Generalized binary arrays from quasi-orthogonal cocycles

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Abstract. Generalized perfect binary arrays (GPBAs) were used by Jedwab to construct perfect binary arrays. A non-trivial GPBA can exist only if its energy is 2 or a multiple of 4. This paper introduces *generalized optimal binary arrays* (GOBAs) with even energy not divisible by 4, as analogs of GPBAs. We give a procedure to construct GOBAs based on a characterization of the arrays in terms of 2-cocycles. As a further application, we determine negaperiodic Golay pairs arising from generalized optimal binary sequences of small length.

1 Introduction

Let $\phi = (\phi(0), \ldots, \phi(n-1)) \in \{\pm 1\}^n$ be a binary sequence of length n. Reading arguments modulo n,

$$R_\phi(w) := \sum_{k=0}^{n-1} \phi(k)\phi(k+w)$$

is the *periodic autocorrelation of* ϕ *at shift* w. The *expansion* of ϕ, denoted ϕ', is the concatenation of ϕ and $-\phi$ (in that order). A pair ϕ_1, ϕ_2 of binary sequences, each of length $2t$, such that $R_{\phi_1'}(w) + R_{\phi_2'}(w) = 0$ for $1 \leq w \leq 2t-1$ (equivalently, for $1 \leq w \leq 4t-1$ and $w \neq 2t$), is a *negaperiodic Golay pair* (NGP). Note that the original definition of NGP in [4] coincides with the definition above by [8, Lemma 2].

We seek good sources of NGPs. This objective is connected to several existence problems in algebraic design theory. For example, Egan showed that NGPs of length $2t$ are equivalent to certain relative $(4t, 2, 4t, 2t)$-difference sets in the dicyclic group Q_{8t} of order $8t$ [8, Theorem 3]. Actually, there is a relative $(4t, 2, 4t, 2t)$-difference set in a central extension E of $\mathbb{Z}_2$ by a group G of order $4t$, relative to $\mathbb{Z}_2$, if and only if there is a Hadamard matrix of order $4t$ whose expanded (group-divisible) design admits a special regular action by E: a *cocyclic* Hadamard matrix over G [6, Theorem 2.4]. By way of [9, Theorem 3.3], Ito [13, p. 370] conjectured that Q_{8t} contains such relative difference sets for all t. Schmidt [16] has verified Ito's conjecture up to $t = 46$. Our recent paper [3] initiated the study of *quasi-orthogonal* cocycles over groups G of even order not divisible by 4, in direct analogy with cocyclic Hadamard matrices. The present paper builds on [3].

It is easy to see that

$$\max_{0<w<n} |R_\phi(w)| \geq \begin{cases} 0 & n \equiv 0 \mod 4 \\ 1 & n \equiv 1 \text{ or } 3 \mod 4 \\ 2 & n \equiv 2 \mod 4. \end{cases} \tag{1}$$

The sequence ϕ is *optimal* if equality holds in (1). In particular, ϕ is *perfect* if $R_\phi(w) = 0$ for $0 < w < n$. No perfect binary sequence of length $n > 4$ is known. Attention consequently turns to the larger class of *perfect binary arrays* (PBAs). Jedwab [14] introduced *generalized perfect binary arrays* (GPBAs) to aid in the construction of PBAs. Hughes [11] subsequently demonstrated the cocyclic nature of GPBAs.

A *generalized perfect binary sequence* (GPBS) is a 1-dimensional GPBA; such ϕ have $R_{\phi'}(w) = 0$ for all w. Each pair of GPBSs is obviously an NGP. However, a GPBS exists only if $n = 2$ [14, Result 4.8]. So let $n > 2$ be even; since $R_{\phi'}(w)$ is divisible by 4, and not every $R_{\phi'}(w)$ is 0, some $|R_{\phi'}(w)|$ must be at least 4. Thus, we will say that ϕ of length $2t$ is a *generalized optimal binary sequence* (GOBS) if $\max_{0<w<2t} |R_{\phi'}(w)| = 4$. Equivalently, ϕ is a GOBS if, for $0 < w < 2t$,

$$|R_{\phi'}(w)| = \begin{cases} 0 & w \text{ odd} \\ 4 & w \text{ even} \end{cases}$$

when t is odd, and

$$|R_{\phi'}(w)| = \begin{cases} 4 & w \text{ odd} \\ 0 & w \text{ even} \end{cases}$$

when t is even. We propose searching for NGPs in the set of GOBs of length $2t$, t odd.

Just as the notion of GPBA extends that of GPBS to dimensions greater than 1, a GOBA (*generalized optimal binary array*) is a higher-dimensional version of a GOBS. Section 3 treats GPBAs and GOBAs from the perspective of [3]. We prove a one-to-one correspondence between GOBAs, quasi-orthogonal cocycles over abelian groups, and abelian relative quasi-difference sets. In Section 4, we outline and apply a method to find NGPs among GOBSs that correspond to quasi-orthogonal cocycles over cyclic groups. The concluding Section 5 looks at an important question for cocyclic designs prompted by the analysis in Section 4.

