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HYPERCYCLIC SEQUENCES OF DIFFERENTIAL AND ANTIDIFFERENTIAL OPERATORS

By

LUIS BERNAL–GONZÁLEZ*

Abstract. In this paper, we provide some extensions of earlier results about hypercyclicity of some operators on the Fréchet space of entire functions of several complex variables. Specifically, we generalize in several directions a theorem about hypercyclicity of certain infinite order linear differential operators with constant coefficients and study the corresponding property for certain kinds of "antidifferential" operators which are introduced in the paper. In addition, the existence of hypercyclic functions for certain sequences of differential operators with additional properties, for instance, boundedness or with some nonvanishing derivatives, is established.

1. INTRODUCTION AND NOTATION

In this paper we denote by $\mathbf{N}$ the set of positive integers, by $\mathbf{C}$ the field of complex numbers and by $\mathbf{N_0}$ the set $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. Let X, Y be two topological spaces, $T_n : X \to Y$ $(n \in \mathbf{N})$ a sequence of continuous mappings and $x \in X$. Then x is said to be *hypercyclic* (or *universal*) for $\{T_n\}$ if its orbit $\{T_n x : n \in \mathbf{N}\}$ under $\{T_n\}$ is dense in Y. The sequence $\{T_n\}$ is *hypercyclic* whenever it has a hypercyclic element. It is clear that, in order that $\{T_n\}$ can be hypercyclic, Y must be separable. If $T : X \to X$ is a continuous selfmapping, then an element $x \in X$ is said to be *hypercyclic* for T if and only if it is hypercyclic for the sequence $\{T^n\}$, where $T^n = T \circ T \circ ... \circ T$ (n times). T is *hypercyclic* when there is a hypercyclic element for T. A subset $A \subset X$ is *invariant* under T when $TA \subset A$. It is evident that x is hypercyclic for T if and only if there is no proper, closed, invariant subset in X containing x. So, hypercyclicity is connected with the problem of the invariant subspace. If X is a linear topological space, we say that T is an *operator* on X whenever T is a continuous linear transformation taking X into itself.

We now furnish a sufficient condition for a sequence $\{T_n\}$ to be hypercyclic. Its proof is an easy application of the Baire Category Theorem and is left to the reader. Several versions of this result have earlier appeared in [13, Section 2], [14, Section 1], [15, Satz 1.2.2], [16] and [19, Theorem 2.1]. Note that, in a Baire space X, a dense G_δ subset is "very large" in X. A subset $A \subset X$ is residual if and only if it contains a dense G_δ subset.

THEOREM 1. *Let X be a linear topological space that is a Baire space, Y a metrizable separable (linear topological space, $D \subset X$ dense in X, $E \subset Y$ dense in Y and $T_n : X \to Y$ ($n \in \mathbf{N}$) a countable family of continuous linear mappings satisfying the following condition:*

> *For every $d \in D$ and every $e \in E$ there is a sequence $\{x_k\} \subset X$ and a subsequence $\{n_k\}$ of positive integers such that $x_k \to d$ and*

$T_{n_k}(x_k) \to e \ (k \to \infty).$

Then $\{T_n\}$ *has a dense* G_δ *subset of hypercyclic vectors.*

The existence of hypercyclic operators on any separable Fréchet space has been recently proved in [1] (see also [7]). B. Beauzamy [2, 3, 4] has constructed examples of linear operators on Hilbert spaces having dense, invariant linear manifolds all of whose nonzero elements are hypercyclic. P. S. Bourdon [11] proved in 1993 that any hypercyclic operator on a complex Banach space has a dense, invariant linear manifold consisting, except for zero, entirely of hypercyclic vectors. In fact (see [1]) this result holds in a more general setting. We state it for future references.

THEOREM 2. *Let* T *be a hypercyclic operator on a complex, separable, locally convex space* X. *Then there is a dense* T*-invariant linear manifold of* X *consisting entirely, except for zero, of vectors that are hypercyclic for* T.

Let G be a nonempty open subset of $\mathbf{C}^N$ $(N \in \mathbf{N})$. G is said to be a domain when, in addition, it is connected. A domain $G \subset \mathbf{C}^N$ is said to be a *Runge domain* if each analytic function on G can be approximated uniformly by polynomials on every compact subset of G (see [18, pp. 52-59] and [20, Chapter 5]). When $N = 1$, the Runge domains are precisely the simply connected domains. Denote by $H(G)$, as usual, the Fréchet space of analytic functions on G endowed with the compact-open topology. G. D. Birkhoff [10] showed in 1929 that every translation operator

τ_a (that is, $\tau_a f(z) = f(z+a)$, where $a \in \mathbf{C} \setminus \{0\}$ is fixed) is hypercyclic on the space $H(\mathbf{C})$ and G. R. MacLane [23] obtained the same conclusion in 1952 for the operator of differentiation $f \mapsto f'$. G. Godefroy and J. H. Shapiro [14, Section 5] demonstrated in 1991 the following strong generalization of the theorems of Birkhoff and MacLane:

THEOREM 3. *If L is an operator on the space $H(\mathbf{C}^N)$ of entire funcions on $\mathbf{C}^N$ that commutes with each of the translation operators τ_a $(a \in \mathbf{C}^N)$, and is not a scalar multiple of the identity, then L has a dense, invariant vector manifold each of whose non-zero elements is hypercyclic for L.*

See also [5, 8, 9, 15 and 22] for other generalizations of Birkhoff-MacLane's theorems. Several works have been made in connection with additional properties imposed to hypercyclicity. For instance, Grosse-Erdmann [16] proved in 1990 that there is no hypercyclic entire function f for the differentiation operator D satisfying $\max_{|z|=r} |f(z)| = O(e^r/r^{1/2})$ $(r \to \infty)$, while there is a D-hypercyclic entire function f such that $\max_{|z|=r} |f(z)| = O(\varphi(r) \cdot e^r/r^{1/2})$ $(r \to \infty)$, $\varphi : (0, +\infty) \to (0, +\infty)$ being a prefixed function such that $\varphi(r) \to \infty$ $(r \to \infty)$. G. Herzog [17] showed in 1994 that there is a D-hypercyclic function f such that f and f' are zero-free. This result has been recently improved by the author [6], which proves that, if $q \in \mathbf{N_0}$ and a nonconstant entire function Φ of subexponential type are given, then the set $A = \{f \in H(\mathbf{C}) : f^{(q)} \text{ and } f^{(q+1)} \text{ are zero-free}\}$ contains a residual subset of $\Phi(D)$-universal functions. The result is sharp in terms of the growth and the type of Φ.

