# INDUCTION, MINIMIZATION AND COLLECTION FOR $\Delta_{n+1}(\mathrm{T})$–FORMULAS

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Abstract. For a theory $\mathbf{T}$, we study relationships among $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. These theories are obtained restricting the schemes of induction, minimization and (a version of) collection to $\Delta_{n+1}(\mathbf{T})$ formulas. We obtain conditions on $\mathbf{T}$ ($\mathbf{T}$ is an extension of $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ or $\Delta_{n+1}(\mathbf{T})$ is closed (in $\mathbf{T}$) under bounded quantification) under which $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ are equivalent.

These conditions depend on $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, the $\Pi_{n+2}$–consequences of $\mathbf{T}$. The first condition is connected with descriptions of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ as $\mathbf{I}\Sigma_n$ plus a class of nondecreasing total $\Pi_n$–functions, and the second one is related with the equivalence between $\Delta_{n+1}(\mathbf{T})$–formulas and bounded formulas (of a language extending the language of Arithmetic). This last property is closely tied to a general version of a well known theorem of R. Parikh.

Using what we call $\Pi_n$–envelopes we give uniform descriptions of the previous classes of nondecreasing total $\Pi_n$–functions. $\Pi_n$–envelopes are a generalization of envelopes (see [10]) and are closely related to indicators (see [12]). Finally, we study the hierarchy of theories $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$, $m \geq n$, and prove a hierarchy theorem.

## 1. Introduction

This paper is devoted to the study of two main topics: the relationship between induction and minimization, and the description of the class of $\Pi_{n+2}$ consequences of a theory.

The first one is on Fragments of Arithmetic obtained restricting the schemes of induction, minimization and collection to $\Delta_{n+1}$–formulas. These schemes for $\Sigma_n$ and $\Pi_n$ formulas have been thoroughly studied by J. Paris, L. Kirby and others (see [17] or [12]). The parameter free versions of those schemes have been studied by R. Kaye, J. Paris and C. Dimitracopoulos (see [11] and [14]). However, the relationships between those schemes for $\Delta_{n+1}$ formulas are not well known. About 1985, H. Friedman claimed that $\mathbf{L}\Delta_{n+1}$ and $\mathbf{I}\Delta_{n+1}$ are equivalent (see [10] pg. 398), but in [6] that equivalence appears as an open problem (problem 34) and it is credited to J. Paris. Here that equivalence will be called the Paris–Friedman's Conjecture. In [19], T. Slaman proves it for $n \geq 1$.

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In sections 2 and 6, we study those schemes restricted to $\Delta_{n+1}(\mathbf{T})$ formulas. If $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ then $\varphi \leftrightarrow \psi$ is a $\Pi_{n+2}$ formula. So, the second topic is related to the first one. In sections 3–5, we analyse the class of $\Pi_{n+2}$ consequences of a theory using a class of $\Pi_n$–functions and extensions of the language of Arithmetic related to that class of functions.

Now we present the main results obtained on these topics in this paper.

**Part I: Induction and minimization for $\Delta_{n+1}(\mathbf{T})$ formulas.**

In order to get a better insight on the Paris–Friedman's Conjecture we consider the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, where

$$\Delta_{n+1}(\mathbf{T}) = \{\varphi(x, \vec{v}) \in \Sigma_{n+1} : \text{ there exists } \psi(x, \vec{v}) \in \Pi_{n+1},\ \mathbf{T} \vdash \varphi \leftrightarrow \psi\}.$$

The idea is to change the semantic part of the axioms schemes on $\Delta_{n+1}$ formulas by a syntactic condition: the equivalence between a $\Sigma_{n+1}$ formula and a $\Pi_{n+1}$ formula is proved in a theory. Thus we obtain a relativization of Paris–Friedman's Conjecture. We study the following problem:

$$(*) \quad \text{Under which conditions on } \mathbf{T} \text{ does } \mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}) \text{ hold?}$$

We first observe that $\Longrightarrow$ always holds. In the other way, let us notice that the usual proof of $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{L}\Sigma_{n+1}$ leans upon the closure of $\Sigma_{n+1}$ under bounded quantification (this property is granted by the collection schemes, $\mathbf{B}\Sigma_{n+1}$). In fact, the closure under bounded quantification of the class of $\Delta_{n+1}$–formulas is the main obstacle in order to adapt the refered proof to obtain that $\mathbf{I}\Delta_{n+1} \Longrightarrow \mathbf{L}\Delta_{n+1}$. So, to answer problem $(*)$ the above remarks suggest two natural properties: $\mathbf{T}$ has $\Delta_{n+1}$–collection (that is, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$), and $\mathbf{T}$ is $\Delta_{n+1}$–closed (that is, $\Delta_{n+1}(\mathbf{T})$ is closed in $\mathbf{T}$ under bounded quantification). We prove that if $\mathbf{T}$ satisfies one of the above conditions then $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$, see theorem 1.4.

We also study relationships among the above schemes, for distinct theories. The following theorem sums up the results obtained.

**Theorem 1.1** *(see 2.1, 2.10, 2.17, 2.18, 6.12, 6.13, 6.14).* *For all $n \in \omega$*

$$\begin{array}{ccccccccc}
\mathbf{I}\Sigma_n & \Longleftrightarrow & \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_n) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n) & \Longleftrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\left.\begin{array}{c}\mathbf{B}\Sigma^-_{n+1},\ \mathbf{I}\Pi^-_{n+1} \\ \mathbf{I}\Delta_{n+1},\ \mathbf{B}\Sigma_{n+1}\end{array}\right\} & \not\Longleftrightarrow & \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) & \mid\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\begin{array}{r}\mathbf{I}\Sigma^-_{n+1} \\ \mathbf{I}\Pi^-_{n+1},\ \mathbf{B}\Sigma^-_{n+1}\end{array} & \begin{array}{c}\not\Longleftarrow \\ \not\Longleftrightarrow\end{array} & \mathbf{L}\Delta_{n+1}(\mathbf{PA}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{PA}) & \mid\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{PA}) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\begin{array}{c}\mathbf{B}\Sigma_{n+1} \\ \mathbf{I}\Pi^-_{n+1} \\ \mathbf{B}\Sigma^-_{n+1}\end{array} & \begin{array}{c}\not\Longleftrightarrow \\ \Longleftarrow\!\mid \\ \not\Longrightarrow\end{array} & \mathbf{L}\Delta_{n+1}(\mathcal{N}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathcal{N}) & \mid\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathcal{N}) \\
 & & & & \underline{\Uparrow} & & \underline{\Uparrow} \\
 & & & & \mathbf{I}\Sigma_{n+1} & \mid\!\Longrightarrow & \mathbf{B}\Sigma_{n+1}
\end{array}$$

*(Some of those relations for parameter free schemes follow from results in* [9]*, see also* [7] *and* [15]*).*

**Part II:** $\Pi_{n+2}$ **consequences of a theory**.

Properties considered in part I ($\mathbf{T}$ has $\Delta_{n+1}$–collection, $\mathbf{T}$ is $\Delta_{n+1}$–closed and others that we call $\Delta_{n+1}$–properties) depend on $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, the class of $\Pi_{n+2}$ consequences of $\mathbf{T}$. Here we give characterizations of these properties in a "functional" way. The idea is to describe $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ using $\mathbf{I}\Sigma_n$ and a class of $\Pi_n$–functions. To this end we introduce the concepts of $\Pi_n$–functional class (which provides a characterization of the theories having $\Delta_{n+1}$–collection) and $\Pi_n$–Parikh pair (which corresponds with $\Delta_{n+1}$–closed theories). Essentially, a $\Pi_n$–functional class is a set of nondecreasing $\Pi_n$–functions. The concept of $\Pi_n$–Parikh pair is suggested by the following well known result.

**Theorem 1.2** *(Parikh). Let $\varphi(x, y) \in \Sigma_1$. If $\mathbf{I}\Delta_0 \vdash \forall x\, \exists y\, \varphi(x, y)$ then there exists $t(x) \in \mathbf{Term}(\mathcal{L})$ such that $\mathbf{I}\Delta_0 \vdash \forall x\, \exists y \leq t(x)\, \varphi(x, y)$.*

As a consequence of this result (see 3.27) each $\Delta_1(\mathbf{I}\Delta_0)$ formula is equivalent (in $\mathbf{I}\Delta_0$) to a $\Delta_0$–formula. So, $\Delta_1(\mathbf{I}\Delta_0)$ is closed (in $\mathbf{I}\Delta_0$) under bounded quantification. We give a general version of this fact. If $\mathbf{T}$ is $\Delta_{n+1}$–closed, then there is a conservative extension of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ (in a language extending the language of Arithmetic) in which each $\Delta_{n+1}(\mathbf{T})$ formula is equivalent to a bounded formula. In particular, if $\mathbf{T}$ has $\Delta_{n+1}$–collection then a strong $\Pi_n$–functional class provides such an extension.

One crutial result that relates the schemes of induction and collection is the Friedman–Paris' conservativeness theorem (see [10] or [12]):

**Theorem 1.3.** *For all $n \in \omega$, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1})$.*

Here we study a similar $\Pi_{n+2}$–conservativeness property, closely tied to $\Delta_{n+1}$–collection: $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$. This property plays a central role in the study of $\Pi_n$–envelopes that will be developed in section 5. Roughly speaking, a $\Pi_n$–envelope is a $\Pi_n$–functional class given in an uniform way and generalizes the concept of envelope (see [10]). In section 6 we use results of sections 4 and 5 to separate the fragments $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$, $m \geq n$ (see theorem 1.1). The following theorem sums up, for a consistent theory, $\mathbf{T}$, the relationships among the concepts introduced.

**Theorem 1.4.** *(see 2.10, 2.11, 3.8, 3.11, 3.28, 4.18, 5.11, 5.21)*

$$\begin{array}{ccccccc}
\mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-PF} & & \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-min.} & & \mathbf{T}\ \textit{is strong}\ \Pi_n\textit{-funct.} & & \mathbf{T}\ \textit{has}\ \Pi_n\textit{-s-env.} \\
\Uparrow & & \Updownarrow & & \Updownarrow & & \Updownarrow^{\mathbf{3}} \\
\mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-closed} & \Longleftarrow & \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-coll.} & \Longleftrightarrow & \mathbf{T}\ \textit{is}\ \Pi_n\textit{-funct.} & \Longleftrightarrow^{\mathbf{2}}_{\mathbf{3}} & \mathbf{T}\ \textit{has}\ \Pi_n\textit{-env.} \\
\Updownarrow & & \Updownarrow & & \Updownarrow_{\mathbf{1}} & & \\
\mathbf{T}\ \textit{is}\ \Pi_n\textit{–Parikh} & & \left\{\begin{array}{l} \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-ind.} \\ \mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-closed} \end{array}\right. & & \mathbf{T}\ \textit{is}\ \Pi^{\mathbf{B}}_{n+2}\textit{-conserv.} & &
\end{array}$$

*Where: $\Longrightarrow_{\mathbf{1}}$ holds if $\mathbf{T}$ is $\Pi_{n+2}$ axiomatizable; $\Longleftarrow^{\mathbf{2}}$ holds if the $\Pi_n$–envelope is given by a $\Pi_n$ formula; and $\Longrightarrow_{\mathbf{3}}$ holds if $\mathbf{T}$ is recursively axiomatizable, and, for $n = 0$, $\mathbf{T} \vdash \mathbf{exp}$.*

In order to simplify the statement of the above theorem we have used there the following notation: $\mathbf{T}$ has $\Pi_n$–envelope ($\Pi_n$–s–envelope) means that there exists a $\Pi_n$–envelope (strong $\Pi_n$–envelope) of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$; and $\mathbf{T}$ is $\Pi^{\mathbf{B}}_{n+2}$–conservative if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.

The analysis of theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ that we develop in this paper is related with the work of L. D. Beklemishev in [2], [3] and [4], on induction and collection as inference rules. Some results in those papers, proved there using Proof Theoretic techniques, are similar to those given here for schemes on $\Delta_{n+1}(\mathbf{T})$–formulas. Now we give a more precise description of the relationship between Beklemishev's work and ours.

In the papers cited above, Beklemishev study the schemes of induction and collection as inference rules. The induction rule for a formula $\varphi(x)$ is:

$$\frac{\varphi(0),\ \forall x\,(\varphi(x) \rightarrow \varphi(x+1))}{\forall x\,\varphi(x)}$$

If $\Gamma$ is a class of formulas, then $\Gamma$–IR is the class of induction rules for each formula in $\Gamma$. Given a theory $\mathbf{T}$, let $\mathbf{T} + \Sigma_{n+1}$–IR be the closure of $\mathbf{T}$ under first order logic and applications of $\Sigma_{n+1}$–IR. We also denote by $[\mathbf{T}, \Sigma_{n+1}\text{–IR}]$ the closure of $\mathbf{T}$ under first order logic and unnested applications of $\Sigma_{n+1}$–IR; that is, the rule of induction can be applied only if the hypothesis of the rule are theorems of $\mathbf{T}$ (in first order logic). The rule of collection for a formula $\varphi(x, y)$ is:

$$\frac{\forall x\,\exists y\,\varphi(x,y)}{\forall z\,\exists\,u\,\forall\,x \leq z\,\exists y \leq u\,\varphi(x,y)}$$

Theories $\mathbf{T} + \Sigma_{n+1}$–CR and $[\mathbf{T}, \Sigma_{n+1}\text{–CR}]$ are defined as for the induction rule. In [4] it is also considered the induction rule for $\Delta_{n+1}$ formulas: for each $\varphi(x) \in \Sigma_{n+1}$ and $\psi(x) \in \Pi_{n+1}$

$$\Delta_{n+1}\text{–IR}: \qquad \frac{\forall x\,(\varphi(x) \leftrightarrow \psi(x))}{\mathbf{I}_{\varphi,x}}$$

As we shall see in 2.19, a theory $\mathbf{T}$ (extension of $\mathbf{I}\Delta_0$) has $\Delta_{n+1}$–collection if and only if $\mathbf{T}$ is closed under $\Sigma_{n+1}$–CR (that is, $[\mathbf{T}, \Sigma_{n+1}\text{–CR}] \Longleftrightarrow \mathbf{T}$). For induction we have that $\mathbf{T}$ has $\Delta_{n+1}$–induction if and only if $\mathbf{T}$ is closed under $\Delta_{n+1}$–IR.

Our analysis of theories with $\Delta_{n+1}$–collection using $\Pi_n$–functional classes is also very similar (for $n = 0$) to the one given by Beklemishev in [2] using what he call monotone formulas. In this way theorem **3.5** can be considered a generalization of theorem **5.4** of [2] (and it is linked with theorem **4.2** of [3]). Nevertheless, we must observe that one of the aims of Beklemishev's work in [3] is to obtain a proof of Friedman–Paris' conservativeness theorem. On the other hand, our analysis goes in a reverse direction, since we take that result as basic (due to its easy model theoretic proof) and relate it with a characterization of $\Pi_n$–envelopes using indicators ($\Pi_n$–IND property, see theorem 5.6).

The relationship of $\Sigma_{n+1}$–IR with the work developed here is not so obvious. But, as Beklemishev has noted (personal communication),

$$\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) \Longleftrightarrow \mathbf{I}\Delta_0 + \Sigma_{n+1}\text{–IR}.$$

This fact is closely tied to a conservativeness theorem of Parsons (see [18])

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_{n+1}) \Longleftrightarrow \mathbf{I}\Delta_0 + \Sigma_{n+1}\text{–IR}.$$

These results are more deeply studied in [8] in connection with axiomatization properties of the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$.

We conclude this section with some basic results and notation that we use through this paper. We work in the first–order language of Arithmetic, $\mathcal{L} = \{0, 1, +, \cdot, <\}$ and $\mathcal{N}$ denotes the standard model of $\mathcal{L}$ whose universe is the set of the natural numbers, $\omega$. As usual, bounded quantifiers are denoted by $\forall x \leq t\, \varphi(x)$ and $\exists x \leq t\, \varphi(x)$ (where $x$ does not occur in $t$). $\Delta_0 = \Sigma_0 = \Pi_0$ is the class of bounded formulas and, for each $n \in \omega$,

$$\Sigma_{n+1} = \{\exists \vec{x}\, \varphi(\vec{x}) :\ \varphi(\vec{x}) \in \Pi_n\} \text{ and } \Pi_{n+1} = \{\forall \vec{x}\, \varphi(\vec{x}) :\ \varphi(\vec{x}) \in \Sigma_n\}.$$

Let $\varphi(x, \vec{v})$ be a formula of $\mathcal{L}$. We shall denote $\varphi(x, \vec{v}) \wedge \forall y < x\, \neg\varphi(y, \vec{v})$ by $\varphi_{\mu,x}(x, \vec{v})$. If $\mathfrak{A} \models \varphi_{\mu,x}(a, \vec{b})$ then we write $\mathfrak{A} \models a = (\mu x)[\varphi(x, \vec{b})]$. If there is no danger of misunderstanding we omit the subscript $x$ and the parameters $\vec{v}$ and we shall write $\varphi_\mu(x)$.

We denote by $\mathbf{P}^-$ a finite set of $\Pi_1$ axioms such that if $\mathfrak{A} \models \mathbf{P}^-$ then $\mathfrak{A}$ is the nonnegative part of a commutative discretely ordered ring (see [12]).

Let $\varphi(x, \vec{v})$ be a formula. The induction and the least number principle axioms for $\varphi(x, \vec{v})$ with respect to $x$ are, respectively, the following formulas

$$\mathbf{I}_{\varphi,x}(\vec{v}) \equiv \varphi(0, \vec{v}) \wedge \forall x\, [\varphi(x, \vec{v}) \rightarrow \varphi(x+1, \vec{v})] \rightarrow \forall x\, \varphi(x, \vec{v}),$$

$$\mathbf{L}_{\varphi,x}(\vec{v}) \equiv \exists x\, \varphi(x, \vec{v}) \rightarrow \exists x\, \varphi_{\mu,x}(x, \vec{v}).$$

Let $\varphi(x, y, \vec{v})$ be a formula. The collection axiom and the strong collection axiom for $\varphi$ with respect to $x, y$ are, respectively, the formulas

$$\mathbf{B}_{\varphi,x,y}(z, \vec{v}) \equiv \forall x \leq z\, \exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y, \vec{v}),$$

$$\mathbf{S}_{\varphi,x,y}(z, \vec{v}) \equiv \exists u\, \forall x \leq z\, [\exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists y \leq u\, \varphi(x, y, \vec{v})].$$

As usual, we write $\mathbf{I}_\varphi$ instead of $\mathbf{I}_{\varphi,x}$ and similarly we use $\mathbf{L}_\varphi$, $\mathbf{B}_\varphi$ and $\mathbf{S}_\varphi$. If $\Gamma$ is a class of formulas of $\mathcal{L}$, then $\mathbf{I}\Gamma = \mathbf{P}^- + \{\mathbf{I}_\varphi :\ \varphi \in \Gamma\}$. The theory $\mathbf{L}\Gamma$ is defined similarly using $\mathbf{L}_\varphi$ instead of $\mathbf{I}_\varphi$. For collection, $\mathbf{B}\Gamma = \mathbf{I}\Delta_0 + \{\mathbf{B}_\varphi :\ \varphi \in \Gamma\}$ and using $\mathbf{S}_\varphi$ instead of $\mathbf{B}_\varphi$ we obtain $\mathbf{S}\Gamma$. Peano Arithmetic is the theory $\mathbf{PA} = \mathbf{P}^- + \{\mathbf{I}_\varphi :\ \varphi \text{ formula}\}$.

Now we consider schemes for parameter free formulas. Let $\Gamma$ be a class of formulas. We write $\varphi(x_1, \ldots, x_n) \in \Gamma^-$ if $\varphi \in \Gamma$ and $x_1, \ldots, x_n$ are all the variables that occur free in $\varphi$. Then $\mathbf{I}\Gamma^- = \mathbf{P}^- + \{\mathbf{I}_{\varphi,x} :\ \varphi(x) \in \Gamma^-\}$ (similarly for $\mathbf{L}\Gamma^-$) and $\mathbf{B}\Gamma^- = \mathbf{I}\Delta_0 + \{\mathbf{B}^-_{\varphi,x,y} :\ \varphi(x, y) \in \Gamma^-\}$, where

$$\mathbf{B}^-_{\varphi,x,y} \equiv \forall x\, \exists y\, \varphi(x, y) \rightarrow \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y).$$

The parameter free version of the strong collection scheme for $\Sigma_n$ formulas is equivalent to $\mathbf{S}\Sigma_n$.

One of the basic functions used to describe metamathematical properties in the language of Arithmetic, such as truth predicates, is the exponential function. Let $\mathbf{E}(x, y, z)$ be a $\Delta_0$–formula that defines in the standard model the exponential function, $\mathbf{I}\Delta_0$ proves its basic properties and $\mathbf{I}\Sigma_1$ proves that it is total (see [10] for details). We shall usually write $x^y = z$ instead of $\mathbf{E}(x, y, z)$ and shall denote by $\mathbf{exp}$ the $\Pi_2$ sentence $\forall x\, \forall y\, \exists z\, \mathbf{E}(x, y, z)$.

We shall write: $\mathbf{T} \Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is an extension of $\mathbf{T}'$; $\mathbf{T} \not\Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is not an extension of $\mathbf{T}'$; $\mathbf{T} \not\Longleftrightarrow \mathbf{T}'$, if $\mathbf{T} \not\Longrightarrow \mathbf{T}'$ and $\mathbf{T}' \not\Longrightarrow \mathbf{T}$; $\mathbf{T} \Longleftrightarrow \mathbf{T}'$, if $\mathbf{T}$ and $\mathbf{T}'$ are equivalent; and $\mathbf{T} \mathrel{|}\!\!\Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is a proper extension of $\mathbf{T}'$.

We recall some definitions and results which are important in the study of the above schemes. Let $\mathfrak{A} \models \mathbf{P}^-$, $n \in \omega$ and $X \subseteq \mathfrak{A}$. Then $\mathcal{K}_n(\mathfrak{A}, X)$ (if $X$ is the empty set, we write $\mathcal{K}_n(\mathfrak{A})$) is the substructure of $\mathfrak{A}$ whose universe is $\{b \in \mathfrak{A} :\ b \text{ is } \Sigma_n \text{ definable in } (\mathfrak{A}, X)\}$. $\mathcal{I}_n(\mathfrak{A}, X)$ is the initial segment of $\mathfrak{A}$ determined by $\mathcal{K}_n(\mathfrak{A}, X)$. It holds the following results.

