Universality of holomorphic functions bounded on closed sets

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Abstract

In this note, the existence of translation-universal entire functions which are bounded on certain closed subsets is characterized in terms of topological and geometrical properties of such subsets. Corresponding results are also stated in the space of holomorphic functions on the unit disk and in the space of harmonic functions on the plane. Moreover, it is shown the existence of entire functions which are bounded on many rays and, simultaneously, are universal with respect to a prescribed infinite-order differential operator.

1 Introduction and notation

Throughout this paper we will use the following notations, most of them being standard: $\mathbb{N}$ is the set of positive integers, $\mathbb{C}$ is the complex plane, $\mathbb{R}$ is the real line, $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ is the open unit disk, $B(a,r)$ ($\overline{B}(a,r)$) is the euclidean open (closed, respectively) ball with center $a \in \mathbb{C}$ and radius $r > 0$. By $\overline{A}$ we

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mean the closure in $\mathbb{C}$ of a subset $A \subset \mathbb{C}$. The symbol $\mathbb{T}$ stands for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Moreover, if G is a domain (:= connected, nonempty open subset) of $\mathbb{C}$ (of $\mathbb{R}^N$, where $N \in \mathbb{N}$), then $H(G)$ ($h(G)$) denotes the space of all holomorphic functions $f : G \to \mathbb{C}$ (of all harmonic functions $u : G \to \mathbb{R}$, respectively). It becomes a separable completely metrizable space (hence a Baire space) when it is endowed with the compact open topology (see [27, pages 238–239]). In particular, $H(\mathbb{C})$ is the space of all entire functions. If F is a closed subset of $\mathbb{C}$ (of $\mathbb{R}^N$) then $A(F)$ ($h(F)$) denotes the class of functions $f : F \to \mathbb{C}$ ($u : F \to \mathbb{R}$) which are continuous on F and holomorphic in the interior of F (which are harmonic in some neighbourhood $V = V_u$ of F, respectively). If G is a domain of $\mathbb{C}$ and $F \subset G$, then F is said to be an Arakelian subset of G if and only if F is a nonempty, proper, (relatively) closed subset of G, and $G_\infty \setminus F$ is connected and locally connected at the ∞-point of $G_\infty :=$ the one-point compactification of G. We define an Arakelian subset of $\mathbb{R}^N$ in an analogous way. Finally, a subset F of a topological space X is called of first category whenever F is the union of countable many nowhere dense subsets of X.

In 1929 Birkhoff [11] constructed an entire function which is 'universal' for translations. In fact, he proved essentially that given $b \in \mathbb{C} \setminus \{0\}$ there exists a function $f \in H(\mathbb{C})$ such that its sequence of translates $\{f(\cdot + nb) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$. Birkhoff's theorem can be observed under the point of view of the operator theory as a universality result; namely, if $\varphi : \mathbb{C} \to \mathbb{C}$ denotes the translation $z \mapsto z + b$ then the composition operator

$$C_\varphi : f \in H(\mathbb{C}) \mapsto f \circ \varphi \in H(\mathbb{C})$$

is universal. In general, if X is a (necessarily separable) topological vector space and (T_n) is a sequence of operators (= continuous linear selfmappings) on X then (T_n) is said to be *universal* (or *hypercyclic*) provided that there exists some vector $x \in X$ –called *universal* for (T_n)– for which the orbit $\{T_n x : n \in \mathbb{N}\}$ of x under (T_n) is dense in Y. And a single operator is called *universal* whenever the sequence of iterates (T^n) (that is, $T^1 = T$, $T^2 = T \circ T$, and so on) is universal; in this case it is easy to see that the set of universal vectors is dense. If X is Baire and metrizable and T is universal then the set of universal vectors for T is residual, that is, its complement is of first category. See [26] for a good account about these concepts and their history.

Since 1929 many papers have dealt with the subject of universality through translations in one complex variable. Let us make a brief report, now in the language of the universality of operators; see also the survey [26] –specially its Section 4a– which contains a rather complete list of references including domains $G \neq \mathbb{C}, \mathbb{D}$ and spaces $X \neq H(G)$. In 1941 Seidel and Walsh [32] were able to construct a function $f \in H(\mathbb{D})$ which is universal in $H(\mathbb{D})$ with respect to the sequence of composition operators generated by the noneuclidean translates $z \mapsto \frac{z+a_n}{1+\overline{a}_n z}$ $(n \in \mathbb{N})$, where $|a_n| \to 1$. In 1976 Luh [29] proved that for a prescribed unbounded sequence $(b_n) \subset \mathbb{C}$ the sequence (C_{φ_n}) is universal on $H(\mathbb{C})$, where φ_n is the translation $z \mapsto z + b_n$. In 1984 Duyos-Ruis [18] showed by functional analysis methods that C_φ $(\varphi(z) = z + b,\, b \in \mathbb{C} \setminus \{0\})$ is universal on $H(\mathbb{C})$ (hence there is a residual subset of universal functions), while the residuality of the (C_{φ_n})-universal entire functions –where the φ_n are the above translations– was observed by Grosse-Erdmann [25] and Gethner and Shapiro [23]. In 1988 Zappa [34] considered the universality of functions of $H(\mathbb{C} \setminus \{0\})$ with respect to 'multiplicative' translations. In 1995 Bernal and Montes [9] characterized the sequences of automorphisms of $\mathbb{C}$ or $\mathbb{D}$ generating universal sequences of composition operators.

Corresponding results in several complex variables can be seen in a number of papers by Godefroy, Shapiro, Abe, Zappa, León, Prado and the first author (see [24], [1], [2], [28], [10] and [8]). As for translation-universality in the space $h(\mathbb{R}^N)$, see [4] by Armitage and Gauthier.

The universality of the differentiation operator

$$D : f \in H(\mathbb{C}) \mapsto f' \in H(\mathbb{C})$$

was stated by MacLane [30] in 1952. Godefroy and Shapiro [24] (see also [26] and [10] for generalizations and improvements) unified both theorems of Birkhoff and MacLane by showing that any infinite-order differential operator $\Phi(D)$ (see Section 2) on $H(\mathbb{C})$ which is not a scalar multiple of the identity is universal.