In this paper we extend Theorem 3 and the result of the latter paragraph about zero-free derivatives to more general domains and sequences of operators and introduce and study a new kind of operators related to antiderivatives. The existence of bounded hypercyclic functions is established for certain domains. We also provide a rather general "eigenvalue test" in order to prove the hypercyclicity of certain kinds of operators and sequences of operators.

2. DIFFERENTIAL AND ANTIDIFFERENTIAL OPERATORS

In order to generalize in Section 4 Godefroy-Shapiro's result stated in Section 1, we adopt the notation of [14, Section 5] and transcribe some preliminaries from it. For $1 \le j \le N$ let D_j denote complex partial differentiation with respect to the jth coordinate. A *multi-index* is an N-tuple $p = (p_1, ..., p_N)$ of nonnegative integers. Denote $|p| = p_1 + ... + p_N$, $p! = p_1! \cdot ... \cdot p_N!$, $D^p = D_1^{p_1} \circ ... \circ D_N^{p_N}$ ($D^0 = I =$ the identity operator) and $z^p = z_1^{p_1} \cdot ... \cdot z_N^{p_N}$ if $z = (z_1, ..., z_N)$. An entire function $\Phi(z) = \sum_{|p| \ge 0} a_p z^p$ on $\mathbf{C}^N$ is said to be of *exponential type* whenever there exist positive constants A and B such that $|\Phi(z)| \le A e^{B|z|}$ ($z \in \mathbf{C}^N$). This happens if and only if there is $R \in (0, +\infty)$ for which

$$|a_p| \leq \frac{R^{|p|}}{p!} \quad (|p| \geq 0).$$

It is shown in [14] that, if Φ is of exponential type, then the mapping $\Phi(D) = \sum_{|p|\geq 0} a_p D^p$ is a well-defined operator on $H(\mathbf{C}^N)$. Note that if Φ is an entire function and $L = \Phi(D)$, then $L^n = \Phi^n(D)$ for all $n \in \mathbf{N}$ ($L^n = L \circ L \circ \ldots \circ L$ but $\Phi^n = \Phi \cdot \Phi \cdot \ldots \cdot \Phi$, n times).

Trivially, every linear differential operator with constant coefficients commutes with translations. In [14] it is shown that the operators on $H(\mathbf{C}^N)$ commuting with translations behave as "infinite order" differential operators.

THEOREM 4. *Let L be an operator on $H(\mathbf{C}^N)$. The following conditions are equivalent:*

a) L commutes with every translation operator τ_a ($a \in \mathbf{C}^N$).

b) L commutes with each of the differentiation operators D_k ($1 \leq k \leq N$).

c) $L = \Phi(D)$, where Φ is an entire function on $\mathbf{C}^N$ of exponential type.

Some additional notations and results are needed in order to prove our theorems. If $a = (a_1, ..., a_N) \in \mathbf{C}^N$ and $r > 0$, we denote by $D(a, r)$ the closed polydisc $D(a, r) = \{z \in \mathbf{C}^N : |z_j - a_j| \leq r, 1 \leq j \leq N\}$. We consider in $\mathbf{C}^N$ the distance $d(z, a) = \max\{|z_1 - a_1|, ..., |z_N - a_N|\}$. If g is a function defined on a subset $B \subset \mathbf{C}^N$, then $||g||_B$ will stand for $\sup\{|g(z)| : z \in B\}$. We say that an entire function $\Phi(z) = \sum_{|p|\geq 0} a_p z^p$ on $\mathbf{C}^N$ is of *subexponential type* whenever, given $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon)$ such that $|\Phi(z)| \leq K e^{\varepsilon|z|}$ $(z \in \mathbf{C}^N)$. A straightforward computation with power series and the Cauchy inequalities [18, p. 27] shows that Φ is of subexponential type if and only if, given $\varepsilon > 0$, there is a positive constant $A = A(\varepsilon)$ such that

$$|a_p| \leq A \cdot \frac{\varepsilon^{|p|}}{p!} \quad (|p| \geq 0).$$

Note that, if $N = 1$, then Φ is of subexponential type if and only if Φ is either of growth order less than one or of growth order one and growth type zero. Each entire function of subexponential type is obviously of exponential type.

THEOREM 5. *If $G \subset \mathbf{C}^N$ is a nonempty open subset and $\Phi(z) = \sum_{|p|\geq 0} a_p z^p$ is an entire function of subexponential type, then the series $\Phi(D) = \sum_{|p|\geq 0} a_p D^p$ defines an operator on $H(G)$.*

Proof. If $G = \mathbf{C}^N$, the result is a particular case of the above considerations. So, we may suppose that $G \neq \mathbf{C}^N$. Fix $f \in H(G)$ and a compact subset $K \subset G$. Let $\varepsilon = \frac{1}{2} d(K, \mathbf{C}^N \setminus G)$. Then there is $A \in (0, +\infty)$ such that $|a_p| \leq A \cdot \frac{(\varepsilon/2)^{|p|}}{p!}$ $(|p| \geq 0)$. Fix a point $a \in K$. The Cauchy formula for derivatives [18, p. 27, Formula 2.2.3] tells that

$$|D^p f(a)| \leq \frac{p! ||f||_{D(a,\varepsilon)}}{\varepsilon^{|p|}} \leq \frac{p! ||f||_{K_1}}{\varepsilon^{|p|}},$$

where K_1 is the compact set $\{z : \, d(z, K) \leq \varepsilon\}$. Note that $K \subset K_1 \subset G$. Therefore

$||D^p f||_K \le p! ||f||_{K_1}/\varepsilon^{|p|}$ and $\sum_{|p|\ge 0} ||a_p D^p f||_K \le \sum_{|p|\ge 0} A \cdot \frac{(\varepsilon/2)^{|p|}}{p!} \cdot \frac{p!\, ||f||_{K_1}}{\varepsilon^{|p|}} = 2^N A \cdot ||f||_{K_1}$, so the series $\sum_{|p|\ge 0} a_p D^p f$ converges uniformly on K and $\Phi(D)$ defines a mapping from $H(G)$ into itself. The linearity is trivial and, since $||\Phi(D)f||_K \le 2^N A \cdot ||f||_{K_1}$ for every $f \in H(G)$, we obtain that $\Phi(D)$ is continuous on $H(G)$. ////

We now introduce a (as far as we know) new kind of operators for $\mathbf{C}^N = \mathbf{C}$, namely, the "infinite order antidifferential operators". They are defined in Theorem 6, after a number of definitions and considerations.