**Theorem 1.5.** (1) *Let $\mathfrak{A} \models \mathbf{I}\Sigma_n$ be nonstandard. Then for all $X \subseteq \mathfrak{A}$*

(a) *$\mathcal{K}_{n+1}(\mathfrak{A}, X) \prec_{n+1} \mathfrak{A}$ and $\mathcal{K}_{n+1}(\mathfrak{A}, X) \models \mathbf{I}\Sigma_n$.*

(b) *$\mathcal{K}_{n+1}(\mathfrak{A}, X) \prec^c_{n+1} \mathcal{I}_{n+1}(\mathfrak{A}, X) \prec^e_n \mathfrak{A}$. ($\subset^c$ and $\subset^e$ mean cofinal and initial substructure, respectively).*

(c) *If $\mathcal{K}_{n+1}(\mathfrak{A}, X)$ is not cofinal in $\mathfrak{A}$ then $\mathcal{I}_{n+1}(\mathfrak{A}, X) \models \mathbf{B}\Sigma_{n+1}$.*

(2) *Let $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$ be nonstandard such that $\mathcal{K}_{n+1}(\mathfrak{A})$ is nonstandard. Then $\mathcal{K}_{n+1}(\mathfrak{A}) \not\models \mathbf{B}\Sigma^-_{n+1}$ and $\mathcal{I}_{n+1}(\mathfrak{A}) \not\models \mathbf{I}\Sigma_{n+1}$.*

Finally, we introduce the axiom schemes for $\Delta_{n+1}$ formulas.

$$\mathbf{I}\Delta_{n+1} = \mathbf{P}^- + \{\forall x\, [\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v})] \rightarrow \mathbf{I}_{\varphi,x}(\vec{v}) :\ \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\}.$$

Using $\mathbf{L}_\varphi$ instead of $\mathbf{I}_\varphi$, we obtain $\mathbf{L}\Delta_{n+1}$. Parameter free schemes, $\mathbf{I}\Delta^-_{n+1}$ and $\mathbf{L}\Delta^-_{n+1}$, are defined similarly. Uniform versions of the above fragments have been introduced by R. Kaye (see [11]). $\mathbf{UI}\Delta_{n+1}$ is $\mathbf{P}^-$ together with, for all $\varphi \in \Sigma_{n+1}$, $\psi \in \Pi_{n+1}$,

$$\forall x\, \forall \vec{v}\, [\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v})] \rightarrow \forall \vec{v}\, \mathbf{I}_{\varphi,x}(\vec{v}).$$

$\mathbf{UL}\Delta_{n+1}$ is defined accordingly using $\mathbf{L}_\varphi$. We introduce a uniform version of collection. $\mathbf{UB}\Delta_{n+1}$ is $\mathbf{I}\Delta_0$ together with, for all $\varphi \in \Pi_n$ and $\psi \in \Sigma_n$,

$$\forall x\, \forall \vec{v}\, [\exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \forall w\, \psi(x, w, \vec{v})] \rightarrow \forall z\, \forall \vec{v}\, \mathbf{B}_{\varphi,x,y}(z, \vec{v}).$$

**Theorem 1.6.** *For all $n \in \omega$, $\mathbf{I}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{UL}\Delta_{n+1}$ and*

$$\begin{array}{ccccccccc}
 & & \mathbf{I}\Pi^-_{n+1} & |\!\!\Longrightarrow & \mathbf{L}\Delta^-_{n+1} & \Longrightarrow & \mathbf{I}\Delta^-_{n+1} & & \\
 & & & & \Uparrow & & \Uparrow & & \\
\mathbf{UB}\Delta_{n+1} & \Longleftrightarrow & \mathbf{B}\Sigma^-_{n+1} & \Longleftrightarrow & \mathbf{UL}\Delta_{n+1} & \Longrightarrow & \mathbf{UI}\Delta_{n+1} & |\!\!\Longrightarrow & \mathbf{I}\Sigma_n \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & \mathbf{B}\Sigma_{n+1} & \Longleftrightarrow & \mathbf{L}\Delta_{n+1} & \Longrightarrow & \mathbf{I}\Delta_{n+1} & &
\end{array}$$

*For $n \geq 1$, $\mathbf{L}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{UI}\Delta_{n+1}$ and $\mathbf{I}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{I}\Sigma_n$, but $\mathbf{I}\Delta^-_1 |\!\!\Longrightarrow \mathbf{I}\Delta_0$.*

R.O. Gandy (see [10]) proved the equivalence between $\mathbf{L}\Delta_{n+1}$ and $\mathbf{B}\Sigma_{n+1}$; and R. Kaye (see [11]) obtained a similar result for the uniform versions. See [9] for $\mathbf{UI}\Delta_{n+1} |\!\!\Longrightarrow \mathbf{I}\Sigma_n$, $\mathbf{L}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{UI}\Delta_{n+1}$ and $\mathbf{I}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{UL}\Delta_{n+1}$; [7] for $\mathbf{I}\Delta_{n+1} |\!\!\Longrightarrow \mathbf{UI}\Delta_{n+1}$; 2.14 for $\mathbf{UB}\Delta_{n+1} \Longleftrightarrow \mathbf{B}\Sigma^-_{n+1}$; and [4] for $\mathbf{UI}\Delta_1 |\!\!\Longrightarrow \mathbf{I}\Delta^-_1$ (there $\mathbf{UI}\Delta_1$ is denoted by $s\mathbf{I}\Delta_1$). The above diagram contains the following open problems:

- **(−):** The Paris–Friedman's Conjecture: $\mathbf{L}\Delta_{n+1} \Longleftrightarrow \mathbf{I}\Delta_{n+1}$.
- **(−):** The Uniform Paris–Friedman's Conjecture: $\mathbf{UL}\Delta_{n+1} \Longleftrightarrow \mathbf{UI}\Delta_{n+1}$.
- **(−):** The Parameter Free Paris–Friedman's Conjecture: $\mathbf{L}\Delta^-_{n+1} \Longleftrightarrow \mathbf{I}\Delta^-_{n+1}$.

Recently, T. Slaman (see [19]) has obtained a partial answer. He has proved that

$$\mathbf{L}\Delta_{n+1} + \mathbf{exp} \Longleftrightarrow \mathbf{I}\Delta_{n+1} + \mathbf{exp}.$$

On the other hand, L. Beklemishev (see [4]) has proved that $\mathbf{I}\Delta_1 + \mathbf{exp}$ is a $\Sigma_3$–conservative extension of $\mathbf{UI}\Delta_1 + \mathbf{exp}$; hence, $\mathbf{UL}\Delta_1 + \mathbf{exp} \Longleftrightarrow \mathbf{UI}\Delta_1 + \mathbf{exp}$. Beklemishev's result seems to be easily extended to $n \geq 1$; so, only the case $n = 0$ seems to be open in the two first problems. However, Slaman's proof rests on the equivalence between $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{L}\Delta_{n+1}$; therefore it can not be adapted to the parameter free problem.

## 2. The theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$

Through this paper $\mathbf{T}$ will denote a consistent theory in the first–order language of Arithmetic. For such a theory we introduce the classes of formulas

$$\Delta_{n+1}(\mathbf{T}) = \{\varphi(x,\vec{v}) \in \Sigma_{n+1} : \text{ there exists } \psi(x,\vec{v}) \in \Pi_{n+1},\ \mathbf{T} \vdash \varphi \leftrightarrow \psi\}.$$

When the schemes of induction and minimization are restricted to these classes of formulas we obtain the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$. We also consider the following version of the collection schemes

$$\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) = \mathbf{I}\Delta_0 + \{\mathbf{B}_{\varphi,x,y}(z,\vec{v}) : \ \varphi \in \Pi_n,\ \exists y\,\varphi(x,y,\vec{v}) \in \Delta_{n+1}(\mathbf{T})\}.$$

*Remark* 2.1. We shall begin with some basic properties of the theories introduced above. First we observe that $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$.

If $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ then $\varphi \leftrightarrow \psi$ is a $\Pi_{n+2}$–formula. So, it follows that (a similar result holds for minimization and collection)

**Claim 2.2.** *If* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}')$ *then* $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}')$.

Let $\Delta^*_{n+1}(\mathbf{T})$ be the dual class of $\Delta_{n+1}(\mathbf{T})$. Since the negation of a $\Delta_{n+1}(\mathbf{T})$ formula (that is, a $\Delta^*_{n+1}(\mathbf{T})$–formula) is equivalent (in $\mathbf{T}$) to a $\Delta_{n+1}(\mathbf{T})$– formula, as in the proof of $\mathbf{I}\Pi_{n+1} \Longleftrightarrow \mathbf{I}\Sigma_{n+1}$ (see lemma **7.5** in [12]), we get that

**Claim 2.3.** $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta^*_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

For each $\psi(x,y) \in \Pi_{n-1}$, $\exists y\,[\psi(x,y) \vee (\neg\exists z\,\psi(x,z) \wedge y = 0)] \in \Delta_{n+1}(\mathbf{T})$. So, as in the proof of $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{I}\Sigma_n$ (see **I.2.15** in [10]), we obtain

**Claim 2.4.** $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$. *Hence, for* $n \geq 1$, $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{B}\Sigma_n$.

Suppose that $\mathbf{T}$ is an extension of $\mathbf{I}\Sigma_n$. Let $\varphi \in \Pi_n$ and $\psi \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\,\varphi(x,y) \leftrightarrow \forall y\,\psi(x,y)$. Let us consider the following formulas

$$\theta_1(x,w) \equiv x \leq w \wedge \exists u\,[\varphi_{\mu,u}(x,u) \wedge (\forall z)_{x\leq z\leq w}\,\exists y \leq u\,\varphi(z,y)],$$
$$\theta_2(x,w) \equiv x \leq w \wedge \forall y\,\psi(x,y) \wedge \forall u\,[\varphi(x,u) \to (\forall z)_{x\leq z\leq w} \exists y \leq u\,\varphi(z,u)].$$

Then $\mathbf{T} \vdash \theta_1(x,w) \leftrightarrow \theta_2(x,w)$ and $\theta_1 \in \Sigma_{n+1}$ and $\theta_2 \in \Pi_{n+1}$ in $\mathbf{T}$ and in $\mathbf{L}\Delta_{n+1}(\mathbf{T})$. From this, as in lemma **I.2.17** in [10], we obtain that

**Claim 2.5.** *If* $\mathbf{T}$ *is an extension of* $\mathbf{I}\Sigma_n$ *then* $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

**Definition 2.6.** *(*$\Delta_{n+1}$ **properties***) We say that*

1. $\mathbf{T}$ *is* $\Delta_{n+1}$*–closed if* $\Delta_{n+1}(\mathbf{T})$ *is closed in* $\mathbf{T}$ *under bounded quantifiers.*
2. $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection if* $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.
3. $\mathbf{T}$ *has* $\Delta_{n+1}$*–minimization if* $\mathbf{T} \Longrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.
4. $\mathbf{T}$ *has* $\Delta_{n+1}$*–induction if* $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.
5. $\mathbf{T}$ *is* $\Delta_{n+1}$*–PF if* $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.

*Remark* 2.7. Let us consider some examples of theories having $\Delta_{n+1}$ properties. Since $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, we get that every theory extending $\mathbf{B}\Sigma_{n+1}$ has $\Delta_{n+1}$–collection. Now we improve this result.

**Claim 2.8.** *If* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma^-_{n+1}$ *then* $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection.*

*Proof of Claim.* Let $\varphi(x, y, v_1, \dots, v_m) \in \Pi^-_n$ and $\psi(x, w, \vec{v}) \in \Sigma^-_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \forall w\, \psi(x, w, \vec{v})$. Let $\theta(x, y) \in \Sigma^-_{n+1}$ be

$$\varphi((x)_0, y, (x)_1, \dots, (x)_m) \vee [y = 0 \wedge \neg\forall w\, \psi((x)_0, w, (x)_1, \dots, (x)_m)].$$

Since $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x, y)$, $\mathbf{T} \vdash \forall z\, \exists u\, \forall x \le z\, \exists y \le u\, \theta(x, y)$. Let $\mathfrak{A} \models \mathbf{T}$ and $a, \vec{b} \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall x \le a\, \exists y\, \varphi(x, y, \vec{b})$ and $c = \langle a, \vec{b}\rangle$. Then there exists $d \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall x \le c\, \exists y \le d\, \theta(x, y)$. Since $a \le c$, then $\mathfrak{A} \models \forall x \le a\, \exists y \le d\, \varphi(x, y, \vec{b})$; hence, $\mathfrak{A} \models \mathbf{B}_\varphi$, as required.

$\square$

There exist theories, e.g. $\mathbf{I}\Sigma_n$ (see 2.17), that have $\Delta_{n+1}$–collection and are not extension of $\mathbf{B}\Sigma^-_{n+1}$. Now we present a case in which both conditions are equivalent.

**Claim 2.9.** *If* $\mathbf{T}$ *is complete and has* $\Delta_{n+1}$*–collection then* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma^-_{n+1}$.

*Proof of Claim.* Let $\mathfrak{A} \models \mathbf{T}$ and $\theta(x, y) \in \Sigma^-_{n+1}$ such that $\mathfrak{A} \models \forall x\, \exists y\, \theta(x, y)$. Since $\mathbf{T}$ is complete, $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x, y)$; so, $\exists y\, \theta(x, y) \in \Delta_{n+1}(\mathbf{T})$. Since $\mathbf{T}$ has $\Delta_{n+1}$–collection, $\mathfrak{A} \models \mathbf{B}_\theta$; hence, $\mathfrak{A} \models \forall z\, \exists u\, \forall x \le z\, \exists y \le u\, \theta(x, y)$. $\square$

Next result was, chronologically, the main reason to introduce the theory $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. This theory became one of the main tools in this work once we came to the concept of $\Pi_n$–functional theory (see subsection 3.1).

**Theorem 2.10.** (1) *If* $\mathbf{T}$ *is* $\Delta_{n+1}$*–closed then* $\mathbf{T}$ *is* $\Delta_{n+1}$*–PF.*
(2) *If* $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection then* $\mathbf{T}$ *is* $\Delta_{n+1}$*–closed.*

*Proof.* (*(1)*): By 2.3, it is enough to see that $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$. Suppose that there exist $\mathfrak{A} \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\varphi(x) \in \Delta_{n+1}(\mathbf{T})$ such that $\mathfrak{A} \models \exists x\, \varphi(x) \wedge \forall x\, \neg\varphi_\mu(x)$. Let $\theta(z) \in \Pi_{n+1}$ be $\forall x \le z\, \neg\varphi(x)$. We have that $\mathfrak{A} \models \theta(0) \wedge [\theta(z) \to \theta(z + 1)]$. Since $\mathbf{T}$ is $\Delta_{n+1}$–closed, $\theta(z) \in \Delta^*_{n+1}(\mathbf{T})$. By 2.3, $\mathfrak{A} \models \mathbf{I}\Delta^*_{n+1}(\mathbf{T})$; so, $\mathfrak{A} \models \forall x\, \neg\varphi(x)$, a contradiction.
(*(2)*): Let $\varphi(x, y) \in \Pi_n$, $\psi(x, y) \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y) \leftrightarrow \forall y\, \psi(x, y)$. By the closure properties under bounded quantification of $\mathbf{B}\Sigma_n$, there exists $\theta(z) \in \Sigma_{n+1}$ such that $\mathbf{B}\Sigma_n \vdash \theta(z) \leftrightarrow \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x, y)$ (for $n = 0$, we do not need $\mathbf{B}\Sigma_n$). The following equivalences hold in the given theories.

$$\begin{array}{rcll} \forall x \le z\, \forall y\, \psi(x, y) & \leftrightarrow & \forall x \le z\, \exists y\, \varphi(x, y) & [\text{in } \mathbf{T}] \\ & \leftrightarrow & \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x, y) & [\text{in } \mathbf{B}^*\Delta_{n+1}(\mathbf{T})] \\ & \leftrightarrow & \theta(z) & [\text{in } \mathbf{B}\Sigma_n] \end{array}$$

Since $\mathbf{T}$ has $\Delta_{n+1}$–collection, all the above equivalences hold in $\mathbf{T}$. Then, as $\forall x \le z\, \forall y\, \psi(x, y) \in \Pi_{n+1}$, $\theta(z) \in \Delta_{n+1}(\mathbf{T})$. So, $\forall x \le z\, \exists y\, \varphi(x, y)$ is equivalent in $\mathbf{T}$ to a $\Delta_{n+1}(\mathbf{T})$ formula. $\square$

*Remark* 2.11. Now we describe others elementary relations among $\Delta_{n+1}$ properties of a theory.

**Claim 2.12.** *If* $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection then* $\mathbf{T}$ *has* $\Delta_{n+1}$*–induction.*

*Proof of Claim.* Suppose that $\mathbf{T} \vdash \exists y\, \varphi(x,y) \leftrightarrow \forall y\, \psi(x,y)$, where $\varphi(x,y) \in \Pi_n$, $\psi(x,y) \in \Sigma_n$. Let $\theta(x,y) \in \Pi_n$ be $\varphi(x,y) \vee \neg\psi(x,y)$. Since $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x,y)$, then $\exists y\, \theta(x,y) \in \Delta_{n+1}(\mathbf{T})$. Now the proof continues as in 2.4. □

**Claim 2.13.** *The following conditions are equivalent*

*(i)* $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection.*

*(ii)* $\mathbf{T}$ *is* $\Delta_{n+1}$*–closed and has* $\Delta_{n+1}$*–induction.*

*(iii)* $\mathbf{T}$ *has* $\Delta_{n+1}$*–minimization.*

*Proof of Claim.* *(i)* $\Longrightarrow$ *(ii)* is 2.10 and 2.12. *(ii)* $\Longrightarrow$ *(iii)* follows from 2.10–*(1)*.
(*(iii)* $\Longrightarrow$ *(i)*): Suppose that $\mathbf{T}$ has $\Delta_{n+1}$–minimization. Then $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$; so, by 2.5, $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. Hence, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. □

For each model $\mathfrak{A}$, $\mathbf{Th}(\mathfrak{A})$ has $\Delta_{n+1}$–collection if and only if $\mathfrak{A} \models \mathbf{UB}\Delta_{n+1}$ (or $\mathfrak{A} \models \mathbf{B}\Sigma^-_{n+1}$, see 2.9); and $\mathbf{Th}(\mathfrak{A})$ has $\Delta_{n+1}$–minimization if only if $\mathfrak{A} \models \mathbf{UL}\Delta_{n+1}$. So, as a consequence of 2.13, we obtain that

**Claim 2.14.** $\mathbf{B}\Sigma^-_{n+1} \Longleftrightarrow \mathbf{UB}\Delta_{n+1} \Longleftrightarrow \mathbf{UL}\Delta_{n+1}$.

*Remark* 2.15 *(*$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ *and* $\Delta_{n+1}$ *properties).* Here we shall see that a theory $\mathbf{T}$ has a $\Delta_{n+1}$–property if and only if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ has that property. This is easily seen for $\Delta_{n+1}$–closed. Now we consider $\Delta_{n+1}$–induction.

**Claim 2.16.** $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$ *if and only if* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

*Proof of Claim.* Let $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ such that $\mathbf{T} \vdash \varphi \leftrightarrow \psi$. Let $\mathbf{I}_{\varphi,\psi}$ be

$$\psi(0) \wedge \forall x\, [\varphi(x) \rightarrow \psi(x+1)] \rightarrow \forall x\, \psi(x).$$

Then, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_\varphi \leftrightarrow \mathbf{I}_{\varphi,\psi}$. Suppose that $\mathbf{T}$ has $\Delta_{n+1}$–induction, then $\mathbf{T} \vdash \mathbf{I}_\varphi$; hence, $\mathbf{T} \vdash \mathbf{I}_{\varphi,\psi}$. Since $\mathbf{I}_{\varphi,\psi} \in \Pi_{n+2}$, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_{\varphi,\psi}$; so, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_\varphi$, as required. □

From this, 2.13 and 2.10 we get a similar result for $\mathbf{L}\Delta_{n+1}(\mathbf{T})$; and from 2.5, using again 2.13, also for $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

**Claim 2.17.** *If* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1})$, $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection. So,* $\mathbf{I}\Sigma_n$, $\mathbf{I}\Delta_{n+1}$ *and* $\mathbf{UI}\Delta_{n+1}$ *have* $\Delta_{n+1}$*–collection.*

*Remark* 2.18. Now we study relations between $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{I}\Sigma_n$, $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{B}\Sigma^-_{n+1}$. By 2.17, $\mathbf{I}\Sigma_n$ has $\Delta_{n+1}$–collection; so, by 2.10 and 2.4, it follows that $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_n) \Longleftrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n)$. From this result, 1.3 and 2.2 we get that

$$\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}) \Longleftrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}).$$

Since $\Sigma_0(\Sigma_n) \subseteq \Delta_{n+1}(\mathbf{B}\Sigma_{n+1})$, the above result gives a generalization of $\mathbf{I}\Sigma_n \iff \mathbf{I}\Sigma_0(\Sigma_n)$ (see [10], theorem **I.2.4**).

Let $\mathbf{T}$ be a theory consistent with $\mathbf{I}\Sigma_{n+1}$, $\mathfrak{A} \models \mathbf{T}+\mathbf{I}\Sigma_{n+1}$ and $a \in \mathfrak{A}$ nonstandard. Then, by 1.5, $\mathcal{K}_{n+1}(\mathfrak{A}, a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}+\mathbf{I}\Sigma_{n+1})$; hence, by 2.15, $\mathcal{K}_{n+1}(\mathfrak{A}, a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T}) + \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. So, again by 1.5, it holds that

$$\mathbf{I}\Sigma_{n+1} \models\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{B}\Sigma_{n+1} \models\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T}).$$

In particular, $\mathbf{B}\Sigma_{n+1} \models\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathcal{N})$.

Now assume that there exists $\mathfrak{A} \models \mathbf{T}+\mathbf{I}\Sigma_{n+1}$ such that $\mathcal{K}_{n+1}(\mathfrak{A})$ is not standard. Then, by 1.5–*(2)*, $\mathcal{K}_{n+1}(\mathfrak{A}) \not\models \mathbf{B}\Sigma^-_{n+1}$. So, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{B}\Sigma^-_{n+1}$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{B}\Sigma^-_{n+1}$.