2 Quasi-orthogonal cocycles and related combinatorial structures

Let G and U be finite groups, with U abelian. A map $\psi : G \times G \to U$ such that $\psi(1,1) = 1$ and

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k) \quad \forall g,h,k \in G \tag{2}$$

is a (normalized) *cocycle* over G. If $\phi : G \to U$ is any map that is normalized (i.e., $\phi(1) = 1$) then $\partial\phi(g,h) = \phi(g)\phi(h)\phi(gh)$ defines a cocycle $\partial\phi$, called a *coboundary*. The set of all cocycles over G forms an abelian group $Z^2(G,U)$, whose quotient by the subgroup $B^2(G,U)$ of coboundaries is the *second cohomology group* $H^2(G,U)$. We display $\psi \in Z^2(G,U)$ as a *cocyclic matrix* $M_\psi = [\psi(g,h)]_{g,h\in G}$. If $U = \mathbb{Z}_2 = \langle -1 \rangle$ and M_ψ is Hadamard then ψ is said to be *orthogonal*.

The *row excess* of a $\{\pm 1\}$-matrix $M = [m_{ij}]$ is

$$RE(M) = \textstyle\sum_{i\geq 2} \left| \sum_{j \geq 1} m_{ij} \right|.$$

The cocycle equation (2) guarantees that ψ is orthogonal if and only if $RE(M_\psi)$ is optimal, i.e., zero.

For the rest of this section, $|G| = 4t + 2 > 2$.

Proposition 1. (i) *If $\psi \in Z^2(G,\mathbb{Z}_2)$ then $RE(M_\psi) \geq 4t$.*
(ii) *If $\psi \in B^2(G,\mathbb{Z}_2)$ then $RE(M_\psi) \geq 8t + 2$.*

Proof. See [3, Proposition 1]. □

In analogy with the definition of orthogonal cocycles, we say that ψ is *quasi-orthogonal* if its matrix has least possible row excess: by Proposition 1, either $\psi \notin B^2(G,\mathbb{Z}_2)$ and $RE(M_\psi) = 4t$, or $\psi \in B^2(G,\mathbb{Z}_2)$ and $RE(M_\psi) = 8t + 2$ (coboundaries were excluded from the notion of quasi-orthogonality in [3]).

Lemma 1. *Let $X_m = \{g \in G \mid \sum_{h\in G}\psi(g,h) = m\}$. Then ψ is quasi-orthogonal if and only if $|X_2 \cup X_{-2}| = 4t + 1$ for $\psi \in B^2(G,\mathbb{Z}_2)$, or $|X_0| = 2t + 1$ and $|X_2 \cup X_{-2}| = 2t$ for $\psi \notin B^2(G,\mathbb{Z}_2)$.*

Proof. See [3, Lemma 2.4]. □

It is not known whether quasi-orthogonal cocycles always exist. Indeed, we do not know of a group G such that $Z^2(G,\mathbb{Z}_2)$ does not contain a quasi-orthogonal element (in contrast, there are several non-existence results for orthogonal cocycles, e.g., due to Ito [12]). We have found quasi-orthogonal coboundaries over many abelian G, but none over non-abelian G such as dihedral groups, apart from the dihedral group of order 6. Thirdly, for all t such that $4t + 1$ is a sum of two squares that we tested, we always found a quasi-orthogonal cocycle ψ over some group of order $4t + 2$ with $|\det(M_\psi)|$

attaining the maximum $2(4t+1)(4t)^{2t}$ established by Ehlich-Wojtas. These existence questions all merit deeper investigation.

Let E be a group with a normal subgroup N of order m and index v. A *relative* (v, m, k, λ)*-difference set in* E *relative to* N (the *forbidden subgroup*) is a k-subset R of a transversal for N in E such that

$$|R \cap xR| = \lambda \quad \forall x \in E \setminus N.$$

Relative $(2s, 2, 2s, s)$-difference sets are especially interesting. If s is even then they are equivalent to cocyclic Hadamard matrices [6, Corollary 2.5], whereas none exist if s is odd [10]. In the latter case there is a natural analog of relative difference set. Suppose that $|E| = 8t+4$ and let $Z \cong \mathbb{Z}_2$ be a normal (hence central) subgroup of E. A *relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set in* E *with forbidden subgroup* Z is a transversal R for Z in E containing a subset $S \subset R \setminus \{1\}$ of size 0 or $2t+1$ such that, for all $x \in E \setminus Z$,

$$|R \cap xR| = \begin{cases} 2t+1 & x \in SZ \\ 2t \text{ or } 2t+2 & \text{otherwise.} \end{cases}$$

We call R *extremal* if $S = \emptyset$. (This modifies the original definition in [3] of relative quasi-difference set, to allow quasi-orthogonal coboundaries).

The next result is mostly Proposition 4.3 in [3]. For each $\psi \in Z^2(G, \mathbb{Z}_2)$ we have a canonical central extension E_ψ with element set $\{(\pm 1, g) \mid g \in G\}$ and multiplication defined by $(u, g)(v, h) = (uv\psi(g, h), gh)$.

Proposition 2. *The cocycle* ψ *is quasi-orthogonal if and only if* $D = \{(1, g) \mid g \in G\}$ *is a relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set in* E_ψ *with forbidden subgroup* $\langle(-1, 1)\rangle$*, where* D *is extremal for* $\psi \in B^2(G, \mathbb{Z}_2)$.

Remark 1. The requisite subset S of D corresponds to the rows of M_ψ with zero sum.

3 Generalized binary arrays with optimal autocorrelation

Jedwab [14] showed that a GPBA is equivalent to an abelian relative difference set, and Hughes [11] identified its underlying orthogonal cocycle. In this section we carry over these ideas into the setting of quasi-orthogonal cocycles.