Firstly, assume that $G \subset \mathbf{C}$ is a simply connected domain and that a is a fixed point in G. If $j \in \mathbf{N}$ and $f \in H(G)$, denote by $D^{-j} f$ the unique antiderivative g of order j of f (i.e., $g^{(j)} = f$) such that $g^{(k)}(a) = 0$ $(k = 0, 1, ..., j-1)$. In fact, we have

$$D^{-j} f(z) = \int_a^z f(t) \cdot \frac{(z-t)^{j-1}}{(j-1)!}\, dt \quad (z \in G) \tag{1}$$

where the integral is taken along any rectifiable curve $\gamma \subset G$ joining a to z. It is easy to verify that each D^{-j} is an operator on $H(G)$. We denote $D^{-0} = I =$ the identity operator. If $\delta \in [0, +\infty)$, then we denote by $S(\delta)$ the set of formal complex power series $\Psi(z) = \sum_{j=0}^{\infty} c_j z^j$ such that

$$\limsup_{j\to\infty} \Big(\frac{|c_j|}{j!}\Big)^{1/j} \le \delta.$$

Note that even for $\delta = 0$ there may be $\Psi \in S(\delta)$ with empty convergence disk: take, e.g., $\Psi(z) = \sum_{j=1}^{\infty} j^{j/2} z^j$. Define the number $\Delta(a, G) = \sup_{z\in G} \inf\{r >$

$0 :$ a is in the connected component of $D(z,r)\cap G$ containing $z\}$ (see [5]). In order to understand the geometric meaning of $\Delta(a,G)$, note that $0 < d(a,\mathbf{C}\setminus G) \leq \sup_{z\in G}|z-a| \leq \Delta(a,G) \leq \text{diameter}\,(G)$ and that G is bounded if and only if $\Delta(a,G)$ is finite. We agree that $d(a,\emptyset)=+\infty$. Note also that if G is starlike with respect to a, then $\Delta(a,G)=\sup_{z\in G}|z-a|$. Furthermore, $d(a,\mathbf{C}\setminus G)=\Delta(a,G)$ if and only if G is $\mathbf{C}$ or an open disk with center a; in this last case, the common value is the radius of G.

THEOREM 6. *If $G\subset\mathbf{C}$ is a simply connected domain, $a\in G$ and $\Psi(z)=\sum_{j=0}^{\infty}c_jz^j \in S(1/\Delta(a,G))$, then the series $\Psi(D^{-1})=\sum_{j=0}^{\infty}c_jD^{-j}$ defines an operator on $H(G)$.*

Proof. Given $z\in G$, there exists a rectifiable arc $\gamma_z\subset G$ joining a to z such that $|z-t|<\Delta(a,G)$ for all $t\in\gamma_z$. Fix a compact set $K\subset G$. Then we may choose a compact set L and the arcs γ_z in such a way that $\bigcup_{z\in K}\gamma_z\subset L\subset G$, $\sup_{z\in K}\text{length}\,(\gamma_z)=\sigma<+\infty$ and $|z-t|\leq M=$ a constant $<\Delta(a,G)$ forall $t\in\gamma_z$ and all $z\in K$. Fix $M_1\in(M,\Delta(a,G))$. Since $\Psi\in S(1/\Delta(a,G))$, there is a constant $A\in(0,+\infty)$ such that $|c_j|/(j-1)!\leq A/M_1^{j-1}$ whenever $j\in\mathbf{N}$. Fix $f\in H(G)$. Let us show that the series $\Psi(D^{-1})=\sum_{j=0}^{\infty}c_jD^{-j}$ converges uniformly on K. By (1) we have

$$|D^{-j}f(z)|=\left|\frac{1}{(j-1)!}\cdot\int_{\gamma_z}(z-t)^{j-1}f(t)\,dt\right|\leq\frac{1}{(j-1)!}\cdot M^{j-1}\cdot||f||_L\cdot\text{length}\,(\gamma_z).$$

Therefore $||D^{-j}f||_K \leq \frac{\sigma M^{j-1}||f||_L}{(j-1)!}$ and

$$\sum_{j=0}^{\infty} ||c_j D^{-j} f||_K \leq \sum_{j=0}^{\infty} |c_j| \, ||D^{-j} f||_K \leq |c_0| \, ||f||_K$$

$$+ \sum_{j=1}^{\infty} |c_j| \, ||D^{-j} f||_K \leq |c_0| \, ||f||_L + \sum_{j=1}^{\infty} A \cdot (M/M_1)^{j-1} \cdot \sigma \cdot ||f||_L,$$

where we have used that $K \subset L$. Summarizing, $\sum_{j=0}^{\infty} ||c_j D^{-j} f||_K \leq B \cdot ||f||_L$, where B is the constant $B = |c_0| + \frac{A\sigma M_1}{M_1 - M}$. So $\sum_{j=0}^{\infty} c_j D^{-j} f$ converges uniformly on K and $\Psi(D^{-1})$ defines a mapping from $H(G)$ into itself. The linearity is trivial and, since $||\Psi(D^{-1})f||_K \leq B \cdot ||f||_L$ for every $f \in H(G)$, we obtain that $\Psi(D^{-1})$ is continuous on $H(G)$. ////

3. AN EIGENVALUE CRITERION FOR HYPERCYCLICITY

The core of the proof of Theorem 3 (see Section 1) in [14, Section 5] is to provide a good supply of eigenvectors of the corresponding operator. We furnish here a rather general criterion of hypercyclicity of sequences of operators based upon the existence of sufficiently many eigenvectors. Recall that, in a linear topological space, a subset is said to be *total* whenever its linear span is dense. If T is an operator and e is an eigenvector, then we denote by $\lambda(T, e)$ its corresponding eigenvalue.