*Remark* 2.19. (**On induction and collection rules**) In this remark we state some relations between $\Delta_{n+1}$–properties and induction and collection rules.

**Claim 2.20.** *The following conditions are equivalent:*

(i) $\mathbf{T}$ *has* $\Delta_{n+1}$*–induction.*

(ii) $\mathbf{T}$ *is closed under* $\Delta_{n+1}$*–IR.*

*Proof of Claim.* The result follows from $[\mathbf{T}, \Delta_{n+1}\text{–IR}] \iff \mathbf{T} + \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

□

**Claim 2.21.** *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Delta_0$. *Then*

$$[\mathbf{T}, \Sigma_{n+1}\text{–}CR] \iff \mathbf{T} + \mathbf{B}^*\Delta_{n+1}(\mathbf{T}).$$

*Proof of Claim.* ($\Longleftarrow$): Let $\varphi(x,y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x,y)$. Then $\exists y\, \varphi(x,y) \in \Delta_{n+1}(\mathbf{T})$. Hence, $\mathbf{T} + \mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall z\, \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x,y)$.

($\Longrightarrow$): Let $\varphi(x,y) \in \Pi_n$ and $\psi(x,y) \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x,y) \leftrightarrow \forall y\, \psi(x,y)$. Then $\mathbf{T} \vdash \forall x\, \exists y\, (\varphi(x,y) \vee \neg\psi(x,y))$. So,

$$[\mathbf{T}, \Sigma_{n+1}\text{–CR}] \vdash \forall z\, \exists u\, \forall x \le z\, \exists y \le u\, (\varphi(x,y) \vee \neg\psi(x,y)).$$

Hence, $[\mathbf{T}, \Sigma_{n+1}\text{–CR}] \vdash \forall x \le z\, \exists y\, \varphi(x,y) \to \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x,y)$.

□

**Claim 2.22.** *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Delta_0$. *The following conditions are equivalent:*

(i) $\mathbf{T}$ *has* $\Delta_{n+1}$*–collection.*

(ii) $\mathbf{T}$ *is closed under* $\Sigma_{n+1}$*–CR.*

*Proof of Claim.* The result follows from 2.21.

□

## 3. Functional character of $\Delta_{n+1}$ properties

In this section we shall see that the $\Delta_{n+1}$–properties of a theory $\mathbf{T}$ are connected with descriptions of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ using $\mathbf{I}\Sigma_n$ and a class of $\Pi_n$ functions.

In what follows $\Gamma$ will denote a class of formulas of $\mathcal{L}$ with two free variables, $x$, $y$ say. For a formula $\varphi(x,y)$, the conjunction of

**(–):** $\forall x\,\forall y_1\,\forall y_2\,[\varphi(x,y_1)\wedge\varphi(x,y_2)\to y_1=y_2]$, and

**(–):** $\forall x_1\,\forall x_2\,\forall y_1\,\forall y_2\,[x_1\le x_2\wedge\varphi(x_1,y_1)\wedge\varphi(x_2,y_2)\to y_1\le y_2]$,

will be denoted by $\mathrm{IPF}(\varphi)$. That is, $\mathrm{IPF}(\varphi)$ expresses that $\varphi(x,y)$ defines an increasing partial function. Let $\mathrm{IPF}(\Gamma)=\{\mathrm{IPF}(\varphi(x,y)):\ \varphi(x,y)\in\Gamma\}$, $\mathrm{Func}(\Gamma)=\{\forall x\,\exists! y\,\varphi(x,y):\ \varphi\in\Gamma\}$ and $\Gamma^*=\mathrm{Func}(\Gamma)+\mathrm{IPF}(\Gamma)$. Let us observe that for $\Gamma\subseteq\Pi_n$, $\Gamma^*\subseteq\Pi_{n+2}$. For a theory $\mathbf{T}$ let $\mathrm{Func}_n(\mathbf{T})=\mathrm{Func}(\mathrm{Gr}_n(\mathbf{T}))$, where $\mathrm{Gr}_n(\mathbf{T})=\{\varphi(x,y)\in\Pi_n^-:\ \mathbf{T}\vdash\forall x\,\exists! y\,\varphi(x,y)\}$.

*Remark* 3.1 *(The language $\mathcal{L}(\Gamma)$).* Let $\mathcal{L}(\Gamma)$ denotes the extension of $\mathcal{L}$ obtained by adding a function symbol $G_\varphi$ for each $\varphi\in\Gamma$. Let $\Delta_0^\Gamma$ be the class of bounded formulas of $\mathcal{L}(\Gamma)$. The classes $\Sigma_n^\Gamma$ and $\Pi_n^\Gamma$, $n\in\omega$, are defined as usually. Let us consider the following theories of language $\mathcal{L}(\Gamma)$.

$$\mathbf{I}\Delta_0^\Gamma=\mathbf{P}^-+\{\mathbf{I}_\varphi:\ \varphi\in\Delta_0^\Gamma\}+\{\varphi(x,y)\leftrightarrow G_\varphi(x)=y:\ \varphi\in\Gamma\},$$
$$\mathbf{I}\Delta_0^{\Gamma^*}=\mathbf{I}\Delta_0^\Gamma+\mathrm{IPF}(\Gamma).$$

Then $\mathbf{I}\Delta_0^\Gamma\iff\mathbf{L}\Delta_0^\Gamma$ and, for $\Gamma\subseteq\Pi_n$, $\mathbf{I}\Delta_0^\Gamma$ and $\mathbf{I}\Delta_0^{\Gamma^*}$ are $\Pi_{n+1}^\Gamma$–axiomatizable.

In general, if $\mathbf{T}$ is a theory in the language of Arithmetic, $\mathbf{T}_\Gamma$ will denote the extension of $\mathbf{T}$ to $\mathcal{L}(\Gamma)$ obtained by adding to $\mathbf{T}$, as new axioms, the formulas $\varphi(x,y)\leftrightarrow G_\varphi(x)=y$, for each $\varphi\in\Gamma$. Observe that $(\mathbf{T}+\mathrm{Func}(\Gamma))_\Gamma$ is a conservative extension of $\mathbf{T}+\mathrm{Func}(\Gamma)$. It holds that

**Claim 3.2.** (i) *If $t(x)\in\mathbf{Term}(\mathcal{L}(\Gamma))$ then $(\mathbf{I}\Delta_0+\Gamma^*)_\Gamma\vdash x\le x'\to t(x)\le t(x')$.*

(ii) *Let $t(\vec{x})\in\mathbf{Term}(\mathcal{L}(\Gamma))$. There is $\varphi(\vec{x},y)\in\Delta_{n+1}(\mathbf{I}\Delta_0+Func(\Gamma))$ such that $(\mathbf{I}\Delta_0+Func(\Gamma))_\Gamma\vdash t(\vec{x})=y\leftrightarrow\varphi(\vec{x},y)$.*

*Remark* 3.3. By a standard argument on contraction of quantifiers (see [12]) for each $\varphi(\vec{x},\vec{y})\in\Pi_n^-$ there exists $\varphi^c(u,v,\vec{x},\vec{y})\in\Pi_n^-$ such that

**(–):** $\mathbf{I}\Delta_0\vdash\forall\vec{x}\,\exists\vec{y}\,\varphi(\vec{x},\vec{y})\leftrightarrow\forall u\,\exists v\,\forall\vec{x}\,\forall\vec{y}\,\varphi^c(u,v,\vec{x},\vec{y})$.

**(–):** $\mathbf{I}\Delta_0\vdash\varphi^c(u,v,\vec{x},\vec{y})\to\vec{x}\le u\wedge\vec{y}\le v$.

**Claim 3.4.** *Let $\psi(x,y)\in\Pi_n^-$. There exists $\psi_f(x,y)\in\Pi_n^-$ such that*

*(i)* $\mathbf{I}\Sigma_n\vdash\psi_f(x,y_1)\wedge\psi_f(x,y_2)\to y_1=y_2$.

*(ii)* $\mathbf{I}\Sigma_n\vdash\exists y\,\psi(x,y)\to\exists y\,\psi_f(x,y)$.

*(iii)* $\mathbf{I}\Sigma_n\vdash\psi_f(x,y)\to\exists z\le y\,\psi(x,z)$.

*Proof of Claim.* For $n=0$, let $\psi_f(x,y)$ be the formula $\psi(x,y)\wedge\forall z<y\,\neg\psi(x,z)$. It is clear that $\psi_f(x,y)$ satisfies the claim. For $n\ge 1$, let $\psi_1(x,y,z)\in\Sigma_{n-1}$ such that $\psi(x,y)$ is $\forall z\,\psi_1(x,y,z)$. Let $\psi_f(x,y)$ be the following formula

$$\mathrm{Seq}(y)\wedge\psi(x,\mathrm{lg}(y))\wedge\forall j<\mathrm{lg}(y)\,[\neg\psi_1(x,j,(y)_j)\wedge\forall z<(y)_j\,\psi_1(x,j,z)].$$

To prove *(ii)* follow the proof of $\mathbf{I}\Sigma_n\Longrightarrow\mathbf{FAC}(\Sigma_n)$ (see lemma **I.2.35** in [10]). Parts *(i)* and *(iii)* are easy. □

From this result, using contraction of quantifiers, it follows that

**Claim 3.5.** *If $\mathbf{T}\Longrightarrow\mathbf{I}\Sigma_n$ then $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})=\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n+Func_n(\mathbf{T}))$.*

Below we shall see that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ can be described, for some theories, using $\mathbf{I}\Sigma_n$ and a family of $\Pi_n$–functions, $\Gamma \subseteq \mathrm{Gr}_n(\mathbf{T})$. In section 3.1, we prove that if $\mathbf{T}$ has $\Delta_{n+1}$–collection then the functions in $\Gamma$ are nondecreasing. In section 3.2, we shall see that if $\mathbf{T}$ is $\Delta_{n+1}$–closed then every $\Delta_{n+1}(\mathbf{T})$ formula is equivalent to a bounded formula of $\mathcal{L}(\mathrm{Gr}_n(\mathbf{T}))$.

### 3.1. $\Pi_n$–functional classes.

**Definition 3.6.** (1) *Let $\Gamma \subseteq \Pi_n$. We say that $\Gamma$ is a $\Pi_n$–functional class if $\mathbf{I}\Sigma_n + \Gamma^*$ is consistent.*

(2) *Let $\mathbf{T}$ be a theory. We say that $\mathbf{T}$ is $\Pi_n$–functional if there exists a $\Pi_n$–functional class $\Gamma$ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*)$. In this case we say that $\Gamma$ is a $\Pi_n$–functional class for $\mathbf{T}$.*

Let us notice that if $\Gamma$ is a $\Pi_n$–functional class for $\mathbf{T}$, then, by 2.2, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$; and, since $\mathbf{I}\Sigma_n + \Gamma^*$ is $\Pi_{n+2}$ axiomatizable, $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n + \Gamma^*$.

**Lemma 3.7.** *Let $\Gamma$ be a $\Pi_n$–functional class. Then*

(1) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*)$.

(2) $\mathbf{I}\Sigma_n + \Gamma^*$ *has $\Delta_{n+1}$–collection. So, it has $\Delta_{n+1}$–induction.*

*Proof.* (*(1)*): Let $\theta(x, y) \in \Sigma_n$ such that $\mathbf{B}\Sigma_{n+1} + \Gamma^* \vdash \forall x\, \exists y\, \theta(x, y)$. Let us suppose that $\mathbf{I}\Sigma_n + \Gamma^* \nvdash \forall x\, \exists y\, \theta(x, y)$. Let $\mathfrak{A} \models \mathbf{I}\Sigma_n + \Gamma^* + \neg\forall x\, \exists y\, \theta(x, y)$ and $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall y\, \neg\theta(a, y)$. Let

$$\mathbf{T} = \mathbf{ED}(\mathfrak{A}) + \{\exists! x\, \psi(x, a) \rightarrow \exists x < \mathbf{c}\, \psi(x, a) : \ \psi(x, y) \in \Sigma_{n+1}\}$$

(where $\mathbf{ED}(\mathfrak{A})$ is the elementary diagram of $\mathfrak{A}$ and $\mathbf{c}$ is a new constant symbol). By compactness, $\mathbf{T}$ is consistent. Let $\mathfrak{B} \models \mathbf{T}$. Then $\mathfrak{A} \prec \mathfrak{B}$, $\mathfrak{B} \models \mathbf{I}\Sigma_n + \Gamma^* + \neg\forall x\, \exists y\, \theta(x, y)$ and, by 1.5, $\mathcal{I}_{n+1}(\mathfrak{B}, a) \prec_n^e \mathfrak{B}$ and it is proper. So, $\mathcal{I}_{n+1}(\mathfrak{B}, a) \models \mathbf{B}\Sigma_{n+1} + \Gamma^*$. Since $\mathcal{I}_{n+1}(\mathfrak{B}, a) \models \forall y\, \neg\theta(a, y)$, this gives the desired contradiction.

(*(2)*): It follows from *(1)*, 2.8 and 2.15. □

**Theorem 3.8.** *Let $\mathbf{T}$ be a consistent theory. Then*

$$\mathbf{T} \text{ has } \Delta_{n+1}\text{–collection} \quad \Longleftrightarrow \quad \mathbf{T} \text{ is } \Pi_n\text{–functional.}$$

*Proof.* ($\Longleftarrow$): Let $\Gamma$ be a $\Pi_n$–functional class for $\mathbf{T}$. Hence, by 3.7–*(2)*, since $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*)$, $\mathbf{T}$ has $\Delta_{n+1}$–collection.

($\Longrightarrow$): Let

$$\Gamma = \{\varphi(x, y) \in \Pi_n^- : \ \mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y), \ \mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\varphi)\}.$$

Let us see that $\Gamma$ is a $\Pi_n$–functional class for $\mathbf{T}$. It is enough to prove that for all $\theta(x, y) \in \Pi_n^-$ if $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x, y)$ then $\mathbf{I}\Sigma_n + \Gamma^* \vdash \forall x\, \exists y\, \theta(x, y)$. We consider the following cases.

<u>Case 1</u>: $n = 0$: Let $\mathcal{C}_\theta(x, y)$ be the formula

$$\forall u \leq x\, \exists v \leq y\, \theta(u, v) \wedge \forall w < y\, \exists u \leq x\, \forall v \leq w\, \neg\theta(u, v).$$

That is, $y = \max(\{v : \ \exists u \leq x\, [v = (\mu z)(\theta(u, z))]\})$. We have that *(i)* $\mathbf{I}\Delta_0 \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y) \to \forall x\, \exists y\, \theta(x,y)$,

*(ii)* $\mathbf{I}\Delta_0 \vdash \mathrm{IPF}(\mathcal{C}_\theta)$, and

*(iii)* $\mathbf{B}^*\Delta_1(\mathbf{T}) \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y) \leftrightarrow \forall x\, \exists y\, \theta(x,y)$. (Since $\exists y\, \theta(x,y) \in \Delta_1(\mathbf{T})$).

Since $\mathbf{T}$ has $\Delta_1$–collection and $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x,y)$, by *(iii)*, $\mathbf{T} \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y)$; hence, by *(ii)*, $\mathcal{C}_\theta(x,y) \in \Gamma$. So, $\mathbf{I}\Delta_0 + \Gamma^* \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y)$, and, by *(i)*, $\mathbf{I}\Delta_0 + \Gamma^* \vdash \forall x\, \exists y\, \theta(x,y)$, as required.

Case 2: $n \geq 1$: Since $\mathbf{T}$ has $\Delta_{n+1}$–collection, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$; hence, $(1 \leq n)$ we can use predicates and functions associated to Gödel's $\beta$ function. For example, we will use the following $\Delta_1(\mathbf{I}\Sigma_1)$ formulas: $\mathrm{Seq}(x)$: "$x$ is a sequence"; $\mathrm{lg}(x) = y$: "$y$ is the length of $x$"; $(x)_j = y$: "$y$ is the $j$–th projection of $x$". Let $\theta'(x,y,z) \in \Sigma_{n-1}$ such that $\theta(x,y)$ is $\forall z\, \theta'(x,y,z)$. Let $\mathcal{C}_\theta(x,y)$ be the formula

$$\left\{\begin{array}{l} \mathrm{Seq}(y) \wedge \mathrm{lg}(y) = 2 \wedge \mathrm{Seq}((y)_1) \wedge \mathrm{lg}((y)_1) = (y)_0 \wedge \\ \forall w < (y)_0\, [\mathrm{Seq}((y)_{1,w}) \wedge \mathrm{lg}((y)_{1,w}) = 2 \wedge (y)_{1,w,0} \leq x] \wedge \\ \forall u \leq x\, \exists v \leq (y)_0\, \forall z\, \theta'(u,v,z) \wedge \\ \forall w < (y)_0 \left\{\begin{array}{l} \forall u < (y)_{1,w,0}\, \exists v \leq w\, \forall z\, \theta'(u,v,z) \wedge \\ \forall v \leq w\, \exists z \leq (y)_{1,w,1}\, \neg\theta'((y)_{1,w,0}, v, z) \wedge \\ \forall t < (y)_{1,w,1}\, \exists v \leq w\, \forall z \leq t\, \theta'((y)_{1,w,0}, v, z) \end{array}\right. \end{array}\right.$$

We give an informal description of $\mathcal{C}_\theta(x,y)$. The interpretation of the first two lines is clear. The other parts of $\mathcal{C}_\theta(x,y)$ are developed to get

$$(y)_0 = (\mu w)[\forall u \leq x\, \exists v \leq w\, \forall z\, \theta'(u,v,z)]$$

$$\forall w < (y)_0 \left\{\begin{array}{l} (y)_{1,w,0} = (\mu u)[\neg\exists v \leq w\, \forall z\, \theta'(u,v,z)] \\ (y)_{1,w,1} = (\mu t)[\forall v \leq w\, \exists z \leq t\, \neg\theta'((y)_{1,w,0}, v, z)] \end{array}\right.$$

Since $\mathcal{C}_\theta(x,y) \in \Pi_n(\mathbf{B}\Sigma_n)$, $\mathcal{C}_\theta(x,y) \in \Pi_n(\mathbf{T}), \Pi_n(\mathbf{I}\Sigma_n)$. We also have that

*(i)* $\mathbf{I}\Sigma_n \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y) \to \forall x\, \exists y\, \theta(x,y)$,

*(ii)* $\mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\mathcal{C}_\theta)$, and

*(iii)* $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall x\, \exists y\, \mathcal{C}_\theta(x,y) \leftrightarrow \forall x\, \exists y\, \theta(x,y)$.

The proofs of *(i)* and *(ii)* are trivial, see the informal description of $\mathcal{C}_\theta(x,y)$ given above. To prove *(iii)* it is enough to see that

**Claim 3.9.** $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall x\, \exists y\, \theta(x,y) \to \forall x\, \exists y\, \mathcal{C}_\theta(x,y)$.

*Proof of Claim.* Let $\mathfrak{A} \models \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ such that $\mathfrak{A} \models \forall x\, \exists y\, \theta(x,y)$ and let $a \in \mathfrak{A}$. Since $\exists y\, \theta(x,y) \in \Delta_{n+1}(\mathbf{T})$, $\mathfrak{A} \models \exists y\, \forall u \leq a\, \exists v \leq y\, \theta(u,v)$. So, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models b = (\mu y)[\forall u \leq a\, \exists v \leq y\, \theta(u,v)]$. For each $d < b$ let

$$c_d = (\mu u)[u \leq a \wedge \neg\exists v \leq d\, \theta(u,v)].$$

So, for every $d < b$, $\mathfrak{A} \models \exists t\, \forall v \leq d\, \exists z \leq t\, \neg\theta'(c_d, v, z)$. Let

$$e_d = (\mu t)[\forall v \leq d\, \exists z \leq t\, \neg\theta'(c_d, v, z)].$$

In what follows we shall see that the elements $\langle c_0, e_0\rangle, \ldots, \langle c_{b-1}, e_{b-1}\rangle$ can be given as a sequence. Let $\varphi(w,p,x) \in \Pi_n(\mathbf{B}\Sigma_n)$ be the formula

$$\left\{\begin{array}{l} \mathrm{Seq}(p) \wedge \mathrm{lg}(p) = 2 \wedge (p)_0 \leq x \wedge \forall u < (p)_0\, \exists v \leq w\, \forall z\, \theta'(u,v,z) \wedge \\ \forall v \leq w\, \exists z \leq (p)_1 \neg\, \theta'((p)_0, v, z) \wedge \forall t < (p)_1\, \exists v \leq w\, \forall z \leq t\, \theta'((p)_0, v, z) \end{array}\right.$$

We have that $\mathfrak{A} \models \forall w < b\, \exists p\, \varphi(w,p,a)$ (if $w < b$, take $p = \langle c_w, e_w\rangle$). Now let $\Psi(x, y', w, p) \in \Sigma_{n+1}(\mathbf{B}\Sigma_n)$ be the formula

$$[y' \le w \wedge p = 0] \vee \begin{cases} [\exists y < y' \, \forall u \le x \, \exists v \le y \, \theta(u,v) \wedge p = 0] \vee \\ [\exists u \le x \, \forall v \le y' \, \neg\theta(u,v) \wedge p = 0] \vee \\ \varphi(w,p,x) \end{cases}$$

Since $\exists p \, \Psi(x, y', w, p) \in \Delta_{n+1}(\mathbf{T})$ and $\mathfrak{A} \models \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, then there exists $\tilde{q} \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall w < b \, \exists p \le \tilde{q} \, \Psi(a, b, w, p)$. Let $s \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathrm{Seq}(s) \wedge \mathrm{lg}(s) = b \wedge \forall j < b \, [(s)_j = \tilde{q}]$. Let $\delta(x, a, s, b) \in \Pi_n(\mathbf{B}\Sigma_n)$ be the formula $a < x \vee \exists y \le \langle b, s\rangle \, \mathcal{C}_\theta(x, y)$. Then

$$\mathfrak{A} \models \delta(0, a, s, b) \wedge [\delta(x, a, s, b) \to \delta(x+1, a, s, b)].$$

So, $\mathfrak{A} \models \forall x \, \delta(x, a, s, b)$; hence, $\mathfrak{A} \models \exists y \le \langle b, s\rangle \, \mathcal{C}_\theta(a, y)$. □

From *(i)*–*(iii)*, as in case $n = 0$, it follows $\mathbf{I}\Sigma_n + \Gamma^* \vdash \forall x \, \exists y \, \theta(x, y)$. □

**Corollary 3.10.** *Let $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$. The following conditions are equivalent:*

(1) *$\mathbf{T}$ is $\Pi_n$–functional.*
(2) *Every total $\Pi_n$–function of $\mathbf{T}$ is bounded by a total increasing function; that is, for every $\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x \, \exists! y \, \varphi(x, y)$ there exists $\mathcal{C}_\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x \, \exists y \, \mathcal{C}_\varphi(x, y) \wedge IPF(\mathcal{C}_\varphi)$ and $\mathbf{T} \vdash \mathcal{C}_\varphi(x, y) \to \exists y' \le y \, \varphi(x, y')$.*

*Remark* 3.11. Here we study the relationship among being $\Pi_n$–functional, axiomatization and conservativeness, for consistent theories.