We start with an adaptation of some material from [11] and [14]. The cyclic group of order m will be written additively, i.e., as $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ under addition modulo m. Let $\mathrm{s} = (s_1, \ldots, s_r)$ be an r-tuple of positive integers greater than 1, and let $G = \mathbb{Z}_{s_1} \times \cdots \times \mathbb{Z}_{s_r}$. A *binary* s*-array* is just a set map $\phi : G \to \{\pm 1\}$; it has *energy* $n := \prod_{i=1}^{r} s_i = |G|$. We view a binary sequence as an s-array with $r = 1$.

Given $\mathbf{s}$ and a *type vector* $\mathbf{z} = (z_1, \ldots, z_r) \in \{0,1\}^r$, let $E = \mathbb{Z}_{(z_1+1)s_1} \times \cdots \times \mathbb{Z}_{(z_r+1)s_r}$. Then

$$H = \{h \in E \mid h_i = 0 \text{ if } z_i = 0, \text{ and } h_i = 0 \text{ or } s_i \text{ if } z_i = 1\},$$
$$K = \{k \in H \mid k \text{ has even weight}\}$$

are elementary abelian 2-subgroups of E. Note that E is a (central) extension of H by G. For $\mathbf{z} \neq \mathbf{0}$ we obtain the short exact sequence

$$1 \longrightarrow \langle -1 \rangle \xrightarrow{\iota} E/K \xrightarrow{\beta} G \longrightarrow 0, \tag{3}$$

where ι maps -1 to the generator of H/K and $\beta(g + K) = g \mod \mathbf{s}$. This sequence determines a cocycle $f_{\mathbf{z}} \in Z^2(G, \langle -1 \rangle)$ after choice of a transversal map $\tau : G \to E/K$. Specifically, set $\tau(x) = x + K$; then

$$f_{\mathbf{z}}(x, y) = \iota^{-1}(\tau(x) + \tau(y) - \tau(x + y)).$$

We can express $f_{\mathbf{z}}$ as a product of cocycles on cyclic groups. Define $\gamma_m \in Z^2(\mathbb{Z}_m, \langle -1 \rangle)$ by $\gamma_m(j, k) = (-1)^{\lfloor (j+k)/m \rfloor}$, evaluating the exponent as an ordinary integer.

Proposition 3 ([11, Lemma 3.1]).

(i) $f_{\mathbf{z}}(x, y) = \prod_{z_i=1} \gamma_{s_i}(x_i, y_i)$.
(ii) $f_{\mathbf{z}} \in B^2(G, \langle -1 \rangle)$ *if and only if* s_i *is odd for all* i *such that* $z_i = 1$.

Each cocycle $\psi \in Z^2(G, \langle -1 \rangle)$ has an associated short exact sequence

$$1 \longrightarrow \langle -1 \rangle \xrightarrow{\iota'} E_\psi \xrightarrow{\beta'} G \longrightarrow 0, \tag{4}$$

where $\iota'(u) = (u, 0)$ and $\beta'(u, x) = x$. The following is standard.

Proposition 4. *If* ψ *and* $f_{\mathbf{z}}$ *are cohomologous, say* $\psi = f_{\mathbf{z}} \partial \phi$, *then* (3) *and* (4) *are equivalent short exact sequences: the isomorphism* Γ *defined by* $(u, x) \mapsto \iota(u\phi(x)) + \tau(x)$ *makes the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle -1 \rangle & \xrightarrow{\iota'} & E_\psi & \xrightarrow{\beta'} & G & \longrightarrow & 0 \\ & & \| & & \Gamma\downarrow & & \| & & \\ 1 & \longrightarrow & \langle -1 \rangle & \xrightarrow{\iota} & E/K & \xrightarrow{\beta} & G & \longrightarrow & 0 \end{array}$$

commute.

We broaden concepts defined earlier only for sequences. The *expansion* of a binary $\mathbf{s}$-array ϕ with respect to a type vector $\mathbf{z}$ is the map ϕ' on E given by

$$\phi'(g) = \begin{cases} \phi(a) & g \in a + K \\ -\phi(a) & g \notin a + K \end{cases}$$

where a denotes g modulo $\mathbf{s}$. For any array $\varphi : A \to \{\pm 1\}$ and $x \in A$, let $R_\varphi(x) = \sum_{a \in A} \varphi(a)\varphi(a + x)$.

Lemma 2. *If $h \in H \setminus K$ then $\phi'(h+g) = -\phi'(g)$, and if $h \in K$ then $\phi'(h+g) = \phi'(g)$.*

Corollary 1. *$R_{\phi'}(g) = |H| \sum_{x \in T} \phi'(x)\phi'(x+g)$ where T is any transversal for H in E.*

Lemma 3. *The isomorphism Γ in Proposition 4 maps $\{(1,x) \mid x \in G\} \subseteq E_\psi$ onto $\{g + K \in E/K \mid \phi'(g) = 1\}$.*

Proof. (Cf. [11, p. 330].) Let $\phi'(g) = 1$ and write a for g modulo $\mathbf{s}$; then $g + K = \iota(\phi(a)) + a + K = \Gamma((1,a))$. Conversely, $\Gamma((1,x)) = h + x + K$ where $h + K$ is the generator of H/K if $\phi(x) = -1$ and $h = 0$ otherwise. By Lemma 2, $\phi'(h+x) = 1$. □

The $\mathbf{s}$-array ϕ is a *GPBA*$(\mathbf{s})$ *of type* $\mathbf{z}$ if

$$R_{\phi'}(g) = 0 \quad \forall g \in E \setminus H.$$

When $\mathbf{z} = 0$, this condition becomes (by Corollary 1)

$$R_\phi(g) = 0 \quad \forall g \in G \setminus \{0\}.$$

In the latter event ϕ is a PBA; which is equivalent to $\partial\phi$ being orthogonal (we return to this case later in the section). More generally, a GPBA$(\mathbf{s})$ is equivalent to a relative difference set in E/K relative to H/K, hence equivalent also to a cocyclic Hadamard matrix over G: see [11, Theorem 5.3] and [14, Theorem 3.2]. So a GPBA can exist only if its energy n is 2 or a multiple of 4. Theorems 1 and 2 below are analogous results for $n \equiv 2 \bmod 4$.

