TITLE: HOLOMORPHIC T–MONSTERS AND STRONGLY OMNIPRESENT OPERATORS. [1]

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Holomorphic T-monsters and strongly omnipresent operators

by

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Abstract

Assume that G is a nonempty open subset of the complex plane and that T is an operator on the linear space of holomorphic functions in G, endowed with the compact–open topology. In this paper we introduce the notions of strongly omnipresent operator and of T–monster, which are related to the wild behaviour of certain holomorphic functions near the boundary of G. T–monsters extend a concept introduced by W. Luh and K.–G. Grosse–Erdmann. After showing that T is strongly omnipresent if and only if the set of T–monsters is residual, it is proved in this paper that certain kinds of infinite order differential and antidifferential operators are strongly omnipresent, which improves some earlier nice results due to the mentioned authors.
Key words and phrases: holomorphic monster, T–monster, strongly omnipresent operator, infinite order differential operator, infinite order antidifferential operator, entire function of subexponential type, affine linear mappings, Laplace transform.

1 Introduction and notation

Throughout this paper G will stand for a nonempty open set in the complex plane $\mathbf{C}$. $\mathbf{N}$ is the set of positive integers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ is the set of all integers, $\mathbf{C}_\infty$ is the extended complex plane $\mathbf{C} \cup \{\infty\}$, ∂G is the (finite) boundary of G, $B(a, r)$ ($\overline{B}(a, r)$) is the euclidean open (closed, respectively) disk with center a and radius r ($a \in \mathbf{C},\ r > 0$) and $\mathbf{D} = B(0, 1)$. If $A \subset \mathbf{C}$ then A^0 is the interior of A and we denote $LT(A) = \{$affine linear transformations $\tau(z) = az + b$ such that $\tau(\mathbf{D}) \subset A\}$.

$H(G)$ denotes, as usual, the linear space of holomorphic functions on G, endowed with the topology τ_{uc} of uniform convergence on each compact subset in G. Let $\mathcal{K}(G)$ be the family of compact subsets of G. It is known that the family

$$\{D(f, K, \epsilon) :\ f \in H(G),\ K \in \mathcal{K}(G),\ \epsilon > 0\}$$

where

$$D(f, K, \epsilon) = \{g \in H(G) :\ |g(z) - f(z)| < \epsilon \text{ for all } z \in K\}$$

is a basis for τ_{uc}.

If K is a compact set of $\mathbf{C}$, then $A(K)$ will stand for the linear space $C(K) \cap H(K^0)$, which becomes a Banach space if it is endowed with the maximum norm.

A topological space X is a Baire space if and only if the intersection of a countable family of open dense subsets is also dense. Baire's theorem asserts that each completely metrizable topological space is a Baire space. Consequently, $H(G)$ is a Baire space. In a Baire space X, a subset is *residual* when it contains a dense G_δ–subset of X; such a subset is "very large" in X. These notions and results can be found, for instance, in [13, pp. 213–214], and [21, pp. 40–41].

If $f \in H(G)$ and $j \in \mathbf{N}_0$ we denote, as usual, by $f^{(j)}$ the derivative of f of order j. The linear operator $D^j : H(G) \to H(G)$ defined by $D^j f = f^{(j)}$ is continuous.

An entire function $\Phi(z) = \sum_{j \geq 0} a_j z^j$ on $\mathbf{C}$ is said to be of *exponential type* whenever there exist positive constants A and B such that

$$|\Phi(z)| \leq A e^{B|z|} \text{ for all } z \in \mathbf{C}.$$

This happens if and only if $\limsup_{j \to \infty} (j! |a_j|)^{\frac{1}{j}}$ is finite (cf. [23, Chapter VII]). Φ is said to be of *subexponential type* if and only if given $\epsilon > 0$, there is a positive constant $A = A(\epsilon)$ such that

$$|\Phi(z)| \leq A e^{\epsilon|z|} \text{ for all } z \in \mathbf{C}.$$

It happens that Φ is of subexponential type if and only if $\limsup_{j \to \infty} (j! |a_j|)^{\frac{1}{j}} = 0$ (cf., e.g., [5, 2.2.9–11]). Trivially, each entire function of subexponential type is also of exponential type. To every entire function Φ we can associate a "formal" infinite order differential operator with constant coefficients $L = \Phi(D)$, that is, $L = \sum_{j=0}^{\infty} a_j D^j$ with $D^0 = I =$ the identity operator. The following statement is easy to prove (see [3], [7, pp. 58–60], [11, Section 5] and [25, p. 35]) and furnishes a sufficient condition in order that $\Phi(D)$ can be an operator.