THEOREM 7. *1) Let X be a separable Fréchet space and $\{T_n\}$ a sequence of operators on X. Assume that there are two subsets $\mathcal{A}, \mathcal{B}$ of X satisfying:*

a) For every pair of finite subsets $\mathcal{F}_1 \subset \mathcal{A}$ *and* $\mathcal{F}_2 \subset \mathcal{B}$ *there is a subsequence* $\{n_k\}$ *of positive integers such that every element in* $\mathcal{F}_1$ *or in* $\mathcal{F}_2$ *is an eigenvector for each* T_{n_k} *in such a way that* $\lim_{k \to \infty} \lambda(T_{n_k}, a) = 0$ *for all* $a \in \mathcal{F}_1$ *and* $\lim_{k \to \infty} \lambda(T_{n_k}, b) = \infty$ *for all* $b \in \mathcal{F}_2$.

b) $\mathcal{A}$ *and* $\mathcal{B}$ *are total in* X.

Then there is a dense G_δ *subset of hypercyclic vectors for* $\{T_n\}$.

2) Let X *be a separable complex Fréchet space and* T *an operator on* X. *Assume that there are two subsets* $\mathcal{A}, \mathcal{B}$ *of* X *satisfying:*

a) Every element in $\mathcal{A}$ *or in* $\mathcal{B}$ *is an eigenvector for each* T *in such a way that* $|\lambda(T, a)| < 1$ *for all* $a \in \mathcal{A}$ *and* $|\lambda(T, b)| > 1$ *for all* $b \in \mathcal{B}$.

b) $\mathcal{A}$ *and* $\mathcal{B}$ *are total in* X.

Then there is a dense G_δ *subset* M *of hypercyclic vectors for* T. *In addition,* M *contains all nonzero vectors of a dense,* T*-invariant, linear submanifold of* X.

Proof. Part 2) is obviously an application of Theorem 2 together with part 1) for $T_n = T^n$ $(n \in \mathbf{N})$, as soon as we realize that $\lambda(T^n, e) = (\lambda(T, e))^n$ $(n \in \mathbf{N})$ whenever e is an eigenvector for T.

We now prove part 1). Let us try to apply Theorem 1. Take $X = Y =$ the Fréchet space given in the hypothesis, $D = \operatorname{span} \mathcal{A}$, $E = \operatorname{span} \mathcal{B}$. Then D and E are dense in X. Fix $d \in D$ and $e \in E$. Then there are scalars $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_q$ and two finite sets of vectors $\mathcal{F}_1 = \{a_1, ..., a_m\} \subset \mathcal{A}$, $\mathcal{F}_2 = \{b_1, ..., b_q\} \subset \mathcal{B}$ such that $d = \alpha_1 a_1 + \ldots + \alpha_m a_m$ and $e = \beta_1 b_1 + \ldots + \beta_q b_q$. By hypothesis, there is a subsequence $\{n_k\}$ of positive integers such that $\lambda(T_{n_k}, a_j) \to 0$ $(k \to \infty)$ for all $j \in \{1, \ldots, m\}$ and $\lambda(T_{n_k}, b_j) \to \infty$ $(k \to \infty)$ for all $j \in \{1, \ldots, q\}$. We may assume that $\lambda(T_{n_k}, b_j) \neq 0$ for all k and all j. Define, for every $k \in \mathbf{N}$, the vector

$$x_k = d + \sum_{j=1}^{q} \frac{\beta_j}{\lambda(T_{n_k}, b_j)} b_j = \sum_{j=1}^{m} \alpha_j a_j + \sum_{j=1}^{q} \frac{\beta_j}{\lambda(T_{n_k}, b_j)} b_j.$$

Then the term $\frac{\beta_j}{\lambda(T_{n_k}, b_j)} b_j \to 0$ $(k \to \infty)$ for every $j \in \{1, \ldots, q\}$, so $x_k \to d$ $(k \to \infty)$. Finally,

$$T_{n_k}(x_k) = \sum_{j=1}^{m} \alpha_j \lambda(T_{n_k}, a_j)\, a_j + \sum_{j=1}^{q} \frac{\beta_j}{\lambda(T_{n_k}, b_j)} \cdot \lambda(T_{n_k}, b_j)\, b_j$$

$$= \sum_{j=1}^{m} \alpha_j\, \lambda(T_{n_k}, a_j)\, a_j + e \to e \quad (k \to \infty),$$

because $\lambda(T_{n_k}, a_j) \to 0$ $(k \to \infty)$ for all $j \in \{1, ..., m\}$. Consequently, the condition in Theorem 1 is fulfilled and the proof is finished. ////

4. HYPERCYCLIC SEQUENCES

In this Section we deal with the hypercyclicity of the operators introduced in Section 2. Results by Godefroy and Shapiro (Theorem 3) and by Herzog [17] and the author [6] (see Section 1) are extended, and some new others are proved.

Before stating our first two results, we point out some ideas about density of certain families of functions. Let $e_a(z) = \exp(a_1 z_1 + \ldots + a_N z_N)$ if $a = (a_1, ..., a_N) \in$

$\mathbf{C}^N$ and $z = (z_1, ..., z_N) \in \mathbf{C}^N$. If $S \subset \mathbf{C}^N$, denote by H_S the linear manifold

$$H_S = \text{span}\,\{e_a : \, a \in S\}.$$

It is well known that the set of all finite linear combinations of functions e_a ($a \in \mathbf{C}^N$) is dense in $H(\mathbf{C}^N)$ [18, p. 97]. In fact, a joint application of Hahn-Banach Theorem, Riesz Representation Theorem and Analytic Continuation Principle (see [14, pp. 259-260]) shows that H_S is dense in $H(\mathbf{C}^N)$ for each nonempty open subset $S \subset \mathbf{C}^N$. The same argument would yield the same result for the case $N = 1$ just by assuming that $S \subset \mathbf{C}$ is a subset with at least one finite accumulation point.

For future references, we state here four conditions that may or may not be satisfied by a sequence $\{\Phi_n\}_1^\infty$ of entire functions on $\mathbf{C}^N$. Recall that if $\Phi(z) = \sum_{|p| \geq 0} a_p z^p \in H(\mathbf{C}^N)$ and Φ is not identically zero, then its *multiplicity* for the zero at the origin is $m(\Phi) = \min\{|p| : \, a_p \neq 0\}$.