Assume that $|G| = 4t + 2 > 2$ unless stated otherwise. Let $s_1/2, s_2, \ldots, s_r$ be odd. Thus, if $z_1 = 0$ then E splits over H by Proposition 3, and so $R_{\phi'}$ is never zero by Corollary 1 and Lemma 2.

Definition 1. A *GOBA*$(\mathbf{s})$ *of type* $\mathbf{z}$ is a binary $\mathbf{s}$-array ϕ such that

(i) $R_{\phi'}(g) \in \{0, \pm 2|H|\} \ \ \forall g \in E \setminus H$,

and if $z_1 = 1$ then

(ii) $|\{g \in E \mid R_{\phi'}(g) = 0\}| = |E|/2$.

A GOBS as defined in Section 1 is a GOBA$(\mathbf{s})$ with $r = z_1 = 1$. When $\mathbf{z} = \mathbf{0}$, Definition 1 reduces to

$$R_\phi(g) = \pm 2 \quad \forall g \in G \setminus \{0\};$$

we call ϕ satisfying this condition an *optimal binary array* (OBA).

Lemma 4 ([14, Lemma 3.1]). *For any array $\varphi : A \to \{\pm 1\}$,*

$$R_\varphi(x) = |A| + 4(d_\varphi(x) - |N_\varphi|)$$

where $N_\varphi = \{a \in A \mid \varphi(a) = -1\}$ and $d_\varphi(x) = |N_\varphi \cap (x + N_\varphi)|$.

Proof. Routine counting. □

Theorem 1. *Let ϕ be a binary $\mathbf{s}$-array, $\mathbf{z}$ be a non-zero type vector, and $D = \{g + K \in E/K \mid \phi'(g) = -1\}$. Then ϕ is a GOBA$(\mathbf{s})$ of type $\mathbf{z}$ if and only if D is a relative $(4t+2, 2, 4t+2, 2t+1)$-quasi-difference set in E/K with forbidden subgroup H/K; furthermore, D is extremal if $z_1 = 0$.*

Proof. We continue with the notation of Lemma 4. By Lemma 3, D is a full transversal for H/K in E/K. Also, $|N_{\phi'}| = |E|/2$ by Lemma 2; thus $|D| = |N_{\phi'}|/|K|$.

For each $g \notin H$, denote $|D \cap (g + K + D)|$ by $d_D(g + K)$: this is the number of $x + K \in D$ such that $x - g + K \in D$. Since $d_D(g + K) = d_{\phi'}(g)/|K|$, Lemma 4 implies that

$$\begin{array}{rcl} R_{\phi'}(g) = -2|H| & \Leftrightarrow & d_D(g+K) = 2t \\ R_{\phi'}(g) = 0 & \Leftrightarrow & d_D(g+K) = 2t+1 \\ R_{\phi'}(g) = 2|H| & \Leftrightarrow & d_D(g+K) = 2t+2. \end{array} \tag{5}$$

Let $S = \{g + K \in D \mid R_{\phi'}(g) = 0\}$. According to (5), Definition 1 (i) holds if and only if

$$d_D(g+K) = \begin{cases} 2t+1 & g + K \in S + H/K \\ 2t \ \text{ or } \ 2t+2 & \text{otherwise.} \end{cases}$$

Lemma 2 yields

$$|S| = \frac{|\{g + K \in E/K \mid R_{\phi'}(g) = 0\}|}{2} = |R_{\phi'}^{-1}(0)|/2|K|.$$

Thus $|S| = 2t + 1$ for $z_1 = 1$ if and only if Definition 1 (ii) holds. □

Remark 2. Theorem 1 remains valid when D is replaced by its complement $\{g + K \in E/K \mid \phi'(g) = 1\}$.

Theorem 2. *A (normalized) binary $\mathbf{s}$-array ϕ is a GOBA$(\mathbf{s})$ of type $\mathbf{z} \neq \mathbf{0}$ if and only if $f_{\mathbf{z}}\partial\phi$ is quasi-orthogonal.*

Proof. This is a consequence of Theorem 1, Remark 2, Proposition 2, and Lemma 3. □

We proceed to formulate 'base' cases of Theorems 1 and 2. Let $\partial\phi \in B^2(G, \mathbb{Z}_2)$. Since $M_{\partial\phi}$ is Hadamard equivalent to a group-developed matrix, and such a matrix has constant row sum, $\partial\phi$ can be orthogonal only if $|G|$ is square. This situation has been extensively studied.

Theorem 3. *Let $|G| = 4u^2$, and let D be a subset of G of size $2u^2 - u$. Define $R = \{(\phi(g), g) \,|\, g \in G\} \subset \mathbb{Z}_2 \times G$ where $\phi : G \to \{\pm 1\}$ is the characteristic function of D. Then the following are equivalent.*

(i) *$\partial\phi$ is orthogonal.*
(ii) *D is a Menon-Hadamard difference set in G.*
(iii) *R is a relative $(4u^2, 2, 4u^2, 2u^2)$-difference set in $\mathbb{Z}_2 \times G$ with forbidden subgroup $\mathbb{Z}_2 \times \{1_G\}$.*
(iv) *ϕ is a perfect nonlinear function.*

If G is abelian then (i) – (iv) *are further equivalent to*

(v) *ϕ is a PBA.*

Proof. See [15, Theorem 1] for (iii) ⇔ (iv). The other equivalences are given by Theorem 2.6 and Lemma 2.10 of [6]. □

Remark 3. In Theorem 3 and Theorem 4 below we may assume that ϕ is normalized, by taking the complement of D (and thus also of R) if necessary.