We want to bring here a question concerning the existence of translation-universal functions when *boundedness* conditions on certain subsets are added. In this setting, the second author [13] defined a universal harmonic function as a function $f \in h(\mathbb{R}^N)$

such that to each $g \in h(\mathbb{R}^N)$ corresponds a sequence $(a_n) \subset \mathbb{R}^N$ depending on g satisfying $\lim_{n\to\infty} f(x + a_n) = g(x)$ uniformly on compact sets. Among other properties, he proved in [13, Theorem 1] the existence of a universal harmonic function which has strong decay (in particular, it is bounded) on any hyperplane strip. M.C. Calderón [14, Theorem 2.1] gave an analogous concept for the space of entire functions. She showed the existence of a universal entire function decaying very fast on every strip and on every sector $\{z : 0 \le \arg z \le \alpha\}$ with $\alpha \in (0, 2\pi)$ (as a matter of fact, she considered the action of certain operators T on $H(\mathbb{C})$, including the identity operator). Very recently, Costakis and Sambarino [15, Theorem 5] proved that there exists an entire function f whose translates $z \mapsto f(z + n)$ $(n \in \mathbb{N})$ are dense in $H(\mathbb{C})$ such that f tends to zero as $z \to \infty$ on every sector $\{z : \varepsilon \le \arg z \le 2\pi(1 - \varepsilon)\}$ with $\varepsilon \in (0, 1)$. They also show (see [15, Theorem 6]) that for a prescribed nowhere dense set $E \subset \mathbb{T}$, there is a D-universal entire function f tending to zero along every ray from the origin passing trough E, that is, $\lim_{r\to+\infty} f(rt) = 0$ for all $t \in E$.

Note that in the concept of universality of [13] and [14] it is equivalent to state that the sequence (a_n) exists independently of g: indeed, fix a countable dense subset (g_k) in $h(\mathbb{R}^N)$ (or in $H(\mathbb{C})$) and consider the sequence $(b_{n,k})_n$ which performs the approximation to g_k; then the adequate sequence (a_n) is made by joining all terms $b_{n,k}$ $(k, n \in \mathbb{N})$ in a single sequence. On the other hand, any euclidean (noneuclidean) translation in $\mathbb{C}$ ($\mathbb{D}$) is in fact an internal law $z \mapsto z * a = z + a$ ($z \mapsto z * a = \frac{z+a}{1+\bar{a}z}$) making $\mathbb{C}$ ($\mathbb{D}$) a topological group, whenever $\mathbb{C}$ ($\mathbb{D}$, respectively) is endowed with the euclidean topology. The last two remarks motivate the following definition.

Definition 1.1. (a) A *plane topological group* (PTG) is a topological group $(G, *)$, such that G is a domain of $\mathbb{C}$, G carries the euclidean topology and for each $a \in G$ the "translation selfmapping" $\tau_a : z \in G \mapsto z * a \in G$ is holomorphic in $\mathbb{D}$.
(b) If $(G, *)$ is a PTG and $f \in H(G)$, then we say that f is *τ-universal* if and only if there exists a sequence $(a_n) \subset G$ such that the set $\{f \circ \tau_{a_n} : n \in \mathbb{N}\}$ is dense in $H(G)$.
(c) If $u : \mathbb{R}^N \to \mathbb{R}$ is harmonic, then u is called *τ-universal* whenever there is a sequence $(a_n) \subset \mathbb{R}^N$ satisfying that the set $\{u(\cdot + a_n) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{R}^N)$.

Observe that we are considering "right" translations τ_a. Of course, one could give analogous concepts (and obtain analogous results) by considering "left translations" if these are defined in a suitable way. We refer to, for instance, the book [6] for the fundamentals about topological groups. We recall that if $(X, *)$ is a metrizable topological group –as, for instance, a PTG– then there exists a distance d on X generating its topology such that d is translation-invariant, that is, $d(x * a, y * a) = d(x, y)$ for all $x, y, a \in X$.

Turning to the question of the boundedness of universal functions, the aim of this paper is to furnish necessary and sufficient conditions for the existence of τ-universal holomorphic functions on a PTG which in addition are bounded on certain prefixed subsets of the domain G, mainly $G = \mathbb{C}$, $\mathbb{D}$. A corresponding result for harmonic functions will be also provided. This will be performed in Section 3. In Section 4 we will deal with the same question for infinite-order differential operators, so completing the above mentioned result on D-universality due to Costakis and Sambarino. Section 2 is devoted to give several results that will reveal useful later, together with some additional terminology.

2 Some auxiliary results

In [7] a geometrical notion was used to characterize the universality of certain sequences of differential operators. Now, we state such notion in a more general setting. Recall that if (X, d) is a metric space then the open (closed) d-ball with center $a \in X$ and radius $r > 0$ is $B_d(a, r) = \{x \in X : d(x, a) < r\}$ ($\overline{B}_d(a, r) = \{x \in X : d(x, a) \leq r\}$, respectively). Recall also that if $A \subset X$ then its diameter is $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Definition 2.1. If (X, d) is a metric space and A is a nonempty subset of X then the *inscribed radius* (or *Tchebychef radius*) of A is defined as the number $\rho_d(A) \in [0, +\infty]$ given by

$$\rho_d(A) = \sup\{r > 0 : \text{there exists a closed ball } B \text{ of radius } r \text{ with } B \subset A\}.$$

It is clear that if A is open in X then $\rho_d(A) > 0$. When d is the euclidean distance on $\mathbb{C}$ or $\mathbb{R}^N$ then the subscript d in ρ_d and in the d-balls will be dropped. Now, we state the following auxiliary geometrical result to be used later.