**Claim 3.12.** *Let $\mathbf{T}$ be such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Then*

(i) *$\mathbf{T}$ is $\Pi_n$–functional.*
(ii) *If $\mathbf{T}'$ is $\Sigma_{n+2}$–axiomatizable then*

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T} + \mathbf{T}') = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{T}').$$

*Proof of Claim.* (*(i)*): By 2.8, $\mathbf{B}\Sigma_{n+1} + \mathbf{T}$ has $\Delta_{n+1}(\mathbf{T})$–collection; so, by the hypothesis, **2.15** and 3.8, $\mathbf{T}$ is $\Pi_n$–functional.
(*(ii)*): Follows from $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. □

**Claim 3.13.** *Let $\mathbf{T}$ be $\Pi_{n+2}$–axiomatizable and let $\mathbf{T}'$ be $\Sigma_{n+2}$–axiomatizable.*

(i) *$\mathbf{T}$ is $\Pi_n$–functional if and only if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*
(ii) *If $\mathbf{T}$ is $\Pi_n$–functional and $\mathbf{T} + \mathbf{T}'$ is consistent, $\mathbf{T} + \mathbf{T}'$ is $\Pi_n$–functional.*

*Proof of Claim.* *(ii)* follows from *(i)* and 3.12. Let us see *(i)*. Let $\Gamma$ be a $\Pi_n$–functional class for $\mathbf{T}$. Then

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}).$$

Where the first identity follows from 3.7–*(1)* and the last one, since $\mathbf{T}$ is $\Pi_{n+2}$–axiomatizable, from $\mathbf{T} \Longleftrightarrow \mathbf{I}\Sigma_n + \Gamma^*$. □

**Claim 3.14.** *If $\mathbf{T}$ is $\Sigma_2$–axiomatizable and $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_0$ then $\mathbf{T}$ is $\Pi_0$–functional. In particular, $\mathbf{I}\Pi_1^-$, $\mathbf{Th}_{\Pi_1}(\mathcal{N})$, $\mathbf{I}\Delta_1^-$ and $\mathbf{L}\Delta_1^-$ are $\Pi_0$–functional.*

By a result of D. Leivant (see [16]), for $n \ge 1$, if $\mathbf{T}$ is a sound and $\Sigma_{n+2}$–axiomatizable theory, then $\mathbf{T}$ does not extend $\mathbf{I}\Sigma_n$. By **2.1** in [8], this is also true for any consistent theory. As $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ extends $\mathbf{I}\Sigma_n$ we get that

**Claim 3.15.** ($n \geq 1$) *If* $\mathbf{T}$ *is* $\Sigma_{n+2}$*–axiomatizable and consistent then* $\mathbf{T}$ *is not* $\Pi_n$*–functional. In particular, for* $n \geq 1$, $\mathbf{I\Pi}^-_{n+1}$, $\mathbf{I\Delta}^-_{n+1}$ *and* $\mathbf{L\Delta}^-_{n+1}$ *are not* $\Pi_n$*–functional.*

*Remark* 3.16. Now we give some examples of $\Pi_{n+2}$–axiomatizable theories which do not have $\Delta_{n+1}$–collection. Suppose that $n \geq 1$, $\mathbf{T} \Longrightarrow \mathbf{I\Sigma}^-_n$ and $\mathbf{Th}_{\Pi_{n+1}}(\mathbf{T}) \neq \mathbf{Th}_{\Pi_{n+1}}(\mathcal{N})$. Then (see [9], theorem **3.7**), it holds that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \not\Longrightarrow \mathbf{L\Delta}^-_{n+1}$. Hence, there exist $\varphi(x) \in \Sigma^-_n$ and $\psi(x) \in \Pi^-_n$ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \nvdash \forall x\,(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \mathbf{L}_\varphi$. Let $\theta \in \Pi_{n+2}$ be the sentence $\forall x\,(\varphi(x) \leftrightarrow \psi(x))$. Then $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \theta$ does not have $\Delta_{n+1}$–minimization; hence, it does not have $\Delta_{n+1}$–collection. From this we get:

**Claim 3.17.** *(*$n \geq 1$*) Every* $\Sigma_1$*–theory,* $\Pi_{n+2}$*–axiomatizable and (consistent) extension of* $\mathbf{I\Sigma}^-_n$ *has a* $\Pi_{n+2}$*–axiomatizable extension that does not have* $\Delta_{n+1}$*–collection.*

We now consider case $n = 0$. By theorem **3.2** in [9], if $\mathbf{T}$ is an extension of $\mathbf{I\Delta}_0$ such that $\mathbf{T} + \mathbf{exp}$ is consistent and $\mathbf{Th}_{\Pi_1}(\mathbf{T} + \mathbf{exp}) \neq \mathbf{Th}_{\Pi_1}(\mathcal{N})$ then $\mathbf{Th}_{\Pi_2}(\mathbf{T}) \not\Longrightarrow \mathbf{L\Delta}^-_1$. This gives the following result (related with some results of Beklemishev (see [2], theorems **6.1** and **6.2**)).

**Claim 3.18.** *Let* $\mathbf{T}$ *be a* $\Sigma_1$*–theory,* $\Pi_2$*–axiomatizable, extension of* $\mathbf{I\Delta}_0$ *such that* $\mathbf{T}+\mathbf{exp}$ *is consistent. Then there exists a* $\Pi_2$*–axiomatizable extension of* $\mathbf{T}$ *which does not have* $\Delta_1$*–collection.*

We now consider $\Delta_{n+1}$–induction. Next result has also been obtained by Beklemishev for $n = 0$ (see [4]).

**Claim 3.19.** *Every theory,* $\mathbf{T}$*,* $\Sigma_{n+1}$*–definable in* $\mathcal{N}$*,* $\Pi_{n+2}$*–axiomatizable and consistent with* $\mathbf{PA} + \mathbf{Th}_{\Pi_n}(\mathcal{N})$ *has a* $\Pi_{n+2}$*–axiomatizable extension which does not have* $\Delta_{n+1}$*–induction.*

*Proof of Claim.* Follows from $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \not\Longrightarrow \mathbf{I\Delta}^-_{n+1}$, see corollary **4.6** in [9]. □

### 3.2. $\Pi_n$–Parikh pairs.

**Definition 3.20.** *Let* $\Gamma \subseteq \Pi_n$ *and* $\Gamma_1 \subseteq \Pi_{n+2}$ *such that Func*$(\Gamma) \subseteq \Gamma_1$*. We say that* $(\Gamma, \Gamma_1)$ *is a* $\Pi_n$*–Parikh pair if* $\mathbf{I\Sigma}_n + \Gamma_1$ *is consistent and*

1. *for all* $\psi(\vec{x}, \vec{y}) \in \Pi_n$ *such that* $\mathbf{I\Sigma}_n + \Gamma_1 \vdash \forall \vec{x}\, \exists \vec{y}\, \psi(\vec{x}, \vec{y})$ *there exists a term of* $\mathcal{L}(\Gamma)$*,* $t(\vec{x})$*, such that* $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \psi(\vec{x}, \vec{y})$*, and*
2. *for all* $\theta(\vec{x}) \in \Delta^\Gamma_0$ *there exists* $\psi(\vec{x}) \in \Delta_{n+1}(\mathbf{I\Sigma}_n + \Gamma_1)$ *such that*
$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \theta(\vec{x}) \leftrightarrow \psi(\vec{x}).$$

**Definition 3.21.** *We say that* $\mathbf{T}$ *is a* $\Pi_n$*–Parikh theory if there exists a* $\Pi_n$*–Parikh pair* $(\Gamma, \Gamma_1)$ *such that* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \Gamma_1)$*.*

*Remark* 3.22. Here we give some basic facts on $\Pi_n$–Parikh pairs. We first observe that $(\mathrm{Gr}_n(\mathbf{T}), \mathrm{Func}_n(\mathbf{T}))$ satisfies condition *(1)* of definition 3.20.

**Claim 3.23.** *Let $\psi(\vec{x},\vec{y}) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall \vec{x}\, \exists \vec{y}\, \psi(\vec{x},\vec{y})$. There is a term of $\mathcal{L}(Gr_n(\mathbf{T}))$, $t(\vec{x})$, such that $(\mathbf{I}\Sigma_n + Func_n(\mathbf{T}))_{Gr_n(\mathbf{T})} \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \psi(\vec{x},\vec{y})$.*

*Proof of Claim.* Let $\psi'(u,v)$ be $\forall \vec{x} \leq u\, \forall \vec{y} \leq v\, \psi^c(u,v,\vec{x},\vec{y})$, where $\psi^c(u,v,\vec{x},\vec{y})$ is as in 3.3. Then $\mathbf{T} \vdash \forall u\, \exists v\, \psi'(u,v)$. Let $\theta(u,v) \in \Pi_n^-$ be $\psi'_f(u,v)$, see 3.4, and let $t(\vec{x})$ be the term $G_\theta(\mathbf{J}_k(x_1,\ldots,x_k))$ (where $\mathbf{J}_k(x_1,\ldots,x_k)$ is a term of $\mathcal{L}(\mathrm{Gr}_n(\mathbf{T}))$ associated to Cantor's function used in contraction of quantifiers). Then $(\mathbf{I}\Sigma_n + \mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})} \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \psi(\vec{x},\vec{y})$. □

In what follows $(\Gamma, \Gamma_1)$ shall denote a $\Pi_n$–Parikh pair.

**Claim 3.24.** *$(n \geq 1)$ Let $\varphi(\vec{x},\vec{y}) \in \Pi_{n-1}$ and $\psi(\vec{x},\vec{y}) \in \Sigma_{n-1}$. Then there exist terms of $\mathcal{L}(\Gamma)$, $t(\vec{x})$, $t'(\vec{x})$, such that*

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists \vec{y}\, \varphi(\vec{x},\vec{y}) \leftrightarrow \exists \vec{y} \leq t(\vec{x})\, \varphi(\vec{x},\vec{y}),$$
$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{y}\, \psi(\vec{x},\vec{y}) \leftrightarrow \forall \vec{y} \leq t'(\vec{x})\, \psi(\vec{x},\vec{y}).$$

*Proof of Claim.* Let $\varphi_1(\vec{x},\vec{y}) \in \Pi_n$ be the formula $\varphi(\vec{x},\vec{y}) \vee (\forall \vec{z}\, \neg\varphi(\vec{x},\vec{z}) \wedge \vec{y} = 0)$. Since $\mathbf{I}\Sigma_n + \Gamma_1 \vdash \forall \vec{x}\, \exists \vec{y}\, \varphi_1(\vec{x},\vec{y})$, by 3.20–*(1)*, there exists a term of $\mathcal{L}(\Gamma)$, $t(\vec{x})$, such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\ \exists \vec{y} \leq t(\vec{x})\, \varphi_1(\vec{x},\vec{y})$; hence,

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists \vec{y}\, \varphi(\vec{x},\vec{y}) \leftrightarrow \exists \vec{y} \leq t(\vec{x})\, \varphi(\vec{x},\vec{y}).$$

For $\psi \in \Sigma_{n-1}$ we obtain the result, from the above one, using $\neg\psi$. □

**Claim 3.25.** *Let $\varphi(\vec{x},\vec{y}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y}\, \varphi(\vec{x},\vec{y})$. There is a term $t(\vec{x})$ of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \varphi(\vec{x},\vec{y})$.*

*Proof of Claim.* By 3.20–*(2)*, there exists $\psi(\vec{x},\vec{y},z) \in \Pi_n$ such that $\exists z\, \psi(\vec{x},\vec{y},z) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$ and $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x},\vec{y}) \leftrightarrow \exists z\, \psi(\vec{x},\vec{y},z)$. Let $t(\vec{x})$ be a term of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y}, z \leq t(\vec{x})\, \psi(\vec{x},\vec{y},z)$. Then $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \varphi(\vec{x},\vec{y})$. □

**Claim 3.26.** *Let $\varphi(\vec{x}) \in \Delta_1^\Gamma((\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma)$. Then there exists $\theta(\vec{x}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \theta(\vec{x})$.*

*Proof of Claim.* Assume $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists y\, \varphi'(\vec{x},y) \leftrightarrow \forall y\, \psi'(\vec{x},y)$, where $\varphi'(\vec{x},y)$ and $\psi'(\vec{x},y)$ are $\Delta_0^\Gamma$ and $\varphi(\vec{x})$ is $\exists y\, \varphi'(\vec{x},y)$. Let $\delta(\vec{x},y) \in \Delta_0^\Gamma$ the formula $\varphi'(\vec{x},y) \vee \neg\, \psi'(\vec{x},y)$. Then $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists y\, \delta(\vec{x},y)$; so, by 3.25, there exists a term $t(\vec{x})$ of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists y \leq t(\vec{x})\, \delta(\vec{x},y)$. Hence, $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \exists y \leq t(\vec{x})\, \varphi'(\vec{x},y)$. □

**Theorem 3.27.** *Let $(\Gamma, \Gamma_1)$ be a $\Pi_n$–Parikh pair, $\varphi(\vec{x}) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$. Then there exists $\theta(\vec{x}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \theta(\vec{x})$.*

*Proof.* For $n = 0$ the result follows from 3.26. Suppose that $n \geq 1$. Let $\varphi_0(\vec{x},\vec{y},\vec{z}_1,\ldots,\vec{z}_n)$, $\psi_0(\vec{x},\vec{y},\vec{z}_1,\ldots,\vec{z}_n) \in \Delta_0$ such that (assume $n$ even)

- **(–):** $\varphi(\vec{x}) \equiv \exists \vec{y}\, \forall \vec{z}_1\, \exists \vec{z}_2 \ldots \exists \vec{z}_n\, \varphi_0(\vec{x},\vec{y},\vec{z}_1,\vec{z}_2,\ldots,\vec{z}_n)$,
- **(–):** $\psi(\vec{x}) \equiv \forall \vec{y}\, \exists \vec{z}_1\, \forall \vec{z}_2 \ldots \forall \vec{z}_n\, \psi_0(\vec{x},\vec{y},\vec{z}_1,\vec{z}_2,\ldots,\vec{z}_n)$, and
- **(–):** $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$.

By 3.24, there exist $t_1(\vec{x},\vec{y})$, $t_2(\vec{x},\vec{y},\vec{z}_1),\dots,t_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})$ terms of $\mathcal{L}(\Gamma)$ such that the following formulas are equivalent in $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$

**(–):** $\forall\vec{z}_1\,\exists\vec{z}_2\dots\exists\vec{z}_n\,\varphi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n)$.

**(–):** $\forall\vec{z}_1\le t_1(\vec{x},\vec{y})\,\exists\vec{z}_2\le t_2(\vec{x},\vec{y},\vec{z}_1)\dots\exists\vec{z}_n\le t_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})\,\varphi_0$.

Let $\varphi'(\vec{x},\vec{y})\in\Delta_0^\Gamma$ be the last formula. Analogously, we get that there exist $t'_1(\vec{x},\vec{y})$, $t'_2(\vec{x},\vec{y},\vec{z}_1),\dots,t'_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})$ terms of $\mathcal{L}(\Gamma)$ such that the following formulas are equivalent in $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$

**(–):** $\exists\vec{z}_1\forall\vec{z}_2\dots\forall\vec{z}_n\,\psi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n)$.

**(–):** $\exists\vec{z}_1\le t'_1(\vec{x},\vec{y})\,\forall\vec{z}_2\le t'_2(\vec{x},\vec{y},\vec{z}_1)\dots\forall\vec{z}_n\le t'_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})\,\psi_0$.

Let $\psi'(\vec{x},\vec{y})\in\Delta_0^\Gamma$ be the last formula. Then

$$(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\leftrightarrow\forall\vec{y}\,\psi'(\vec{x},\vec{y}).$$

So, $\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\in\Delta_1^\Gamma((\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma)$ and, by 3.26, there is $\theta(\vec{x})\in\Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\leftrightarrow\theta(\vec{x})$; hence, $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\varphi(\vec{x})\leftrightarrow\theta(\vec{x})$, as required. □

**Theorem 3.28.** *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Sigma_n$. *Then*

$$\mathbf{T}\text{ is }\Pi_n\text{–Parikh}\quad\Longleftrightarrow\quad\mathbf{T}\text{ is }\Delta_{n+1}\text{–closed.}$$

*Proof.* ($\Longrightarrow$): Let $\varphi(x,\vec{v})\in\Delta_{n+1}(\mathbf{T})$ and $t(\vec{v})\in\mathbf{Term}(\mathcal{L})$. Let us see that $\forall x\le t(\vec{v})\,\varphi(x,\vec{v})\in\Delta_{n+1}(\mathbf{T})$. Let $(\Gamma,\Gamma_1)$ be a $\Pi_n$–Parikh pair for $\mathbf{T}$. Then, using 3.27 and 3.20–*(2)*, there exist $\theta(x,\vec{v})\in\Delta_0^\Gamma$ and $\psi(\vec{v})\in\Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$ proves $\varphi(x,\vec{v})\leftrightarrow\theta(x,\vec{v})$ and $\forall x\le t(\vec{v})\,\varphi(x,\vec{v})\leftrightarrow\psi(\vec{v})$.

($\Longleftarrow$): Let us prove that $(\mathrm{Gr}_n(\mathbf{T}),\mathrm{Func}_n(\mathbf{T}))$ is a $\Pi_n$–Parikh pair for $\mathbf{T}$. By 3.23, we only need to prove 3.20–*(2)*. The proof is by induction on the length of $\Delta_0^\Gamma$–formulas. Let $\theta(\vec{x})\in\Delta_0^\Gamma$, we only consider the case where $\theta(\vec{x})$ is $\exists y\le t(\vec{x})\,\theta_0(\vec{x},y)$. By induction hypothesis there exists $\psi_0(\vec{x},y)\in\Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})}\vdash\psi_0(\vec{x},y)\leftrightarrow\theta_0(\vec{x},y)$. Then, by 3.2–*(ii)*, there exists $\delta(\vec{x},v)\in\Delta_{n+1}(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))$ such that

$$(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})}\vdash\exists v\,[\delta(\vec{x},v)\wedge\exists y\le v\,\psi_0(\vec{x},y)]\leftrightarrow\exists y\le t(\vec{x})\,\theta_0(\vec{x},y).$$

As $\mathbf{T}$ is $\Delta_{n+1}$–closed, there exists $\psi(\vec{x},v)\in\Delta_{n+1}(\mathbf{T})$ such that

$$\mathbf{T}\vdash\exists y\le v\,\psi_0(\vec{x},y)\leftrightarrow\psi(\vec{x},v).$$

Since $\exists v\,[\delta(\vec{x},v)\wedge\psi(\vec{x},v)]\in\Delta_{n+1}(\mathbf{T})$, this proves the result. □

## 4. Extended Parikh's Theorem

In this section, we shall see that for some kind of $\Pi_n$–functional class $\Gamma$ there exists an extension of $\mathcal{L}$ such that each $\Delta_{n+1}(\mathbf{I}\Sigma_n+\Gamma^*)$ formula is equivalent to a bounded formula of $\mathcal{L}(\Gamma)$.

### 4.1. $\Delta_0^\Gamma$ formulas as $\Delta_{n+1}$ formulas.

**Lemma 4.1.** *Let* $\Gamma\subseteq\Pi_n$ *and* $\Gamma_1\subseteq\Pi_{n+2}$ *such that* Func$(\Gamma)\subseteq\Gamma_1$ *and*

*for all* $s(\vec{x}),t(\vec{x},y)\in\mathbf{Term}(\mathcal{L}(\Gamma))$ *there exists* $t_s(\vec{x})\in\mathbf{Term}(\mathcal{L}(\Gamma))$ *such that* $(\mathbf{B}\Sigma_n+\Gamma_1)_\Gamma\vdash y\le s(\vec{x})\to t(\vec{x},y)\le t_s(\vec{x})$.

(1) *Let* $\varphi(\vec{x}) \in \Delta_0^\Gamma$. *Then there exist* $\psi(\vec{x}, z) \in \Sigma_n$, $\theta(\vec{x}, z) \in \Pi_n$ *and* $t(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ *such that*

$$(\mathbf{B}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall z \geq t(\vec{x}) \, [\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}, z) \leftrightarrow \theta(\vec{x}, z)].$$

(2) *Let* $\varphi(\vec{x}) \in \Delta_0^\Gamma$. *Then there exists* $\delta(\vec{x}) \in \Delta_{n+1}(\mathbf{B}\Sigma_n + \Gamma_1)$ *such that* $(\mathbf{B}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \delta(\vec{x})$.

*For* $n = 0$, $\mathbf{B}\Sigma_0$ *can be replaced by* $\mathbf{I}\Delta_0$ *(collection is not needed).*

*Proof.* By induction on the length of $\varphi(\vec{x})$ as in lemma **I.1.30** in [10]. □

*Remark* 4.2. Let $\Gamma$ be a $\Pi_n$–functional class. We have the following results.