Theorem 1.1 *Let $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ an entire function of subexponential type. Then $\sum_{j=0}^{\infty} |a_j| \sup_{z \in L} |f^{(j)}(z)| < +\infty$ for every $L \in \mathcal{K}(G)$ and every $f \in H(G)$, and $\Phi(D)$ is a well–defined continuous linear operator on $H(G)$. If $G = \mathbf{C}$, the same assertion holds just by assuming that Φ is of exponential type.*

Recall now the "infinite order antidifferential operators". They were studied by the first author (see [4]). Firstly, assume that $G \subset \mathbf{C}$ is a simply connected domain and that a is a fixed point in G. If

$j \in \mathbf{N}$ and $f \in H(G)$, denote by $D^{-j}f$ the unique antiderivative g of order j of f (i.e., $(D^{-j}f)^{(j)} = f$) such that $g^{(k)}(a) = 0 \quad (k = 0, 1, \ldots, j-1)$. In fact, we have

$$D^{-j}f(z) = \int_a^z f(t)\frac{(z-t)^{j-1}}{(j-1)!}dt \quad (z \in G)$$

where the integral is taken along any rectifiable curve $\gamma \subset G$ joining a to z. Each D^{-j} is a continuous linear operator on $H(G)$. If $\delta \in [0, +\infty)$, then we denote by $S(\delta)$ the set of formal complex power series $\Psi(z) = \sum_{j=0}^{\infty} c_j z^j$ such that $\limsup_{j\to\infty} \left(\frac{|c_j|}{j!}\right)^{\frac{1}{j}} \leq \delta$. For fixed $c \in G$ we use the notation $\Delta(c, G) = \sup_{z \in G} \inf\{r > 0 : \ c$ is in the connected component of $B(z, r) \cap G$ containing $z\}$ (see [2, 6]). The following result can be found in [4, Theorem 6].

Theorem 1.2 *If* $\Psi(z) = \sum_{j=0}^{\infty} c_j z^j \in S\left(\frac{1}{\Delta(a, G)}\right)$ *then the series* $\Psi(D^{-1}) = \sum_{j=0}^{\infty} c_j D^{-j}$ *defines a continuous linear operator on* $H(G)$.

W. Luh has studied in a series of papers the problem of existence of functions with wild behaviour at every boundary point of certain open sets (see [14, 15, 16 and 17]). The following strong result can be found in [17].

Theorem 1.3 *Let* $G \subset \mathbf{C}$, $G \neq \mathbf{C}$, *be an open set with simply connected components. Then there exists a function* $f \in H(G)$ *with the following properties:*

(1) For every $t \in \partial G$, *every compact subset* K *with connected complement and every* $g \in A(K)$, *there exist affine linear mappings* $\tau_n(z) = a_n z + b_n$ *with* $\tau_n(K) \subset G \quad (n \in \mathbf{N})$ *and* $a_n \to 0$,

$b_n \to t$ $(n \to \infty)$ such that $f(\tau_n(z)) \to g(z)$ $(n \to \infty)$ uniformly on K.

(2) In addition, each derivative $f^{(j)}$ $(j \in \mathbf{N})$ of f and each antiderivative of f of arbitrary order has the boundary behaviour described in (1).

Functions satisfying (1) and (2) are called "holomorphic monsters" by Luh. He shows in [17] that its set is dense in $H(G)$. In addition, he proves that every f in $H(G)$ satisfying (1) also satisfies the next two properties:

(a) For every bounded open set $U \subset \mathbf{C}$ with simply connected components, every $g \in H(U)$ and every $t \in \partial G$, there exist affine linear mappings $\tau_n(z) = a_n z + b_n$ with $\tau_n(U) \subset G$ $(n \in \mathbf{N})$ and $a_n \to 0$, $b_n \to t$ $(n \to \infty)$ such that $f(\tau_n(z)) \to g(z)$ in $H(U)$.

(b) For every bounded Lebesgue–measurable set $S \subset \mathbf{C}$, every Lebesgue–measurable function $g : S \to \mathbf{C}_\infty$ and every $t \in \partial G$, there exist affine linear mappings $\tau_n(z) = a_n z + b_n$ with $\tau_n(S) \subset G$ $(n \in \mathbf{N})$ and $a_n \to 0$, $b_n \to t$ $(n \to \infty)$ such that $f(\tau_n(z)) \to g(z)$ almost everywhere in S.

The properties exhibited in Theorem 1.3 and in (a) and (b) can be expressed in terms of the maximality of certain generalized cluster sets introduced in [17].

A little later, in 1987, K.–G. Grosse–Erdmann showed [12, Kapitel 3] that if $f \in H(G)$ then it is a monster in the sense of Luh if and only if every derivative and every antiderivative of f of arbitrary order, say F, satisfies that for each Jordan domain $\Omega \subset \mathbf{C}$, each $t \in \partial G$ and each $g \in H(\Omega)$, there exist two sequences $\{a_n\}_{n\in\mathbf{N}}$

and $\{b_n\}_{n\in\mathbf{N}}$ in $\mathbf{C}$ such that $a_n \to 0$, $b_n \to t$ $(n \to \infty)$, $a_n z + b_n \in G$ $(z \in \Omega,\ n \in \mathbf{N})$ and $F(a_n z + b_n) \to g(z)$ $(n \to \infty)$ uniformly on compact subsets in Ω. In fact [12, Satz 3.0.2], it suffices to take $\Omega = \mathbf{D}$ in the latter property. By using the fact that being a monster is equivalent to the universality with respect to a certain family of composition–differentiation–antidifferentiation operators, he also proves [12, Satz 3.1.8] that the set of monsters on G is not only dense but residual in $H(G)$.