Lemma 2.1. *Let (X, d) be a connected metric space with $\delta(X) = +\infty$. If A is a subset of X with $\rho_d(A) = +\infty$ and B is any closed d-ball, then $\rho_d(A \setminus B) = +\infty$.*

Proof. Here we will also delete the subscript d. Hence we have that $\rho(A) = +\infty$ and that $B = \overline{B}(a, R)$ for certain $a \in X$, $R > 0$. Since $\delta(X) = +\infty$, given $x_0 \in X$ we have that the mapping $x \in X \mapsto d(x, x_0) \in [0, +\infty)$ is not bounded. But d is continuous and X is connected. Therefore, given $x_0 \in X$ and $r > 0$, there exists $x \in X$ such that $d(x, x_0) = r$.

Fix $M > 0$. We are looking for a point $c \in X$ with $\overline{B}(c, M) \subset A \setminus B$. By hypothesis, there exists $b \in X$ satisfying $\overline{B}(b, 3M + 2R + 1) \subset A$. At this point we distinguish two cases. If $d(a, b) > M + R$ and $c := b$ then $\overline{B}(c, M) \subset \overline{B}(b, 3M + 2R + 1)$ and, by the triangle inequality, $\overline{B}(c, M) \cap \overline{B}(a, R) = \emptyset$, whence we are done. If, on the contrary, $d(a, b) \leq M + R$, then again by the triangle inequality we obtain $\overline{B}(a, R) \subset \overline{B}(b, M + 2R)$. Choose $c \in X$ with $d(b, c) = 2M + 2R + 1$. We claim that $\overline{B}(c, M) \subset A \setminus B$. Indeed, a further use of the triangle inequality gives $\overline{B}(c, M) \subset \overline{B}(b, 3M + 2R + 1) \subset A$. Finally, by way of contradiction, let us suppose that there exists some point $x \in \overline{B}(c, M) \cap B$. Then $d(x, c) \leq M$ and $d(x, a) \leq R$, hence $d(c, a) \leq M + R$, which yields $d(b, c) \leq d(c, a) + d(b, a) \leq 2M + 2R$, that is absurd. □

Next, we enunciate the Arakelyan theorem that can be found in [19, pages 153–154].

Lemma 2.2. *If F is a relatively closed subset of a domain G in $\mathbb{C}$, then F is an Arakelian subset if and only if for every $g \in A(F)$ and $\varepsilon > 0$ there is a holomorphic function f in G such that $|f(z) - g(z)| < \varepsilon$ for all $z \in F$.*

We now establish as lemmas two powerful results about tangential approximation of holomorphic or harmonic functions. The first one is a variant for $G = \mathbb{C}$ of the Arakelian theorem, see [3] or [19, pages 160–162]. The second one is a harmonic version by Armitage and Goldstein of this theorem, see [5, Theorem 1.1] or [20, Corollary 5.10]. If $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ then $\|x\|$ denotes its norm $\|x\| = (x_1^2 + \cdots + x_N^2)^{1/2}$.

Lemma 2.3. *If F is a closed subset of $\mathbb{C}$, then F is an Arakelian subset if and only if for every $g \in A(F)$ and every continuous function $\varepsilon : [0, +\infty) \to (0, +\infty)$ with $\int_1^{+\infty} t^{-3/2} \log(1/\varepsilon(t))\, dt < +\infty$, there is an entire function f such that $|f(z) - g(z)| < \varepsilon(|z|)$ for all $z \in F$.*

Lemma 2.4. *If F is an Arakelian subset of $\mathbb{R}^N$, then for each $v \in h(F)$ and each choice of positive numbers a and ε, there exists $u \in h(\mathbb{R}^N)$ such that $|u(x) - v(x)| < \varepsilon(1 + \|x\|)^{-a}$ for all $x \in F$.*

The following result is purely topological and its content can be found in [21, Section 5, pages 242–243]. Given a relatively closed subset F of a domain $G \subset \mathbb{C}$, we set $\widehat{F} := F \cup c(F)$, where $c(F)$ denotes the union of the connected components of $G \setminus F$ having compact closure in G, that is, the union of the 'holes' of F. So $c(F) = \emptyset$ if F has no holes.

Lemma 2.5. *Let $G \subset \mathbb{C}$ be a domain and F be a relatively closed subset of G. Then we have:*
(a) *F is Arakelian in G if and only if $c(F) = \emptyset$ and $c(F \cup K)$ is relatively compact in G for every compact subset K of G.*
(b) *If F is Arakelian in G and $K \subset G$ is compact then $\widehat{F \cup K}$ is Arakelian in G.*
(c) *If F is Arakelian in G and K is a compact subset of G with connected complement such that $F \cap K = \emptyset$, then $F \cup K$ is Arakelian in G.*

To finish, recall that if $\Phi(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function of exponential type –that is, there are positive finite constants A, B satisfying $|\Phi(z)| \le A e^{B|z|}$ $(z \in \mathbb{C})$– then the formal expression $\Phi(D) = \sum_{k=0}^{\infty} a_k D^k$ (where $D^0 = I =$ the identity operator) defines in fact an operator –in general, an "infinite-order differential operator"– on $H(\mathbb{C})$. The entire function Φ is said to be of subexponential type whenever given $\varepsilon > 0$ there is a constant $A = A_\varepsilon \in (0, +\infty)$ satisfying $|\Phi(z)| \le A e^{\varepsilon |z|}$ $(z \in \mathbb{C})$. In other words, Φ is of exponential (subexponential) type if and only if it has either growth order < 1 or growth order 1 and finite growth type (it has either growth order < 1 or growth order 1 and growth type 0, respectively). Obviously, if Φ is of subexponential type then it is also of exponential type. The content of the following lemma can be found in [12].

Lemma 2.6. *An entire function $\Phi(z) = \sum_{k=0}^{\infty} a_k z^k$ is of exponential type if and only if* $\lim_{k\to\infty}(k!|a_k|)^{1/k} = 0$.

