_3	2	3	4	5	6
I_3	0	1	2	3	4

c_4	1	2	3	4	5	6	7
I_4	0	1	2	3	4	5	6

Only 36 of the 105 hypothetical solutions are real solutions, distributed in the following way:

(I_2, I_3, I_4)	$(0,0,2)$	$(0,1,1)$	$(0,1,2)$	$(0,1,3)$	$(0,1,4)$	$(0,2,2)$	$(0,2,3)$	$(0,2,4)$	$(0,2,5)$
$\|M_f\|$	8	6	18	18	6	12	12	24	12
(I_2, I_3, I_4)	$(0,3,3)$	$(0,3,4)$	$(1,0,3)$	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,1,5)$	$(1,2,1)$
$\|M_f\|$	4	8	12	18	12	8	4	6	4
(I_2, I_3, I_4)	$(1,2,2)$	$(1,2,3)$	$(1,2,4)$	$(1,2,5)$	$(1,3,2)$	$(1,3,3)$	$(1,3,4)$	$(1,4,3)$	$(2,1,2)$
$\|M_f\|$	24	72	24	4	12	32	20	20	2
(I_2, I_3, I_4)	$(2,2,1)$	$(2,2,2)$	$(2,2,3)$	$(2,2,4)$	$(2,3,2)$	$(2,3,3)$	$(2,1,3)$	$(2,1,4)$	$(2,1,5)$
$\|M_f\|$	6	6	2	12	24	12	12	24	12

For $t = 5$, we formally have 945 solutions for the system $\begin{cases} c_2 - I_2 = 4 \\ c_3 - I_3 = 3 \\ c_4 - I_4 = 2 \\ c_5 - I_5 = 1 \end{cases}$ coming from the combination of any solution of each of the equations

c_2	4	5	6
I_2	0	1	2

c_3	3	4	5	6	7
I_3	0	1	2	3	4

c_4	2	3	4	5	6	7	8
I_4	0	1	2	3	4	5	6

c_5	1	2	3	4	5	6	7	8	9
I_4	0	1	2	3	4	5	6	7	8

Only 153 of the 945 hypothetical solutions are real solutions,

(I_2, I_3, I_4, I_5)	$(0,0,1,3)$	$(0,0,2,3)$	$(0,1,1,4)$	$(0,1,2,2)$	$(0,1,2,3)$	$(0,1,2,4)$
$\|M_f\|$	4	12	3	6	12	21
(I_2, I_3, I_4, I_5)	$(0,1,2,5)$	$(0,1,3,2)$	$(0,1,3,3)$	$(0,1,3,4)$	$(0,1,3,5)$	$(0,1,3,6)$
$\|M_f\|$	6	6	12	21	21	9
(I_2, I_3, I_4, I_5)	$(0,1,3,7)$	$(0,1,4,3)$	$(0,1,4,4)$	$(0,1,4,5)$	$(0,1,4,6)$	$(0,1,4,7)$
$\|M_f\|$	3	6	3	9	18	6
(I_2, I_3, I_4, I_5)	$(0,2,0,2)$	$(0,2,1,2)$	$(0,2,1,3)$	$(0,2,1,4)$	$(0,2,2,2)$	$(0,2,2,3)$
$\|M_f\|$	2	4	2	2	4	12
(I_2, I_3, I_4, I_5)	$(0,2,2,4)$	$(0,2,3,2)$	$(0,2,3,3)$	$(0,2,3,4)$	$(0,2,3,5)$	$(0,2,3,6)$
$\|M_f\|$	4	8	8	24	14	2
(I_2, I_3, I_4, I_5)	$(0,2,4,4)$	$(0,2,4,5)$	$(0,2,4,6)$	$(0,2,5,4)$	$(0,3,2,5)$	$(0,3,3,2)$
$\|M_f\|$	12	26	4	4	2	2
(I_2, I_3, I_4, I_5)	$(0,3,3,3)$	$(0,3,3,4)$	$(0,3,3,5)$	$(0,3,4,2)$	$(0,3,4,3)$	$(0,3,4,4)$
$\|M_f\|$	7	6	6	1	4	5
(I_2, I_3, I_4, I_5)	$(0,3,4,5)$	$(0,3,5,4)$	$(1,0,1,4)$	$(1,0,2,2)$	$(1,0,2,3)$	$(1,0,2,4)$
$\|M_f\|$	6	1	2	4	2	4
(I_2, I_3, I_4, I_5)	$(1,0,4,2)$	$(1,0,4,3)$	$(1,0,4,4)$	$(1,0,5,4)$	$(1,1,1,3)$	$(1,1,1,4)$
$\|M_f\|$	4	2	4	2	6	3
(I_2, I_3, I_4, I_5)	$(1,1,2,3)$	$(1,1,2,4)$	$(1,1,2,5)$	$(1,1,2,6)$	$(1,1,3,2)$	$(1,1,3,3)$
$\|M_f\|$	7	14	22	8	4	16
(I_2, I_3, I_4, I_5)	$(1,1,3,4)$	$(1,1,3,5)$	$(1,1,3,6)$	$(1,1,3,7)$	$(1,1,4,3)$	$(1,1,4,4)$
$\|M_f\|$	20	8	16	16	5	10
(I_2, I_3, I_4, I_5)	$(1,1,4,5)$	$(1,1,4,6)$	$(1,1,5,3)$	$(1,1,5,4)$	$(1,2,1,3)$	$(1,2,1,4)$
$\|M_f\|$	18	8	2	1	4	8

