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# Universal entire functions for affine endomorphisms of $\mathbb{C}^N$

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## Abstract

In this paper the affine endomorphisms of  $\mathbb{C}^N$  which support compositionally universal entire functions are completely characterized.

## 1 Introduction

Throughout this paper  $\mathbb{N}$  will denote the set of all positive integers, and if  $N \in \mathbb{N}$  then  $\mathbb{C}^N$  will stand for the  $N$ -dimensional complex space. In particular,  $\mathbb{C}^1$  is the complex plane  $\mathbb{C}$ . The closed polydisk of radius  $r \geq 0$  centered at the origin is  $D(r) = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \|z\| \leq r\}$ , where  $\|z\| = \max_{1 \leq j \leq N} |z_j|$ . A domain  $G$  of  $\mathbb{C}^N$  is a nonempty connected open subset of  $\mathbb{C}^N$ . By  $H(G)$  we denote the space of holomorphic functions  $f : G \rightarrow \mathbb{C}$ , endowed with the topology of uniform convergence in compacta. In particular,  $H(\mathbb{C}^N)$  is the space of entire functions of  $N$  complex variables. It is well known that  $H(G)$  becomes a separable Fréchet space under the above topology. The symbol  $\text{Aut}(G)$  will stand for the group of all automorphisms (= biholomorphic bijective selfmappings) on  $G$ .

In 1929 Birkhoff [8] constructed an entire function which is ‘universal’ for translations. In fact, he proved essentially that given  $b \in \mathbb{C} \setminus \{0\}$  there exists

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a function  $f \in H(\mathbb{C})$  such that its sequence of translates  $\{f(\cdot + nb) : n \in \mathbb{N}\}$  is dense in  $H(\mathbb{C})$ .

Birkhoff's theorem can be observed under the point of view of the operator theory as a universality result; namely, if  $\varphi : \mathbb{C} \rightarrow \mathbb{C}$  denotes the translation  $z \mapsto z + b$  then the composition operator

$$C_\varphi : f \in H(\mathbb{C}) \mapsto f \circ \varphi \in H(\mathbb{C})$$

is universal. In general, if  $X$  is a (necessarily separable) topological vector space and  $T$  is an operator (= continuous linear selfmapping) on  $X$  then  $(T_n)$  is said to be *universal* (or *hypercyclic*) provided that there exists some vector  $x \in X$  –called *universal* for  $T$ – for which the orbit  $\{T^n x : n \in \mathbb{N}\}$  of  $x$  under  $T$  is dense in  $X$ . Here  $(T^n)$  represents the sequence of iterates  $T^1 = T, T^2 = T \circ T, \dots$  of  $T$ . It is easy to see that the set of universal vectors is dense. The operator  $T$  is called *hereditarily universal* if and only if given a sequence  $\{n_1 < n_2 < \dots\} \subset \mathbb{N}$  there is a vector  $x \in X$  such that  $\{T^{n_k} x : n \in \mathbb{N}\}$  is dense in  $X$ . If  $X$  is Baire and metrizable and  $T$  is universal (hereditarily universal) then the set of universal vectors for  $T$  (for each sequence  $(T^{n_k})$ , respectively) is residual, that is, its complement is of first category. These notions can be easily extended to a sequence  $(T_n)$  of operators. See [16] for a good account about these concepts and their history.

Since 1929 many papers have dealt with the subject of universality through translations in one (complex) variable, but only a few ones in several variables. Let us make a brief report, now in the language of the universality of operators; see also the survey [16] –specially its Section 4a– which contains a rather complete list of references including domains  $G \neq \mathbb{C}$  and spaces  $X \neq H(G)$ . In 1976 Luh [21] proved that for a prescribed unbounded sequence  $(b_n) \subset \mathbb{C}$  the sequence  $(C_{\varphi_n})$  is universal on  $H(\mathbb{C})$ , where  $\varphi_n$  is the translation  $z \mapsto z + b_n$ . In 1984 Duyos-Ruis [11] showed by functional analysis methods that  $C_\varphi$  ( $\varphi(z) = z + b, b \in \mathbb{C} \setminus \{0\}$ ) is universal on  $H(\mathbb{C})$  (hence there is a residual subset of universal functions), while the residuality of the  $(C_{\varphi_n})$ -universal entire functions (where the  $\varphi_n$  are the above translations) was observed by Grosse-Erdmann [15] and Gethner and Shapiro [12]. In 1995 Bernal and Montes [6] were able to show the same result for a sequence  $\{\varphi_n(z) = a_n z + b_n : n \in \mathbb{N}\} \subset \text{Aut}(\mathbb{C})$ ; recall that  $\varphi \in \text{Aut}(\mathbb{C})$  if and only if  $\varphi$  is a nonconstant affine endomorphism of  $\mathbb{C}$ , that is, there are  $a, b \in \mathbb{C}$  with  $a \neq 0$  and  $\varphi(z) = az + b$ . Specifically, they proved that  $(C_{\varphi_n})$  is universal