3 Bounded τ-universality

In this section, our results about the existence of a τ-universal function which is bounded on a certain prescribed subset F will be listed and proved. The inscribed radius will play a crucial role. Specifically, a necessary condition for such existence on general PTGs is shown in Theorem 3.1. In Theorem 3.3, a complete geometrical and topological characterization of the possible subsets F is given for $\mathbb{C}$. If we assume that F is Arakelian, a similar statement of existence is shown for $\mathbb{D}$ in Theorem 3.4. Finally, a corresponding result of 'bounded' τ-universality in the setting of harmonic functions on $\mathbb{R}^N$ is stated in Theorem 3.5.

Theorem 3.1. *Let $(G,*)$ be a PTG. Assume that d is a distance on G satisfying the following two properties:*
(i) *d is translation-invariant and generates the topology of G.*
(ii) *Every closed d-ball is compact.*

Let F be a nonempty, proper subset of G. If there exists a τ-universal function $f \in H(G)$ such that f is bounded on F then $\rho_d(G \setminus F) = +\infty$.

Proof. Suppose, by way of contradiction, that $R := \rho_d(G \setminus F) < +\infty$ and that there exists a τ-universal function $f \in H(G)$ which is bounded on F. Then there exist a sequence $(a_n) \subset \mathbb{C}$ and a constant $M \in (0,+\infty)$ such that the sequence $(f \circ \tau_{a_n})$ of translates of f is dense in $H(G)$ and $|f(z)| \le M$ for all $z \in F$. We have that $\overline{B}_d(a_n, R+1) \cap F \neq \emptyset$ for all $n \in \mathbb{N}$. Observe that due to (i) it holds that $\overline{B}_d(a,r) = \{z * a : z \in \overline{B}_d(e,r)\}$ $(a \in \mathbb{C},\ r > 0)$, where e is the neutral element of $(G,*)$. Therefore we can find a sequence $(b_n) \subset \overline{B}_d(e, R+1)$ with $b_n * a_n \in F$ for all $n \in \mathbb{N}$. Let us consider the constant function $g(z) := M + 1$. Then there exists a sequence $\{n(1) < n(2) < \cdots\} \subset \mathbb{N}$ such that $f \circ \tau_{a_{n(j)}} \to g$ as $j \to \infty$ uniformly on compacta in G. Thus, (ii) yields that, in particular,

$$\lim_{j\to\infty} \sup_{z \in \overline{B}_d(e,R+1)} |f(z * a_{n(j)}) - M - 1| = 0,$$

which is absurd, since $|f| \leq M$ on F and we have for any $n \in \mathbb{N}$ that

$$\sup_{z \in \overline{B}_d(e,R+1)} |f(z * a_n) - M - 1| \geq |f(b_n * a_n) - M - 1| \geq M + 1 - |f(b_n * a_n)| \geq 1.$$

This concludes the proof. □

Remarks 3.2. 1. If in particular we set $G = \mathbb{C}$, Theorem 3.1 shows that if there exists some translation-universal entire function that is bounded on a prefixed set F, then $\rho(\mathbb{C} \setminus F) = +\infty$; indeed, take as $*$ the usual sum and as d the euclidean distance. With an analogous proof, the last theorem also holds for $h(\mathbb{R}^N)$ when $\mathbb{R}^N$ is endowed with the ordinary sum and with the euclidean distance. If $G = \mathbb{D}$ and the disk is endowed with the hyperbolic (or Poincaré) distance $d_P(z, w) := \tanh^{-1} |\frac{z-w}{1-\bar{z}w}|$ (see [17] for a quite complete description) and with the internal law $z * w = \frac{z+w}{1+\bar{z}w}$ $(z, w \in \mathbb{D})$, then Theorem 3.1 also applies (recall that d_P is invariant under the automorphisms of $\mathbb{D}$ and generates the usual topology on $\mathbb{D}$, and that the d_P-balls are euclidean balls in $\mathbb{D}$; specifically, $B_{d_P}(a, R) = B(\frac{(1-\tanh^2 R)a}{1-|a|^2 \tanh^2 R}, \frac{(1-|a|^2)\tanh R}{1-|a|^2 \tanh^2 R})$ for all $a \in \mathbb{D}$ and all $R \in (0, +\infty)$) yielding that if F is a subset of $\mathbb{D}$ and there exists a translation-universal function $f \in H(\mathbb{D})$ which is bounded on F, then $\rho_{d_P}(\mathbb{D} \setminus F) = +\infty$.

2. We illustrate with an example in $\mathbb{C}$ that some kind of restriction on the subset F is necessary: The set $F := \mathbb{C} \setminus \bigcup_{n\geq 3} B(2^n, n)$ is a closed subset which is not contained in any Arakelian subset of $\mathbb{C}$, and satisfies $\rho(\mathbb{C} \setminus F) = +\infty$; an application of the Maximum Modulus Principle shows that if an entire function f is bounded on F then it must be bounded on $\mathbb{C}$, so f is constant and therefore it cannot be τ-universal.

3. Condition (ii) in Theorem 3.1 cannot be derived from (i). For instance, the distance $d(z, w) := \frac{|z-w|}{1+|z-w|}$ is a translation-invariant distance on the PTG $(\mathbb{C}, +)$ generating the topology of $\mathbb{C}$, but $\overline{B}_d(0, 1) = \mathbb{C}$, which is not compact in $\mathbb{C}$.

Theorem 3.3. *Let F be a closed subset of $\mathbb{C}$. Then the following conditions are equivalent:*

(a) *There exists a τ-universal entire function f such that f is bounded on F.*

(b) *There exists an Arakelian subset F_0 of $\mathbb{C}$ such that $F \subset F_0$ and $\rho(\mathbb{C}\setminus F_0) = +\infty$.*

Proof. Assume that (a) is true and define $F_0 := \{z \in \mathbb{C} : |f(z)| \le \sup_F |f|\}$. Then F_0 is Arakelian by [16, Theorem 1]. Obviously, $F \subset F_0$ and f is bounded on F_0. By Theorem 3.1 we have that $\rho(\mathbb{C} \setminus F_0) = +\infty$.