Monsters with additional properties were constructed by Luh [18] and I. Schneider [24], see also [19].

On the other hand, the first author [1] introduced in 1992 the "omnipresent holomorphic operators". Let us recall this notion. Let T be a continuous mapping $T : H(G) \to H(G)$. T need not be linear. Denote $O(\partial G) = \{V \subset \mathbf{C} :\ V$ is open and $V \cap \partial G \neq \emptyset\}$. Recall that T is said to be omnipresent if each subset $R(T, V, W) = \{f \in H(G) : \text{there exists } z \in G \cap V \text{ such that } Tf(z) \in W\}$ is dense in $H(G)$ for all $V \in O(\partial G)$ and all nonempty open subsets $W \subset \mathbf{C}$. In [1] it is shown that each differential and each antidifferential operator is omnipresent. These properties will be strengthened in Theorem 3.1 and, partially, in Theorem 4.2.

In this paper we introduce two original concepts which are connected with each other: the T–monsters and the strongly omnipresent operators. With these, we will be following the ways opened by Luh and Grosse–Erdmann from the different point of view presented by the first author in [1]. In particular, operators of type $\Phi(D)$ and $\Psi(D^{-1})$ are going to be studied.

2 Definitions

From now on, we denote by G an arbitrary open subset of $\mathbf{C}$, $G \neq \mathbf{C}$.

Let $T : H(G) \to H(G)$ be a continuous operator, which need not be linear, as it was supposed before. If $g \in H(\mathbf{D})$, $\epsilon > 0$, $r \in (0,1)$ and $V \in O(\partial G)$, then $U(T, g, \epsilon, r, V)$ will stand for the set

$$U(T, g, \epsilon, r, V) = \{f \in H(G) : \text{ there exists } \tau \in LT(V \cap G) \text{ such that}$$

$$|(Tf)(\tau(z)) - g(z)| < \epsilon \text{ for all } z \in r\overline{\mathbf{D}}\}.$$

It is evident that each of these sets is open in $H(G)$.

We will say that T is *strongly omnipresent* in G if and only if every set $U(T, g, \epsilon, r, V)$ ($g \in H(\mathbf{D})$, $\epsilon > 0$, $r \in (0,1)$, $V \in O(\partial G)$) is dense in $H(G)$. It is evident that each strongly omnipresent operator on $H(G)$ is also omnipresent. By Theorem 1(c) of [1] –applied on $T =$ the identity operator and $g(z) = \exp z$– we have that $f \mapsto \exp f$ is a (nonlinear) omnipresent operator on $H(G)$ which is not strongly omnipresent by Hurwitz's theorem and Lemma 2.1 together with Theorem 2.2 below. Up to date, we do not know whether a *linear* omnipresent non-strongly omnipresent operator can exist or not.

Inspired by [12], we will say that a function $f \in H(G)$ is a T–*monster* in G if and only if for each Jordan domain $\Omega \subset \mathbf{C}$, each $g \in H(\Omega)$ and each $t \in \partial G$, there exist two sequences $\{a_n\}_{n \in \mathbf{N}}$ and $\{b_n\}_{n \in \mathbf{N}}$ in $\mathbf{C}$ such that

$$a_n \to 0, \quad b_n \to t \quad (n \to \infty);$$

$$a_n z + b_n \in G \text{ for all } n \in \mathbf{N} \text{ for all } z \in \Omega$$

and

$$(Tf)(a_n z + b_n) \to g(z) \quad (n \to \infty)$$

uniformly on compact subsets in Ω. Note that a function $f \in H(G)$ is a monster in the sense of Luh if and only if f is a D^j-monster for every $j \in \mathbf{Z}$ (it is easy to see that if a j–antiderivative of f satisfies condition (1) in Theorem 1.3 then *every* j–antiderivative of f also satisfies (1)).

Denote by $\mathcal{M}(T, G)$ the set of T–monsters in G. Let us see now (Theorem 2.2) that both concepts are closely connected. Before this, we need the following auxiliary lemma. The proof of it is a simple adaptation of the first part of the proof of [12, Satz 3.0.2], and so it is left to the interested reader.

Lemma 2.1 *Let $\{g_i\}_{i\in\mathbf{N}}$ be a dense countable subset of $H(\mathbf{D})$ and $\{t_k\}_{k\in\mathbf{N}}$ a dense countable subset of ∂G. Then, $f \in H(G)$ is a T–monster in G if and only if for each $i \in \mathbf{N}$ and each $k \in \mathbf{N}$ there exist two sequences $\{a_n\}_{n\in\mathbf{N}}$, $\{b_n\}_{n\in\mathbf{N}}$ in $\mathbf{C}$ such that $a_n \to 0$, $b_n \to t_k$ ($n \to \infty$), $a_n z + b_n \in G$ for all $n \in \mathbf{N}$ and all $z \in \mathbf{D}$ and $(Tf)(a_n z + b_n) \to g_i(z)$ ($n \to \infty$) locally uniformly in Ω.*

