UNIVERSALITY OF SEQUENCES OF OPERATORS RELATED TO TAYLOR SERIES

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Abstract. In this note, the universality of a sequence of operators associated to the partial sums of the Taylor series of a holomorphic function is investigated. The emphasis is put on the fact that the Taylor series are evaluated at a prescribed point and the variable is the center of the expansion. The dynamics of the sequence of operators linked to the partial sums of a power series that is not generated by an entire function is also studied.

1. Introduction, preliminaries and background

Universal Taylor series and universal sequences of differential operators have been largely investigated along the last decades; see [5–7, 17, 18, 24], [1, Chapter 3] and the references contained in them. This paper deals with specific points inside both topics, which are, in a certain sense, connected. We will use notation that is mostly standard, so that the reader who is already acquainted with it may skip the next three paragraphs.

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}_\infty$ and $B(z_0, r)$ will represent, respectively, the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the field of rationals, the real line, the complex plane, the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, the extended complex plane $\mathbb{C} \cup \{\infty\}$, and the open ball $\{z \in \mathbb{C} : |z - z_0| < r\}$ with center z_0 and radius r. By a domain we mean a nonempty connected open set $G \subset \mathbb{C}$. We say that a domain G is simply connected whenever $\mathbb{C}_\infty \setminus G$ is connected. For any domain G, the vector space $H(G)$ of holomorphic functions $G \to \mathbb{C}$ is endowed with the topology of uniform convergence on compact subsets of G. It is well known (see, e.g. [20]) that, under this topology, $H(G)$ becomes an F-space, that is, a complete metrizable topological vector space. Moreover, $H(G)$ is separable. If K is a compact subset of $\mathbb{C}$, then $A(K)$ will stand for the space of all continuous functions $K \to \mathbb{C}$ that are holomorphic in the interior K° of K. The set $A(K)$ becomes a separable Banach space under the norm $\|f\|_\infty = \max_{z \in K} |f(z)|$, that generates

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the topology of uniform convergence on K. By $\overline{A}$ we denote the closure in a topological space X of a subset $A \subset X$.

Some additional terminology, borrowed from the theories of lineability and of linear chaos, will be needed. For background on them, the reader may consult [1,2,6,12,13,21,24,38]. Assume that X and Y are (Hausdorff) topological vector spaces. Then a subset $A \subset X$ is said to be *dense-lineable* (*spaceable,* resp.) in X whenever there is a dense (a closed infinite dimensional, resp.) vector subspace M of X such that $M \setminus \{0\} \subset A$.

Let us denote by $L(X,Y)$ the space of all continuous linear mappings $X \to Y$, and by $L(X)$ the space $L(X,X)$ of all operators on X. A sequence $(T_n)_n \subset L(X,Y)$ is said to be *hypercyclic* (or *universal*) provided that there is a vector $x_0 \in X$ –called hypercyclic or universal for $(T_n)_n$– such that the orbit $\{T_n x_0 : n \in \mathbb{N}\}$ of x_0 under $(T_n)_n$ is dense in Y. An operator $T \in L(X)$ is said to be *hypercyclic* if the sequence $(T^n)_n$ of its iterates is hypercyclic. The corresponding sets of hypercyclic vectors will be respectively denoted by $HC((T_n)_n)$ and $HC(T)$. A sequence $(T_n)_n \subset L(X,Y)$ is said to be *transitive* (*mixing,* resp.) provided that, given two nonempty open sets $U \subset X, V \subset Y$, there is $n_0 \in \mathbb{N}$ such that $T_{n_0}(U) \cap V \neq \varnothing$ (such that $T_n(U) \cap V \neq \varnothing$ for all $n \geq n_0$, resp.). From Birkhoff transitivity theorem (see, e.g., [24]), we have that, provided that X and Y are F-spaces and Y is separable, a sequence $(T_n)_n \subset L(X,Y)$ is transitive if and only if $HC((T_n)_n)$ is residual (in fact, a dense G_δ subset) in X. Moreover, $(T_n)_n$ is mixing if and only if any subsequence $(T_{n_k})_k$ is transitive.