As for the converse, suppose that F_0 is Arakelian, $F \subset F_0$ and $\rho(\mathbb{C} \setminus F_0) = +\infty$. From Lemma 2.1 we can select a sequence of pairwise disjoint closed balls $\overline{B}(a_n, n+1)$ such that $\overline{B}(a_n, n+1) \cap F_0 = \emptyset$ for all $n \in \mathbb{N}$. Set $B_n := \overline{B}(a_n, n)$ $(n \in \mathbb{N})$. It is easy to check that the set

$$\widetilde{F_0} := F_0 \cup \bigcup_{n=1}^{\infty} B_n$$

is an Arakelian subset of $\mathbb{C}$. Let $\{P_n : n \in \mathbb{N}\}$ be a countable subset of $H(\mathbb{C})$, for instance, the set of polynomials whose coefficients have rational real and imaginary parts. We can construct a sequence (Q_n) whose members are in $\{P_n : n \in \mathbb{N}\}$ such that every function P_m occurs infinitely many times in (Q_n).

Next, we consider the function $g : \widetilde{F_0} \to \mathbb{C}$ given by

$$g(z) = \begin{cases} 0 & \text{if} \quad z \in F \\ Q_n(z - a_n) & \text{if} \quad z \in B_n. \end{cases}$$

Then, obviously, $g \in A(\widetilde{F_0})$. By Lemma 2.3 (with $\varepsilon(t) := \exp(-t^{1/4})$), there exists an entire function f such that

$$|f(z) - g(z)| < \exp(-|z|^{1/4}) \quad \text{for all } z \in \widetilde{F_0}. \tag{1}$$

In particular, $|f(z) - 0| < 1$ for all $z \in F_0$, so f is bounded on F_0, hence on F.

Finally, we show that the sequence $(f \circ \tau_{a_n})$ is dense in $H(\mathbb{C})$, so making f a τ-universal function. For this, fix a basic open set $D(h, R, \varepsilon) := \{\varphi \in H(\mathbb{C}) : |\varphi(z) -$

$h(z)| < \varepsilon$ for all $z \in \overline{B}(0,R)\}$ for the topology of $H(\mathbb{C})$, where $\varphi \in H(\mathbb{C})$, $R > 0$ and $0 < \varepsilon < 1$. It must be proved that there is a number $n \in \mathbb{N}$ such that

$$f \circ \tau_{a_n} \in D(h, R, \varepsilon). \tag{2}$$

Observe first that

$$\lim_{n\to\infty} \min_{z\in B_n} |z| = +\infty. \tag{3}$$

Indeed, if this were not true, then one could choose two sequences $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$, $z_j \in B_{n_j}$ $(j \in \mathbb{N})$ and a positive finite constant M with $|z_j| \le M$ for all $j \in \mathbb{N}$. Hence some subsequence of (z_j) would converge to a finite point, which is absurd because $|z_j - z_k| \ge 1$ for all $j, k \in \mathbb{N}$ with $j \ne k$. Therefore (3) holds. By (3) and the denseness of polynomials, there exist $n_0, m \in \mathbb{N}$ such that $\min_{w\in B_n} |w| > (\log(2/\varepsilon))^4$, $n > R$ for all $n > n_0$ and $|P_m(z) - h(z)| < \varepsilon/2$ for all $z \in \overline{B}(0,R)$. Fix $n > n_0$ satisfying $Q_n = P_m$. Consequently, from (1) we obtain for all $z \in \overline{B}(0,R)$ that

$$\begin{aligned} |f(z+a_n) - h(z)| &\le |f(z+a_n) - g(z+a_n)| + |g(z+a_n) - h(z)| \\ &\le \exp(-(\min_{w\in B_n} |w|)^{1/4}) + |Q_n(z) - h(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and this is (2). □

Theorem 3.4. *Let F be an Arakelian subset of $\mathbb{D}$. Then the following conditions are equivalent:*

(a) *There exists a τ-universal function $f \in H(\mathbb{D})$ such that f is bounded on F.*

(b) $\rho_{d_P}(\mathbb{D} \setminus F) = +\infty$.

Proof. By Theorem 3.1 we have again that (a) implies (b). As for the converse, suppose that $\rho_{d_P}(\mathbb{D}\setminus F) = +\infty$. Consider again a countable dense subset $\{P_n : n \in \mathbb{N}\}$ of $H(\mathbb{D})$, and let (Q_n) be a sequence such that every function P_m occurs infinitely many times in (Q_n). Recall that in the PTG $\mathbb{D}$ the origin is the neutral element for $*$ and that the $*$-inverse element for each $z \in \mathbb{D}$ is $-z$.

Fix $n \in \mathbb{N}$ and consider the ball $B_n := \overline{B}(0, n/(n+1)) \subset \mathbb{D}$. Since F is Arakelian and B_n is compact, Lemma 2.5 says us that $H_n := \widehat{F \cup B_n}$ is also an Arakelian subset of $\mathbb{D}$. Observe that

$$\rho_{d_P}(\mathbb{D} \setminus H_n) = +\infty. \tag{4}$$

Indeed, by Lemma 2.5 the set $c(F \cup B_n)$ is relatively compact in $\mathbb{D}$, so $B_n \cup c(F \cup B_n)$ is also relatively compact. Therefore there is a closed d_P-ball B with $B_n \cup c(F \cup B_n) \subset B$. Hence $H_n \subset F \cup B$. But, from the hypothesis, $\rho_{d_P}(\mathbb{D} \setminus F) = +\infty$, whence $\rho_{d_P}(\mathbb{D} \setminus H_n) \geq \rho_{d_P}(\mathbb{D} \setminus (F \cup B)) = \rho_{d_P}((\mathbb{D} \setminus F) \setminus B) = +\infty$, where the last equality follows from Lemma 2.1. Then (4) holds. Consequently, there is a ball $S_n := \overline{B}_{d_P}(a_n, n) = \overline{B}_{d_P}(0, n) * a_n \subset \mathbb{D} \setminus H_n$. Again by Lemma 2.5, $F_n := S_n \cup H_n$ is an Arakelian subset of $\mathbb{D}$.