Theorem 2.2 *Let $T : H(G) \to H(G)$ be a continuous mapping. Then T is strongly omnipresent in G if and only if $\mathcal{M}(T, G)$ is residual in $H(G)$.*

Proof. Fix a dense countable subset $\{g_i\}_{i\in\mathbf{N}}$ of $H(\mathbf{D})$ and a dense countable subset $\{t_k\}_{k\in\mathbf{N}}$ of ∂G. From the fact that $H(G)$ is a Baire space, the statement of the theorem will become obvious as soon as we prove that

$$\mathcal{M}(T, G) = \bigcap_{i,k,l\in\mathbf{N}} U(T, g_i, \frac{1}{l}, 1 - \frac{1}{l}, B(t_k, \frac{1}{l}))$$

For this, fix $f \in \mathcal{M}(T, G)$. For each $i \in \mathbf{N}$ and each $k \in \mathbf{N}$ there are two sequences $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$ in $\mathbf{C}$ such that $a_n \to 0$,

$b_n \to t_k$ $(n \to \infty)$; $a_n z + b_n \in G$ for all $n \in \mathbf{N}$ and for all $z \in \mathbf{D}$ and $(Tf)(a_n z + b_n) \to g_i$ $(n \to \infty)$ uniformly on compacta in $\mathbf{D}$. So, for every $l \in \mathbf{N}$ there exists $m \in \mathbf{N}$ such that

$$|a_m| < \frac{1}{2l} \ , \qquad |b_m - t_k| < \frac{1}{2l},$$

$$|(Tf)(a_m z + b_m) - g_i(z)| < \frac{1}{l} \text{ for all } z \in \overline{B}(0, 1 - \frac{1}{l}).$$

Then, $\tau(z) := a_m z + b_m \in LT(B(t_k, \frac{1}{l}) \cap G)$ and

$$f \in U(T, g_i, \frac{1}{l}, 1 - \frac{1}{l}, B(t_k, \frac{1}{l})).$$

Now, let $f \in \bigcap_{i,k,l \in \mathbf{N}} U(T, g_i, \frac{1}{l}, 1 - \frac{1}{l}, B(t_k, \frac{1}{l}))$. Then, for each $i \in I$ and each $k \in \mathbf{N}$ there is a sequence $\{\tau_l(z) = a_l z + b_l\}_{l \in \mathbf{N}}$ of affine linear mappings such that

$$\tau_l(\mathbf{D}) \subset G \cap B(t_k, \frac{1}{l}), \tag{1}$$

$$|(Tf)(\tau_l(z)) - g_i(z)| < \frac{1}{l} \text{ for all } z \in \overline{B}(0, 1 - \frac{1}{l}). \tag{2}$$

By (1), $a_l z + b_l \in G$ for all $l \in \mathbf{N}$ and all $z \in \mathbf{D}$, and $|a_l z + b_l - t_k| < \frac{1}{l}$ for every $z \in \mathbf{D}$. So, by taking $z = 0$ and $z = 1/2$, we get $b_l \to t_k$ and $a_l \to 0$ $(l \to \infty)$. Finally, (2) gives that $(Tf)(a_l z + b_l) \to g_i$ $(l \to \infty)$ uniformly on compact subsets in $\mathbf{D}$. Apply Lemma 2.1 to obtain that $f \in \mathcal{M}(T, G)$, as required. The proof is finished.

The following problem remains open: Is T strongly omnipresent just by assuming that $\mathcal{M}(T, G)$ is nonempty?

3 Infinite order differential operators

Let $\Phi(z) = \sum_{n \geq 0} a_n z^n$ be a nonidentically zero entire function of subexponential type and $G \subset \mathbf{C}$ an open subset with $G \neq \mathbf{C}$. Consider the continuous linear operator $T = \Phi(D)$. Since T can be defined on the space $H(\mathbf{C})$ of all entire functions and commutes with the translations, it is surjective from $H(\mathbf{C})$ onto $H(\mathbf{C})$ (see [9], [11, Section 5] and [20]). In particular, for every polynomial P, there exists an entire function f such that $Tf = P$. We are now ready to state our next result, which improves strongly Theorem 2(a) of [1].

Theorem 3.1 *The operator $T = \Phi(D)$ is strongly omnipresent in G.*

Proof. Fix $g \in H(\mathbf{D})$, $\epsilon > 0$, $r \in (0,1)$ and $V \in O(\partial G)$. We have to prove that $U(T, g, \epsilon, r, V)$ is dense in $H(G)$.

Let K be a compact subset of G, $\delta > 0$ and $f \in H(G)$. It is evident that we can find a point a, a positive real number s and a set L satisfying:

1. L is compact, $K \subset L \subset G$ and each connected component of $\mathbf{C}_\infty \setminus L$ contains some connected component of $\mathbf{C}_\infty \setminus G$.
2. $\overline{B}(a,s) \subset V \cap (G \setminus L)$.