**Claim 4.3.** *For every* $t(\vec{x}, y), s(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ *there exists a term* $t_s(\vec{x})$ *such that* $(\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma \vdash y \leq s(\vec{x}) \rightarrow t(\vec{x}, y) \leq t_s(\vec{x})$. *So, lemma 4.1 holds for* $(\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma$ *and* $(\Gamma, \Gamma^*)$ *satisfies part (2) of definition 3.20.*

*Proof of Claim.* By 3.2–*(i)*, the result follows taking $t_s(\vec{x})$ as $t(\vec{x}, s(\vec{x}))$. □

**Claim 4.4.** $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \Longrightarrow \mathbf{I}\Delta_0^{\Gamma^*}$.

*Proof of Claim.* Let $\varphi(x) \in \Delta_0^\Gamma$ and $\mathfrak{A} \models (\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma$ such that $\mathfrak{A} \models \exists x \, \varphi(x)$. By 4.1–*(1)*, there exist $\psi(x, z) \in \Sigma_n$ and $t(x)$ term of $\mathcal{L}(\Gamma)$ such that $(\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma \models \forall z \geq t(x) \, [\varphi(x) \leftrightarrow \psi(x, z)]$. Let $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(a)$ and let $b = t(a)$. Then $\mathfrak{A} \models \psi(a, b)$. Since $\mathfrak{A} \models \mathbf{L}\Sigma_n$, there is $c \in \mathfrak{A}$ such that $\mathfrak{A} \models c = (\mu x)[\psi(x, b)]$. Since $\Gamma$ is a $\Pi_n$–functional class, by 3.2–*(i)* $\mathfrak{A} \models c = (\mu x)[\varphi(x)]$; hence, $\mathfrak{A} \models \mathbf{L}_\varphi$. □

*Remark* 4.5. Here we prove that $\Pi_0$–functional classes provide examples of $\Pi_0$–Parikh pairs. In the next subsection we shall see that for $n \geq 1$ this is also true for some kind of $\Pi_n$–functional classes. In what follows $\Gamma$ shall denote a $\Pi_0$–functional class. As in 4.4, using 4.1–*(1)* for $n = 0$, we get

**Claim 4.6.** $\mathbf{I}\Delta_0^{\Gamma^*} \Longleftrightarrow (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$.

**Claim 4.7** *(Parikh's theorem). Let* $\Gamma' \subseteq \Pi_1^\Gamma$. *For each* $\varphi(\vec{x}, \vec{y}) \in \Delta_0^\Gamma$ *such that* $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall \vec{x} \, \exists \vec{y} \, \varphi(\vec{x}, \vec{y})$ *there exists a term* $t(\vec{x})$ *of* $\mathcal{L}(\Gamma)$ *such that* $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall \vec{x} \, \exists \vec{y} \leq t(\vec{x}) \, \varphi(\vec{x}, \vec{y})$.

**Claim 4.8.** $(\Gamma, \Gamma^*)$ *is a* $\Pi_0$*–Parikh pair.*

*Proof of Claim.* By 4.7, part *(1)* of definition 3.20 holds for $(\Gamma, \Gamma^*)$. So, the result follows from 4.3. □

**4.2. $\Delta_{n+1}$ formulas as $\Delta_0^\Gamma$ formulas. Strong $\Pi_n$–functional classes.** In order to improve 4.1, 4.4 and 4.6–4.8, we consider a special kind of $\Pi_n$–functional classes. Let $\Gamma$ be a $\Pi_n$–functional class and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$. We shall also denote by $\mathfrak{A}$ the expansion of $\mathfrak{A}$ to $\mathcal{L}(\Gamma)$ given by: for every $a, b \in \mathfrak{A}$ and $\varphi \in \Gamma$

$$\mathfrak{A}(G_\varphi(a)) = b \quad \Longleftrightarrow \quad \mathfrak{A} \models \varphi(a, b).$$

Let $a_1, \ldots, a_k \in \mathfrak{A}$. The simple initial segment of $\mathfrak{A}$ determined by $a_1, \ldots, a_k$ is $\mathcal{S}_\Gamma(\mathfrak{A}, a_1, \ldots, a_k) = \{b : \; b \le t(\vec{a}), \; t(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))\}$. Observe that if $\mathfrak{A} \models (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$ then $\mathcal{S}_\Gamma(\mathfrak{A}, \vec{a})$ is an $\mathcal{L}(\Gamma)$–structure.

**Definition 4.9.** *Let $\Gamma$ be a $\Pi_n$–functional class. We say that $\Gamma$ is a strong $\Pi_n$–functional class if for every $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$ and every $\mathfrak{I}$*

*if $\mathfrak{I} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$ structures then $\mathfrak{I} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures.*

Let us observe that every $\Pi_0$–functional class is a strong $\Pi_0$–functional class. Moreover, if $\Gamma$ is a strong $\Pi_n$–functional class and $\Gamma'$ is a $\Pi_n$–functional class such that $\Gamma \subseteq \Gamma'$, then $\Gamma'$ is a strong $\Pi_n$–functional class.

**Lemma 4.10.** *$(n \ge 1)$ Let $\Gamma$ be a strong $\Pi_n$–functional class. Then for every $k < n$, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{k+2} + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_k + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Delta_0 + \Gamma^*)$.*

*Proof.* Suppose that $\mathbf{B}\Sigma_{k+2} + \Gamma^* \vdash \forall x \, \exists y \, \varphi(x, y)$, where $\varphi(x, y) \in \Pi_n$ and $\mathbf{I}\Sigma_k + \Gamma^* \nvdash \forall x \, \exists y \, \varphi(x, y)$. By compacteness, $\mathbf{T}$ is consistent, where

$$\mathbf{T} = (\mathbf{I}\Sigma_k + \Gamma^*)_\Gamma + \forall y \, \neg\varphi(\mathbf{c}, y) + \{t(\mathbf{c}) < \mathbf{d} : \; t(x) \text{ term of } \mathcal{L}(\Gamma)\}.$$

Let $\mathfrak{A} \models \mathbf{T}$, $a = \mathfrak{A}(\mathbf{c})$ and $\mathfrak{B} = \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Since $\Gamma$ is a $\Pi_n$–functional class, $\mathfrak{B} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures and, by the last group of axioms of $\mathbf{T}$, it is proper. Also, for all $\theta(x, y) \in \Gamma$ and $b \in \mathfrak{B}$ there exists $d \in \mathfrak{B}$ such that $\mathfrak{A} \models \theta(b, d)$. Since $\Gamma$ is a strong $\Pi_n$–functional class and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$, $\mathfrak{B} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures. So, from $\mathfrak{A} \models \forall y \, \neg\varphi(a, y)$ we get that $\mathfrak{B} \models \forall y \, \neg\varphi(a, y)$. Since, $\mathfrak{B} \models \mathbf{I}\Delta_0 + \Gamma^*$; and, for $k < n$, $\mathfrak{B} \prec_{k+1}^e \mathfrak{A}$, then $\mathfrak{B} \models \mathbf{B}\Sigma_{k+2}$. So, $\mathfrak{B} \models \mathbf{B}\Sigma_{k+2} + \Gamma^*$ and $\mathfrak{B} \models \exists y \, \varphi(a, y)$. Contradiction. This proves the first identity. The second one follows from the first by induction on $k < n$. □

*Remark* 4.11. (Strength of 4.4, 4.6, 3.7, 4.1) In what follows let $\Gamma$ be a strong $\Pi_n$–functional class.

**Claim 4.12.** *(i)* $\mathbf{I}\Delta_0^{\Gamma^*} \iff (\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \iff (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$.
*(ii)* $\mathbf{I}\Delta_0 + \Gamma^* \iff \mathbf{I}\Sigma_n + \Gamma^*$.

*Proof of Claim.* By 4.4, $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \implies \mathbf{I}\Delta_0^{\Gamma^*} \implies (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$. Let $\theta \in \Sigma_n$. Then $\mathbf{B}\Sigma_{n+1} \vdash \mathbf{I}_\theta$; so, by 4.10 (for $k = n - 1$), $\mathbf{I}\Delta_0 + \Gamma^* \vdash \mathbf{I}_\theta$. This proves *(i)*. Part *(ii)* follows from *(i)*. □

By 4.12, we can rewrite 3.7 as follows

**Claim 4.13.** *(i)* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Delta_0 + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*)$.
*(ii)* $\mathbf{I}\Delta_0 + \Gamma^* \implies \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \implies \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$.

**Claim 4.14.** *(i) Let $\varphi(\vec{x}) \in \Delta_0^\Gamma$. There are $\psi(\vec{x}, z) \in \Sigma_n$, $\theta(\vec{x}, z) \in \Pi_n$ and $t(\vec{x})$ such that $\mathbf{I}\Delta_0^{\Gamma^*} \vdash \forall z \ge t(\vec{x}) \, [\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}, z) \leftrightarrow \theta(\vec{x}, z)]$.*
*(ii) Let $\varphi(\vec{x}) \in \Delta_0^\Gamma$. Then there exists $\delta(\vec{x}) \in \Delta_{n+1}(\mathbf{B}\Sigma_n + \Gamma^*)$ such that $\mathbf{I}\Delta_0^{\Gamma^*} \vdash \varphi(\vec{x}) \leftrightarrow \delta(\vec{x})$.*

*Proof of Claim.* For $n \geq 1$, $\mathbf{I}\Sigma_n \Longrightarrow \mathbf{B}\Sigma_n$. So, $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \Longrightarrow (\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma$. Then the result follows from 4.1 and 4.12. □

**Theorem 4.15** *(Extended Parikh's theorem (Strength of 4.7)).*
*Let $\Gamma$ be a strong $\Pi_n$–functional class and $\Gamma' \subseteq \Pi_{n+1} \cup \Pi_1^\Gamma$. For each $\varphi(x,y) \in \Pi_n \cup \Delta_0^\Gamma$ such that $(\mathbf{B}\Sigma_{n+1} + \Gamma' + \Gamma^*)_\Gamma \vdash \forall x\, \exists y\, \varphi(x,y)$ there exists a term $t(x)$ of $\mathcal{L}(\Gamma)$ such that $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall x\, \exists y \leq t(x)\, \varphi(x,y)$.*

*Proof.* Deny the proposition's conclusion. We proceed as in 4.10. By compacteness the following theory is consistent ($\mathbf{c}$ and $\mathbf{d}$ are new constants)

$$\mathbf{T} = \left\{ \begin{array}{lcl} \mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' & + & \{\forall y \leq t(\mathbf{c})\, \neg\varphi(\mathbf{c},y) :\ t(x) \text{ term of } \mathcal{L}(\Gamma)\} \\ & + & \{t(\mathbf{c}) < \mathbf{d} :\ t(x) \text{ term of } \mathcal{L}(\Gamma)\} \end{array} \right.$$

Let $\mathfrak{A} \models \mathbf{T}$, $a = t(\mathbf{c})$ and $\mathfrak{B} = \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Since $\Gamma$ is a $\Pi_n$–functional class, $\mathfrak{B} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures. Then $\mathfrak{B} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures. So, $\mathfrak{B} \models \forall y\, \neg\varphi(a,y)$ and, since $\mathfrak{A} \models \mathbf{I}\Sigma_n$ and $\mathfrak{A}$ is a proper extension of $\mathfrak{B}$ (last set of axioms of $\mathbf{T}$), $\mathfrak{B} \models (\mathbf{B}\Sigma_{n+1} + \Gamma' + \Gamma^*)_\Gamma$. Contradiction. □

**Corollary 4.16.** *(Strength of 4.8) If $\Gamma$ is a strong $\Pi_n$–functional class then $(\Gamma, \Gamma^*)$ is a $\Pi_n$–Parikh pair.*

### 4.3. Existence theorems of strong $\Pi_n$–functional classes.

**Theorem 4.17.** $(n \geq 1)$ *There is a strong $\Pi_n$–functional class, $\mathbb{H}_n$, such that*
*for all $\varphi \in \mathbb{H}_n$, $\mathbf{I}\Sigma_{n-1} \vdash IPF(\varphi)$ and $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Sigma_n + \mathbb{H}_n^*$.*

*Proof.* For each $\theta(v,y) \in \Pi_{n-1}^-$ let $\theta'(x,w)$ be the following formula

$$\left\{ \begin{array}{l} [\neg\exists v \leq x\, \exists y\, \theta(v,y) \wedge w = 0] \vee \\ \exists w_1, w_2 \leq w \left\{ \begin{array}{l} w = \langle w_1, w_2 \rangle \wedge w_1 \leq x \wedge \\ \forall v \leq x\, [\exists y\, \theta(v,y) \rightarrow \exists y \leq w_2\, \theta(v,y)] \wedge \\ \theta_{\mu,w_2}(w_1,w_2) \wedge \forall v \leq x\, [\theta_{\mu,w_2}(v,w_2) \rightarrow v \leq w_1] \end{array} \right. \end{array} \right.$$

It is clear that there is $\theta^*(x,w) \in \Pi_n$ such that $\mathbf{I}\Sigma_{n-1} \vdash \theta'(x,w) \leftrightarrow \theta^*(x,w)$. Let $\mathbb{H}_n = \{\theta^*(x,w) :\ \theta(v,y) \in \Pi_{n-1}\}$. Let $\theta(v,y) \in \Pi_{n-1}$. It holds that $\mathbf{I}\Sigma_{n-1} \vdash \mathrm{IPF}(\theta^*)$ and $\mathbf{I}\Sigma_n \vdash \forall x\, \exists w\, \theta^*(x,w)$; so, $\mathbb{H}_n$ is a $\Pi_n$–functional class and $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Sigma_n + \mathbb{H}_n^*$. Let us observe that $\mathbb{H}_1 \subseteq \mathbb{H}_2 \subseteq \cdots \subseteq \mathbb{H}_n \subseteq \ldots$. Now, by induction on $n \geq 1$, we prove that $\mathbb{H}_n$ is a strong $\Pi_n$–functional class. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_n^*$ and $\mathfrak{I} \subset^e \mathfrak{A}$ such that

$(*)$ for all $\varphi(x,w) \in \mathbb{H}_n$, $a \in \mathfrak{I}$ there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \varphi(a,b)$.

By induction on $n \geq 1$, using Tarski–Vaught's test, we prove that $\mathfrak{I} \prec_n \mathfrak{A}$.
$\underline{(n = 1)}$: Let us see that $\mathfrak{I} \prec_1 \mathfrak{A}$. Let $\theta(v,y) \in \Pi_0$ and $a \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y\, \theta(a,y)$. Since $\theta^*(x,w) \in \mathbb{H}_1$ and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_1^*$, then $\mathfrak{A} \models \forall x\, \exists y\, \theta^*(x,w)$. Since $a \in \mathfrak{I}$, by $(*)$, there exists $d \in \mathfrak{I}$ such that $\mathfrak{A} \models \theta^*(a,d)$. Since $\mathbf{I}\Delta_0 \vdash \theta'(x,w) \leftrightarrow \theta^*(x,w)$, $\mathfrak{A} \models \theta'(a,d)$; so, there exists $b \in \mathfrak{A}$ such that $b \leq d$ and $\mathfrak{A} \models \theta(a,b)$. Since $d \in \mathfrak{I}$ and $\mathfrak{I} \subset^e \mathfrak{A}$, then $b \in \mathfrak{I}$, as required.
$\underline{(n \rightarrow n+1)}$: Since $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_n^*$ and, by induction hypothesis, $\mathbb{H}_n$ is a strong $\Pi_n$–functional class, we get that $\mathfrak{A} \models \mathbf{I}\Sigma_n + \mathbb{H}_n^*$. Let $\theta(x,y) \in \Pi_n$ and $a \in \mathfrak{I}$ such that

$\mathfrak{A} \models \exists y\, \theta(a, y)$. Now as in the case $n = 1$, using that $\mathfrak{A} \models \mathbf{I\Sigma}_n + \mathbb{H}_n^*$, we obtain that there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \theta(a, b)$. $\square$

**Proposition 4.18** *(Strength of 3.8). If $\mathbf{T}$ has $\Delta_{n+1}$–collection, there is a strong $\Pi_n$–functional class $\Gamma$ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Delta}_0 + \Gamma^*)$.*

*Proof.* Suppose that $n \geq 1$. By 3.8, there is a $\Pi_n$–functional class $\Gamma_1$ such that $\mathbf{Th}_{n+2}(\mathbf{T}) = \mathbf{Th}_{n+2}(\mathbf{I\Sigma}_n + \Gamma_1^*)$. Let $\Gamma = \mathbb{H}_n + \Gamma_1$. Then, $\Gamma$ is a strong $\Pi_n$–functional class; so, by 4.17 and 4.12, $\mathbf{I\Sigma}_n + \Gamma_1^* \iff \mathbf{I\Delta}_0 + \Gamma^*$. Hence $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \Gamma_1^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Delta}_0 + \Gamma^*)$. $\square$

**Lemma 4.19.** *Let $\mathbf{T} \Longrightarrow \mathbf{I\Sigma}_n$, $\mathfrak{A} \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ and $a \in \mathfrak{A}$. If $(\Gamma, \Gamma_1)$ and $(\Gamma', \Gamma_1')$ are $\Pi_n$–Parikh pairs for $\mathbf{T}$ then $\mathcal{S}_\Gamma(\mathfrak{A}, a) = \mathcal{S}_{\Gamma'}(\mathfrak{A}, a)$.*

*Proof.* Let $b \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. There are $t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $b \leq t(a)$ and $\varphi(x, y) \in \Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash t(x) = y \leftrightarrow \varphi(x, y)$. Let $s(x)$ be a term of $\mathcal{L}(\Gamma')$ such that $(\mathbf{I\Sigma}_n + \Gamma_1')_{\Gamma'} \vdash \forall x\, \exists y \leq s(x)\, \varphi(x, y)$. So, $b \leq s(a)$; hence, $b \in \mathcal{S}_{\Gamma'}(\mathfrak{A}, a)$. $\square$

**Theorem 4.20.** *Let $\mathbf{T}$ be an extension of $\mathbf{I\Sigma}_n$ and $(\Gamma, \Gamma_1)$ a $\Pi_n$–Parikh pair for $\mathbf{T}$ (so, $\mathbf{T}$ is $\Delta_{n+1}$–closed). The following conditions are equivalent*

1. *$\mathbf{T}$ has $\Delta_{n+1}$–collection.*
2. *$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \Longrightarrow \mathbf{I\Delta}_0^\Gamma$.*
3. *For each $s(\vec{v}), t(\vec{v}, x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ there exists $t_s(\vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash x \leq s(\vec{v}) \to t(\vec{v}, x) \leq t_s(\vec{v})$.*
4. *$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B\Sigma}_{n+1} + \Gamma_1)$.*
5. *For every $\mathfrak{A} \models (\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma$ and $a \in \mathfrak{A}$, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures and $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*

*Proof.* From 2.8 and 2.15, it follows *(4) $\Longrightarrow$ (1)*.

(*(1) $\Longrightarrow$ (5)*): By 4.19 and 4.16 we may assume that $\Gamma$ is a strong $\Pi_n$–functional class for $\mathbf{T}$ (and that $\Gamma_1 = \Gamma^*$). So, by 4.9, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures. Let $\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y)$. Then there exists $t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma^*)_\Gamma \vdash \forall x\, \exists y \leq t(x)\, \varphi(x, y)$. Let $b \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Then, it holds that there exist $c \in \mathfrak{A}$ and $s(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $\mathfrak{A} \models c \leq t(b) \wedge \varphi(b, c) \wedge b \leq s(a)$. Since $\Gamma$ is $\Pi_n$–functional, $c \leq t(b) \leq t(s(a))$; hence, $c \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. So, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \exists y\, \varphi(b, y)$.

(*(5) $\Longrightarrow$ (4)*): Let $\varphi(x, y) \in \Pi_n$ such that $\mathbf{B\Sigma}_{n+1} + \Gamma_1 \vdash \forall x\, \exists y\, \varphi(x, y)$. Suppose that $\mathbf{I\Sigma}_n + \Gamma_1 \nvdash \forall x\, \exists y\, \varphi(x, y)$. Let $\mathbf{T}'$ be the theory

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma + \forall y\, \neg\varphi(\mathbf{c}, y) + \{t(\mathbf{c}) < \mathbf{d} :\ t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))\}.$$

By compacteness, $\mathbf{T}'$ is consistent. Let $\mathfrak{A} \models \mathbf{T}'$ and $a = \mathfrak{A}(\mathbf{c})$. Since $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures and is proper, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \forall y\, \neg\varphi(a, y)$ and $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{B\Sigma}_{n+1}$. Then, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{B\Sigma}_{n+1} + \Gamma_1$. Contradiction.

(*(1) $\Longrightarrow$ (2)*): Let $\varphi(x) \in \Delta_0^\Gamma$. There exists $\psi(x) \in \Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \varphi(x) \leftrightarrow \psi(x)$. Since $\mathbf{T}$ has $\Delta_{n+1}$–induction, by 2.15, $\mathbf{I\Sigma}_n + \Gamma_1$ has $\Delta_{n+1}$–induction; hence, $\mathbf{I\Sigma}_n + \Gamma_1 \vdash \mathbf{I}_\psi$. So, $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \mathbf{I}_\varphi$.

*((2) $\Longrightarrow$ (1))*: Let $\varphi(x) \in \Delta_{n+1}(\mathbf{T})$. By 3.27, there exists $\theta(x) \in \Delta_0^\Gamma$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \theta(x) \leftrightarrow \varphi(x)$. Then, by *(2)*, $\mathbf{T} \vdash \mathbf{I}_\varphi$; so, $\mathbf{T}$ has $\Delta_{n+1}$–induction. Since $\mathbf{T}$ is $\Delta_{n+1}$–closed, by 2.13, $\mathbf{T}$ has $\Delta_{n+1}$–collection.

*((1) $\Longrightarrow$ (3))*: Let $s(\vec{v}), t(\vec{v}, x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$. By 3.2 there exist $\varphi(\vec{v}, x)$, $\theta(\vec{v}, x, z) \in \Delta_{n+1}(\mathbf{I\Sigma}_n + \Gamma_1)$ such that

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash [s(\vec{v}) = x \leftrightarrow \varphi(\vec{v}, x)] \wedge [t(\vec{v}, x) = z \leftrightarrow \theta(\vec{v}, x, z)].$$

Let $\Gamma_0$ be a strong $\Pi_n$–functional class for $\mathbf{T}$. By 4.16, there exist $t_0(\vec{v}, x)$ and $s_0(\vec{v})$ terms of $\mathcal{L}(\Gamma_0)$ and $\psi(\vec{v}, z) \in \Delta_{n+1}(\mathbf{T})$ such that

$$(\mathbf{I\Sigma}_n + \Gamma_0^*)_{\Gamma_0} \vdash \forall \vec{v}\, \exists x \leq s_0(\vec{v})\, \varphi(\vec{v}, x) \ \wedge\ \forall \vec{v} \forall x\, \exists z \leq t_0(\vec{v}, x)\, \theta(\vec{v}, x, z).$$

and $(\mathbf{I\Sigma}_n + \Gamma_0^*)_{\Gamma_0} \vdash t_0(\vec{v}, s_0(\vec{v})) = z \leftrightarrow \psi(\vec{v}, z)$. Then

$$\mathbf{I\Sigma}_n + \Gamma_0^* \vdash \varphi(\vec{v}, x') \wedge x \leq x' \wedge \theta(\vec{v}, x, z) \rightarrow \exists z' \,(\psi(\vec{v}, z') \wedge z \leq z').$$

Since $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \Gamma_0^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \Gamma_1)$, then

$$\mathbf{I\Sigma}_n + \Gamma_1 \vdash \varphi(\vec{v}, x') \wedge x \leq x' \wedge \theta(\vec{v}, x, z) \rightarrow \exists z' \,(\psi(\vec{v}, z') \wedge z \leq z').$$

Since $\mathbf{T} \vdash \forall \vec{v}\, \exists z\, \psi(\vec{v}, z)$, there exists $t_s(\vec{v})$ such that

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \forall \vec{v}\, \exists z \leq t_s(\vec{v})\, \psi(\vec{v}, z).$$

So, $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash x \leq s(\vec{v}) \rightarrow t(\vec{v}, x) \leq t_s(\vec{v})$, as required.