Next, we use an induction procedure following ideas of Sinclair (see [33] and [22]). Assume that for some $n \in \mathbb{N}$ the functions $f_0 := 0, f_1, \ldots, f_{n-1} \in H(\mathbb{D})$ have been defined. Consider the function $g_n : F_n \to \mathbb{C}$ given as

$$g_n(z) = \begin{cases} f_{n-1} & \text{if} \quad z \in H_n \\ Q_n(z * (-a_n)) & \text{if} \quad z \in S_n. \end{cases}$$

From Lemma 2.2, there exists $f_n \in H(\mathbb{D})$ such that $|f_n - g_n| < 1/2^n$ on F_n. In particular, we obtain for each $n \in \mathbb{N}$ that

$$|f_n(z) - f_{n-1}(z)| < 1/2^n \text{ for all } z \in B_n \cup F, \tag{5}$$

and

$$|f_n(z) - Q_n(z * (-a_n))| < 1/2^n \text{ for all } z \in S_n. \tag{6}$$

Since every compact subset of $\mathbb{D}$ is eventually contained in all except a finite number of balls $\overline{B}(n/(n+1))$, it is deduced from (5) that the series $\sum_{n=0}^{\infty}(f_n - f_{n-1})$ converges uniformly on compacta in $\mathbb{D}$, so it defines a function $f \in H(\mathbb{D})$. Again by (5), $|f| < \sum_{n=0}^{\infty} 1/2^n = 1$ on F, whence f is bounded on F.

It remains to show that f is τ-universal. In fact, we have that the sequence $(f \circ \tau_{a_n})$ is dense in $H(\mathbb{D})$, where $\tau_a(z) = z * a$. To see this, observe that for every n we can write f as $f = f_n + \sum_{k=n+1}^{\infty}(f_k - f_{k-1})$. Fix a basic open set $D(h, R, \varepsilon) := \{\varphi \in H(\mathbb{C}) : |\varphi(z) - h(z)| < \varepsilon \text{ for all } z \in \overline{B}(0, R)\}$ for the topology of $H(\mathbb{D})$, where $\varphi \in H(\mathbb{D})$ and $R, \varepsilon \in (0, 1)$. It must be proved that $f \circ \tau_{a_n} \in D(h, R, \varepsilon)$ for some $n \in \mathbb{N}$. We have that there exist $m, n_0 \in \mathbb{N}$ such that $|P_m(z) - h(z)| < \varepsilon/3$ for all

$z \in \overline{B}(0,R)$ and $\overline{B}(0,R) \subset \overline{B}_{d_P}(0,n)$ for all $n > n_0$. Fix $n > n_0$ with $Q_n = P_m$ and $1/2^n < \varepsilon/3$. Then $f \circ \tau_{a_n} \in D(h,R,\varepsilon)$ because for every $z \in \overline{B}(0,R)$ we have from (5) and (6) that

$$|f(z * a_n) - h(z)| \leq |f_n(z * a_n) - h(z)| + \sum_{k=n+1}^{\infty} |f_k(z) - f_{k-1}(z)|$$

$$\leq |f_n(z * a_n) - Q_n(z)| + |Q_n(z) - h(z)| + \frac{1}{2^n} < \frac{1}{2^n} + \frac{1}{2^n} + \frac{\varepsilon}{3} + < \varepsilon.$$

Thus the translates $f \circ \tau_{a_n}$ are dense in $H(\mathbb{D})$, as required. □

Two opposite examples are furnished in the following corollary.

Corollary 3.5. *Let $D_0 \subset \mathbb{D}$ be an open ball which is tangent to $\mathbb{T}$ and S be a small sector $S := \{t_0 z : \pi - \alpha < \arg(z-1) < \pi + \alpha,\ |z-1| < \beta\}$ ($\alpha \in (0,\pi/2)$, $\beta \in (0,1)$) with vertex at a point $t_0 \in \mathbb{T}$. Then we have:*
(a) *There exists a τ-universal function $f \in H(\mathbb{D})$ such that f is bounded in $\mathbb{D} \setminus D_0$.*
(b) *There exists no τ-universal function $f \in H(\mathbb{D})$ such that f is bounded in $\mathbb{D} \setminus S$.*

Proof. It is evident that $\mathbb{D} \setminus D_0$ and $\mathbb{D} \setminus S$ are Arakelian sets in $\mathbb{D}$. In view of Theorem 3.4, our unique task is to show that $\rho_{d_P}(D_0) = +\infty > \rho_{d_P}(S)$. Since d_P is invariant under rotations, we may suppose without loss of generality that $D_0 = B(a, 1-a)$ for some $a \in (0,1)$ and that $t_0 = 1$.

The ball $B(1, 1-a)$ is a neighbourhood of 1 intersecting D_0. Take any sequence $(x_n) \subset (0,1)$ tending to 1. Since $x_n \to 1$, there exists $n_1 \in \mathbb{N}$ such that $\overline{B}(0,1/2) * x_{n_1} \subset B(1, 1-a)$. But $\overline{B}(0,1/2) * x_{n_1}$ is an euclidean ball with center at $(0,1)$, so it is a subset of D_0. By induction, we get an increasing sequence $(n_k) \subset \mathbb{N}$ satisfying $B_k := \overline{B}(0, k/(k+1)) * x_{n_k} \subset D_0$ for all $k \in \mathbb{N}$. But, since d_P is invariant under automorphisms of $\mathbb{D}$, we have that the Poincaré radius of B_k equals the Poincaré radius of $\overline{B}(0,k/(k+1))$, which tends to $+\infty$ because the increasing sequence of balls $\overline{B}(0,k/(k+1))$ exhausts $\mathbb{D}$ and the Poincaré diameter $\delta(\mathbb{D})$ of $\mathbb{D}$ is $+\infty$. This proves (a). As for (b), from the rotation-invariance of d_P together with the fact that the subsets which are far from $\mathbb{T}$ do not contribute to make ρ_{d_P} large, it is enough to find a finite constant M such that $\delta(B) \leq M$ for any ball $B \subset S$ of the form