Consider the affine linear mapping $\tau(z) = sz + a$. Then

$$\tau(\mathbf{D}) = B(a,s) \subset V \cap G \quad \text{and} \quad \tau(r\overline{\mathbf{D}}) = \overline{B}(a, rs). \tag{1}$$

On the other hand, by taking $r' \in (r,1)$, a positive constant A can be found in such a way that

$$|a_n| \leq A \frac{(\frac{1}{2}(r'-r)s)^n}{n!} \text{ for all } n \geq 0, \tag{2}$$

because Φ is of subexponential type.

Since $g \circ \tau^{-1} \in H(B(a,s))$, we can obtain a polynomial $P(z)$ with

$$|P(z) - g \circ \tau^{-1}(z)| < \frac{\epsilon}{2} \text{ for all } z \in \overline{B}(a, r's) \tag{3}$$

and, as noted just above this theorem, there exists an entire function Q such that

$$\Phi(D)\,Q = P. \tag{4}$$

Choose an open subset W of G satisfying $L \subset W$ and $W \cap B(a,s) = \emptyset$. If $S = L \cup \overline{B}(a, r's)$ and $U = W \cup B(a,s)$, then S is compact, U is open, $S \subset U \subset G$ and each "hole" of S contains at least one "hole" of G. Define φ on U by

$$\varphi(z) = \begin{cases} f(z) & \text{if } z \in W \\ Q(z) & \text{if } z \in B(a,s). \end{cases}$$

Trivially $\varphi \in H(U)$. By Runge's Theorem, there exists a rational function h, with poles outside G (hence $h \in H(G)$), such that

$$|\varphi(z) - h(z)| < \min\left(\delta, \frac{r'-r}{4Ar'}\epsilon\right) \text{ for all } z \in S.$$

Then

$$|f(z) - h(z)| < \delta \text{ on } K. \tag{5}$$

Furthermore, if $\gamma = \{t : \ |t-a| = r's\}$, Cauchy's formula gives

$$|h^{(n)}(z) - Q^{(n)}(z)| = \left| \frac{n!}{2\pi i} \oint_\gamma \frac{h(t) - Q(t)}{(t-z)^{n+1}} dt \right| \leq$$

$$\frac{n!}{2\pi} \frac{\text{length}(\gamma)}{((r'-r)s)^{n+1}} \sup_{\overline{B}(a,r's)} |h(t) - Q(t)| \leq$$

$$\frac{n!\epsilon}{4A((r'-r)s)^n} \text{ for all } n \geq 0 \text{ and all } z \in \overline{B}(a, rs). \tag{6}$$

Therefore, by (2), (4) and (6),

$$|\Phi(D)h(z) - P(z)| = |\Phi(D)h(z) - \Phi(D)Q(z)| =$$

$$\left| \sum_{n \geq 0} a_n (h^{(n)}(z) - Q^{(n)}(z)) \right| \leq \sum_{n \geq 0} |a_n||h^{(n)}(z) - Q^{(n)}(z)| \leq$$

$$\frac{\epsilon}{4}\sum_{n\geq 0}\left(\frac{1}{2}\right)^n = \frac{\epsilon}{2} \text{ for all } z \in \overline{B}(a, rs). \tag{7}$$

So, by (1), (3) and (7), there exists $\tau \in LT(G \cap V)$ such that

$$|\Phi(D)h(z) - g \circ \tau^{-1}(z)| \leq$$

$$|\Phi(D)h(z) - P(z)| + |P(z) - g \circ \tau^{-1}(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

in $\overline{B}(a, rs)$. Thus,

$$|(Th)(\tau(z)) - g(z)| < \epsilon \text{ for all } z \in r\overline{\mathbf{D}}. \tag{8}$$

Consequently, by (5) and (8),

$$h \in U(T, g, \epsilon, r, V) \cap D(f, K, \delta)$$

i.e., $U(T, g, \epsilon, r, V)$ is dense in $H(G)$ and T is strongly omnipresent.

Remark 3.2 Although we have proved that T is strongly omnipresent, we could give a similar proof to see, directly, that $\mathcal{M}(T, G)$ is residual in $H(G)$. In fact, the proof would only be changed in its first part.

4 Infinite order antidifferential operators

In [1, Theorem 3], the first author showed that the operator $S : H(G) \to H(G)$ defined by

$$Sf(z) = \int_a^z \varphi(z, t) f(t)\, dt \qquad (z \in G)$$

is omnipresent, where $G \subset \mathbf{C}$ is a simply connected domain, $a \in G$ is fixed, $\varphi : G \times G \to \mathbf{C}$ is a function which is not identically zero and holomorphic with respect to both variables. The integral is taken along any rectifiable curve in G joining a to z.

We can improve the latter result, at least when the kernel $\varphi(z, t)$ is an entire function depending only on the difference $z - t$ and does not grow too much. Note that, in our main result (Theorem 4.2), T is something like a Volterra operator of the second kind in the complex plane (S would be of the first kind). In the proof of the theorem the following elementary lemma will be needed.