The next theorem is an analog of the previous one for $|G| \equiv 2 \bmod 4$ (recall that we have not found quasi-orthogonal coboundaries over non-abelian G at orders greater than 6).

Theorem 4. *Let G be abelian of order $4t + 2$, and let D be a k-subset of G with characteristic function $\chi : G \to \mathrm{GF}(2)$. Define $R = \{(\phi(g), g) \,|\, g \in G\} \subset \mathbb{Z}_2 \times G$ where $\phi(x) = (-1)^{\chi(x)}$. Then the following are equivalent.*

(i) *$\partial\phi$ is quasi-orthogonal.*
(ii) *D is a $(4t+2, k, k-(t+1), (4t+1)(k-t)-k(k-1))$-almost difference set in G.*
(iii) *R is an extremal relative $(4t+2, 2, 4t+2, 2t+1)$-quasi-difference set in $\mathbb{Z}_2 \times G$ with forbidden subgroup $\mathbb{Z}_2 \times \{1_G\}$.*
(iv) *ϕ is an OBA.*

If a difference set with parameters $(n, \frac{n \pm \sqrt{3n-2}}{2}, \frac{n+2\pm 2\sqrt{3n-2}}{4})$ does not exist, then (i) – (iv) *are further equivalent to*

(v) *χ has optimal nonlinearity $(t+1)/(2t+1)$.*

Proof. Put $|G| = n$.

(i) ⇔ (iv): Lemma 1 and the fact that $\phi(g)R_\phi(g)$ is the sum of row g in $M_{\partial\phi}$.

(i) ⇔ (ii): by Lemma 4, $R_\phi(g) = 2$ or -2 if and only if $d_\phi(g) = k - t - 1$ or $k - t$, respectively. Identity (19) of [5] then accounts for this part.

(i) ⇔ (iii): Proposition 2 together with the isomorphism $E_{\partial\phi} \to \mathbb{Z}_2 \times G$ defined by $(u, g) \mapsto (u\phi(g), g)$; cf. Proposition 4.

(ii) ⇔ (v): see [5, Theorem 25]. □

Remark 4. The condition attached to (v) is only needed for (v) $\Rightarrow$ (ii). No difference sets with the stated parameters are known; see [5, Remark II, p. 224].

We end this section with a discussion of calculating GOBAs. Label the elements of G as $g_1 = 0, g_2, \dots, g_{4t+2}$, and let $\delta_k \colon G \to \{\pm 1\}$ be the characteristic function of $\{g_k\}$. Up to relabeling, $\{\partial_2, \dots, \partial_{4t+1}\}$ is a basis of $B^2(G, \langle -1 \rangle)$, where $\partial_k := \partial \delta_k$ is an *elementary coboundary*. Choose $\mathbf{z} \neq \mathbf{0}$. We first try to find quasi-orthogonal $\psi \in Z^2(G, \langle -1 \rangle)$ such that $f_{\mathbf{z}} \psi \in B^2(G, \langle -1 \rangle)$. Straightforward linear algebra gives the decomposition $\psi = f_{\mathbf{z}} \prod_k \partial_k^{i_k}$. Then $\phi = \prod_k \delta_k^{i_k}$ is a GOBA($\mathbf{s}$) of type $\mathbf{z}$ over G.

Example 1. The maps $\phi_1 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\phi_2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, $\phi_3 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ on $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$ are GOBA$(2,3)$s of type $\mathbf{z}_1 = (1,0)$, $\mathbf{z}_2 = (0,1)$, $\mathbf{z}_3 = (1,1)$, respectively. We display each quasi-orthogonal cocycle $f_{\mathbf{z}_i} \partial \phi_i$ as a Hadamard (componentwise) product:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Note that $f_{\mathbf{z}_2} \partial \phi_2$ is a quasi-orthogonal coboundary; as are all the $\partial \phi_i$.

Example 2. The map $\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$ on $\mathbb{Z}_6 \times \mathbb{Z}_3 = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is a GOBA$(6,3)$ of type $\mathbf{z} = (1,0)$. Its quasi-orthogonal cocycle is $f_{\mathbf{z}} \partial_4 \partial_8 \partial_{10} \partial_{13}$.

4 Negaperiodic Golay pairs

In this section we explore how GOBSs can be used to construct NGPs.

Proposition 5 ([8, Theorem 3]). *Binary sequences ϕ_1, ϕ_2 of length $2t$ form an NGP if and only if $\{x^i \mid \phi'_1(i) = 1\} \cup \{x^i y \mid \phi'_2(i) = 1\}$ is a relative $(4t, 2, 4t, 2t)$-difference set in the dicyclic group $Q_{8t} = \langle x, y \mid x^{2t} = y^2, y^4 = 1,\, y^{-1}xy = x^{-1}\rangle$.*

Remark 5. By Proposition 5 and [2, Theorems 5.6 and 5.7], NGPs of length $(q+1)/2$ exist for all prime powers $q \equiv 3 \mod 4$.