$B = \overline{B}(a, r)$, with $a \in ((2/c) - 1, 1)$, $r = (1 - a)\sin\alpha$, where c is a fixed number with $1 < c < \csc\alpha$; that is, B is a maximal ball in S which is near the boundary. Let us set $M := \tanh^{-1}(c\sin\alpha)$. Finally, the rotation-invariance of d_P and the fact that the lines passing through the origin are hyperbolic geodesics between points which are aligned with the origin drive us to assert that $\delta(B)$ is the d_P-distance between the points $a - r$, $a + r$, hence

$$\delta(B) = \tanh^{-1}\left|\frac{(a+r)-(a-r)}{1-(a+r)(a-r)}\right| < \tanh^{-1}\frac{2(1-a)\sin\alpha}{1-a^2} = \tanh^{-1}\frac{2\sin\alpha}{1+a} < M,$$

as required. □

Theorem 3.6. *Let F be an Arakelian subset of $\mathbb{R}^N$. Then the following conditions are equivalent:*

(a) *There exists a τ-universal function $u \in h(\mathbb{R}^N)$ such that u is bounded on F.*

(b) $\rho(\mathbb{R}^N \setminus F) = +\infty$.

Proof. The statement can be proved along the same lines as the proof of Theorem 3.1, where this time Lemma 2.4 (with for instance $a = 1$) should be used instead of Lemma 2.3. The details are left to the interested reader. □

4 Infinite-order differential operators

We recall that given a subset $A \subset \mathbb{C}$ with some finite accumulation point, the set $H_A := \mathrm{span}\{e_a : a \in A\}$ where $e_a(z) := \exp(az)$ is dense in $H(\mathbb{C})$ [24, Section 5]. In particular, if Φ is a nonconstant entire function, we have that $H_{\Phi^{-1}(\{|z|>1\})}$ and $H_{\Phi^{-1}(\{|z|>1\})}$ are dense in $H(\mathbb{C})$.

Lemma 4.1. *Assume that a number $\varepsilon > 0$, a compact subset $K \subset \mathbb{C}$, a nonconstant entire function Φ of exponential type and a function $g \in H_A$ have been prescribed, where $A := \Phi^{-1}(\{|z| > 1\})$. Then there exists an $n_0 \in \mathbb{N}$ with the property that for every $n \geq n_0$ there is an entire function $F_n \in H_A$ satisfying $\Phi(D)^n F_n = g$ on $\mathbb{C}$ and $|F_n| < \varepsilon$ on K.*

Proof. We can write $g = \sum_{j=1}^N c_j e_{a_j}$ for some $N \in \mathbb{N}$ and certain complex constants c_j, a_j such that $|\Phi(a_j)| > 1$ for all $j = 1, \ldots, N$. Since $\Phi(D)e_a = \Phi(a)e_a$ $(a \in \mathbb{C})$, it is enough to take

$$F_n = \sum_{j=1}^N \frac{c_j}{\Phi^n(a_j)} e_{a_j},$$

for all $n \geq n_0$ where n_0 is chosen so that

$$(\min_{1\leq j\leq N} |\Phi(a_j)|)^{n_0} > \frac{N}{\varepsilon} \max_{1\leq j\leq N} \{|c_j| \sup_{z\in K} \exp|a_j z|\},$$

which is possible because $\lim_{n\to\infty}(\min_{1\leq j\leq N} |\Phi(a_j)|)^n = +\infty$. □

We finish the paper with the promised extension of Costakis-Sambarino's result on bounded D-universality. Observe that the conditions as on the set E as on the kind of operators have been weakened.

Theorem 4.2. *Let T be the operator on $H(\mathbb{C})$ given by $T = \Phi(D)$, where Φ is a nonconstant entire function of subexponential type. Let E be a prescribed subset of first category in $\mathbb{T}$. Then there exists an entire T-universal function f satisfying*

$$\lim_{r\to+\infty} f(rt) = 0 \text{ for each } t \in E.$$

Proof. By hypothesis, there are countably many closed subsets E_n $(n \in \mathbb{N})$ with empty interior in $\mathbb{T}$ such that $E \subset \cup_{n=1}^\infty E_n$. Let $F = \cup_{n=1}^\infty F_n$, where $F_n := \{rt : r \geq n \text{ and } t \in E_n\}$. Each set F_n is closed in $\mathbb{C}$ with empty interior. Since $\mathbb{C}$ is a Baire space, F has also empty interior. Since a convergent sequence of points of F must be included in finitely many sets F_n, we get that F is also closed in $\mathbb{C}$.

Consider the set

$$\mathcal{A} = \{g \in H(\mathbb{C}) : |g(z)| < \varepsilon(|z|) \text{ for all } z \in F\}$$

where $\varepsilon(t) := \exp(-t^{1/4})$. Hence $\varepsilon(t)$ is a positive, continuous function decreasing to zero as $t \to +\infty$. We will use later that it also satisfies the integrability condition of Lemma 2.3.

Thus, we would be done if we were able to find a function $f \in \mathcal{A}$ which is also universal for T.

Note that the null function is in $\mathcal{A}$ and that $\mathcal{A}$ can be written as the intersection of the open sets $A_k := \{g \in H(\mathbb{C}) : |g(z)| < \varepsilon(|z|) \text{ for all } z \in \overline{B}(0,k) \cap F\}$ $(k \in \mathbb{N})$. Consequently, $\mathcal{A}$ is a nonempty G_δ subset of $H(\mathbb{C})$. Hence, by Alexandroff's theorem, $\mathcal{A}$ endowed with the topology inherited from $H(\mathbb{C})$ is a completely metrizable space, so a Baire space. Thus, we can apply Baire's category arguments to prove that there exists a residual (so nonempty) subset in $\mathcal{A}$ consisting of universal functions for T.