if and only if the sequence  $\{\min\{|b_n|, |b_n/a_n|\} : n \in \mathbb{N}\}$  is unbounded, from which they derived that if  $\varphi$  is an affine endomorphism of  $\mathbb{C}$  then

*$C_\varphi$  is universal if and only if  $\varphi$  is a translation,*

that is,  $\varphi(z) \equiv z + b$  for some  $b \in \mathbb{C} \setminus \{0\}$ . As for several variables, the history is not too long. In 1941 Seidel and Walsh [23] proved a non-Euclidean version of Birkhoff's theorem for the unit disk, and in 1979 Chee [10] extended this to the unit polidisk and ball of  $\mathbb{C}^N$ . León [20] has recently characterized the corresponding universal sequences of automorphisms in both domains. In the case of the euclidean translations  $\varphi(z) = z + b$  ( $z \in \mathbb{C}^N$ ,  $b \in \mathbb{C} \setminus \{0\}$ ) in several variables –where  $z = (z_1, \dots, z_N)$ ,  $b = (b_1, \dots, b_N)$ , and these will be the standard representations of  $z, b$  along the current paper– the natural extensions of the theorems of Birkhoff and Luh are covered by [13, Section 5] and by the recent papers [1], [3, Section 2] and [7] (see also [2] and [3] for a matricial extension of Zappa's result [24], which in turn is a multiplicative version in  $\mathbb{C} \setminus \{0\}$  of Birkhoff's theorem).

In view of the above discoveries, it is natural to pose the problem of characterizing those mappings  $\varphi \in \text{Aut}(\mathbb{C}^N)$  such that  $C_\varphi$  is universal on  $H(\mathbb{C}^N)$ . Nevertheless, a complete description of  $\text{Aut}(\mathbb{C}^N)$  (see for instance [4], [5] and [22] for a study of some subfamilies of it) is unknown up to date. Although there are plenty of automorphisms of  $\mathbb{C}^N$ , the simplest among them are with no doubt the affine linear mappings (or 'affine endomorphisms') from  $\mathbb{C}^N$  into itself which are invertible. Each affine endomorphism  $S = S(A, b)$  is biunivocally determined by a pair  $(A, b)$ , where  $A := [a_{ij}]_{i,j=1,\dots,N}$  is a matrix with complex entries and  $b$  is a fixed vector of  $\mathbb{C}^N$ ; so  $S$  is given by

$$S(z) = Az + b \quad \text{for all } z \in \mathbb{C}^N.$$

Observe that as we make the calculation  $S(z) = Az + b$  it is convenient to consider the vectors  $z$  and  $b$  as 'column' vectors. It is clear that  $S \in \text{Aut}(\mathbb{C}^N)$  if and only if  $S$  is one-to-one if and only if  $S$  is onto if and only if  $\det(A) \neq 0$ .

Hence the main aim of this paper is to characterize the universality of the composition operator  $C_S : H(\mathbb{C}^N) \rightarrow H(\mathbb{C}^N)$  generated by an affine endomorphism  $S = S(A, b)$  in terms of the matrix  $A$  and the vector  $b$ . This will be accomplished in Section 3 where we prove, among other things, that  *$C_S$  is universal if and only if  $S$  is univalent and has no fixed point.* In Section 2 we present a number of statements that will reveal useful for our goal.

## 2 Several auxiliary results

This section is devoted to background material on  $N$ -dimensional complex approximation that will be needed for the work of Section 3.