Lemma 4.1 *If $G \subset \mathbf{C}$ is a simply connected domain with $G \neq \mathbf{C}$, $b \in \mathbf{C} \setminus G$ and $N \in \mathbf{N}$, then the set $\mathcal{P}(b, N)$ of polynomials P such that b is a zero of P with multiplicity not less than N is dense in $H(G)$.*

The proof of the latter lemma is an easy application of Runge's theorem, so it is left to the reader.

Theorem 4.2 *Assume that $G \subset \mathbf{C}$ is a simply connected domain, $G \neq \mathbf{C}$, $a \in G$ and h is an entire function of exponential type, h not identically zero. Let $\lambda \in \mathbf{C}$ and consider the linear operator T on $H(G)$ defined by*

$$Tf(z) = \lambda f(z) + \int_a^z h(z - t) f(t)\, dt \quad (z \in G)$$

where the integral is taken along any rectifiable curve in G joining a to z. Then T is strongly omnipresent.

Proof. In order to cause no confusion, we will keep the letter z to denote the points of $\mathbf{D}$. Let us set $h(w - t) = \varphi(w, t)$. Then $\varphi : G \times G \to \mathbf{C}$ is a function not identically zero and holomorphic with respect to both variables. We may do this because the first part of the proof is not based on the fact that the kernel only depends upon $w - t$.

In order to show that T is strongly omnipresent, fix $f \in H(G)$, $\epsilon > 0$, a compact subset $K \subset G$, $V \in O(\partial G)$, $\delta > 0$, $r \in (0,1)$ and $g \in H(\mathbf{D})$. It is clear that we can find $b,\ B,\ \gamma,\ \tau$ and L satisfying:

1. L is compact, $\{a\} \cup K \subset L \subset G$ and the complement of L is connected,
2. B is a closed ball with $B = \tau(\overline{\mathbf{D}}) \subset V \cap G \setminus L$, τ being a nonconstant affine linear mapping,
3. γ is a rectifiable Jordan arc in G joining a to b and $\gamma = \gamma_1 + \gamma_2$, where γ_k $(k = 1,2)$ are rectifiable Jordan arcs too, $\gamma_1 = \gamma \cap L$, b is the end point of γ_2 and $\gamma \cap B = \{b\}$.

Assume that we have proved the following property:

$$\left\{ \begin{array}{c} \text{The linear operator } S : H_b(\mathbf{C}) \to H(B^0) \text{ given by} \\ \\ (S\psi)(w) = \lambda\psi(w) + \int_b^w \psi(t)\varphi(w,t)dt \quad (w \in B^0) \\ \\ \text{has dense range,} \end{array} \right\} (P)$$

where $H_b(\mathbf{C}) := \{\psi \in H(\mathbf{C}) :\ \psi(b) = 0\}$, which is a closed subspace of $H(\mathbf{C})$. Each integration curve may unambiguously be chosen as a rectifiable Jordan arc joining b to w contained in B, for instance, the segment $[b,w]$.

Denote $\gamma(w) = \gamma + [b,w]$. Observe that $\text{length}(\gamma(w)) \le \text{length}(\gamma) + 2s$ for all $w \in B$, where s is the radius of B. Define the constant α by

$$\alpha = \sup\{|\varphi(w,t)| :\ w \in B,\ t \in \gamma \cup B\}.$$

We can suppose that γ_2 has a parametrization $u \in [0,1] \mapsto \beta(u)$, where β is injective. If $\tilde{f} : \gamma \to \mathbf{C}$ is defined as

$$\tilde{f}(w) = \begin{cases} f(w) & \text{if } w \in \gamma_1 \\ f(\beta(0))(1-u) & \text{if } w = \beta(u) \in \gamma_2, \end{cases}$$

then $\tilde{f}$ is continuous on γ.

Note that $g \circ \tau^{-1}$ is defined and holomorphic on B^0 because g is holomorphic on $\mathbf{D} = \tau^{-1}(B^0)$. The same is true for the function $w \mapsto \int_\gamma \varphi(w,t)\tilde{f}(t)\,dt$. From property (P), there exists an entire function Ψ such that $\Psi(b) = 0$ and

$$\left|(S\Psi)(w) - \left(g(\tau^{-1}(w)) - \int_\gamma \varphi(w,t)\tilde{f}(t)dt\right)\right| < \frac{\delta}{3} \text{ for all } w \in B_1 \tag{1}$$

where $B_1 := \tau(r\overline{\mathbf{D}})$. Let $K_0 = L \cup \gamma \cup B$. Define the function $f_1 : K_0 \to \mathbf{C}$ as

$$f_1(w) = \begin{cases} f(w) & \text{if } w \in L \\ f(\beta(0))(1-u) & \text{if } w = \beta(u) \in \gamma_2 \\ \Psi(w) & \text{if } w \in B. \end{cases}$$

Then K_0 is a compact set whose complement is connected, f_1 is continuous on K_0 and holomorphic in its interior $K_0^0 = L^0 \cup B^0$. By Mergelyan's Theorem ([10, pp. 97–109] or [22, Chap. 20]), there is a polynomial P such that

$$|P(z) - f_1(z)| < \frac{\min(\epsilon, \delta, 1)}{1 + 3|\lambda| + 3\alpha(\text{length}(\gamma) + 2s)} \text{ for all } z \in K_0. \tag{2}$$