(P) There are two nonempty open subset A, B of $\mathbf{C}^N$ such that for every pair of finite subset $F_1 \subset A$ and $F_2 \subset B$ there exists a subsequence $\{n_k\}$ of positive integers such that $\lim_{k \to \infty} \Phi_{n_k}(a) = 0$ for all $a \in F_1$ and $\lim_{k \to \infty} \Phi_{n_k}(b) = \infty$ for all $b \in F_2$.

(P') There are two subsets A, B of $\mathbf{C}^N$ each of them with at least one finite accumulation point such that for every pair of finite subsets $F_1 \subset A$ and $F_2 \subset B$ there exists a subsequence $\{n_k\}$ of positive integers such that $\lim_{k\to\infty} \Phi_{n_k}(a) = 0$ for all $a \in F_1$ and $\lim_{k\to\infty} \Phi_{n_k}(b) = \infty$ for all $b \in F_2$.

(Q) $m(\Phi_n) \to \infty$ $(n \to \infty)$ and there is a nonempty open subset $B \subset \mathbf{C}^N$ such that for every finite subset $F \subset B$ there exists a subsequence $\{n_k\}$ of positive integers satisfying $\lim_{k\to\infty} \Phi_{n_k}(b) = \infty$ for all $b \in F$.

(Q') $m(\Phi_n) \to \infty$ $(n \to \infty)$ and there is a subset $B \subset \mathbf{C}^N$ with at least one finite accumulation point such that for every finite subset $F \subset B$ there exists a subsequence $\{n_k\}$ of positive integers satisfying $\lim_{k\to\infty} \Phi_{n_k}(b) = \infty$ for all $b \in F$.

Trivially (P) implies (P') and (Q) implies (Q'). For instance, the sequence $\Phi_n(z) = z^n$ $(z \in \mathbf{C};\ n \in \mathbf{N})$ satisfies all four properties; the sequence $\Phi_n(z) = n^n z^n$ satisfies (Q) but does not (P) ($n^n z^n \to \infty$ as $n \to \infty$ for every $z \in \mathbf{C} \setminus \{0\}$); the sequence $\Phi_n(z) = ne^{nz} + \frac{z^n}{n^2}$ satisfies (P) (take $A = \{z : |z| < 1, \operatorname{Re} z < 0\}$ and $B = \{z : |z| < 1, \operatorname{Re} z > 0\}$) but not (Q).

THEOREM 8. *Suppose that G is a Runge domain of $\mathbf{C}^N$ and Φ, Φ_n $(n \in \mathbf{N})$ are entire functions on $\mathbf{C}^N$. Assume that Φ is not a constant and denote $L_n = \Phi_n(D)$ $(n \in \mathbf{N})$.*

a) Suppose that every Φ_n is of subexponential (exponential, resp.) type and the sequence $\{\Phi_n\}$ satisfies (P). Then there is a dense G_δ subset of $H(G)$ $(H(\mathbf{C}^N)$, resp.)

all of whose elements are hypercyclic functions for $\{L_n\}$.

b) For $N = 1$ *the statement of a) still holds if (P) is changed to (P').*

c) Suppose that Φ *is of subexponential type and let* $L = \Phi(D)$. *Then there is a dense* G_δ *subset* M *of* $H(G)$ *all of whose elements are hypercyclic functions for* L. *In addition,* M *contains all nonzero functions of a dense,* L*-invariant, linear submanifold of* $H(G)$.

d) Suppose that L *is an operator on* $H(\mathbf{C}^N)$ *that commutes with each of the translation operators* τ_a $(a \in \mathbf{C}^N)$, *and is not a scalar multiple of the identity. Then* L *has a dense* G_δ *subset* M *of hypercyclic functions. In addition,* M *contains all nonzero functions of a dense,* L*-invariant, linear submanifold of* $H(\mathbf{C}^N)$.

Proof. a) Firstly, by Theorem 5 and the initial considerations of Section 2, every L_n is an operator defined on $H(G)$ (even on $H(\mathbf{C}^N)$ if Φ_n is of exponential type). From now on, G may be $\mathbf{C}^N$ or not. Note that $D_j e_a = a_j e_a$ for each $j \in \{1, ..., N\}$ and each $a \in \mathbf{C}^N$, so $D^p e_a = a^p e_a$ for every multi-index p. Then $L_n e_a = \Phi_n(D) e_a = \Phi_n(a) e_a$ $(a \in \mathbf{C}^N, n \in \mathbf{N})$. Observe that each function e_a is an eigenvector for every L_n with eigenvalue $\Phi_n(a)$.

Consider the open subsets A and B provided by the condition (P). Fix a compact subset $K \subset G$, a funcion $f \in H(G)$ and $\varepsilon > 0$. Since G is a Runge domain, a polynomial $P(z)$ of N complex variables can be found in such a way that $|f(z) - P(z)| < \varepsilon/2$ for all $z \in K$. There exists $h \in H_S$ $(S = A$ or $B)$ with

$|P(z) - h(z)| < \varepsilon/2$ for all $z \in K$. Therefore $|f(z) - h(z)| < \varepsilon$ for all $z \in K$. This shows that H_A and H_B are also dense subsets of $H(G)$. It now suffices to apply part 1) of Theorem 7 on $X = H(G)$, $\mathcal{A} = \{e_a : a \in A\}$, $\mathcal{B} = \{e_b : b \in B\}$ and $T_n = L_n$ $(n \in \mathbf{N})$.

b) This part is obvious from a), together with the remark for the case $N = 1$ at the beginning of this section. We would have anew that H_A and H_B are dense in $H(\mathbf{C})$, so in $H(G)$ as well.

c) By Theorem 5, L is an operator defined on $H(G)$. Since Φ is a nonconstant entire function, the sets $A = \Phi^{-1}(|z| < 1)$ and $B = \Phi^{-1}(|z| > 1)$ are nonempty open subsets. Now use part 2) of Theorem 7 with $X = H(G)$, $T = L = \Phi(D)$, $\mathcal{A} = \{e_a : a \in A\}$ and $\mathcal{B} = \{e_b : b \in B\}$. Note that, like in part a), every function e_a $(a \in \mathbf{C}^N)$ is an eigenvector for T with eigenvalue $\lambda(T, e_a) = \Phi(a)$.

d) This is essentially Theorem 3. It has been put here for the sake of completeness. It is derived as c) ($G = \mathbf{C}^N$ here) by using Theorem 4. It should be noted that, if $L = \Phi(D)$, then Φ is nonconstant if and only if L is not a scalar multiple of the identity. ////

For instance, we have that there is a dense G_δ subset of entire functions f on $\mathbf{C}$ such that each entire function can be uniformly approximated on compact sets by functions of the form $nf(z+n) + \frac{f^{(n)}(z)}{n^2}$ $(n \in \mathbf{N})$. Indeed, it suffices to consider the sequence $\Phi_n(z)$ of the third example just before the latter theorem. Note that $e^{aD} = \tau_a$ for every $a \in \mathbf{C}$.