From now on  $G$  will represent a domain in  $\mathbb{C}^N$ . Let us denote by  $H(G, G)$  the set of all holomorphic selfmappings  $\varphi = (\varphi_1, \dots, \varphi_n)$  on  $G$ , that is,  $\varphi(G) \subset G$  and each component  $\varphi_j : G \rightarrow \mathbb{C}$  ( $j = 1, \dots, N$ ) is a holomorphic function. Then if  $\varphi \in H(G, G)$  the composition operator  $C_\varphi : H(G) \rightarrow H(G)$  is well defined. Of course,  $\text{Aut}(G) \subset H(G, G)$ . Our first lemma prevents us to use non-injective selfmappings to obtain  $C_\varphi$ -universality.

**Lemma 2.1.** *If  $\varphi \in H(G, G)$  and  $C_\varphi$  is universal on  $H(G)$  then  $\varphi$  is one-to-one and has no fixed points.*

*Proof.* The results contained in this lemma are well known in the one-dimensional context, see for instance [9, pages 3 and 10]. The proof given on page 10 of that monograph for the necessity of univalence works, word for word, in several variables. Indeed, if  $\varphi$  identifies two distinct points  $a$  and  $b$  of  $G$ , then so does the  $n$ -th component  $\varphi_n$  of  $\varphi$ , and so does  $f \circ \varphi_n$  for each  $n$  and each  $f \in H(G)$ . Thus if  $g$  is a limit point of the  $C_\varphi$ -orbit of  $f$ , then  $g(a) = g(b)$ , hence (because some  $g \in H(G)$ , namely an appropriate coordinate function, takes different values at  $a$  and  $b$ ) no  $f$  holomorphic on  $G$  can include every function in  $H(G)$  in the closure of its orbit. Hence  $C_\varphi$  is not universal.

Finally, if  $a \in G$  were a fixed point for  $\varphi$  and  $f \in H(G)$  were  $C_\varphi$ -universal then by considering the compact set  $K = \{a\}$  the closure in  $\mathbb{C}^N$  of the set  $\{f(\varphi^n(a)) : n \in \mathbb{N}\} = \{f(a)\}$  would be dense in  $\mathbb{C}$ , which is absurd.  $\square$

A set  $B \subset \mathbb{C}^N$  is said to be  $H(\mathbb{C}^N)$ -convex (see [17] or [19]) whenever  $\tilde{B} = B$ , where

$$\tilde{B} := \{z \in \mathbb{C}^N : |f(z)| \leq \sup_{t \in B} |f(t)| \text{ for all } f \in H(\mathbb{C}^N)\}.$$

The next generalization of Runge's approximation theorem for several complex variables is a special case of a statement that can be found in [17, Theorem 4.3.2 and following note].

**Proposition 2.2.** *Let  $f$  be a holomorphic function in a neighborhood of an  $H(\mathbb{C}^N)$ -convex compact subset  $K$  of  $\mathbb{C}^N$ . Then there is a sequence  $(f_j) \subset H(\mathbb{C}^N)$  such that  $f_j \rightarrow f$  ( $j \rightarrow \infty$ ) uniformly on  $K$ .*

In 1965 Kallin [18] proved an important separation lemma in several variables, from which the following proposition –that will be crucial for our approximation problem– is a particular case. The word “convex” means “geometrically convex”, that is, a set  $B \subset \mathbb{C}^N$  is convex whenever  $z, w \in B$  implies  $\lambda z + (1 - \lambda)w \in B$  for all  $\lambda \in [0, 1]$ .

**Proposition 2.3.** *If  $K$  and  $L$  are disjoint convex compact sets in  $\mathbb{C}^N$  then  $K \cup L$  is  $H(\mathbb{C}^N)$ -convex.*

In connection with the last proposition we point out that it is not known yet whether the disjoint union of 4 closed balls in  $\mathbb{C}^N$  is  $H(\mathbb{C}^N)$ -convex.