(I_2,I_3,I_4,I_5)	$(1,2,2,2)$	$(1,2,2,3)$	$(1,2,2,4)$	$(1,2,2,5)$	$(1,2,3,2)$	$(1,2,3,3)$
$\lvert M_f\rvert$	4	16	28	16	24	20
(I_2,I_3,I_4,I_5)	$(1,2,3,4)$	$(1,2,3,5)$	$(1,2,4,2)$	$(1,2,4,3)$	$(1,2,4,4)$	$(1,2,4,5)$
$\lvert M_f\rvert$	32	56	4	12	24	24
(I_2,I_3,I_4,I_5)	$(1,2,5,4)$	$(1,3,1,2)$	$(1,3,1,3)$	$(1,3,1,4)$	$(1,3,2,2)$	$(1,3,2,3)$
$\lvert M_f\rvert$	8	1	3	2	8	16
(I_2,I_3,I_4,I_5)	$(1,3,2,4)$	$(1,3,2,5)$	$(1,3,3,3)$	$(1,3,3,4)$	$(1,3,3,5)$	$(1,3,4,2)$
$\lvert M_f\rvert$	10	6	24	32	16	8
(I_2,I_3,I_4,I_5)	$(1,3,4,3)$	$(1,3,4,4)$	$(1,3,4,5)$	$(1,3,5,2)$	$(1,3,5,3)$	$(1,3,5,4)$
$\lvert M_f\rvert$	16	22	10	3	9	6
(I_2,I_3,I_4,I_5)	$(1,4,1,3)$	$(1,4,2,3)$	$(1,4,2,4)$	$(1,4,3,3)$	$(1,4,3,4)$	$(1,4,4,3)$
$\lvert M_f\rvert$	2	6	6	16	8	6
(I_2,I_3,I_4,I_5)	$(1,4,4,4)$	$(1,4,5,3)$	$(2,1,2,3)$	$(2,1,2,4)$	$(2,1,2,5)$	$(2,1,2,6)$
$\lvert M_f\rvert$	6	2	2	1	3	6
(I_2,I_3,I_4,I_5)	$(2,1,2,7)$	$(2,1,3,2)$	$(2,1,3,3)$	$(2,1,3,4)$	$(2,1,3,5)$	$(2,1,3,6)$
$\lvert M_f\rvert$	2	2	4	7	7	3
(I_2,I_3,I_4,I_5)	$(2,1,3,7)$	$(2,1,4,2)$	$(2,1,4,3)$	$(2,1,4,4)$	$(2,1,4,5)$	$(2,1,5,4)$
$\lvert M_f\rvert$	1	2	4	7	2	1
(I_2,I_3,I_4,I_5)	$(2,2,1,4)$	$(2,2,2,4)$	$(2,2,2,5)$	$(2,2,2,6)$	$(2,2,3,2)$	$(2,2,3,3)$
$\lvert M_f\rvert$	4	12	26	4	8	8
(I_2,I_3,I_4,I_5)	$(2,2,3,4)$	$(2,2,3,5)$	$(2,2,3,6)$	$(2,2,4,2)$	$(2,2,4,3)$	$(2,2,4,4)$
$\lvert M_f\rvert$	24	14	2	4	12	4
(I_2,I_3,I_4,I_5)	$(2,2,5,2)$	$(2,2,5,3)$	$(2,2,5,4)$	$(2,2,6,2)$	$(2,3,1,4)$	$(2,3,2,2)$
$\lvert M_f\rvert$	4	2	2	2	3	3
(I_2,I_3,I_4,I_5)	$(2,3,2,3)$	$(2,3,2,4)$	$(2,3,2,5)$	$(2,3,3,2)$	$(2,3,3,3)$	$(2,3,3,4)$
$\lvert M_f\rvert$	12	15	18	6	21	18
(I_2,I_3,I_4,I_5)	$(2,3,3,5)$	$(2,3,4,5)$	$(2,4,2,3)$			
$\lvert M_f\rvert$	18	6	12			