*((3) $\Longrightarrow$ (1))*: Let $\varphi(x, y, \vec{v}) \in \Pi_n^-$ such that $\exists y\, \varphi(x, y, \vec{v}) \in \Delta_{n+1}(\mathbf{T})$. Then there exist $\theta(x, \vec{v}), \varphi_0(x, y, \vec{v}) \in \Delta_0^\Gamma$ such that

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash [\exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \theta(x, \vec{v})] \ \wedge\ [\varphi(x, y, \vec{v}) \leftrightarrow \varphi_0(x, y, \vec{v})].$$

Let $\psi(x, \vec{v}, y) \in \Delta_0^\Gamma$ be $(\theta(x, \vec{v}) \wedge \varphi_0(x, y, \vec{v})) \vee (\neg\theta(x, \vec{v}) \wedge y = 0)$. Then, by 3.25, there exists $t(x, \vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \forall x \forall \vec{v}\, \exists y \leq t(x, \vec{v})\, \psi(x, \vec{v}, y)$. By *(3)*, there exists $t'(u, \vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash x \leq u \rightarrow t(x, \vec{v}) \leq t'(u, \vec{v})$. So,

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \forall u \forall \vec{v}\, [\forall x \leq u\, \exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists u' \,\forall x \leq u\, \exists y \leq u'\, \varphi(x, y, \vec{v})].$$

That is, $\mathbf{T} \vdash \mathbf{B}_{\varphi, x, y}$. So, $\mathbf{T}$ has $\Delta_{n+1}$–collection. □

## 5. $\Pi_n$–envelopes

**5.1. General properties of $\Pi_n$–envelopes. Initial segments.** In this section we introduce the concept of $\Pi_n$–envelope. This generalizes the concept of envelope (see [10]) and is closely related to indicators (see [12]). Some results in this section are generalizations of results on indicators that appear in chapter 14 of [12]. However, $\Pi_n$–envelopes will provide us with $\Pi_n$–functional classes defined uniformely. This is why we include these results here. In particular, we will obtain $\Pi_n$–envelopes that will be used in section 6 to prove the hierarchy theorem.

For each formula $\varphi(u, x, y)$ let $\Gamma_\varphi = \{\varphi(k, x, y) :\ k \in \omega\}$.

**Definition 5.1.** *Let $\varphi(u, x, y) \in \Sigma_{n+1}^-$. We say that*

1. *$\varphi(u, x, y)$ is a $\Pi_n$–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ if $\mathbf{T} \vdash \Gamma_\varphi^*$, and for all $k \in \omega$, $\mathbf{T}_0 \vdash \varphi(k+1, x, y) \rightarrow \exists z < y\, \varphi(k, x, z)$.*

(2) $\varphi(u,x,y)$ *satisfies* $\Pi_n$*–ENV for* $\mathbf{T}$ *and* $\mathbf{T}_0$ *if for each* $\psi(x,y) \in \Pi_n^-$ *such that* $\mathbf{T} \vdash \forall x\, \exists y\, \psi(x,y)$, *there exists* $k \in \omega$ *such that*

$$\mathbf{T}_0 \vdash \varphi(k,x,y) \to \exists z < y\, \psi(x,z).$$

(3) $\varphi(u,x,y)$ *is a* $\Pi_n$*–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$ *if* $\varphi(u,x,y)$ *is a* $\Pi_n$*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$ *and satisfies* $\Pi_n$*–ENV for* $\mathbf{T}$ *and* $\mathbf{T}_0$.

*Remark* 5.2. Now we shall give some basic properties of envelopes. Let $\varphi(u,x,y) \in \Sigma_{n+1}$ a $\Pi_n$–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$. By contraction of quantifiers, part *(2)* of definition 5.1 is also true for $\psi(x,y) \in \Sigma_{n+1}^-$. We also have that

**Claim 5.3.** *(i) If* $\mathbf{T} \Longrightarrow \mathbf{T}_0$ *then* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}_0 + \Gamma_\varphi^*)$.
*(ii) If* $\varphi \in \Pi_n$ *and* $\mathbf{T} + \mathbf{I}\Sigma_n$ *is consistent then* $\Gamma_\varphi$ *is a* $\Pi_n$*–functional class.*

**Definition 5.4.** *Let* $\varphi(u,x,y) \in \Sigma_{n+1}$. *We say that* $\varphi(u,x,y)$ *satisfies* $\Pi_n$*–IND for* $\mathbf{T}$ *and* $\mathbf{T}_0$ *if for every* $\mathfrak{A} \models \mathbf{T}_0$ *countable, nonstandard and* $a, b \in \mathfrak{A}$, *the following conditions are equivalent:*

**(IND-(i))**: *For all* $k \in \omega$, $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$.
**(IND-(ii))**: *There exists* $\mathfrak{I} \models \mathbf{T}$ *such that* $\mathfrak{I} \prec_n^e \mathfrak{A}$ *and* $a < \mathfrak{I} < b$.

*Remark* 5.5. Let $\varphi(u,x,y) \in \Sigma_{n+1}$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(k,x,y)$, for all $k \in \omega$. Then for all theory $\mathbf{T}_0$ we have that: **IND-(ii)** $\Longrightarrow$ **IND-(i)**. So, if $\varphi(u,x,y)$ is a $\Pi_n$–q–envelope, then in order to prove that $\varphi(u,x,y)$ satisfies $\Pi_n$–IND it is enough to establish that: **IND-(i)** $\Longrightarrow$ **IND-(ii)**.

Now we shall study conditions under which it holds that $\Pi_n$–ENV is equivalent to $\Pi_n$–IND. Let us note, however, that the proof of part $\Longleftarrow$ of next theorem shows that, if $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$, then every $\Pi_n$–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ satisfying $\Pi_n$–IND is a $\Pi_n$–envelope.

**Theorem 5.6.** *($n \geq 1$) Suppose that* $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$ *and*
*(i)* $\mathbf{T}$ *is recursively axiomatizable, and*
*(ii)* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.
*Let* $\varphi(u,x,y) \in \Sigma_{n+1}$ *be a* $\Pi_n$*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$. *Then with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$

$$\varphi(u,x,y) \text{ satisfies } \Pi_n\text{–ENV} \quad \Longleftrightarrow \quad \varphi(u,x,y) \text{ satisfies } \Pi_n\text{–IND}.$$

*Proof.* ($\Longleftarrow$): Let $\psi(x,y) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \psi(x,y)$ and suppose that for all $k \in \omega$, $\mathbf{T}_0 \nvdash \varphi(k,x,y) \to \exists z < y\, \psi(x,z)$. For all $k \in \omega$ let

$$\mathbf{T}_k = \mathbf{T}_0 + \{\exists y < \mathbf{d}\, \varphi(j,\mathbf{c},y) \wedge \forall z < \mathbf{d}\, \neg\psi(\mathbf{c},z) : \ j < k\}.$$

Since, for all $k \in \omega$, $\mathbf{T}_k$ is consistent, $\mathbf{T}^* = \bigcup_{k \in \omega} \mathbf{T}_k$ is consistent. Let $\mathfrak{A}^* \models \mathbf{T}$ countable nonstandard, $\mathfrak{A} = \mathfrak{A}^*_{|\mathcal{L}}$, $a = \mathfrak{A}^*(\mathbf{c})$ and $b = \mathfrak{A}^*(\mathbf{d})$. Then $\mathfrak{A} \models \mathbf{T}_0$ and for all $k \in \omega$, $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$. Since $\varphi$ satisfies $\Pi_n$–IND for $\mathbf{T}$ and $\mathbf{T}_0$, there exists $\mathfrak{I} \models \mathbf{T}$ such that $\mathfrak{I} \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I} < b$. So, there exists $e \in \mathfrak{I}$ such that $\mathfrak{I} \models \psi(a,e)$; hence, $e < b$ and $\mathfrak{A} \models \psi(a,e)$. But $\mathfrak{A}^* \models \forall z < \mathbf{d}\, \neg\psi(\mathbf{c},z)$; hence, $\mathfrak{A} \models \forall z < b\, \neg\psi(a,z)$. So, $\mathfrak{A} \models \neg\psi(a,e)$, a contradiction.

($\Longrightarrow$): By 5.5, it is enough to prove **IND-(i)** $\Longrightarrow$ **IND-(ii)**. We follow the proof of theorem **11.7** in [12]. Let $\mathfrak{A} \models \mathbf{T}_0$ countable, nonstandard and $a, b \in \mathfrak{A}$ such that $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$, for all $k \in \omega$. Let

$$\mathbf{T}' = \mathbf{T} + \mathbf{B}\Sigma_{n+1} + \{\forall \vec{z}\, \psi(\mathbf{c}, \vec{z}) :\ \psi(x, \vec{z}) \in \Sigma_n,\ \mathfrak{A} \models \forall \vec{z} \leq b\, \psi(a, \vec{z})\}.$$

By *(ii)* it follows that $\mathbf{T}'$ is consistent. Since $\mathfrak{A} \models \mathbf{I}\Sigma_n$ and $n \geq 1$, the $\Sigma_n$–type of $a, b$ in $\mathfrak{A}$ belongs to $\mathbf{SSy}(\mathfrak{A})$ (the standard system of $\mathfrak{A}$); hence, $\{\ulcorner \forall \vec{z}\, \psi(\mathbf{c}, \vec{z}) \urcorner :\ \psi \in \Sigma_n,\ \mathfrak{A} \models \forall \vec{z} \leq b\, \psi(a, \vec{z})\} \in \mathbf{SSy}(\mathfrak{A})$. So, by *(i)*, $\mathbf{T}' \in \mathbf{SSy}(\mathfrak{A})$. Since $\mathbf{SSy}(\mathfrak{A})$ is a Scott system, there exists $\mathfrak{B} \models \mathbf{T}'$ countable which is $\mathbf{SSy}(\mathfrak{A})$–saturated; hence, $\mathfrak{B}$ is recursively saturated. Let $c = \mathfrak{B}(\mathbf{c})$. Then, for each $\theta(x, \vec{z}) \in \Pi_n$, if $\mathfrak{B} \models \exists \vec{z}\, \theta(c, \vec{z})$ then $\mathfrak{A} \models \exists \vec{z} \leq b\, \theta(a, \vec{z})$. So, by Friedman's theorem, there exists $H : \mathfrak{B} \precsim_n^e \mathfrak{A}$ such that $H(c) = a$ and $b \notin H(\mathfrak{B})$. Let $\mathfrak{I} = H(\mathfrak{B})$. Then $\mathfrak{I} \models \mathbf{T}$, $\mathfrak{I} \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I} < b$. □

*Remark* 5.7. Condition *(ii)* in 5.6 cannot be deleted. We have used it there in order to prove that: **IND-(i)** $\Longrightarrow$ **IND-(ii)**. Even more, suppose that $\mathbf{T} \Longrightarrow \mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$ and $\varphi(u, x, y) \in \Sigma_{n+1}$ is a $\Pi_n$–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ that satisfies $\Pi_n$-IND for these theories. Let $\psi(x, y) \in \Pi_n$ be such that $\mathbf{T} + \mathbf{B}\Sigma_{n+1} \vdash \forall x\, \exists y\, \psi(x, y)$. Then, it holds that there is $k \in \omega$ such that $\mathbf{T}_0 \vdash \varphi(k, x, y) \rightarrow \exists z < y\, \psi(x, z)$. So, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.

*Remark* 5.8. For $\Pi_0$–envelopes we have the following form of 5.6.

**Claim 5.9.** *Suppose that* $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Delta_0 + \mathbf{exp}$, $\mathbf{T}$ *is recursively axiomatizable and* $\mathbf{Th}_{\Pi_2}(\mathbf{T}) = \mathbf{Th}_{\Pi_2}(\mathbf{T} + \mathbf{B}\Sigma_1)$. *Let* $\varphi(u, x, y) \in \Sigma_1$ *a* $\Pi_0$*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$. *Then, with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$,

$$\varphi(u, x, y) \text{ satisfies } \Pi_0\text{–ENV} \iff \varphi(u, x, y) \text{ satisfies } \Pi_0\text{–IND}.$$

In some cases this result is also true even though $\mathbf{T}_0$ is not an extension of $\mathbf{I}\Delta_0 + \mathbf{exp}$. Using methods that appears in [1], mainly the superexponential function (see the proof of lemma **3** there), it can be proved that

**Claim 5.10.** *Suppose that* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma_1 + \mathbf{exp} \Longrightarrow \mathbf{T}_0 \Longrightarrow \mathbf{I}\Delta_0$, *and* $\mathbf{T}$ *is recursively axiomatizable. Let* $\varphi(u, x, y) \in \Delta_0$ *be a* $\Pi_0$*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$. *Then, with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$,

$$\varphi(u, x, y) \text{ satisfies } \Pi_0\text{–ENV} \iff \varphi(u, x, y) \text{ satisfies } \Pi_0\text{–IND}.$$

**5.2. Existence theorems of $\Pi_n$–envelopes.** In this and in the next subsection we are going to use formulas in the language and in the metalanguage. In order to write expressions that are easier to read we shall use uppercase Greek letters for formulas in the metalanguage (real formulas) and lowercase Greek letters for formulas in the language (elements of a model that it thinks that are formulas). We shall use $\sigma, \tau, \ldots$ as variables (in the language of Arithmetic) for formulas, and $p$ as variable (in the language of Arithmetic) for proofs.

**Theorem 5.11.** *If* $\mathbf{T}$ *is recursively axiomatizable,* $\Pi_n$*–functional and, for* $n = 0$, $\mathbf{T} \vdash \mathbf{exp}$, *then there exists a* $\Pi_n$*–envelope of* $\mathbf{T}$ *in* $\mathbf{I}\Sigma_n$.

*Proof.* Since $\mathbf{T}$ is $\Pi_n$–functional, $\mathbf{T}$ has $\Delta_{n+1}$–collection and, for $n \geq 1$, $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n \Longrightarrow \mathbf{B}\Sigma_n$. Let us consider the following cases:

Case A: $n \geq 1$. Since $\mathbf{T}$ is recursively axiomatizable, there is $\mathbf{Prf_T}(x,y) \in \Sigma_1$ that represents to $\{(\sigma,p) \in \omega^2 : \ p \text{ is a proof of } \sigma \text{ in } \mathbf{T}\}$ in $\mathbf{P}^-$. Let $\Phi'(u,x,y) \in \Pi_n$ be a formula equivalent in $\mathbf{B}\Sigma_n$ (so, also in $\mathbf{T}$), to

$$\forall p, \tau \leq u \left\{ \begin{array}{r} \mathbf{Form}_{\Pi_n^-}(\tau(v_0,v_1)) \wedge \mathbf{Prf_T}(\forall v_0\, \exists v_1\, \tau(v_0,v_1), p) \ \rightarrow \\ \rightarrow \forall x_0 \leq x\, \exists y_0 \leq y\, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}_0, \dot{y}_0)) \end{array} \right\}$$

Where $\mathbf{Sat}_{\Pi_n}(v)$ is a truth definition in $\mathbf{I}\Sigma_1$ for $\Pi_n$–formulas.

Let $\Phi(u,x,y) \in \Sigma_{n+1}$ be a formula equivalent in $\mathbf{B}\Sigma_n$ (so, also in $\mathbf{T}$), to

$$\exists y' \leq y\, [y = y' + u \wedge \Phi'(u,x,y') \wedge \forall y'' < y'\, \neg\Phi'(u,x,y'')].$$

Let $k \in \omega$. Since $\mathbf{T}$ has $\Delta_{n+1}$–collection, $\mathbf{T} \vdash \forall x\, \exists y\, \Phi(k,x,y)$. Moreover, as $y = y' + k$, $\mathbf{I}\Sigma_n \vdash \Phi(k+1,x,y) \rightarrow \exists z < y\, \Phi(k,x,z)$ and $\mathbf{T} \vdash \mathrm{IPF}(\Phi(k,x,y))$.

Let $\Psi(x,y) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \Psi(x,y)$ and $k > \ulcorner \Psi(x,y) \urcorner$. Then $\mathbf{I}\Sigma_n \vdash \Phi(k,x,y) \rightarrow \exists z < y\, \Psi(x,z)$. So, $\Phi(u,x,y)$ satisfies $\Pi_n$–ENV for $\mathbf{T}$ and $\mathbf{I}\Sigma_n$.

Case B: $n = 0$. Since $\mathbf{T}$ is recursively axiomatizable, there is $\mathbf{Prf_T}(x,y,w) \in \Delta_0$ such that $\exists w\, \mathbf{Prf_T}(x,y,w)$ represents to $\{(\sigma,p) : \ p \text{ is a proof of } \sigma \text{ in } \mathbf{T}\}$ in $\mathbf{P}^-$. Let $\Phi'(u,x,y) \in \Sigma_0$ be

$$\forall p, \rho, w \leq u \left[ \begin{array}{l} \mathbf{Form}_{\Pi_0^-}(\rho(v_0,v_1)) \wedge \mathbf{Prf_T}(\forall v_0\, \exists v_1\, \rho(v_0,v_1), p, w) \ \rightarrow \\ \quad \forall z, z' \leq y \left\{ \begin{array}{l} y = \langle z, z' \rangle \ \rightarrow \\ \quad \rightarrow \left\{ \begin{array}{l} z = 2^{(x+z'+2)^{c^u}} \ \wedge \\ \forall x_0 \leq x\, \exists y_0 \leq z'\, \mathcal{V}_0(\rho, \langle x_0, y_0 \rangle, z) \end{array} \right\} \end{array} \right\} \end{array} \right]$$

Where $\mathcal{V}_0(v_1,v_2,v_3) \in \Delta_0$ is a truth definition in $\mathbf{I}\Delta_0 + \mathbf{exp}$ for $\Delta_0$ formulas and $c \in \omega$ is a constant which depends upon the explicit definition of $\mathcal{V}_0(v_1,v_2,v_3)$ (see [10], **V.5.4**). Let $\Phi(u,x,y) \in \Sigma_0$ defined as in case A. Now, as there, it is proved that $\Phi(u,x,y)$ is a $\Pi_0$–envelope of $\mathbf{T}$ in $\mathbf{I}\Delta_0$. □

*Remark* 5.12. Let $\mathbf{T}$ be $\Pi_n$–functional and $\Phi'(u,x,y,w) \in \Pi_n$ such that $\exists w\, \Phi'(u,x,y,w)$ is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. Let us see that there exists a $\Pi_n$ formula which is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.

Let $\Psi(u,x,y) \in \Pi_n$ be $\exists w, y' \leq y\, [y = \langle w, y' \rangle \wedge \Phi'(u,x,y',w)]$. For each $k \in \omega$, let $\Psi_k(x,y)$ be $\Psi(k,x,y)$. Then $\mathbf{T} \vdash \forall x\, \exists y\, \Psi_k(x,y)$. Let $\mathcal{C}_{\Psi_k}(x,y)$ be as in the proof of 3.8. The definition of $\mathcal{C}_{\Psi_k}(x,y)$ is uniform in $k$; so, using $k$ as a parameter we obtain $\mathcal{C}_{\Psi}(u,x,y) \in \Pi_n$. Let $\Theta(u,x,y) \in \Pi_n$ be

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = u + 1 \wedge \forall j \leq u\, \mathcal{C}_{\Psi}(j, x, (y)_j).$$

Then $\Theta(u,x,y)$ is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.

**Theorem 5.13.** (1) *For all $m \geq n$ ($m \geq 1$, for $n = 0$) there exists a $\Pi_n$–envelope of $\mathbf{I}\Sigma_m$ in $\mathbf{I}\Sigma_n$, $\Phi(u,x,y) \in \Pi_n$, such that*
- (a) $\mathbf{I}\Sigma_{m+1} \vdash \forall u\, \forall x\, \exists y\, \Phi(u,x,y)$.
- (b) $\mathbf{I}\Sigma_{m+1} \vdash \forall u, x, y_1, y_2\, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \rightarrow y_1 < y_2]$.

(2) *For all $n \in \omega$ there exists a $\Pi_n$–envelope, $\Phi(u,x,y) \in \Pi_n$, of $\mathbf{PA}$ in $\mathbf{I}\Sigma_n$ such that*
- (a) $\mathbf{Th}(\mathcal{N}) \vdash \forall u\, \forall x\, \exists y\, \Phi(u,x,y)$.
- (b) $\mathbf{Th}(\mathcal{N}) \vdash \forall u, x, y_1, y_2\, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \rightarrow y_1 < y_2]$.

*Proof.* Let $1 \leq n \leq m$. We will prove that the $\Pi_n$–envelope obtained in 5.12, from the one given in 5.11, satisfies the properties of *1–(a)* and *1–(b)*. Let $\Phi'(u,x,y) \in \Pi_n$ the formula

$$\forall p, \tau \leq u \left\{ \begin{array}{c} \mathbf{Form}_{\Pi_n^-}(\tau(v_0, v_1)) \wedge \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\forall v_0 \, \exists v_1 \, \tau(v_0, v_1), p) \;\rightarrow \\ \rightarrow \forall x_0 \leq x \, \exists y_0 \leq y \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}_0, \dot{y}_0)) \end{array} \right\}$$

It is enough to prove that $\mathbf{I}\Sigma_{m+1} \vdash \forall u \, \forall x \, \exists y \, \Phi'(u, x, y)$.