Let $G \subset \mathbb{C}$ be a domain with $G \neq \mathbb{C}$, $\zeta \in G$ and $f \in H(G)$. Then f is said to be a *universal Taylor series* with center ζ provided that it satisfies the following property: For every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a (strictly increasing) sequence $(\lambda_n) \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \sup_{z\in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0,$$

where $S(N, f, \zeta)$ represents the Nth partial Taylor sum of f at ζ, that is,

$$S(N, f, \zeta)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta)}{j!}(z-\zeta)^j \qquad (z \in \mathbb{C},\ N \in \mathbb{N}_0).$$

This concept dates back to Nestoridis [32], who studied a kind of universality which was slightly stronger than the one considered by Luh [25,26] and Chui and Parnes [19] (where K is supposed not to cut $\overline{G}$). The set of universal Taylor series in G with center ζ is denoted by $U(G,\zeta)$. It is proved in [32] that $U(\mathbb{D},0)$ is a dense G_δ subset of $H(\mathbb{D})$, and this is generalized in [33] by showing that $U(G,\zeta)$ is a dense G_δ subset of $H(G)$ for any simply connected domain G and any $\zeta \in G$. Now, for a domain $G \subset \mathbb{C}$, let $U(G)$ denote the family of all functions $f \in H(G)$ satisfying that, for every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a sequence $(\lambda_n) \subset \mathbb{N}_0$ such that, for every compact set $L \subset G$, one has

$$\lim_{n\to\infty} \sup_{\zeta\in L} \sup_{z\in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0.$$

Obviously, $U(G) \subset U(G,\zeta)$ for all $\zeta \in G$. It is shown in [33] that $U(G)$ is a dense G_δ subset of $H(G)$ if G is simply connected, in [28] that $U(G) = \varnothing$ if G is not simply connected, and in [31] that $U(G,\zeta) = U(G)$ if G is simply connected and ζ is any point of G.

According to [40], Nestoridis posed the question of whether the universality of Taylor series is preserved if we fix the point of evaluation z (without loss of generality, we may assume $z = 0$) and the center ζ of expansion is variable. To be more specific, the question is whether the set

$$\mathcal{S}(G) := \big\{ f \in H(G) : \{\widetilde{T_n} f\}_{n\geq 0} \text{ is dense in } H(G) \big\}$$

is not empty, where

$$(\widetilde{T_n} f)(\zeta) := \sum_{j=0}^{n} \frac{f^{(j)}(\zeta)}{j!} (-\zeta)^j \quad (\zeta \in G,\ n \geq 0). \tag{1}$$

We remark the connection: $\mathcal{S}(G) = HC((\widetilde{T_n})_n)$, where we are considering $\widetilde{T_n} \in L(H(G))$ $(n \geq 0)$. It is proved in [40, Section 4] that $\mathcal{S}(G)$ is always a G_δ subset of $H(G)$ (the proof is there given for a simply connected domain G, but it can be extended to any domain, just by replacing the dense sequence (p_j) of polynomials by a dense sequence in $H(G)$, which exists thanks to the separability of $H(G)$), that $\mathcal{S}(G) = \varnothing$ if $0 \in G$ and that, if G is simply connected, then $\mathcal{S}(G)$ is either empty or dense (so either empty or residual). In [40] the broader class

$$\mathcal{S}_t(G) := \big\{ f \in H(G) : \overline{\{\widetilde{T_n} f\}_{n\geq 0}} \supset \{\text{constants}\} \big\}$$

is also considered, and it is shown to be a G_δ subset of $H(G)$. Once again, $\mathcal{S}_t(G) = \varnothing$ if $0 \in G$. Moreover, if G is simply connected and $0 \notin G$, then $\mathcal{S}_t(G)$ is dense (hence residual) in $H(G)$. Recently, Panagiotis [34] has answered the conjecture by Nestoridis (see [40]) in the affirmative by proving that $\mathcal{S}(G) \neq \varnothing$ in the special case where G is an open disc not containing 0.

In this paper, we prove –with methods that are rather different from those in [34]– that the condition $0 \notin G$ *characterizes* the non-vacuousness of $\mathcal{S}(G)$ if G is simply connected. In fact, in Section 2, we shall study the universality of sequences that are more general than $(\widetilde{T_n})$. Finally, in Section 3, the dynamics of the sequence of differential operators generated by a *power series* with finite radius of convergence is investigated, and lineability properties of the corresponding sets of universal functions are shown.

2. Universality of Taylor-like series

In this section, the hypercyclicity of the sequence of operators $\widetilde{T_n}$ $(n \geq 0)$ given by (1) will be studied. In order to tackle the problem, we shall adopt a slightly general point of view, by considering the following more general families of operators.

For each $(a, n, f, z) \in \mathbb{C} \times \mathbb{N}_0 \times H(G) \times G$, we set

$$(T_{a,n} f)(z) := \sum_{j=0}^{n} \frac{f^{(j)}(z)}{j!} \cdot (az)^j. \tag{2}$$

Note that $\widetilde{T_n} = T_{-1,n}$. From the continuity of the derivative operator D $(Df := f')$, it follows that every $T_{a,n}$ is a well defined continuous linear mapping $H(G) \to H(G)$, that is, $(T_{a,n})_n \subset L(H(G))$ for all $a \in \mathbb{C}$. We start with a necessary condition for universality. As usual, ∂A represents the boundary of a set $A \subset \mathbb{C}$.