In view of the result on growth of Grosse-Erdmann [16] for entire functions (see Section 1), it is natural to ask what is the minimal growth allowed for a D-hypercyclic function on a bounded domain in $\mathbf{C}$. The answer for Runge domains is almost trivial and is provided in Corollary 2. We denote, as usual, by $g|_S$ the restriction of a function g to a subset S.

COROLLARY 1. *Assume that* $G \subset \mathbf{C}^N$ *is a Runge domain and that* L *is an operator on* $H(\mathbf{C}^N)$ *that commutes with each of the translation operators* τ_a $(a \in \mathbf{C}^N)$*, and is not a scalar multiple of the identity. Then the set*

$$M = \{f|_G : f \text{ is entire and } \{(L^n f)|_G\}_1^\infty \text{ is dense in } H(G)\}$$

is dense in $H(G)$*.*

Proof. The assertion is evident from part d) of Theorem 8 and from the fact that $H(\mathbf{C}^N)$ is dense in $H(G)$. ////

COROLLARY 2. *If* $G \subset \mathbf{C}^N$ *is a bounded Runge domain, then there exists a dense subset* M *in* $H(G)$ *such that, for every* $f \in M$*, each derivative* $f^{(n)}$ $(n \in \mathbf{N_0})$ *is bounded and the orbit* $\{f^{(n)}\}_1^\infty$ *is dense in* $H(G)$*.*

Proof. Just apply Corollary 1 with $L = D$. ////

THEOREM 9. *Suppose that G is a Runge domain of $\mathbf{C}^N$ and Φ, Φ_n $(n \in \mathbf{N})$ are entire functions on $\mathbf{C}^N$. Denote $L_n = \Phi_n(D)$ $(n \in \mathbf{N})$.*

a) Suppose that every Φ_n is of subexponential (exponential, resp.) type and the sequence $\{\Phi_n\}$ satisfies (Q). Then there is a dense G_δ subset of $H(G)$ ($H(\mathbf{C}^N)$, resp.) all of whose elements are hypercyclic functions for $\{L_n\}$.

b) For $N = 1$ the statement of a) still holds if (Q) is changed to (Q').

Proof. We can also apply part 1) of Theorem 7. Take $X = Y = H(G)$, $T_n = L_n$ $(n \in \mathbf{N})$, $\mathcal{A} = \{z^p : p \in \mathbf{N_0}^N\}$, $\mathcal{B} = \{e_b : b \in B\}$ where the set B is furnished by hypothesis (Q) (or by (Q') if $N = 1$). Observe that span $\mathcal{A}$ (= {polynomials}) is dense in $H(G)$. Each function e_b is an eigenvector for every T_n with eigenvalue $\lambda(T_n, b) = \Phi_n(b)$. Fix two finite subsets $\mathcal{F}_1 = \{z^{p_1}, ..., z^{p_r}\} \subset \mathcal{A}$ and $\mathcal{F}_2 = \{e_{b_1}, ..., e_{b_s}\} \subset \mathcal{B}$. From (Q) (or (Q')), a subsequence $\{n_k\}$ of positive integers can be found for the finite set $F = \{b_1, ..., b_s\} \subset B$ in such a way that $\lim_{k\to\infty} \lambda(T_{n_k}, b_j) = \lim_{k\to\infty} \Phi_{n_k}(b_j) = \infty$ for all $j \in \{1, ..., s\}$. On the other hand, if $\alpha = \max\{|p_1|, ..., |p_r|\}$, there is $n_0 \in \mathbf{N}$ such that $m(\Phi_n) > \alpha$ for all $n > n_0$, so $\Phi_n(D)\, z^{p_j} = 0$ for all $j \in \{1, ..., r\}$ because $D^p z^{p_j} = 0$ for all $j \in \{1, ..., r\}$ and for every multi-index p with $|p| > \alpha$. Consequently, each z^{p_j} is an eigenvector for T_{n_k} (we can assume $n_k > n_0$ for all k) with eigenvalue $\lambda(T_{n_k}, z^{p_j}) = 0$, which trivially tends to zero as $k \to \infty$. ////

Unfortunately, one cannot expect any hypercyclicity result for an antidifferential operator $\Psi(D^{-1})$.

THEOREM 10. *Assume that* $G \subset \mathbf{C}$ *is a simply connected domain. Fix a point* $a \in G$ *and consider the corresponding antiderivative operator* D^{-1}. *Suppose that* Ψ *and* Ψ_n $(n \in \mathbf{N})$ *are in* $S(1/\Delta(a,G))$ *and that* $\Psi_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$. *Let* L, L_n *be the operators* $L = \Psi\,(D^{-1})$, $L_n = \Psi_n\,(D^{-1})$ $(n \in \mathbf{N})$. *We have:*

a) If $\{L_n\}$ *is hypercyclic, then the sequence* $\{c_0^{(n)} : n \in \mathbf{N}\}$ *is dense in* $\mathbf{C}$.

b) L *is not hypercyclic.*

Proof. L and L_n $(n \in \mathbf{N})$ are well defined operators by Theorem 6. If $f \in H(G)$ is hypercyclic for $\{L_n\}$ then, given $b \in \mathbf{C}$, some subsequence $\{L_{n_k} f\}$ of $\{L_n f\}$ must approximate the constant function $g(z) \equiv b$ on the compact set $\{a\}$. But $(L_n f)(a) = (\sum_{j=0}^{\infty} c_j^{(n)} D^{-j} f)(a) = c_0^{(n)} f(a)$ because $D^{-j} f(a) = 0$ for all $j \in \mathbf{N}$. This implies that $c_0^{(n_k)} f(a) \to b$ $(k \to \infty)$, so $\{c_0^{(n)} : n \in \mathbf{N}\}$ is dense in $\mathbf{C}$. This proves a). Part b) is an unpleasant consequence of a): indeed, assume that $\Psi(z) = \sum_{j=0}^{\infty} c_j z^j$ and put $L_n = L^n$. Then $c_0^{(n)} = c_0^n$ and for each $c_0 \in \mathbf{C}$ the sequence $\{c_0^n\}$ is not dense in $\mathbf{C}$. ////

Nevertheless, a sort of "pseudo-hypercyclicity" is true, as our next theorem shows. For this, let us cite the following result of W. Luh [21]: For every simply connected domain $G \subset \mathbf{C}$ there exists a sequence $\{C_n\}_1^{\infty} \subset \mathbf{C}$ with the property that for every $\varphi \in H(G)$ the set $\{Q_n(z) = D^{-n}\varphi(z) + \sum_{j=0}^{n-1} \frac{C_{n-j}}{j!} z^j \,:\, n \in \mathbf{N}\}$ is dense in H(G). Note that the coefficients C_n's do not depend upon φ.