The following lemma settles the question of which sequences of automorphisms are adequate to generate universality. Following [6], we say that a sequence  $(\varphi_n) \subset H(G, G)$  is *run-away* if and only if given a compact set  $K \subset G$  there exists  $n_0 = n_0(K) \in \mathbb{N}$  such that  $K \cap \varphi_{n_0}(K) = \emptyset$ ; and we say that a function  $\varphi \in H(G, G)$  is *non-recurrent* whenever its sequence  $(\varphi^n)$  is run-away.

**Lemma 2.4.** *Suppose that  $\varphi$  is an affine automorphism of  $\mathbb{C}^N$ . Then  $C_\varphi$  is universal on  $H(\mathbb{C}^N)$  if and only if  $\varphi$  is non-recurrent.*

*Proof.* Let us suppose that  $C_\varphi$  is universal and that, by way of contradiction,  $\varphi$  is not non-recurrent. Then there is a compact set  $K$  such that  $K \cap \varphi^n(K) \neq \emptyset$  for all  $n \in \mathbb{N}$ . Choose a sequence  $(z_n) \subset K$  with  $(\varphi_n(z_n)) \subset K$  and a  $C_\varphi$ -universal function  $f \in H(G)$ . Consider the constant function  $g(z) = 1 + \max_K |f|$ . We have that, for every  $n \in \mathbb{N}$ ,

$$\max_{z \in K} |g(z) - f(\varphi^n(z))| \geq \|g(z) - f(\varphi^n(z_n))\| = 1 + \max_K |f| - |f(\varphi^n(z_n))| \geq 1,$$

which is a contradiction.

Conversely, assume that  $\varphi$  is non-recurrent. Our final goal is to show that the set  $M$  of universal functions for  $C_\varphi$  is residual (so nonempty). Since  $H(G)$  is a second-countable Baire space, Birkhoff’s transitivity theorem (see for instance [14, 9.20]) asserts that  $M$  is a dense  $G_\delta$ -subset (hence residual) if and only if for every pair of nonempty open subsets  $A$  and  $B$  of  $H(\mathbb{C}^N)$  there exists some  $m \in \mathbb{N}$  with

$$(C_\varphi)^m(A) \cap B = \emptyset. \tag{1}$$

Note that  $(C_\varphi)^n = C_{\varphi^n}$  ( $n \in \mathbb{N}$ ). Fix  $A, B$  as before. Then there exists  $\varepsilon > 0, R > 0$  and  $f, h \in H(\mathbb{C}^N)$  such that  $A \supset A_1 := \{g \in H(\mathbb{C}^N) : \max_{z \in D(R)} |g(z) - f(z)| < \varepsilon\}$  and  $B \supset B_1 := \{g \in H(\mathbb{C}^N) : \max_{z \in D(R)} |g(z) - h(z)| < \varepsilon\}$ . Since  $\varphi$  is non-recurrent, there exists  $m \in \mathbb{N}$  such that  $D(R) \cap \varphi^m(D(R)) = \emptyset$ . But  $\varphi^m(D(R))$  is a convex compact set because  $\varphi^m$  is continuous and convex-preserving (that is,  $\varphi^m(C)$  is convex whenever  $C$  is convex; this is true because  $\varphi$  is affine). Then  $L := D(R) \cup \varphi^m(D(R))$  is  $H(\mathbb{C}^N)$ -convex by Proposition 2.3. Fix  $U$  and  $V$  open subsets in  $\mathbb{C}^N$  such that  $D(R) \subset U, \varphi^m(D(R)) \subset V$  (so  $U \cup V \supset L$ ) and  $U \cap V = \emptyset$ . Define the function  $F : U \cup V \rightarrow \mathbb{C}$  as

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ h(\varphi^{-m}(z)) & \text{if } z \in V, \end{cases}$$

which is holomorphic on  $U \cup V$ . Hence by Proposition 2.2 there exists an entire function  $g$  satisfying

$$|F(z) - g(z)| < \varepsilon \text{ for all } z \in L.$$

But from the definition of  $F$  we get

$$|g(z) - f(z)| < \varepsilon \text{ for all } z \in D(R).$$

and

$$|g(z) - h(\varphi^{-m}(z))| < \varepsilon \text{ for all } z \in \varphi^m(D(R)).$$