Proposition 5 ties NGPs into the mainstream theory of cocyclic Hadamard matrices: by [9, Proposition 6.5], existence of a $(4t, 2, 4t, 2t)$-difference set in Q_{8t} is equivalent to existence of certain orthogonal cocycles over the dihedral group D_{4t} of order $4t$. (Incidentally, this gives another justification of Remark 5, via Ito's Hadamard groups of quadratic residue type [12, pp. 986–987].) These cocycles lie in a single cohomology class, with representative labeled $(A, B, K) = (1, -1, -1)$ in [9]; A, B are 'inflation' variables and K is the 'transgression' variable in a Universal Coefficients theorem decomposition of $H^2(D_{4t}, \mathbb{Z}_2)$.

The next theorem makes Proposition 5 more explicit. It shows how to translate directly between cocycles and NGPs. When the latter are complementary GOBSs, this implies existence of orthogonal cocycles if there exist quasi-orthogonal cocycles at half the order (unfortunately, the process does not reverse).

Theorem 5. *Let $G = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle \cong D_{2n}$ with elements ordered as $1, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b$. Also let ϕ_1, ϕ_2 be binary sequences of length n, and define $j_{k,i}$ to be 1 or 0 depending on whether $\phi_i(k) = -1$ or 1, respectively. Then (ϕ_1, ϕ_2) is an NGP if and only if $\lambda \prod_{k=1}^{n} \partial_k^{j_{k,1}} \partial_{n+k}^{j_{k,2}}$ is an orthogonal cocycle over G, where λ is the cohomology class representative labeled $(A, B, K) = (1, -1, -1)$ in* [9, Section 6].

Proof. The center of $\langle x, y \mid x^n = y^2, y^4 = 1,\, y^{-1}xy = x^{-1}\rangle \cong Q_{4n}$ is $\langle x^n\rangle$. Since $G \cong Q_{4n}/\langle x^n\rangle$, we may define a transversal map $\sigma : G \to Q_{4n}$ by

$$a^i \mapsto x^{i+n\delta_{\phi_1(i),-1}}, \qquad a^i b \mapsto x^{i+n\delta_{\phi_2(i),-1}} y$$

where δ is the Kronecker delta. Assuming that ϕ_1 and ϕ_2 are normalized, let ψ be the cocycle for σ, i.e., $\psi(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. By Proposition 5 and [6, Corollary 2.5], ψ is orthogonal if and only if (ϕ_1, ϕ_2) is an NGP.

Set $\varphi(a^i) = \phi_1(i)$ and $\varphi(a^i b) = \phi_2(i)$. Then $\lambda = \psi\partial\varphi$ has matrix

$$\begin{bmatrix} A & A \\ B & -B \end{bmatrix}$$

where $A = [(-1)^{\lfloor (i+j)/n \rfloor}]_{0 \le i,j \le n-1}$ is back negacyclic, and B is A with rows r and $n - r + 1$ swapped for $1 \le r \le n$. Furthermore, $\partial\varphi = \prod_{k=1}^{n} \partial_k^{j_{k,1}} \partial_{n+k}^{j_{k,2}}$ under the stipulated ordering of G. □

We now undertake a case study of quasi-orthogonal cocycles over cyclic groups. Let $G = \mathbb{Z}_{4t+2}$ and index matrices by $1, \ldots, 4t+2$ in this order. The set $\mathcal{B} = \{\gamma, \partial_i \mid 2 \le i \le 4t+2\}$ where $\gamma = \gamma_{4t+2}$ (as defined before Proposition 3) is a basis of $Z^2(G, \mathbb{Z}_2)$. We get an elementary coboundary matrix $M_i := M_{\partial_i}$ by normalizing the back circulant matrix whose first row is 1s except for the ith entry. Also, M_γ is the back negacyclic matrix N of order $4t+2$.

Lemma 5. *Let $\psi \in Z^2(G, \mathbb{Z}_2) \setminus B^2(G, \mathbb{Z}_2)$, say $M_\psi = M_{i_1} \circ \cdots \circ M_{i_w} \circ N$. Then*

(i) *up to sign, M_ψ has ith row sum equal to its $(4t+4-i)$th row sum.*
(ii) *The $(2t+2)$th row sum of M_ψ is 0.*
(iii) *ψ is quasi-orthogonal if and only if the ith row sum of M_ψ is 0 for even i and ± 2 for odd $i > 1$.*

Proof. If $\psi \in \mathcal{B}$ then row $i > 2t+2$ of M or its negation is row $(4t+4-i)$ cycled $4t+4-i-1$ positions to the right. Part (i) then follows. For (ii), observe that row $2t+2$ in N is $[1 \overset{2t+1}{\cdots} 1 \; -1 \overset{2t+1}{\cdots} -1]$, whereas the first half of row $2t+2$ in M_i is identical to the second half. Finally, (iii) holds because the number of -1s in any row of M_i is even; and the rows of N indexed by an even (respectively, odd) integer have an odd (respectively, even) number of -1s. □

We use an approach borrowed from [1] to count the negative entries in a G-cocyclic matrix. Negating row i of M_i gives a *generalized coboundary matrix* $\overline{M}_i$, with exactly two -1s in each non-initial row r: these are in columns i and $[i - r + 1]_{4t+2}$, where $[m]_n \in \{1, \ldots, n\}$ denotes the residue of m modulo n. (Although $\overline{M}_i$ is not cocyclic, row negation preserves row excess.) Hence the two generalized coboundary matrices with -1 in position (r, c) are $\overline{M}_c$ and $\overline{M}_{[r+c-1]_{4t+2}}$.