Proposition 2.1. *Let $a \in \mathbb{C}$. Assume that $G \subset \mathbb{C}$ is a domain, and that the sequence of operators $T_{a,n} : H(G) \to H(G)$ $(n \in \mathbb{N})$ defined by (2) is universal. Then we have:*

(a) $0 \notin G$, *and*
(b) $|a| \geq \sup_{z \in G} \dfrac{\text{dist}\,(z, \partial G)}{|z|}$.

Proof. (a) By hypothesis, there is $f \in HC((T_{a,n})_n)$. Proceeding by way of contradiction, assume that $0 \in G$. Consider the constant function $g(z) := 1 + f(0)$. Then there would exist a sequence $(n_k) \subset \mathbb{N}$ such that $T_{n_k} f \to g$ $(k \to \infty)$ uniformly on every compact set $K \subset G$. In particular, for $K = \{0\}$, we would obtain

$$f(0) = \frac{f^{(0)}(0)}{0!} = (T_{n_k} f)(0) \longrightarrow g(0) = 1 + f(0) \quad \text{as} \quad k \to \infty,$$

which is clearly absurd.
(b) We proceed, again, by way of contradiction, so that we are simultaneously assuming $|a| < \sup_{z \in G} \frac{\text{dist}\,(z,\partial G)}{|z|}$ and the existence of an $f \in HC((T_{a,n})_n)$. Then there exists $z_0 \in G$ such that $|a| < \frac{R}{|z_0|}$, where $R := \text{dist}\,(z_0, \partial G)$. Therefore $B(z_0, R) \subset G$. Consequently, the Taylor expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ holds in $B(z_0, R)$ for our function f. Due to the hypercyclicity of f, some subsequence of $(T_{a,n} f)_n$ should tend in the compact set $K = \{z_0\} \subset G$ to any prescribed constant, in particular, to the constant $1 + f((a+1)z_0)$: this is, indeed, a well defined number because

$|(a+1)z_0 - z_0| = |az_0| < R$, and so $(a+1)z_0 \in B(z_0, R) \subset G$. However,

$$(T_{a,n}f)(z_0) = \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}(az_0)^j = \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}((a+1)z_0 - z_0)^j$$
$$\longrightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}((a+1)z_0 - z_0)^n = f((a+1)z_0)$$

as $n \to \infty$, which is the sought-after contradiction. $\square$

Remarks 2.2. 1. In the case $a = -1$, condition (a) above was already obtained in [40], and (b) is always satisfied as soon as $0 \notin G$, because we would have $|-1| \cdot |z| = |z| = |z - 0| \geq \operatorname{dist}(z, \partial G)$ for all $z \in G$.
2. From condition (a) in Proposition 2.1 one derives as in the last remark that $|z| = |z - 0| \geq \operatorname{dist}(z, \partial G)$ for all $z \in G$. Then we have $\sup_{z \in G} \frac{\operatorname{dist}(z, \partial G)}{|z|} \leq 1$. Therefore, according to (b), if $|a| < 1$ and G is a domain such that some sequence $(z_n) \subset G$ satisfies $\lim_{n\to\infty} \frac{\operatorname{dist}(z_n, \partial G)}{|z_n|} = 1$ (for instance $G = B(c, |c|)$, where $c \in \mathbb{C} \setminus \{0\}$), then $(T_{a,n})$ is not universal on $H(G)$. Another example in which $(T_{a,n})$ is not universal (even though $0 \notin G$) is obtained when G is a sector $\{re^{i\theta} : r > 0, 0 < \theta < \alpha\}$ $(0 < \alpha < 2\pi)$ and $|a| < \sin \frac{\alpha}{2}$.

In order to provide sufficient conditions for universality, we distinguish two cases, namely, $a \neq -1$ and $a = -1$. The reason is that the approaches of the proofs are rather different. Note that we obtain in fact (see Theorem 2.10 below) a characterization of universality in the case $a = -1$: this follows from Proposition 2.1 and the fact that the condition $G \cap (a+1)G = \varnothing$ given in the next theorem means $0 \notin G$ in that case. As usual, we have set $c\,S := \{cz : z \in S\}$ for $c \in \mathbb{C}$, $S \subset \mathbb{C}$.

The auxiliary results contained in the next lemma are needed to face the case $a \neq -1$. If $M \subset \mathbb{N}_0$ is an infinite set and $G \subset \mathbb{C}$ is a domain, then we denote by $U(G, M)$ the family of all functions $f \in H(G)$ satisfying that, for every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a strictly increasing sequence $(\lambda_n) \subset M$ such that, for every compact set $L \subset G$, one has

$$\lim_{n\to\infty} \sup_{\zeta \in L} \sup_{z \in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0.$$

Note that $U(G, \mathbb{N}_0) = U(G)$.