The last display is clearly equivalent to

$$|g(\varphi^m(z)) - h(z)| < \varepsilon \text{ for all } z \in D(R).$$

In other words,  $g \in A_1$  and  $(C_\varphi)^m g \in B_1$ . Thus,  $g \in A$  and  $(C_\varphi)^m g \in B$ , so (1) holds.  $\square$

**Remark 2.5.** The proof of Lemma 2.4 can be easily modified to obtain the following extension: Suppose that  $G$  is a convex domain of  $\mathbb{C}^N$  and that  $(\varphi_n)$  is a sequence in  $\text{Aut}(G)$  of convex-preserving mappings. Then  $(C_{\varphi_n})$  is universal if and only if  $(\varphi_n)$  is run-away. Moreover, in this case there exists a residual subset of universal functions. Suffice to say that if  $G$  is convex then the polidisks  $D(R)$  of the proof of Lemma 2.4 can be replaced to convex

compact subsets of  $G$  and that Birkhoff's transitivity theorem also works with a sequence  $(T_n)$  of mappings from a Baire space into a second-countable space, see [16, Theorem 1]. It might be interesting to investigate whether the last lemma can be extended to non-convex domains or to sequences of automorphisms that do not preserve convexity.

### 3 Universal functions for endomorphisms

From now on  $A$  will represent an  $(N \times N)$ -matrix with complex entries and  $b$  will be a fixed vector in  $\mathbb{C}^N$ .

Lemma 2.4 focuses attention on the dynamics of affine mappings of  $\mathbb{C}^N$ . In order to characterize when they generate universal composition operators, we need to know which of such mappings are non-recurrent. We are now ready to state our main result.

**Theorem 3.1.** *Assume that  $S : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is an affine endomorphism, say  $Sz = Az + b$  ( $z \in \mathbb{C}^N$ ). Consider the composition operator  $C_S$  generated by  $S$ . Then the following properties are equivalent:*

- (a)  $S$  has no fixed point in  $\mathbb{C}^N$  and  $\det(A) \neq 0$ .
- (b) The vector  $b$  is not in  $\text{ran}(A - I)$  and  $\det(A) \neq 0$ .
- (c)  $C_S$  is universal.
- (d)  $C_S$  is hereditarily universal.

*Proof.* (a)  $\iff$  (b): Simply observe that  $b \in \text{ran}(A - I)$  if and only if there exists  $z_0 \in \mathbb{C}^N$  such that  $(A - I)z_0 = b$  if and only if  $Az_0 + b = z_0$  for some  $z_0 \in \mathbb{C}^N$  if and only if  $Sz_0 = z_0$  for some  $z_0 \in \mathbb{C}^N$ .

(d)  $\implies$  (c): This is trivial.

(c)  $\implies$  (a): If  $C_S$  is universal then  $S$  is one-to-one (hence  $\det(A) \neq 0$ ) and has no fixed point by Lemma 2.1.

(b)  $\implies$  (d): Since  $\det(A) \neq 0$ ,  $S$  is an affine automorphism of  $\mathbb{C}^N$ . Let us prove that  $S$  is non-recurrent. Denote by  $J$  the Jordan matrix of  $A$ . Then there is an invertible  $(N \times N)$ -matrix  $Q$  such that  $A = QJQ^{-1}$ . Define

$$c := Q^{-1}b \quad \text{and} \quad Mz := Jz + c \quad (z \in \mathbb{C}^N).$$



It is easy to see that  $b \notin \text{ran}(A - I)$  if and only if  $c \notin \text{ran}(J - I)$ . On the other hand, non-recurrence is preserved by similarities; more precisely, if  $\varphi \in H(\mathbb{C}^N, \mathbb{C}^N)$  and  $\psi \in \text{Aut}(\mathbb{C}^N)$ , then  $\varphi$  is non-recurrent if and only if  $\psi \circ \varphi \circ \psi^{-1}$  is non-recurrent. Since  $S = Q \circ M \circ Q^{-1}$  we obtain that  $S$  is non-recurrent if and only if  $M$  is.