A set $\{\overline{M}_{i_j} : 1 \le j \le w\}$ defines an *r-walk* if there is an ordering $\overline{M}_{l_1}, \ldots, \overline{M}_{l_w}$ of its elements such that $\overline{M}_{l_i}$ and $\overline{M}_{l_{i+1}}$ both have -1 in row r and column l_{i+1}, for $1 \le i \le w$. The walk is an *r-path* if its initial (equivalently, final) element shares a -1 in row r with a generalized coboundary matrix not in the walk itself. Clearly, the number of -1s in row r of $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w}$ is $2\mathcal{C}_r$ where $\mathcal{C}_r$ is the number of maximal r-paths in $\{\overline{M}_{i_1}, \ldots, \overline{M}_{i_w}\}$. To calculate $\mathcal{C}_r$ we set up a bipartite graph on vertex sets $S = \{i_1, \ldots, i_w\}$ and $T = \{[i_1 - r + 1]_{4t+2}, \ldots, [i_w - r + 1]_{4t+2}\}$. Draw an edge between $i_j \in S$ and $l \in T$ if $i_j = l$ or $l = [i_j - r + 1]_{4t+2} \in S$. The number of maximal paths in this bipartite graph is $\mathcal{C}_r$.

Next, let $\mathcal{I}_r$ be the number of columns where N and $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w}$ share a -1 in row r. These column indices comprise the intersection of $\{4t+4-r, \ldots, 4t+2\}$ and the set of endpoints of the previously calculated maximal r-paths.

Theorem 6 (cf. [1, Proposition 1]). *A $\mathbb{Z}_{4t+2}$-cocyclic matrix $M_{i_1} \circ \cdots \circ M_{i_w} \circ N$ is quasi-orthogonal if and only if, for $2 \leq r \leq 2t+1$,*

$$\begin{array}{ll} \mathcal{C}_r \in \{\mathcal{I}_r + t + \frac{1-r}{2}, \mathcal{I}_r + t + \frac{3-r}{2}\} & r \text{ odd} \\ \mathcal{C}_r = \mathcal{I}_r + t + 1 - \frac{r}{2} & r \text{ even.} \end{array}$$

Proof. The number of -1s in row r of $\overline{M}_{i_1} \circ \cdots \circ \overline{M}_{i_w} \circ N$ is $2\mathcal{C}_r + r - 1 - 2\mathcal{I}_r$, so Lemma 5 gives the result. □

Corollary 2. *Let $\psi = \gamma \prod_{j=1}^{w} \partial_{i_j}$ with $\partial_{i_j} \in \mathcal{B}$. If ψ is quasi-orthogonal then $t \leq w \leq 3t+1$.*

Proof. We have $\mathcal{I}_2 = 0$, and $\mathcal{C}_2 = t$ by Theorem 6. Thus $t \leq w$. On the other hand, since the basis of coboundaries forms a 2-path, at least $t-1$ coboundaries must be removed to get t 2-paths. Hence $w \leq 4t - (t-1)$. □

Corollary 2 is equivalent to

Lemma 6. *If $\phi\colon \mathbb{Z}_{4t+2} \to \{\pm 1\}$ is a GOBS containing w occurrences of -1 then $t \leq w \leq 3t+1$.*

Proof. Negating all odd index entries or all even index entries of a GOBS produces another GOBS. So it may be assumed that $\phi(0) = \phi(4t+1) = 1$. □

We search for NGPs in the set of quasi-orthogonal cocycles over $\mathbb{Z}_{4t+2}$, motivated by the ubiquity of these cocycles and the optimal autocorrelation of each map in the resulting pair. Computer-aided searches found the NGPs in Table 1.

k	ϕ_1	ϕ_2
3	$1^2, 4$	$2, 1, 3$
5	$2, 1^3, 5$	$3, 1, 2, 1, 3$
7	$2, 1, 5, 1^3, 3$	$2, 1, 4, 2, 1^2, 3$
9	$3, 1, 2, 1^3, 3, 1, 5$	$2, 1, 2, 3, 2, 1^3, 5$
13	$3, 3, 2, 2, 1, 2, 1, 2, 1^4, 6$	$3, 3, 1, 3, 1, 2, 1, 2, 1^4, 6$
15	$3, 2, 4, 1^2, 2, 2, 1, 2, 1^5, 7$	$3, 2, 3, 2, 1, 2, 2, 1, 2, 1^5, 7$

Table 1. NGPs (ϕ_1, ϕ_2) from quasi-orthogonal cocycles over $\mathbb{Z}_{2k}$

Each sequence in Table 1 starts with 1 and is designated by an integer string, where i in the string means a run of i identical entries in the sequence, and 1^j is an alternating subsequence of length j. There are no NGPs among the sequences coming from quasi-orthogonal cocycles over $\mathbb{Z}_{22}$ (however, as we know, NGPs of length 22 exist). This gap could be related to the maximal determinant problem: the Ehlich-Wojtas bound is not attainable because 21 is not a sum of two squares.

Egan [8] classified NGPs of length $2k$ for $k \leq 10$ up to equivalence with respect to five elementary operations as defined in [4]. The set of NGPs that come from GOBSs is invariant under each elementary operation. Table 2 records the number $\hat{n}(k)$ of such NGPs of length $2k$, and the number $\hat{d}(k)$ of their equivalence classes. To compare against [8, Table 2], we have included the total number $n(k)$ of NGPs of length $2k$ and the number $d(k)$ of their equivalence classes.

k	$n(k)$	$\hat{n}(k)$	$d(k)$	$\hat{d}(k)$
3	576	576	1	1
5	11200	4800	3	2
7	90944	18816	5	1
9	1041984	62208	20	2

Table 2. Enumeration of NGPs and their equivalence classes

5 Normal cocyclic matrices

This section is essentially independent of the main thrust of the paper. Nonetheless, it addresses a fundamental question in algebraic design theory, which we answer in special cases that were the focus of Section 4.