Attending to the tables above, we conclude that, for $2 \leq t \leq 5$, there is a large density of cocyclic Hadamard matrices in the case $c_i = t$ for $2 \leq i \leq t$, that is, $(I_2, \ldots, I_t) = (1, \ldots t-1)$. We call this case the *central distribution* for intersections and i-paths on D_{4t}.

We include now a table comparing the number *central* of cocyclic Hadamard matrices in the central distribution with the proportion $\% = \frac{|M_f|}{cases}$ of the amount $|M_f|$ of cocyclic Hadamard matrices over D_{4t} by the total number *cases* of valid distributions of intersections $(I_2, \ldots, I_t)$. The last column contains the number of cocyclic Hadamard matrices of the most prolific case:

t	*cases*	$\lvert M_f\rvert$	%	*central*	*best*
2	3	16	5.33	10	10
3	9	72	8	20	20
4	36	512	14.22	72	72
5	153	1400	9.15	32	56

It seems then reasonable trying to constraint the search for cocyclic Hadamard matrices over D_{4t} to the central distribution case.

The search space in the central distribution $(I_2, \ldots, I_t) = (1, \ldots, t-1)$ may be represented as a forest of two rooted trees of depth $t-1$. We identify each level of the tree to the correspondent row of the cocyclic matrix at which intersections are being counted, so that the roots of the trees are located at level 2 (corresponding to the intersections created at the second row of the cocyclic matrix).

This way the level i contains those coboundaries which must be added to the father configuration in order to get the desired $i-1$ intersections at the i^{th}-row, for $2 \leq i \leq t$.

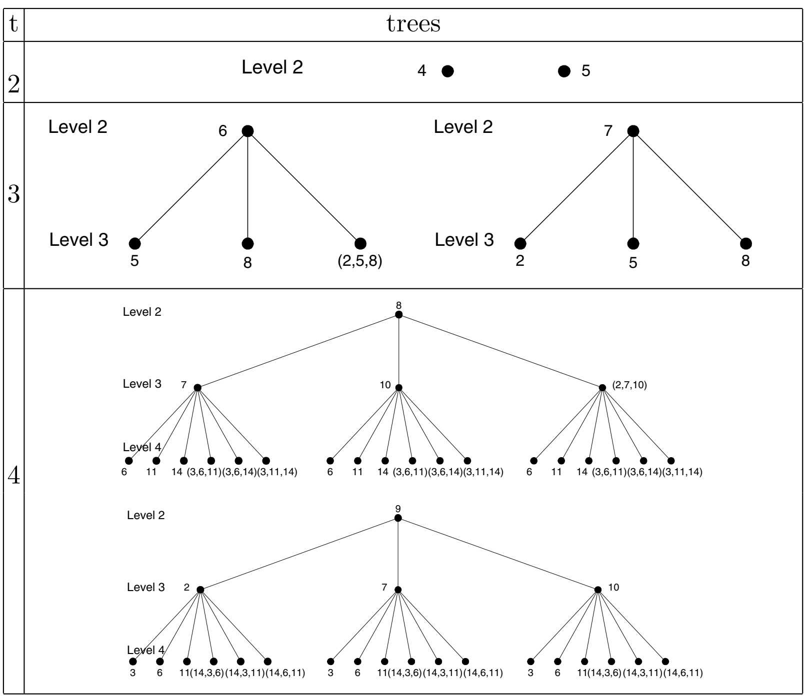
The root of the first tree is ∂_{2t}, whereas the root of the second tree is ∂_{2t+1}, since from Lemma 1 these are the only coboundaries which may give an intersection at the second row.