We have, for every $w \in B_1$,

$$(TP)(w) - g(\tau^{-1}(w)) = \lambda P(w) + \int_{\gamma(w)} \varphi(w,t)P(t)dt - g(\tau^{-1}(w)) =$$

$$\lambda(P(w) - \Psi(w)) + \int_{\gamma(w)} \varphi(w,t)(P(t) - f_1(t))dt + \int_\gamma \varphi(w,t)\tilde{f}(t)dt +$$

$$\lambda\Psi(w) + \int_b^w \varphi(w,t)\Psi(t)dt - g(\tau^{-1}(w)) =$$
$$I + J + M,$$

where

$$I = \int_{\gamma(w)} \varphi(w,t)(P(t) - f_1(t))\, dt,$$
$$J = (S\Psi)(w) - \left(g(\tau^{-1}(w)) - \int_\gamma \varphi(w,t)\tilde{f}(t)\, dt\right),$$
$$M = \lambda(P(w) - \Psi(w)).$$

Inequality (2) yields

$$|I| < \frac{\delta}{3}$$

and

$$|M| < \frac{\delta}{3}.$$

From (1) we obtain $|J| < \frac{\delta}{3}$, so $|(TP)(w) - g(\tau^{-1}(w))| < \delta$ because of the triangle inequality. A change of variables shows that

$$|(TP)(\tau(z)) - g(z)| < \delta \quad \text{for all } z \in r\overline{\mathbf{D}}. \tag{3}$$

But (2) also gives that

$$|P(w) - f(w)| < \epsilon \ \text{ on } K \tag{4}$$

because $K \subset L \subset K_0$. Then (3) and (4) tell us that

$$P \in D(f,K,\epsilon) \cap U(T,g,\delta,r,V).$$

Hence $U(T,g,\delta,r,V)$ is dense in $H(G)$ and T is strongly omnipresent, as required.

Thus, we should prove property (P). At this point the shape $\varphi(w,t) = h(w-t)$ of φ is crucial. A "Laplace transform" method will be used in this part of the proof. From now on, the letter z may be any complex number, not necessarily in $\mathbf{D}$.

By applying Lemma 4.1 on $G = B^0$ and taking into account that every polynomial can be written as a finite linear combination of powers $(z-b)^m$, it suffices to show that there is $m_0 \in \mathbf{N}$ such that for each $m \geq m_0$ a corresponding function $\Psi \in H(\mathbf{C})$ can be found in such a way that $\Psi(b) = 0$ and

$$\lambda\Psi(z) + \int_b^z \Psi(t)h(z-t)\,dt = (z-b)^m \text{ for all } z \in \mathbf{C}.$$

By the analytic continuation principle and a simple change of variables, we should be done whenever we are able to prove the existence of $m_0 \in \mathbf{N}$ such that for every $m \geq m_0$ there is an entire solution $\Psi(z)$ with $\Psi(0) = 0$ of the functional equation

$$\lambda\Psi(x) + (\Psi \star h)(x) - x^m = 0 \quad (x \in (0, +\infty)), \tag{5}$$

where $\Psi \star h$ is the convolution product

$$(\Psi \star h)(x) = \int_0^x \Psi(t)h(x-t)\,dt.$$

Let us choose $m_0 = 2 +$ [the multiplicity of h for the zero at the origin] (the multiplicity may be, of course, zero) and fix $m \geq m_0$.

Since h is of exponential type, its Laplace transform

$$(Lh)(z) = \int_0^{+\infty} h(t)e^{-zt}dt$$

is defined and holomorphic on a certain half plane $\{\mathrm{Re}\, z > x_0\}$. In fact, if $\sum_{j=0}^{\infty} h_j z^j$ is the Taylor series of h, then $\rho := \limsup_{j\to\infty}(j!|h_j|)^{\frac{1}{j}} < +\infty$ and the series $\sum_{j=0}^{\infty} \frac{j!h_j}{z^{j+1}}$ converges on $\{|z| > \rho\}$. We have that $(Lh)(z) = \sum_{j=0}^{\infty} \frac{j!h_j}{z^{j+1}}$ on $\mathrm{Re}\, z > \rho$. We have used that the Laplace transform of each function x^m is $\frac{m!}{z^{m+1}}$. Consider the expression

$$F^\star(z) = \frac{m!z^{m+1}}{\lambda + \sum_{j=1}^{\infty}(j-1)!h_{j-1}z^j}.$$

Note that the series in the denominator defines a holomorphic function on $B(0, \frac{1}{\rho})$ whose zero at the origin has multiplicity not greater than $m_0 - 1$, so $F^\star(0) = 0$ (with multiplicity at least 2) and there is $\mu \in (0, \frac{1}{\rho})$ such that $F^\star \in H(B(0, \mu))$. Then the function $F(z) := F^\star(\frac{1}{z})$ is holomorphic in the neighbourhood $|z| > \frac{1}{\mu}$ of ∞ and has a zero at this point. Consequently, it has an expansion

$$F(z) = \sum_{j=0}^{\infty} \frac{c_j}{z^j} \qquad (|z| > \frac{1}{\mu}),$$

with $c_0 = c_1 = 0$.