For this, observe that $\mathcal{A} \cap \{T-\text{universal functions}\}$ is the intersection of the sets

$$E(s,j,m) = \bigcup_{n \in \mathbb{N}} \{g \in \mathcal{A} : \ |T^n g - h_j(z)| < \frac{1}{s} \text{ for all } z \in \overline{B}(0,m)\} \quad (s,j,m \in \mathbb{N}),$$

where $\{h_j\}_{j \in \mathbb{N}}$ is a countable dense subset of H_A and $A := \Phi^{-1}(\{|z| > 1\})$. Therefore, it is enough to see that each set $E(s,j,m)$ is open (this is easy by the continuity of each operator T^n) and dense in $\mathcal{A}$. In order to prove the denseness, fix $s,j,m \in \mathbb{N}$ together with a function $g_0 \in A$ and numbers δ, $R > 0$. Because of the nature of the topology of $\mathcal{A}$ we must find a positive integer n and a function $f \in \mathcal{A}$ so that

$$\sup_{|z| \le R} |f(z) - g_0(z)| < \delta \quad \text{and} \quad \sup_{|z| \le m} |(T^n f)(z) - h_j(z)| < 1/s. \tag{7}$$

We may assume without loss of generality that $R \ge m$. Let γ be defined as

$$\gamma = \inf_{z \in F \cap \overline{B}(0,R+2)} \{\varepsilon(|z|) - |g_0(z)|\}$$

and choose any constant β such that $0 < \beta < \min\{\delta, \gamma\}$; note that this implies that we also have $\beta < \inf_{z \in F \cap \overline{B}(0,R+2)} \varepsilon(|z|)$. Since H_B where $B := \Phi^{-1}(\{|z| < 1\})$ is dense in $H(\mathbb{C})$, we may find a function $q_0 \in H_B$ such that

$$\sup_{|z| \le R+1} |q_0(z) - g_0(z)| < \beta/4. \tag{8}$$

Recall that $h_j \in H_A$. From Lemma 4.1 and from the facts that $q_0 \in H_B$ and $T^n e_a = \Phi^n(a) e_a \to 0$ $(n \to \infty)$ for all $a \in B$, it is derived the existence of an $n \in \mathbb{N}$ and of an entire function H such that $T^n H = h_j$ on $\mathbb{C}$, and

$$|H(z)| < \beta/4 \quad \text{and} \quad |T^n q_0(z)| < 1/2s \quad \text{for all } z \in \overline{B}(0,R+1). \tag{9}$$

Let $q := q_0 + H$. Then, by the triangle inequality,

$$\sup_{|z|\leq R+1} |q(z) - g_0(z)| < \beta/2. \tag{10}$$

Since $T^n = \Phi^n(D)$ and Φ^n is also of subexponential type, by Lemma 2.6 we derive that $|a_k| \leq M(1/2)^k/k!$ $(k \geq 0)$ for some finite constant $M > 0$, where the a_k's are the Taylor coefficients of Φ^n. If we use Cauchy's estimates then for every $z \in \overline{B}(0, R)$ and every $h \in H(\mathbb{C})$ we obtain that

$$|(T^n h)(z)| \leq \sum_{k=0}^{\infty} |a_k h^{(k)}(z)| \leq M \sum_{k=0}^{\infty} \frac{(1/2)^k}{k!} k! \frac{\sup_{|w-z|=1} |h(w)|}{1^k} \leq 2M \sup_{\overline{B}(0,R+1)} |h|.$$

Consequently, if h is entire and $\sup_{\overline{B}(0,R+1)} |h| < 1/(4Ms) =: \beta_1$, then

$$\sup_{|z|\leq R} |(T^n h)(z)| < 1/2s. \tag{11}$$

Finally, set $\widetilde{F} := \overline{B}(0, R+1) \cup F$ and define the function $g : \widetilde{F} \to \mathbb{C}$ by

$$g(z) = \begin{cases} q(z) & \text{if} \quad z \in \overline{B}(0, R+1) \\ 0 & \text{if} \quad z \in \{|w| \geq R+2\} \cap F \\ (1-t)q(z) & \text{if} \quad z \in \{|w| = R+1+t\} \cap F, \ 0 \leq t \leq 1. \end{cases}$$

Observe that $\widetilde{F}$ is an Arakelian set and that g is continuous on it and holomorphic in its interior. It is elementary to construct a continuous positive function $\varepsilon_1(t)$ on $[0, +\infty)$ such that $\varepsilon_1(t) \leq \varepsilon(t)$ for all $t \geq 0$, $\varepsilon_1(t) < \min\{\beta/2, \beta_1\}$ for $0 \leq t \leq R+2$ and such that $\varepsilon_1(t)$ still satisfies the integrability condition given in Lemma 2.3. Consequently, there exists an entire function f with $|f(z) - g(z)| < \varepsilon_1(|z|)$ for every $z \in \widetilde{F}$. Putting all inequalities (9) to (11) together, we get in a similar way to the final part of the proof of Theorem 6 in [15] that $f \in \mathcal{A}$ and that (7) is fulfilled. Suffice it to say that (11) should be applied on $h := f - q$. The (cumbersome, but easy) details are left to the interested reader. □

Remarks 4.3. 1. If T is as in the last theorem, then there exist a set $E \subset \mathbb{T}$ with full linear measure and a T-universal entire function f such $\lim_{r\to\infty} f(rt) = 0$ for each $t \in E$. Indeed, it suffices to choose $E = \bigcup_{n=1}^{\infty} E_n$ where E_n is a Cantor set in $\mathbb{T}$ of measure $2\pi - (1/n)$.

2. The statement of Theorem 4.2 is sharp, at least in terms of growth order and growth type. Indeed, if Φ is allowed to be only of *exponential* type then the universal entire function f of the statement may not exist: Take for instance $E := \{1\}$ and $\Phi(z) := e^z$; then $\Phi(D)$ becomes the translation operator $f(\cdot) \mapsto f(\cdot + 1)$ and, if f were universal, then the limit $\lim_{r \to +\infty} f(r)$ could not exist.

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