A matrix M is *normal* if it commutes with its transpose (possibly up to row or column permutations), i.e., $\mathrm{Gr}(M) = \mathrm{Gr}(M^\top)$, where $\mathrm{Gr}(M)$ denotes the Grammian $MM^\top$. Many kinds of pairwise combinatorial designs are normal matrices (the defining pairwise constraint on rows implies the same constraint on columns; see [7, Chapter 7]). We also note that the matrix of a quasi-orthogonal cocycle is normal [3, Remark 6]. Thus, by the following lemma derived from (2), a cocycle ψ is quasi-orthogonal if and only if M_ψ has optimal column excess.

Lemma 7. *For any group G and $\psi \in Z^2(G, \mathbb{Z}_2)$,*

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(g_i g_j^{-1}, g_j) \sum_{g \in G} \psi(g_i g_j^{-1}, g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \psi(g_i, g_i^{-1}g_j) \sum_{g \in G} \psi(g, g_i^{-1}g_j).$$

We use Lemma 7 to prove that cocyclic matrices for two familiar classes of indexing groups are normal.

Proposition 6. *Let G be abelian or dihedral of order $2m$, m odd, and let $\psi \in Z^2(G,\mathbb{Z}_2)$ where $\psi \notin B^2(G,\mathbb{Z}_2)$ if G is dihedral. Then M_ψ is normal (under the same indexing of rows and columns by the elements of G).*

Proof. We suppose that G is generated by a and b, with $a^m = b^2 = 1$, and index rows and columns by the elements of G under the ordering $1, a, \ldots, a^{m-1}, b, ab, \ldots, a^{m-1}b$. A representative β for the non-identity element of $H^2(G,\mathbb{Z}_2)$ has matrix

$$\begin{bmatrix} J & J \\ J & -J \end{bmatrix}.$$

Thus, if G is abelian then M_ψ is symmetric and so trivially normal.

Henceforth G is dihedral. Let $\psi = \beta\partial\phi$. We collect together some basic properties of M_ψ.

(i) For each i, $\{\partial\phi(a^ib, a^j) \mid 1 \le j \le m\} = \{\partial\phi(a^ib, a^jb) \mid 1 \le j \le m\}$; and for each j, $\{\partial\phi(a^i, a^jb) \mid 1 \le i \le m\} = \{\partial\phi(a^ib, a^jb) \mid 1 \le i \le m\}$. Thus, if $k > m$ then the kth row sum and kth column sum of M_ψ are zero.

(ii) Since $\{\partial\phi(a^i, a^jb) \mid 1 \le j \le m\} = \{\partial\phi(a^jb, a^i) \mid 1 \le j \le m\}$, the kth row sum of M_ψ equals its kth column sum for $k \le m$.

Now we consider the Grammian quadrants in turn.

If $1 \le i \le m$ and $m+1 \le j \le 2m$ then

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(a^{i+j-2}b, a^{j-1}b) \sum_{g \in G} \psi(a^{i+j-2}b, g) = 0$$

by Lemma 7 and (i); $\mathrm{Gr}(M_\psi^\top)_{ij} = 0$ similarly.

Let $1 \le i \le m$ and $1 \le j \le m$. Then

$$\mathrm{Gr}(M_\psi)_{ij} = \partial\phi(a^{i-j}, a^{j-1}) \sum_{g \in G} \partial\phi(a^{i-j}, g) = \phi(a^{j-1})\phi(a^{i-1}) \sum_{g \in G} \phi(g)\phi(a^{i-j}g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \phi(a^{j-1})\phi(a^{i-1}) \sum_{g \in G} \phi(g)\phi(ga^{j-i}).$$

These entries are equal by the identity $\sum_{k=1}^m \phi(a^k)\phi(a^{k+1}) = \sum_{k=1}^m \phi(a^k)\phi(a^{k-1})$.

Finally, let $m+1 \leq i, j \leq 2m$. Then

$$\mathrm{Gr}(M_\psi)_{ij} = \psi(a^{i-j}, a^{j-1}b) \sum_{g \in G} \psi(a^{i-j}, g)$$

and

$$\mathrm{Gr}(M_\psi^\top)_{ij} = \psi(a^{i-1}b, a^{i-j}) \sum_{g \in G} \psi(g, a^{i-j}).$$

Since $\psi(a^{i-1}b, a^{i-j}) = \partial\phi(a^{i-1}b, a^{i-j}) = \partial\phi(a^{i-j}, a^{j-1}b) = \psi(a^{i-j}, a^{j-1}b)$, we are done by (ii). □

Remark 6. There are plenty of examples of non-normal cocyclic matrices M_ψ for $\psi \notin B^2(G, \mathbb{Z}_2)$ and $|G|$ divisible by 4.

Acknowledgments. The authors thank Kristeen Cheng for reading the manuscript, and Víctor Álvarez for his assistance with computations. We are also grateful to Ronan Egan, who shared his insights on NGPs with us. Remark 5 and reference [2] were kindly provided by one of the referees. This research has been supported by project FQM-016 funded by JJAA (Spain).

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