Thus, F is representable by means of an absolutely convergent Laplace integral. In fact (see [8, p. 66]), the function

$$\Psi(z) = \sum_{j=1}^{\infty} \frac{c_{j+1}}{j!} z^j \qquad (z \in \mathbf{C})$$

is entire and of exponential type, and satisfies $\Psi(0) = 0$ and

$$F(z) = \int_0^\infty \Psi(t) e^{-zt} dt \quad (\operatorname{Re} z > R)$$

for certain $R \in (0, +\infty)$. Hence

$$\left(\lambda + \sum_{j=0}^{\infty} \frac{j! h_j}{z^{j+1}} \right) \int_0^\infty \Psi(t) e^{-zt} dt = \frac{m!}{z^{m+1}} \quad (\operatorname{Re} z > R),$$

that is,

$$\lambda (L\Psi)(z) + (Lh)(z) \cdot (L\Psi)(z) = (L\sigma)(z) \ \ (\operatorname{Re} z > R),$$

where $\sigma(x) := x^m$. But, by Borel's theorem, $L(\Psi \star h)(z) = L(\Psi)(z) \cdot (Lh)(z)$, so, by linearity,

$$L(\lambda \Psi + \Psi \star h - \sigma)(z) = 0 \quad (\operatorname{Re} z > R).$$

Since $\lambda\Psi + \Psi \star h - \sigma$ is continuous on the interval $(0, +\infty)$, we obtain $\lambda\Psi + \Psi \star h - \sigma = 0$ on that interval (see [8, pp. 53–54]). But this is (5). This completes the proof.

We obtain immediately the following corollary, which of course can be applied on a nonzero entire function Ψ.

Corollary 4.3 *Let $G \subset \mathbf{C}$ be a simply connected domain with $G \neq \mathbf{C}$. Fix a point $a \in G$ and let $\Psi(z) := \sum_{j=0}^{\infty} c_j z^j \in H(B(0,R))$ for some $R > 0$, Ψ being not identically zero. Consider the operator $T = \Psi(D^{-1})$ on $H(G)$. Then T is strongly omnipresent.*

Proof. Recall that, in the definition of $\Psi(D^{-1})$, a point $a \in G$ has been fixed. We will see that the hypotheses of Theorem 3.1 or Theorem 4.2 are satisfied. We have that

$$\limsup_{j \to \infty} (|c_j|^{\frac{1}{j}}) < +\infty \tag{6}$$

and, trivially,

$$\limsup_{j \to \infty} \left(\frac{|c_j|}{j!} \right)^{\frac{1}{j}} = 0. \tag{7}$$

By (7) and Theorem 1.2, T is a linear continuous operator on $H(G)$ and for any $f \in H(G)$

$$Tf(z) = \lambda f(z) + \int_a^z h(z-t) f(t) dt \quad (\text{for all } z \in G)$$

where $\lambda \equiv c_0 \in \mathbf{C}$ and $h(w) = \sum_{j=1}^{\infty} \frac{c_j}{(j-1)!} w^{j-1}$. Observe that h is an entire function of exponential type, because of (6).

Consequently, either h is identically zero (so $\lambda \neq 0$, because Ψ is not identically zero) and we get all hypotheses of Theorem 3.1 just by taking $\Phi(z) \equiv \lambda$, or h is not identically zero and we apply Theorem 4.2. In any case T is strongly omnipresent and the proof is finished.

The following final result is a straightforward consequence of Theorems 2.2, 3.1 and Corollary 4.3, and of the fact that, in a Baire space, the intersection of a countable family of residual subsets is also residual. In fact, the Baire category approach is now a classical tool in the study of universality. Note that, for simply connected domains, Luh's and Grosse–Erdmann's results follow from the special case $\Phi_j(z) = z^{|j|}$ $(j \in \mathbf{Z})$.

Theorem 4.4 *Assume that G is a simply connected domain of $\mathbf{C}$, $G \neq \mathbf{C}$. Fix a point $a \in G$ and let D^{-1} be the corresponding antidifferential operator. Suppose that $\{\Phi_j\}_{j \in \mathbf{Z}}$ is a family of nonidentically zero entire functions such that Φ_j is of subexponential type for $j \in \mathbf{N}_0$. Denote $T_j = \Phi_j(D)$ whenever $j \in \mathbf{N}_0$ and $T_j = \Phi_j(D^{-1})$ whenever $-j \in \mathbf{N}$. Then there is a residual subset in $H(G)$ consisting of functions f satisfying the following property: For each $j \in \mathbf{Z}$, each $t \in \partial G$, each compact subset $K \subset \mathbf{C}$ with connected complement and each $g \in A(K)$, there exist affine linear transformations τ_n with $\tau_n(K) \subset G$ $(n \in \mathbf{N})$ and $a_n \to 0$, $b_n \to t$ $(n \to \infty)$ such that $(T_j f) \circ \tau_n \to g$ $(n \to \infty)$ uniformly on K.*

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