Just a remark before the theorem. Let $G \subset \mathbf{C}$ be a simply connected domain and fix a point $a \in G$. If $\Psi(z) = \sum_{j=0}^{\infty} c_j z^j$ is a formal power series, then $\Psi \in S(1/\Delta(a,G))$ if and only if $\alpha(\Psi,\delta)$ is finite for all $\delta \in (0, \Delta(a,G))$, where we have set

$$\alpha(\Psi,\delta) = |c_0| + \sup_{j\in\mathbf{N}} \frac{|c_j|\,\delta^{j-1}}{(j-1)!}.$$

THEOREM 11. *Assume that $G \subset \mathbf{C}$ is a simply connected domain. Fix a point $a \in G$ and consider the corresponding antiderivative operator D^{-1}. Then there exists a sequence $\{C_n\}_1^{\infty} \subset \mathbf{C}$ satisfying the following property: For every $f \in H(G)$ and every sequence $\{\Psi_n(z)\}_1^{\infty} \subset S(1/\Delta(a,G))$ of formal power series for which*

$$\alpha(\Psi_n,\delta) \to 0 \;\; (n \to \infty) \quad \textit{for all } \delta \in (0, \Delta(a,G)),$$

the sequence $\{\Psi_n(D^{-1})\, f(z) + \sum_{j=0}^{n-1} \frac{C_{n-j}}{j!} z^j \,:\, n \in \mathbf{N}\}$ is dense in H(G).

Proof. Denote $L_n = \Psi_n(D^{-1})$ and assume that $\Psi_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$ $(n \in \mathbf{N})$. From Theorem 6 we have that every L_n is an operator on $H(G)$. We apply the mentioned result of Luh [21] to the domain G and the function $\varphi = 0$. We obtain that there is a sequence $\{C_n\}_1^{\infty} \subset \mathbf{C}$ such that the set $\{H_n\}_1^{\infty}$ given by $H_n(z) = \sum_{j=0}^{n-1} \frac{C_{n-j}}{j!} z^j$ is dense in $H(G)$. Fix a function $f \in H(G)$ and a compact set $K \subset G$.

If the final steps of the proof of Theorem 6 are watched then one sees that there are a compact set $L \subset G$ and positive constants σ, M, M_1 with $M_1 \in (M, \Delta(a, G))$ such that $||L_n f||_K \leq B_n \cdot ||f||_L$, where

$$B_n = |c_0^{(n)}| + \frac{\sigma M_1}{M_1 - M} \cdot \sup_{j \in \mathbf{N}} \frac{|c_j^{(n)}| \, M_1^{j-1}}{(j-1)!} \quad \text{for all } n \in \mathbf{N}.$$

But $\alpha(\Psi_n, M_1) \to 0$ $(n \to \infty)$ by hypothesis, so $\lim_{n\to\infty} B_n = 0$ and $||L_n f||_K \to 0$ $(n \to \infty)$. Hence $\{L_n f\}_1^\infty$ converges uniformly to zero on compact sets. Since $\{H_n\}_1^\infty$ is dense in $H(G)$, we have that $\{L_n f + H_n\}_1^\infty$ is also dense in $H(G)$, as required. ////

By using [21, Lemma 3] one can easily establish that for every compact set $B \subset \mathbf{C}$ with connected complement and every function g which is continuous on B and holomorphic in the interior of B, there is a subsequence of $\{L_n f + H_n\}_1^\infty$ converging to g uniformly on B and, in addition, for every Lebesgue-measurable set $E \subset G$ and every Lebesgue measurable function $g : E \to \mathbf{C} \cup \{\infty\}$, there is a subsequence of $\{L_n f + H_n\}_1^\infty$ converging almost everywhere to g on E. Theorem 11 together with this remark generalizes [5, Theorem 5]: in fact there we dealt with the case $\Psi_n(z) = c_n z^n$, where $\{c_n\}_1^\infty \subset \mathbf{C}$ is a sequence such that $\limsup_{n\to\infty} (\frac{|c_n|}{n!})^{1/n} \leq 1/\Delta(a, G)$.

We propose here as an open problem to give conditions on $\{\Psi_n\}$ which guarantee the hypercyclicity of $\{\Psi_n(D^{-1})\}$. Note that this sequence can certainly be hypercyclic. Indeed, for $a=0$ the constant function $f\equiv 1$ is $\{\Psi_n(D^{-1})\}$-universal if and only if the set $\{\sum_{j=0}^{\infty}(c_j^{(n)}/j!)z^j\,:\,n\in\mathbf{N}\}$ is dense in $H(G)$. Appropriate $c_j^{(n)}$ can always be found.

To finish, we establish a result about hypercyclicity of functions with the additional property that certain derivatives do not vanish on the domain. Notice that there is no D-hypercyclic entire function f such that $f\cdot f'\cdot f''$ is zero-free, since $\{f\in H(\mathbf{C})\,:\,f\cdot f'\cdot f''$ is zero-free$\}=\{e^{\alpha z+\beta}\,:\,\alpha,\beta\in\mathbf{C},\ \alpha\neq 0\}$ (see [12, p. 433] and [24]). If $q\in\mathbf{N_0}$, let us denote $A(q)=\{f\in H(G)\,:\,f^{(q)}(z)f^{(q+1)}(z)\neq 0$ for all $z\in G\}$. Since $\exp\in\bigcap_{q\in\mathbf{N_0}}A(q)$, every $A(q)$ is nonempty. If $\mathcal{L}=\{L_n\,:\,n\in\mathbf{N}\}$ is a sequence of continuous mappings from X into Y, then we denote by $HC(\mathcal{L})$ the set of hypercyclic elements for $\mathcal{L}$. If $A\subset X$, denote $\mathcal{L}|_A=\{L_n|_A\,:\,n\in\mathbf{N}\}$. We are now ready to state our theorem on hypercyclicity. Herzog's result [17] is the special case $q=0$, $G=\mathbf{C}$, $\mathcal{L}=\{D^n\,:\,n\in\mathbf{N}\}$ while the result of the author in [6] is the special case $G=\mathbf{C}$, $\mathcal{L}=\{L^n\,:\,n\in\mathbf{N}\}$ with $L=\Phi(D)$.