As soon as one of these coboundaries is used, the other one is forbidden, since otherwise a second intersection would be introduced at the second row.

Now one must add some coboundaries to get two intersections at the third row. Since ∂_{2t} is already used and ∂_{2t+1} is forbidden, there are only 3 coboundaries left (those boxed in the table of Lemma 3).

Successively, in order to construct the nodes at level k, one must add some of the correspondent boxed coboundaries of the table of Lemma 3, since the remaining coboundaries are either used or forbidden.

We include the forests corresponding to the cases $t = 2, 3, 4$ for clarity.



Every branch ending at level t gives a cocyclic matrix meeting the desired distribution of intersections $(I_2, \ldots, I_t)$. Now one has to check whether any of the 16 possible combinations with the free intersection coboundaries $\{\partial_t, \partial_{t+1}, \partial_{3t}, \partial_{3t+1}\}$ of Lemma 2 gives rise to a cocyclic Hadamard matrix (that is, to the desired distribution of i-paths, $(c_2, \ldots, c_t) = (t, \ldots, t)$).

We now give some properties of the trees above.

Proposition 4. *In the circumstances above, it may be proved that*

1. *The skeleton (i.e., the branches, forgetting about the indexes of the coboundaries used) of the trees related to D_{4t} are preserved to form the levels from 2 to t corresponding to the trees of $D_{4(t+1)}$.*
2. *Among the boxed coboundaries $\{\partial_{2t-k+2}, \partial_{k-1}, \partial_{4t-k+2}, \partial_{2t+k-1}$ to be added at level k of the trees, precisely one of them removes an intersection, whereas the remaining three adds one intersection each.*
3. *At each level, either just one or exactly three boxed coboundaries must be used, there is no other possible choice in order to meet the desired amount of intersections.*
4. *Consequently, a branch may be extended from level k to level $k+1$ if and only if $k - h_k \in \{-1, 1, 3\}$, where h_k denotes the number of intersections inherited from level k to level $k+1$.*
5. *Branches ending at levels above level t will never give rise to cocyclic Hadamard matrices meeting the central distribution. This will be more frequent the greater t is.*
6. *Both trees may have branches ending at level t which may not produce any cocyclic Hadamard matrix at all. This will be more frequent the greater t is.*

Proof

Most of the properties are consequences of the results explained in Lemma 1 through Lemma 3, and are left to the reader.

Property 3 comes as a result of a parity condition: there must be an odd number of intersections at odd levels, and an even number of intersections at even levels. Since boxed coboundaries either add an intersection each, or just one of them removes an intersection (added by a coboundary previously used), the parity condition leads to the result.

Concerning Property 5, we give a branch not reaching the last level for $t = 9$:

level	2	3	4	5	6	7	8
cob.	18	17	21	4, 22, 33	32	6, 13, 24	12, 25, 30

Concerning Property 6, we give a branch reaching the last level for $t = 9$, which do not give rise to any cocyclic Hadamard matrix at all:

level	2	3	4	5	6	7	8	9
cob.	19	17	21	22	14	6, 24, 31	12, 25, 30	29

□

So far, it is evident that the above trees reduce the search space for cocyclic Hadamard matrices over D_{4t}, constraining the solutions to the central distribution case.

There is only one question left. Is the new proportion $ratio_c$ of cocyclic Hadamard matrices in the central distribution case by the size of the reduced space greater than the proportion $ratio_g$ of general cocyclic Hadamad matrices by the size of the general search space?

It seems so, attending to the table below (we have followed the calculations of [2] about the size of the general search space in D_{4t}).

t	$\|M_f\|$	g. size	$ratio_g$	$\|M_f\|$	central size	$ratio_c$
2	16	32	0.5	10	16	0.625
3	72	492	0.146	20	96	0.208
4	512	8008	0.063	72	576	0.125

We claim that developing a heuristic search in the forest described above will produce some cocyclic Hadamard matrices over D_{4t} more likely than any other technique applied till now to the general case.

This will be the goal of our work in the near future.

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