The matrix  $J$  is a direct sum of Jordan blocks  $J_j$ . The vector  $c$  has components corresponding to each of these blocks and to say  $c \notin \text{ran}(J - I)$  is to say that at least one of these component, say  $c_j$  is not in  $\text{ran}(J_j - I_j)$ . Here  $I_j$  is the identity matrix having the same dimension as  $J_j$ . Let  $M_j$  be the restriction of  $M$  to the direct summand of  $\mathbb{C}^N$  on which  $J_j$  acts, i.e.,  $M_j z_j = J_j z_j + c_j$ . It suffices to prove that  $S_j$  is non-recurrent (this follows from the fact that if  $\mathbb{C}^N$  is represented as a product space, then each compact subset of  $\mathbb{C}^N$  is contained in a product of compact subsets corresponding to the factors of  $\mathbb{C}^N$ ). Hence we may drop the subscripts and assume that  $J$  itself is a Jordan block with  $c \notin \text{ran}(J - I)$ . In particular,  $J - I$  is not invertible. Thus the spectrum of  $J$  is the singleton  $\{1\}$ , so  $J$  itself has just 1's on the main diagonal and the first superdiagonal, and zeros elsewhere. But, by induction, one obtains

$$M^n z = J^n z + \sum_{k=0}^{n-1} J^k c \quad (z \in \mathbb{C}^N, n \in \mathbb{N}).$$

The “1” in the  $(N, N)$  position is crucial here; it is also the  $(N, N)$  entry of any power of  $M$ , and in all these powers the rest of the  $N$ -th row consists of zeros. Thus  $(M^n z)_N$ , the  $N$ -th component of  $M^n z$ , is  $z_N + n c_N$ . Now  $c \notin \text{ran}(J - I)$  means  $c_N \neq 0$ , hence as  $n \rightarrow \infty$  the  $N$ -th component of  $M^n z$  goes to infinity uniformly on each compact subset of  $\mathbb{C}^N$ . Hence  $\|S^n z\| \rightarrow +\infty$  ( $n \rightarrow \infty$ ) in the same way. It is easy to see that because of this  $M$  is recurrent.

Consequently,  $S$  is non-recurrent and, due to Lemma 2.4,  $C_S$  is universal. Finally, observe that if we fix a sequence  $\{n_1 < n_2 < \dots\} \subset \mathbb{N}$  then the same reasoning above –replacing  $n$  to  $n_j$ – shows that  $(M^{n_j})$  (hence  $(S^{n_j})$ ) is run-away, which by Remark 2.5 proves that  $(C_{S^{n_j}})$  is universal. In other words,  $C_S$  is hereditarily universal.  $\square$

**Remarks 3.2.** 1. The proof of Theorem 3.1 reveals that, even in the case that  $S$  is not invertible, we have:  $S$  has no fixed point if and only if  $b \notin$

$\text{ran}(A - I)$  if and only if  $S$  is non-recurrent. Observe also that, by Rouché–Frobenius’ theorem, the middle condition can be expressed as  $\text{rank}(A - I) < \text{rank}(A - I|b)$ , where  $A - I|b$  represents the  $(N \times (N + 1))$ -matrix obtained from  $A - I$  by adding the column  $b$  to  $A - I$ . In addition, note that if  $S$  is non-recurrent then 1 is an eigenvalue of  $A$ .

2. There are other popular automorphisms of  $\mathbb{C}^N$  which are not affine. For example there are the “shears” introduced by Rosay and Rudin [22], and defined by

$$\sigma(z) = z + f(\Lambda(z))u \quad (z \in \mathbb{C}^N),$$

where  $\Lambda$  is a linear functional on  $\mathbb{C}^N$  with  $\Lambda(u) = 0$ ,  $u$  is a fixed non-zero and  $f$  is an entire function of one complex variable that is never zero. Since  $\sigma^n(z) = z + nf(\Lambda(z))u$ , it is clear that  $\sigma$  is non-recurrent, hence if  $\sigma$  preserved convexity then the induced composition operator would be universal. Shears are of interest because the group they generate is known to be dense in the full automorphism group of  $\mathbb{C}^N$  [5]. The example of Rosay–Rudin can be generalized: Let  $\Lambda$  be a linear map on  $\mathbb{C}^N$  onto a proper subspace, and let  $f$  be an entire function non-zero on  $\text{ran } \Lambda$ . Then  $\sigma(z) := z + f(\Lambda(z))u$  defines an automorphism of  $\mathbb{C}^N$  with the same iteration properties as the one above.

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