Let $\mathfrak{A} \models \mathbf{I}\Sigma_{m+1}$, $c$, $p$, $\tau \in \mathfrak{A}$, $p, \tau \leq c$, such that

$$\mathfrak{A} \models \mathbf{Form}_{\Pi_n^-}(\tau(v_0, v_1)) \wedge \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\forall v_0 \, \exists v_1 \, \tau(v_0, v_1), p).$$

Then $\mathfrak{A} \models \forall x \, \exists p \, \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\exists v_1 \, \tau(\dot{x}, v_1), p)$. By reflexion, see [10],

$$\mathbf{I}\Sigma_{m+1} \vdash \mathbf{Sent}_{\Sigma_{n+1}}(\sigma) \wedge \exists p \, \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\sigma, p) \rightarrow \mathbf{Sat}_{\Sigma_{n+1}}(\sigma).$$

Hence, $\mathfrak{A} \models \forall x \, \mathbf{Sat}_{\Sigma_{n+1}}(\exists v_1 \, \tau(\dot{x}, v_1))$. So, $\mathfrak{A} \models \forall x \, \exists y \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}, \dot{y}))$. Since $\mathfrak{A} \models \mathbf{B}\Pi_n$, for all $a \in \mathfrak{A}$ there is $b \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \forall x \leq a \, \exists y \leq b \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}, \dot{y})).$$

Then $\mathfrak{A} \models \forall x \, \exists y \, \Phi'(c, x, y)$. □

### 5.3. Strong $\Pi_n$–envelopes.

**Definition 5.14.** *Let $\Phi(u, x, y) \in \Pi_n$ a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{T}_0$. We say that $\Phi(u, x, y)$ is a strong $\Pi_n$–envelope if $\Gamma_\Phi$ is a strong $\Pi_n$–functional class.*

**Lemma 5.15.** *($n \geq 1$) Suppose that $\mathbf{T}$ is recursively axiomatizable and $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$. Let $\Phi(u, x, y) \in \Pi_n$ be a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. If $\mathbf{I}\Delta_0 + \Gamma_\Phi^* \Longrightarrow \mathbf{I}\Sigma_n$ then $\Phi(u, x, y)$ is a strong $\Pi_n$–envelope.*

*Proof.* By 5.3, $\Gamma_\Phi$ is a $\Pi_n$–functional class; hence, $\mathbf{I}\Sigma_n + \Gamma_\Phi^*$ is consistent. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma_\Phi^*$ and $\mathfrak{I} \prec_0^e \mathfrak{A}$ such that for all $k \in \omega$ and $a \in \mathfrak{I}$ there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \Phi(k, a, b)$. Let us see, using Tarski–Vaught test's, that $\mathfrak{I} \prec_n \mathfrak{A}$. Let $\Theta(x, y) \in \Pi_{n-1}$, $a \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y \, \Theta(a, y)$ and $c \in \mathfrak{A}$ such that $\mathfrak{I} < c$. Then for all $k \in \omega$, $\mathfrak{A} \models \exists y < c \, \Phi(k, a, y)$. So, by 5.6, there exists $\mathfrak{I}_1 \models \mathbf{T}$ such that $\mathfrak{I}_1 \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I}_1 < c$. Hence, $\mathfrak{I}_1 \models \exists y \, \Theta(a, y)$ and $\mathfrak{A} \models \exists y < c \, \Theta(a, y)$. Then, by underspill ($\mathfrak{A} \models \mathbf{I}\Sigma_n$), there exists $d \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y < d \, \Theta(a, y)$. So, there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \Theta(a, b)$. □

**Theorem 5.16.** *($n \geq 1$) There exists a formula $\mathbb{E}_n(u, x, y) \in \Pi_n$ such that*

1. *For every $k \in \omega$, $\mathbf{I}\Sigma_{n-1} \vdash$ IPF$(\mathbb{E}_n(k, x, y))$.*
2. *$\mathbb{E}_n(u, x, y)$ is a $\Pi_n$–q–envelope of $\mathbf{I}\Sigma_n$ in $\mathbf{I}\Sigma_n$.*
3. *$\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_0 + \Gamma_{\mathbb{E}_n}^*$.*
4. *$\Gamma_{\mathbb{E}_n}$ is a strong $\Pi_n$–functional class.*
5. *Let $\mathbb{K}_n(x) = y$ be the formula $\mathbb{E}_n(x, x, y)$ and $\Gamma_n = \{\mathbb{K}_n(x) = y\}$. Then*
   (a) *$\mathbf{I}\Sigma_n \vdash$ IPF$(\mathbb{K}_n)$.*
   (b) *$\mathbf{I}\Sigma_n \vdash \forall x \, \exists y \, [\mathbb{K}_n(x) = y]$.*
   (c) *$\Gamma_n$ is a strong $\Pi_n$–functional class.*

*Proof.* The idea is to define $\mathbb{E}_n(u, x, y)$ as the strong $\Pi_n$–functional class of 4.17, $\mathbb{H}_n$, in an uniform way. A $\Sigma_{n+1}$ formula similar to $\mathbb{K}_n(x) = y$ has been considered by R. Kaye in [11] and [13]. However, we need a $\Pi_n$ formula.

The proof of the theorem is by induction on $n \geq 1$.

<u>$(n = 1)$</u>: Let $\Theta_1'(\sigma, x, w, z) \in \Pi_1$ be the following formula

$$\left\{\begin{array}{l} \neg\exists v \leq x\, \exists y\, \exists z'\, [2^{(x+v+y+2)^{c^\sigma}} \leq z' \wedge \mathcal{V}_0(\sigma, \langle v, y\rangle, z') \wedge w = 0] \vee \\ \exists w_1, w_2 \leq w \left\{\begin{array}{l} w = \langle w_1, w_2\rangle \wedge w_1 \leq x \wedge \mathcal{V}_0(\sigma_{\mu,w_2}, \langle w_1, w_2\rangle, z) \wedge \\ \forall v \leq x \left\{\begin{array}{l} \forall y\, \forall z' \left\{\begin{array}{c} 2^{(x+v+y+2)^{c^\sigma}} \leq z' \wedge \mathcal{V}_0(\sigma, \langle v, y\rangle, z') \;\rightarrow \\ \rightarrow \exists y \leq w_2\, \mathcal{V}_0(\sigma, \langle v, y\rangle, z) \end{array}\right\} \\ \wedge\ (\mathcal{V}_0(\sigma_{\mu,w_2}, \langle v, w_2\rangle, z) \rightarrow v \leq w_1) \end{array}\right. \end{array}\right. \end{array}\right.$$

where $\mathcal{V}_0(\sigma_{\mu,w_2}, \langle w_1, w_2\rangle, z)$ is the formula

$$\mathcal{V}_0(\sigma, \langle w_1, w_2\rangle, z) \wedge \forall w < w_2 \neg\mathcal{V}_0(\sigma, \langle w_1, w\rangle, z).$$

Let $\Theta_1(\sigma, x, y)$ and $\mathbb{E}_1(u, x, y)$ be the following $\Pi_1$–formulas

$$\exists z, w \leq y \left\{\begin{array}{l} y = \langle w, z\rangle \wedge z = 2^{(x+w+2)^{c^\sigma}} \;\wedge \\ \left\{\begin{array}{l} [\mathbf{Form}_{\Pi_0^-}(\sigma(v_0, v_1)) \wedge \Theta_1'(\sigma, x, w, z)] \vee \\ [\neg\mathbf{Form}_{\Pi_0^-}(\sigma(v_0, v_1)) \wedge w = 0] \end{array}\right. \end{array}\right.$$

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = u + 1 \wedge \forall j \leq u\, \Theta_1(j, x, (y)_j).$$

The proof of *(1)* is as in 4.17, and *(2)* follows easily from the definition of $\mathbb{E}_1(u, x, y)$. We also have that

**Claim 5.17.** $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1} \Longrightarrow \mathbf{S}\Pi_0^-$.

*Proof of Claim.* Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$, $\Psi(x, y) \in \Pi_0^-$ and $a \in \mathfrak{A}$. Let us see that

$$\mathfrak{A} \models \exists w\, \forall x \leq a\, [\exists y\, \Psi(x, y) \rightarrow \exists y \leq w\, \Psi(x, y)].$$

Let $\psi = \ulcorner\Psi\urcorner$. Since $\mathfrak{A} \models \Gamma^*_{\mathbb{E}_1}$, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathbb{E}_1(\psi, a, b)$. Then $\mathfrak{A} \models \Theta(\psi, a, (b)_\psi)$; so, $\mathfrak{A} \models \Theta_1'(\psi, a, b', b'')$, where $(b)_\psi = \langle b', b''\rangle$. Let $d \in \mathfrak{A}$ such that $d \leq a$ and $\mathfrak{A} \models \exists y\, \Psi(d, y)$. Then

$$\mathfrak{A} \models \exists y\, \exists z'\, [2^{(a+d+y+2)^{c^\psi}} \leq z' \wedge \mathcal{V}_0(\psi, \langle d, y\rangle, z')].$$

So, $\mathfrak{A} \models \exists y \leq (b')_2\, \mathcal{V}_0(\psi, \langle d, y\rangle, b'')$. That is, $\mathfrak{A} \models \exists y \leq b\, \Psi(d, y)$. □

Since $\mathbf{S}\Pi_0^- \Longleftrightarrow \mathbf{S}\Pi_0 \Longleftrightarrow \mathbf{I}\Sigma_1$, then $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1} \Longrightarrow \mathbf{I}\Sigma_1$. This proves *(3)*. To prove *(4)*, follow the proof of 4.17. Now we prove *(5)*. From the definition of $\mathbb{K}_1(x) = y$, $\mathbf{I}\Sigma_1 \vdash \mathrm{IPF}(\mathbb{K}_1(x) = y)$. This gives *5–(a)*. We have that

**Claim 5.18.** $\mathbf{I}\Sigma_1 \vdash \forall x\, \exists y\, [\mathbb{K}_1(x) = y]$.

*Proof of Claim.* Since $\mathbf{I}\Sigma_1 \Longleftrightarrow \mathbf{S}\Pi_0$, then

$$\mathbf{I}\Sigma_1 \vdash \exists z_1\, \forall v \leq x \left[\begin{array}{l} \exists y, z\, (\mathcal{V}_0(\sigma, \langle v, y\rangle, z) \wedge 2^{(v+y+2)^{c^\sigma}} \leq z) \;\rightarrow \\ \quad \rightarrow \exists y, z \leq z_1 (\mathcal{V}_0(\sigma, \langle v, y\rangle, z) \wedge 2^{(v+y+2)^{c^\sigma}} \leq z) \end{array}\right]$$

By properties of $\mathcal{V}_0(v_1, v_2, v_3)$, (see theorem **V.5.4** of [10]), we have that

$$\mathbf{I}\Sigma_1 \vdash 2^{(v+y+2)^{c^\sigma}} \leq z_1, z_2 \rightarrow [\mathcal{V}_0(\sigma, \langle v, y\rangle, z_1) \leftrightarrow \mathcal{V}_0(\sigma, \langle v, y\rangle, z_2)].$$

Then, as in 4.17, we get that $\mathbf{I}\Sigma_1 \vdash \forall \sigma, x\, \exists w, z\, \Theta_1'(\sigma, x, w, z)$. □

This completes the proof of *5–(b)*. Now we prove *5–(c)*.

Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$ and $\mathfrak{I} \subset^e \mathfrak{A}$ such that for all $a \in \mathfrak{I}$ there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{K}_1(a) = b$. Let $k \in \omega$, $a \in \mathfrak{I}$ and $c = \max(k, a)$. Then $c \in \mathfrak{I}$; hence, there exists

$d \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{K}_1(c) = d$; that is, $\mathfrak{A} \models \mathbb{E}_1(c, c, d)$. Since $\mathbf{I}\Delta_0 \vdash \mathbb{E}_1(u, x, y) \to \forall v < u\, \exists z < y\, \mathbb{E}_1(v, x, z)$, there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{E}_1(k, a, b)$. Since $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$, by *(4)*, $\mathfrak{I} \prec_1 \mathfrak{A}$.

This proves 5–*(c)* and completes the proof of the theorem for $n = 1$.

<u>$(\leq n \to n+1)$</u>: For each $m$, $1 \leq m \leq n$, let $\mathbb{E}_m(u, x, y)$ be a $\Pi_m$ formula that satisfies *(1)–(5)*. Let $\Theta'_{n+1}(\sigma, x, w) \in \Pi_{n+1}(\mathbf{I}\Sigma_n)$ the formula

$$\left\{ \begin{array}{l} [\neg \exists v \leq x\, \exists y\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot v, \dot y)) \wedge w = 0] \vee \\ \exists w_1, w_2 \leq w \left\{ \begin{array}{l} w = \langle w_1, w_2 \rangle \wedge w_1 \leq x \wedge \mathbf{Sat}_{\Pi_n}(\sigma_{\mu, w_2}(\dot w_1, \dot w_2)) \wedge \\ \forall v \leq x \left\{ \begin{array}{l} \exists y\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot v, \dot y)) \to \exists y \leq w_2\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot v, \dot y)) \wedge \\ \mathbf{Sat}_{\Pi_n}(\sigma_{\mu, w_2}(\dot v, \dot w_2)) \to v \leq w_1 \end{array} \right. \end{array} \right. \end{array} \right.$$

(where $\mathbf{Sat}_{\Pi_n}(\sigma_{\mu, v_2}(\dot v_1, \dot v_2))$ is $\mathbf{Sat}_{\Pi_n}(\sigma(\dot v_1, \dot v_2)) \wedge \forall y < v_2\, \neg \mathbf{Sat}_{\Pi_n}(\sigma(\dot v_1, \dot y)))$. Let $\Theta_{n+1}(u, x, y)$ and $\mathbb{E}_{n+1}(u, x, y)$ be the following $\Pi_{n+1}$–formulas

$$\mathrm{Seq}(y) \wedge \lg(y) = u + 1 \wedge \forall j \leq u\, \Theta'_{n+1}(j, x, (y)_j),$$
$$\mathrm{Seq}(y) \wedge \lg(y) = n + 1 \wedge [\textstyle\bigwedge_{1 \leq m \leq n} \mathbb{E}_m(u, x, (y)_{m-1})] \wedge \Theta_{n+1}(u, x, (y)_n).$$

It is clear that for all $k \in \omega$, $\mathbf{I}\Sigma_{n+1} \vdash \forall x\, \exists y\, \Theta'_{n+1}(k, x, y)$. So, for all $k \in \omega$, $\mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\mathbb{E}_{n+1}(k, x, y))$ and $\mathbb{E}_{n+1}(u, x, y)$ is a $\Pi_{n+1}$–q–envelope of $\mathbf{I}\Sigma_{n+1}$ in $\mathbf{I}\Sigma_{n+1}$. This proves *(1)* and *(2)* for $\mathbb{E}_{n+1}(u, x, y)$. We also have that

**Claim 5.19.** $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}} \Longrightarrow \mathbf{I}\Sigma_n$.

*Proof of Claim.* Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}}$. By induction on $m$, $1 \leq m \leq n$, let us see that $\mathfrak{A} \models \mathbf{I}\Sigma_m$.

<u>$(m = 1)$</u>: By *(1)*, for $n = 1$, $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$. So, by *(3)* (for $n = 1$), $\mathfrak{A} \models \mathbf{I}\Sigma_1$.

<u>$(m \to m + 1 \leq n)$</u>: By induction hypothesis (on $n$, using *(1)* for $m + 1$), for all $k \in \omega$, $\mathbf{I}\Sigma_m \vdash \mathrm{IPF}(\mathbb{E}_{m+1}(k, x, y))$. By induction hypothesis (on $m$) $\mathfrak{A} \models \mathbf{I}\Sigma_m$, so $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{m+1}}$. Then, by induction hypothesis (on $n$, using *(3)* for $m + 1 \leq n$), $\mathfrak{A} \models \mathbf{I}\Sigma_{m+1}$. □

**Claim 5.20.** $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}} \Longrightarrow \mathbf{I}\Sigma_{n+1}$.

*Proof of Claim.* Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}}$. By 5.19, $\mathfrak{A} \models \mathbf{I}\Sigma_n$; so, $\Theta'_{n+1}(u, x, y)$ is $\Pi_{n+1}$ in $\mathfrak{A}$. Now, as in 5.17, we get that $\mathfrak{A} \models \mathbf{S}\Pi_n^-$; so, $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$. □

This proves *(3)*. The proofs of *(4)* and *(5)* are as for $n = 1$. □

**Theorem 5.21.** *$(n \geq 1)$ If $\mathbf{T}$ is recursively axiomatizable and $\Pi_n$–functional then there is a $\Pi_n$–formula which is a strong $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.*

*Proof.* By 5.11 and 5.12, there exists $\Theta(u, x, y) \in \Pi_n$ which is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. Using 5.16 we get $\mathbb{E}_n(u, x, y) \in \Pi_n$ which is a $\Pi_n$–q–envelope of $\mathbf{I}\Sigma_n$ in $\mathbf{I}\Sigma_n$. Let $\Phi(u, x, y) \in \Pi_n$ the following formula

$$\mathrm{Seq}(y) \wedge \lg(y) = 2 \cdot (u + 1) \wedge \forall j \leq u\, [\mathbb{E}_n(j, x, (y)_{2j}) \wedge \Theta(j, x, (y)_{2j+1})].$$

Since $\Theta(u, x, y)$ is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$, $\Phi(u, x, y)$ is a $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. By 5.16–*(3)*, $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_n} \Longleftrightarrow \mathbf{I}\Sigma_n$; hence, $\mathbf{I}\Delta_0 + \Gamma^*_\Phi \Longrightarrow \mathbf{I}\Sigma_n$. So, by 5.15, $\Phi(u, x, y)$ is a strong $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. □

**Corollary 5.22.** (1) *($n \geq 1$) For all $m \geq n$ there exists a strong $\Pi_n$–envelope of $\mathbf{I\Sigma}_m$ in $\mathbf{I\Sigma}_n$, $\Phi(u,x,y) \in \Pi_n$, such that*
   (a) $\mathbf{I\Sigma}_{m+1} \vdash \forall u\, \forall x\, \exists y\, \Phi(u,x,y)$.
   (b) $\mathbf{I\Sigma}_{m+1} \vdash \forall u,x,y_1,y_2\, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \to y_1 < y_2]$.

(2) *For all $n \in \omega$ there is a strong $\Pi_n$–envelope, $\Phi(u,x,y) \in \Pi_n$, of $\mathbf{PA}$ in $\mathbf{I\Sigma}_n$ such that*
   (a) $\mathbf{Th}(\mathcal{N}) \vdash \forall u\, \forall x\, \exists y\, \Phi(u,x,y)$.
   (b) $\mathbf{Th}(\mathcal{N}) \vdash \forall u,x,y_1,y_2\, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \to y_1 < y_2]$.

## 6. The hierarchy $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_m)$, $m \geq n$

For each formula $\psi(x,\vec{y})$ and term $t(x)$ of $\mathcal{L}(\Gamma)$, let $[\psi,t](z;\vec{y})$ denotes

$$z \leq t(\max(\vec{y})) \wedge \psi(z,\vec{y}) \wedge \forall x\, (\psi(x,\vec{y}) \to x = z).$$

**Definition 6.1.** *Let $\Gamma$ be a $\Pi_n$–functional class, $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$ and $\emptyset \neq X \subseteq \mathfrak{A}$. Let $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ be the substructure of $\mathfrak{A}$ whose universe is*

$$\{a \in \mathfrak{A} : \ \mathfrak{A} \models [\psi,t](a;\vec{b}),\ \psi \in \Delta_0^\Gamma,\ t \in \mathbf{Term}(\mathcal{L}(\Gamma)),\ \vec{b} \in X\},$$

*and let $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$ be the initial segment of $\mathfrak{A}$ given by $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$.*

*Remark* 6.2. Let $\Gamma$ be a $\Pi_n$–functional class, $\mathfrak{A} \models \mathbf{I\Delta}_0^{\Gamma^*}$ and $\emptyset \neq X \subseteq \mathfrak{A}$. The structures $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ and $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$ have similar properties, with respect to $\mathcal{L}(\Gamma)$, that $\mathcal{K}_1(\mathfrak{A};X)$ and $\mathcal{I}_1(\mathfrak{A};X)$. In particular, $\mathcal{K}_0^\Gamma(\mathfrak{A};X) \prec_0 \mathfrak{A}$ and $\mathcal{K}_0^\Gamma(\mathfrak{A};X) \prec_1^e \mathcal{I}_0^\Gamma(\mathfrak{A};X) \prec_0^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures.

*Remark* 6.3. Let $\Gamma$ be a strong $\Pi_n$–functional class, $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$ and $\emptyset \neq X \subseteq \mathfrak{A}$. Here we prove some basic facts on $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ and $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$. Let us first observe that since $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$, by 4.12, $\mathfrak{A} \models \mathbf{I\Delta}_0^{\Gamma^*}$.