THEOREM 12. *Assume that $G\subset\mathbf{C}$ is a simply connected domain and that $\{\Phi_n\,:\,n\in\mathbf{N}\}$ is a sequence of entire functions of subexponential type satisfying (P') or (Q'). Fix $q\in\mathbf{N_0}$ and set $A=A(q)$, $\mathcal{L}=\{L_n\,:\,n\in\mathbf{N}\}$, where L_n is the operator on $H(G)$ given by $L_n=\Phi_n(D)$ $(n\in\mathbf{N})$. Then the set $HC(\mathcal{L}|_A)$ is residual in A.*

Proof. The proof is very similar to that in [6], so we merely indicate some necessary changes. The closed discs $D(0,k)$ $(k \in \mathbf{N})$ in the proof of Theorem 5 of [6] should be replaced to the closure $\overline{G_k}$ of G_k, where $\{G_k : k \in \mathbf{N}\}$ is a sequence of simply connected domains such that every $\overline{G_k}$ is compact, $\overline{G_k} \subset G_{k+1}$ and $G = \bigcup_{k=1}^{\infty} G_k$. The metric $d(f,g)$ in that paper is here changed to

$$d(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||f-g||_j}{1+||f-g||_j} \quad (f,g \in H(G)),$$

where $||h||_j = \max_{z \in \overline{G_j}} |h(z)|$. From Theorems 8 and 9, $HC(\mathcal{L})$ is residual in $H(G)$ and an adequate application of Theorem 2.1 of [17] will give the result if one takes into account that $A = \bigcap_{k=1}^{\infty} A_k$ where $A_k = \{f \in H(G) : \min_{\overline{G_k}} |f^{(q)} \cdot f^{(q+1)}| > 0\}$. If $k \in \mathbf{N}$ is fixed, then there is a simply connected subdomain $U \subset G$ such that $\overline{G_k} \subset U$ and $f^{(q)}(z) f^{(q+1)}(z) \neq 0$ for all $z \in U$. The existence of an approximating sequence $\{P_m\}$ of polynomials on $D(0, k+\varepsilon)$ in [6, Theorem 5] is here guaranteed by Runge's theorem, which should be applied on $\overline{V}$, V being a simply connected domain such that $\overline{V}$ is compact and $\overline{G_k} \subset V \subset \overline{V} \subset U$. We left the details to the reader, which should find no difficulty if he follows step by step the proof of the cited reference. ////

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REFERENCES

1. S. I. Ansari, Existence of hypercyclic operators on topological vector spaces, *J. Funct. Anal. to appear.*
2. B. Beauzamy, Un opérateur, sur l'espace de Hilbert, dont tous les polynômes sont hypercyclic, *C. R. Acad. Sci. Paris, Sér. I Math.* **303** (1986), 923-927.
3. B. Beauzamy, An operator in a separable Hilbert space with many hypercyclic vectors, *Studia Math.* **87** (1987), 71-78.
4. B. Beauzamy, An operator on a separable Hilbert space with all polynomials hypercyclic, *Studia Math.* **96** (1990), 81-90.
5. L. Bernal-González, Derivative and antiderivative operators and the size of complex domains. *Annales Pol. Math.* **59**, 267-274 (1994).
6. L. Bernal-González, On universal entire functions with zero-free derivatives, *Archiv Math.* **68** (1997), 145-150.
7. L. Bernal-González, On hypercyclic operators on Banach spaces, *Proc. Amer. Math. Soc. to appear.*
8. L. Bernal-González and A. Montes-Rodríguez, Universal functions for composition operators, *Complex Variables* **27** (1995), 47-56.
9. L. Bernal-González and A. Montes-Rodríguez, Non-finite dimensional closed vector spaces of universal functions for composition operators, *J. Approx. Theory* **82** (1995), 375-391.
10. G. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* 189 (1929), 473-475.
11. P. S. Bourdon, Invariant manifolds of hypercyclic vectors, *Proc. Amer. Math. Soc.* **118** (1993), 845-847.
12. R. B. Burckel, "An Introduction to Classical Complex Analysis", Vol. 1, Birkhäurser, Basel, 1979.
13. R. M. Gethner and J. H. Shapiro, Universal vectors for operators on spaces of

holomorphic functions, *Proc. Amer. Math. Soc.* **100** (1987), 281-288.

14. G. Godefroy and J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.* **98** (1991), 229-269.
15. K. G. Grosse-Erdmann, Holomorphe Monster und universelle Funktionen, Mitt. Math. Sem. Giessen **176** (1987).
16. K. G. Grosse-Erdmann, On the universal functions of G. R. MacLane, *Complex Variables* **15** (1990), 193-196.
17. G. Herzog, On zero-free universal entire functions, *Arch. Math.* **63** (1994), 329-332.
18. L. Hormander, "An Introduction to Complex Analysis in Several Variables", Van Nostrand, Princeton, NJ, 1966.
19. C. Kitai, "Invariant Closed Sets for Linear Operators", Thesis, University of Toronto, 1982.
20. S. Krantz, "Function Theory of several complex variables", J. Wiley, New York, 1982.
21. W. Luh, Approximation by antiderivatives, *Constr. Approx.* **2** (1986), 179-187.
22. W. Luh, Multiply Universal Holomorphic Functions, *J. Approx. Theory* **89** (1997), 135-155.
23. G. R. MacLane, Sequences of derivatives and normal families, *J. Analyse Math.* **2** (1952), 72-87.
24. W. Saxer, Über die Picardschen Ausnahmewerte sukzessiver Derivierten, *Math. Zeit.* **17** (1923), 206-227.

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