**Claim 6.4.** *$\mathcal{I}_0^\Gamma(\mathfrak{A},X) \prec_n^e \mathfrak{A}$, as $\mathcal{L}$–structures, and $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$.*

*Proof of Claim.* The first part follows from 6.2. As a consequence, since $\Gamma \subseteq \Pi_n$, $\mathcal{I}_0^\Gamma(\mathfrak{A},X)$ is a model of $\mathbf{I\Delta}_0$ and for all $\varphi \in \Gamma$ it satisfies $\mathrm{IPF}(\varphi)$ and the definition axiom of $G_\varphi$. So, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$. □

**Claim 6.5.** *$\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_{n+1} \mathcal{I}_0^\Gamma(\mathfrak{A},X)$ and $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_n \mathfrak{A}$, as $\mathcal{L}$–structures. Also $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$.*

*Proof of Claim.* Let $\psi(x,\vec{w}) \in \Pi_n$ and $\vec{b} \in \mathcal{K}_0^\Gamma(\mathfrak{A},X)$ such that $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \exists x\, \psi(x,\vec{b})$. Let $\theta(x,\vec{w}) \in \Delta_0^\Gamma$ such that $\mathbf{I\Delta}_0^{\Gamma^*} \vdash \psi(x,\vec{w}) \leftrightarrow \theta(x,\vec{w})$. Then, by 6.4, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \exists x\, \theta(x,\vec{b})$. So, by 6.2, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \exists x\, \theta(x,\vec{b})$. Let $a \in \mathcal{K}_0^\Gamma(\mathfrak{A},X)$, such that $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \theta(a,\vec{b})$. Then $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \theta(a,\vec{b})$. Hence, by 6.4, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \psi(a,\vec{b})$. By Tarski–Vaught's test, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_{n+1} \mathcal{I}_0^\Gamma(\mathfrak{A},X)$. From this and 6.4 we get that $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_n \mathfrak{A}$. So, as in the second part of 6.4, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$. □

**Claim 6.6.** *If $\mathcal{K}_0^{\Gamma}(\mathfrak{A};X)$ is not cofinal in $\mathfrak{A}$ then $\mathcal{I}_0^{\Gamma}(\mathfrak{A},X) \models (\mathbf{B}\Sigma_{n+1} + \Gamma^*)_{\Gamma}$.*

*Proof of Claim.* Since $\Gamma$ is a strong $\Pi_n$–functional class, $\mathfrak{A} \models \mathbf{I}\Sigma_n$. We also have that $\mathcal{I}_0^{\Gamma}(\mathfrak{A},X) \prec_n^e \mathfrak{A}$ and is proper; hence, $\mathcal{I}_0^{\Gamma}(\mathfrak{A};X) \models (\mathbf{B}\Sigma_{n+1} + \Gamma^*)_{\Gamma}$. □

*Remark* 6.7. Let $\varphi(u,x,y) \in \Pi_n$ be a strong $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{T}$, where $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$. We shall denote by $G_k$ the function symbol of $\mathcal{L}(\Gamma_\varphi)$ associated with $\varphi(k,x,y)$ and by $[\psi,k](x;\vec{y})$ the formula $[\psi,G_k](x;\vec{y})$. Let $\mathfrak{A} \models \mathbf{T}_{\Gamma_\varphi}$ and $a \in \mathfrak{A}$ nonstandard. We have that

**Claim 6.8.** *$\{G_k(a) : \ k \in \omega\}$ is cofinal in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$.*

*Proof of Claim.* Let $b \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$. Then there are $\psi(x,y) \in \Delta_0^{\Gamma_\varphi}$ and a term $t(x)$ of $\mathcal{L}(\Gamma_\varphi)$ such that $\mathfrak{A} \models [\psi,t](b;a)$. By 4.14 there exists $\theta(x,z) \in \Sigma_{n+1}$ such that $(\mathbf{I}\Delta_0 + \Gamma_\varphi^*)_\varphi \vdash t(x) = z \leftrightarrow \theta(x,z)$. Since $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_0 + \Gamma_\varphi^*$, then $\mathbf{T} \vdash \forall x\, \exists z\, \theta(x,z)$; so, there is $k \in \omega$ such that $\mathbf{T} \vdash \varphi(k,x,y) \rightarrow \exists z \leq y\, \theta(x,z)$. Hence, $(\mathbf{T} + \Gamma_\varphi^*)_{\Gamma_\varphi} \vdash t(x) \leq G_k(x)$. So, $\mathfrak{A} \models b \leq t(a) \leq G_k(a)$. □

**Claim 6.9.** *Suppose that $\mathfrak{A} \models \forall u,x,y_1,y_2\, [\varphi(u,x,y_1) \wedge \varphi(u+1,x,y_2) \rightarrow y_1 < y_2]$ and $\mathfrak{A} \models \forall u\, \forall x\, \exists y\, \varphi(u,x,y)$. Then $\omega$ is definable in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ by the formula $\exists y\, \varphi(u,a,y)$; that is, by a $\Sigma_{n+1}$ formula with parameters.*

*Proof of Claim.* Let us see that $\{c \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) : \ \mathcal{K}_0^{\Gamma_\varphi} \models \exists y\, \varphi(c,a,y)\} \subseteq \omega$. Let $c,d \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ such that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \varphi(c,a,d)$. Since $\varphi \in \Pi_n$, by 6.5, $\mathfrak{A} \models \varphi(c,a,d)$. By 6.8, there exists $k \in \omega$ such that $\mathfrak{A} \models d \leq G_k(a)$. Then, $c \leq k$; hence, $c \in \omega$. □

**Theorem 6.10.** *Let $\mathbf{T}$ be a $\Pi_n$–functional theory (if $n = 0$ we assume that $\mathbf{T} \vdash \mathbf{exp}$), $\varphi(u,x,y) \in \Pi_n$ a strong $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{T}$, $\mathfrak{A} \models \mathbf{T}_{\Gamma_\varphi}$ and $a \in \mathfrak{A}$ nonstandard. Then*

1. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_1^c \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_0^e \mathfrak{A}$, as $\mathcal{L}(\Gamma_\varphi)$–structures.*
2. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_{n+1}^c \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_n^e \mathfrak{A}$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_n \mathfrak{A}$, as $\mathcal{L}$–structures.*
3. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_0^{\Gamma_\varphi}$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{B}\Sigma_{n+1}$.*
4. *$\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Sigma_{n+1}$.*
5. *If $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ is not cofinal in $\mathfrak{A}$ then $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{B}\Sigma_{n+1}$.*
6. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*
7. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$.*

*Proof.* Part *(1)* follows from 6.2 and part *(2)* from 6.3.
(*(3)*): By 6.5 it is only necessary to prove that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{B}\Sigma_{n+1}$. Let $b \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$. Then there exist $\psi(x,y) \in \Delta_0^{\Gamma_\varphi}$ and $k \in \omega$ such that $\mathfrak{A} \models [\psi,k](b;a)$. By 4.14, there exist $\theta(x,y,z) \in \Pi_n$ and a term $t(x,y)$ of $\mathcal{L}(\Gamma_\varphi)$ such that $\mathbf{I}\Delta_0^{\Gamma_\varphi} \vdash \forall z \geq t(x,y)\, [\psi(x,y) \leftrightarrow \theta(x,y,z)]$. Let $c_1,c_2 \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ such that $c_1 \geq b$ and $c_2 \geq t(c_1,a)$. Then $\mathfrak{A} \models \psi(x,a) \leftrightarrow [\theta(x,a,c_2) \wedge x \leq c_1]$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},X) \models \psi(x,a) \leftrightarrow [\theta(x,a,c_2) \wedge x \leq c_1]$. Let $\theta'(x,y,z_1,z_2)$ be the formula $\theta(x,y,z_2) \wedge x \leq z_1$. We consider the following cases.

Case A: $n = 0$. For all $\delta(y_1, \dots, y_m) \in \Delta_0$ there exists $r \in \omega$ such that

$$\mathbf{I}\Delta_0 \vdash 2^{(\max(y_1,\dots,y_m)+2)^r} \le u \to [\delta(y_1, \dots, y_m) \leftrightarrow \mathcal{V}_0(\ulcorner\delta\urcorner, \langle y_1, \dots, y_m\rangle, u)].$$

Since $\mathbf{T} \vdash \mathbf{exp}$, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \mathcal{V}_0(\ulcorner\theta'\urcorner, \langle b, a, c_1, c_2\rangle, 2^{(\max(b,a,c_1,c_2)+2)^r})$. Let $d \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ nonstandard. Then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ satisfies the following formula

$$\exists\sigma \le d \left\{ \begin{array}{l} \mathbf{Form}_{\Delta_0}(\sigma) \wedge \\ \exists z_1\, \exists z_2 \left\{ \begin{array}{l} \mathcal{V}_0(\sigma, \langle b, a, z_1, z_2\rangle, 2^{(\max(b,a,z_1,z_2)+2)^a}) \wedge \\ \forall x < b\, \neg\mathcal{V}_0(\sigma, \langle x, a, z_1, z_2\rangle, 2^{(\max(x,a,z_1,z_2)+2)^a}) \end{array} \right. \end{array} \right.$$

Since $b$ is an arbitrary element of $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$, then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ satisfies

$$\forall u \le d+1\, \exists\sigma \le d \left\{ \begin{array}{l} \mathbf{Form}_{\Delta_0}(\sigma) \wedge \\ \exists z_1\, \exists z_2 \left\{ \begin{array}{l} \mathcal{V}_0(\sigma, \langle u, a, z_1, z_2\rangle, 2^{(\max(u,a,z_1,z_2)+2)^a}) \wedge \\ \forall x < u\, \neg\mathcal{V}_0(\sigma, \langle x, a, z_1, z_2\rangle, 2^{(\max(x,a,z_1,z_2)+2)^a}) \end{array} \right. \end{array} \right.$$

Let $\gamma(d, a)$ denotes this formula. Then $\gamma(d, a) \in \Sigma_1$ (in $\mathbf{B}\Sigma_1$). Assume that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ is a model of $\mathbf{B}\Sigma_1$. Since $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \prec_0 \mathfrak{A}$, $\mathfrak{A} \models \gamma(d, a)$. So, there is in $\mathfrak{A}$ an one–one $\Sigma_1$–mapping from $(\le d+1)$ to $(\le d)$, a contradiction.

Case B: $n \ge 1$. We proceed as in case A but now we use $\mathbf{Sat}_{\Pi_n}(x)$. Let $\gamma(d, a)$ be the following formula

$$\forall u \le d+1\, \exists\sigma \le d \left\{ \begin{array}{l} \mathbf{Form}_{\Pi_n}(\sigma) \wedge \\ \exists z_1\, \exists z_2 \left\{ \begin{array}{l} \mathbf{Sat}_{\Pi_n}(\sigma(\dot{u}, \dot{a}, \dot{z}_1, \dot{z}_2)) \wedge \\ \forall x < u\, \neg\mathbf{Sat}_{\Pi_n}(\sigma(\dot{x}, \dot{a}, \dot{z}_1, \dot{z}_2)) \end{array} \right. \end{array} \right.$$

We have that $\gamma \in \Sigma_{n+1}(\mathbf{B}\Sigma_{n+1})$. Assume that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \mathbf{B}\Sigma_{n+1}$. Since $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \prec_n \mathfrak{A}$, $\mathfrak{A} \models \gamma(d, a)$. So, there exists in $\mathfrak{A}$ an one–one $\Sigma_{n+1}$–mapping from $(\le d+1)$ to $(\le d)$, a contradiction.

(*(4)*): Suppose that $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \mathbf{I}\Sigma_{n+1}$. By *(2)* we have that for all $k \in \omega$, $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \exists y\, [\varphi(k, a, y) \wedge \forall u < k\, \exists z < y\, \varphi(u, a, z)]$. Then by $\Sigma_{n+1}$–overspill there exists $c \in \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ nonstandard such that

$$\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \exists y\, [\varphi(c, a, y) \wedge \forall u < c\, \exists z < y\, \varphi(u, a, z)].$$

Let $b \in \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ such that $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \varphi(c, a, b) \wedge \forall u < c\, \exists z < y\, \varphi(u, a, z)$. Then for all $k \in \omega$, $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models G_k(a) < b$, a contradiction (see 6.8).

(*(5)*): This follows from 6.6.

(*(6)*): Let $\psi(x, z) \in \Pi_n$ and $k \in \omega$ such that $\mathbf{T} \vdash \forall x\, \exists z\, \psi(x, z)$ and

$$\mathbf{T} \vdash \forall x\, \forall y\, [\varphi(k, x, y) \to \exists z \le y\, \psi(x, z)].$$

Let $\psi'(x, z) \in \Delta_{n+1}(\mathbf{B}\Sigma_n)$ be $\psi(x, z) \wedge \forall w < z\, \neg\psi(x, w)$. For all $b \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ there is $c \in \mathfrak{A}$ such that $\mathfrak{A} \models \psi'(b, c)$. Since $\Gamma_\varphi$ is a strong $\Pi_n$–envelope, by 4.12, $\psi'(x, z) \in \Delta_{n+1}(\mathbf{I}\Delta_0 + \Gamma_\varphi^*)$; hence, there exists $\theta(x, z) \in \Delta_0^{\Gamma_\varphi}$ such that $\mathbf{I}\Delta_0^{\Gamma_\varphi} \vdash \psi'(x, z) \leftrightarrow \theta(x, z)$. Since $\mathfrak{A} \models \mathbf{I}\Delta_0^{\Gamma_\varphi}$, $\mathfrak{A} \models [\theta, k](c; b)$ and $c \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$. Since $\theta \in \Delta_0^{\Gamma_\varphi}$, by *(1)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \theta(b, c)$. So, by *(3)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \psi'(b, c)$; hence, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \psi(b, c)$. So, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a) \models \forall x\, \exists z\, \psi(x, z)$. This proves *(6)*.

(*(7)*): This follows from 2.13 ($\mathbf{T}$ is $\Pi_n$–functional and 3.8) and *(6)*. $\square$

**Theorem 6.11** *(The Hierarchy Theorem). Let $\mathbf{T}$ be a $\Pi_n$–functional theory (if $n=0$ we assume that $\mathbf{T} \vdash \mathbf{exp}$), $\varphi(u,x,y)$ a strong $\Pi_n$–envelope of $\mathbf{T}$ in $\mathbf{T}$ and $\mathbf{T}'$ an extension of $\mathbf{T}$ such that $\mathbf{T}' \vdash \forall u \, \forall x \, \exists y \, \varphi(u,x,y)$, and $\mathbf{T}' \vdash \varphi(u,x,y_1) \wedge \varphi(u+1,x,y_2) \rightarrow y_1 < y_2$. Then*

(1) *For each $\mathfrak{A} \models \mathbf{T}'_{\Gamma_\varphi}$ and $a \in \mathfrak{A}$ nonstandard, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Delta_{n+1}(\mathbf{T}')$.*

(2) $\mathbf{I}\Delta_{n+1}(\mathbf{T}') \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

*Proof.* Part *(2)* follows from *(1)*. By 6.10–*(7)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$. Since $\exists y \, \varphi(u,x,y) \in \Delta_{n+1}(\mathbf{T}')$ and, by 6.9, $\exists y \, \varphi(u,a,y)$ defines $\omega$ in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$, then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Delta_{n+1}(\mathbf{T}')$. □

**Theorem 6.12.** (1) *For all $m \leq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m) \Longleftrightarrow \mathbf{I}\Sigma_n$.*

(2) *For all $m \geq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1}) \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$.*

(3) $\mathbf{I}\Delta_{n+1}(\mathcal{N}) \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{PA})$.

*Proof.* *(1)* follows from 2.18. Let us see *(2)*. By 5.22–*(1)*, for every $m \geq n$ there exists a strong $\Pi_n$–envelope that satisfies the hypothesis of 6.11 for $\mathbf{T} = \mathbf{I}\Sigma_m$ and $\mathbf{T}' = \mathbf{I}\Sigma_{m+1}$; hence, *(2)* follows from 6.11–*(1)*. Part *(3)* is proved in a similar way using 5.22–*(2)*. □

**Lemma 6.13.** *For every $m \geq n$, $\mathbf{B}\Sigma_{n+1} \not\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1})$.*

*Proof.* Since $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ is a $\Pi_{n+2}$–axiomatizable theory (see [8], theorem **1.1**, or [7], [15]) and, by 6.12, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) \models\!\!\!\Longrightarrow \mathbf{I}\Sigma_n$, the result follows from 1.3. □

**Theorem 6.14.** (1) *For all $m \geq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_{m+1})$.*

(2) $\mathbf{I}\Delta_{n+1}(\mathbf{PA}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{PA})$.

(3) $\mathbf{I}\Delta_{n+1}(\mathcal{N}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathcal{N})$.

*Proof.* First observe that for every theory $\mathbf{T}$, $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, and if $\mathbf{T}$ has $\Delta_{n+1}$–collection then, by 2.10, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. Since $\mathbf{I}\Sigma_{m+1}$, $m \geq n$, $\mathbf{PA}$ and $\mathbf{Th}(\mathcal{N})$ have $\Delta_{n+1}$–collection, then *(1)*, *(2)* and *(3)* follow from 6.13. □

## 7. Remarks and open questions

The main problem we have considered in this work is the Paris–Friedman's Conjecture in three versions

(1) Paris–Friedman's Conjecture: $\mathbf{I}\Delta_{n+1} \Longleftrightarrow \mathbf{L}\Delta_{n+1}$.
(2) Uniform Paris–Friedman's Conjecture: $\mathbf{UI}\Delta_{n+1} \Longleftrightarrow \mathbf{UL}\Delta_{n+1}$.
(3) Parameter Free Paris–Friedman's Conjecture: $\mathbf{I}\Delta^-_{n+1} \Longleftrightarrow \mathbf{L}\Delta^-_{n+1}$.

From Slaman's result, it holds $\mathbf{I}\Delta_{n+1} \Longleftrightarrow \mathbf{L}\Delta_{n+1}$, for $n \geq 1$. We have studied here the relativization of these problems to $\Delta_{n+1}$ formulas in a theory $\mathbf{T}$. This gives a new version of the Conjecture.

*4.* Relativized Paris–Friedman's Conjecture: $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.

We have proved that if $\mathbf{T}$ satisfies some conditions then the relativized Paris–Friedman's Conjecture for $\mathbf{T}$ holds. So, we consider the following strong forms of these Conjectures.

*Problem* 1. Does it hold that for all $\mathbf{T}$ extension of $\mathbf{I}\Sigma_n$

(1) If $\mathbf{T}$ is $\Delta_{n+1}$–closed then $\mathbf{T}$ has $\Delta_{n+1}$–collection?
(2) If $\mathbf{T}$ has $\Delta_{n+1}$–induction then $\mathbf{T}$ has $\Delta_{n+1}$–collection?
(3) If $\mathbf{T}$ is $\Delta_{n+1}$–PF then $\mathbf{T}$ has $\Delta_{n+1}$–collection?

Let us observe that if every (complete) extension of $\mathbf{I}\Sigma_n$ satisfies 1–*(2)* then the Uniform Paris–Friedman's Conjecture holds.

Condition 3.10–*(2)* is related with the Uniform Paris-Friedman's Conjecture. Let $\mathfrak{A} \models \mathbf{UI}\Delta_{n+1}$ and $\varphi(x, y) \in \Pi_n^-$ such that $\mathfrak{A} \models \forall x\, \exists y\, \varphi(x, y)$. Let $F_\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$ be defined by: $F_\varphi(a) = (\mu y)[\varphi(a, y)]$. Let $F_\varphi^*$ be, the bounding map of $F_\varphi$, defined by

$$F_\varphi^*(a) = (\mu x)_{\leq a}[\forall u \leq a\, (F_\varphi(u) \leq F_\varphi(x))]$$

**Claim.** Let $\mathfrak{A} \models \mathbf{UI}\Delta_{n+1}$. If for each $\varphi(x, y) \in \Pi_n^-$ such that $\mathfrak{A} \models \forall x\, \exists! y\, \varphi(x, y)$, it holds that $F_\varphi^*$ is a total function on $\mathfrak{A}$ then $\mathfrak{A} \models \mathbf{UL}\Delta_{n+1}$.

Let us consider the following question.

*Problem* 2. In the above conditions. Is $F_\varphi^*$ a total function?

In 3.12 we have obtained a conservativeness property, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$, under which $\mathbf{T}$ is $\Pi_n$–functional and, hence, satisfies the Relativized Paris–Friedman's Conjecture. We have also extended this result in 3.13 for $\Sigma_{n+2}$ extensions of $\Pi_{n+2}$ axiomatizable theories. Let us consider the following problems.

*Problem* 3.

(1) Let $\mathbf{T}$ be a theory such that $\mathbf{T}+\mathbf{B}\Sigma_{n+1}$ is consistent. Are the following conditions equivalent?
   (a) $\mathbf{T}$ is $\Pi_n$–functional.
   (b) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.
   (c) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1}^-)$.
(2) Let $\mathbf{T}$ be a $\Pi_{n+2}$–axiomatizable extension of $\mathbf{I}\Sigma_n$ and let $\mathbf{T}'$ be $\Sigma_{n+2}$–axiomatizable such that $\mathbf{T} + \mathbf{T}'$ is consistent. Does it hold that

$$\mathbf{T} \text{ is } \Pi_n\text{–functional} \iff \mathbf{T} + \mathbf{T}' \text{ is } \Pi_n\text{–functional?}$$

In 5.11 it is proved that if $\mathbf{T}$ is $\Pi_n$–functional and recursively axiomatizable then $\mathbf{T}$ has a $\Pi_n$–envelope in $\mathbf{I}\Sigma_n$ (for $n = 0$ we add that $\mathbf{T} \vdash \mathbf{exp}$). For all $n \in \omega$, $\mathbf{Th}_{\Pi_{n+2}}(\mathcal{N})$ is $\Pi_n$–functional and proves $\mathbf{exp}$. Nevertheless, $\mathbf{Th}_{\Pi_{n+2}}(\mathcal{N})$ does not have a $\Pi_n$–envelope in $\mathbf{I}\Sigma_n$. So, it cannot be omitted that $\mathbf{T}$ is recursively axiomatizable.

Now, we will consider if $\mathbf{T} \vdash \mathbf{exp}$ could be eliminated for $n = 0$. The theory $\mathbf{I}\Pi_1^-$ has $\Pi_0$–collection, is recursively axiomatized and $\mathbf{I}\Pi_1^- \nvdash \mathbf{exp}$. It holds that if $\varphi(x, y) \in \Delta_0^-$ and $\mathbf{I}\Pi_1^- \vdash \forall x\, \exists y\, \varphi(x, y)$ then there exists $k \in \omega$ such that $\mathbf{I}\Pi_1^- \vdash \exists z\, \forall x\, [z < x \rightarrow \exists y < x^k\, \varphi(x, y)]$ (see [5]).

From this it follows that $\varphi(u, x, y) \equiv x^u + u = y$ is a $\Pi_0$–envelope of $\mathbf{I}\Pi_1^-$ in $\mathbf{Th}_{\Pi_1}(\mathcal{N})$. Let us consider the following problem.

*Problem* 4. Is there a $\Pi_0$–envelope of $\mathbf{I\Pi}_1^-$ in $\mathbf{I\Delta}_0$?

In section 6 the models $\mathcal{K}_0^\Gamma(\mathfrak{A}, a)$ have been used to separate the fragments $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_m)$, $m \geq n$. Theorem 1.1 sums up results obtained using these models. Let us consider the following problem.

*Problem* 5. Is strict the following chain of theories?

$$\mathbf{B}^*\Delta_{n+1}(\mathcal{N}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{PA}) \Longrightarrow \ldots \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I\Sigma}_{n+1}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I\Sigma}_n)$$

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