Lemma 2.3. *Let $G \subset \mathbb{C}$ be a simply connected domain with $G \neq \mathbb{C}$, and $M \subset \mathbb{N}_0$ be an infinite subset. Then the following holds:*

(a) *$U(G, M)$ is a dense G_δ subset of $H(G)$.*
(b) *$U(G)$ is dense-lineable in $H(G)$.*
(c) *$U(G)$ is spaceable in $H(G)$.*

Proof. Part (a) is a refinement of an assertion of [33] given in Section 1, and it is a consequence of Theorem 3.4 in [28] just by choosing $A =$ the infinite unit matrix there.

Part (b) can be derived from Theorem 6 in [5]. In fact, we only need the conclusion (ii) of such theorem (for $l = 0$), together with the property that –thanks to Mergelyan's approximation theorem (see, e.g., [22])– the set of entire functions is dense in $A(K)$, provided that K is a compact subset of $\mathbb{C}$ with connected complement.

Part (c) follows from the just mentioned denseness property together with Theorem 4.2 in [29] (see also [16]). We need only the conclusion (i) (for $l = 0$) of this theorem. □

Remark 2.4. In 2005, Bayart established the dense-lineability ([3]) and the spaceability ([4]) of $U(\mathbb{D})$.

Theorem 2.5. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $T_{a,n} : H(G) \to H(G)$ $(n \in \mathbb{N})$ defined by (2), where $a \in \mathbb{C} \setminus \{-1\}$. If $G \cap (a+1)G = \varnothing$ then we have:*

(a) *The sequence $(T_{a,n})$ is mixing (hence universal).*
(b) *The set $HC((T_{a,n}))$ is dense-lineable and spaceable in $H(G)$.*

Proof. (a) To show that $(T_{a,n})$ is mixing, we are going to prove that, for every fixed sequence $M = \{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}_0$, the set $HC((S_k)_{k\geq 1})$ is residual in $H(G)$, where we have set $S_k := T_{a,n_k}$. According to Lemma 2.3(a), it is enough to prove that $U(G, M) \subset HC((S_k)_{k\geq 1})$ or, equivalently, that for each $f \in U(G, M)$ the orbit $\{S_k f : k \in \mathbb{N}\}$ is dense in $H(G)$. Since G is simply connected, the set of polynomials is dense in $H(G)$. Therefore it is sufficient to exhibit, for every fixed polynomial P, a sequence $(k(l))_l \subset M$ such that $S_{k(l)} f \longrightarrow P$ $(l \to \infty)$ uniformly on compacta in G. Choose an increasing sequence of compact sets $\{L_l\}_{l\geq 1}$ such that $G = \bigcup_{l\geq 1} L_l$ and every set $\mathbb{C} \setminus L_l$ is connected; this is possible due to the simple connectedness of G (see, e.g., [37, Chapter 13]). Then every compact set $L \subset G$ is contained in some $L_{l(L)}$.

Fix f and P as above. Since $a + 1 \neq 0$, the set $(a+1)G$ is a simply connected domain contained in $\mathbb{C} \setminus G$. Moreover, each set $K_l := (a+1)L_l$ is compact, $\mathbb{C} \setminus K_l$ is connected and $K_l \subset \mathbb{C} \setminus G$. In addition, every mapping $z \in K_l \longmapsto P(\frac{z}{a+1}) \in \mathbb{C}$ belongs to $A(K_l)$. Thus, there is $m_l = n_{k(l)} \in M$ such that

$$\sup_{\zeta\in L_l} \sup_{z\in K_l} \left|S(m_l, f, \zeta)(z) - P\big(\frac{z}{a+1}\big)\right| < \frac{1}{l}.$$

It is evident that (m_l) can be selected so as to be strictly increasing. Notice that we have, in particular, that $|S(m_l, f, z)((a+1)z) - P(z)| < 1/l$ for all $z \in L_l$. But

$$S(m_l, f, z)((a+1)z) = \sum_{j=0}^{m_l} \frac{f^{(j)}(z)}{j!}((a+1)z - z)^j = (S_{k(l)} f)(z).$$

On the other hand, given a compact set $L \subset G$, there is $l_0 \in \mathbb{N}$ such that $L \subset L_l$ for all $l \geq l_0$. This yields $\sup_{z \in L} |(S_{k(l)} f)(z) - P(z)| < 1/l$ for all $l \geq l_0$ and, consequently, $\lim_{l \to \infty} \sup_{z \in L} |(S_{k(l)} f)(z) - P(z)| = 0$, which proves the desired uniform convergence.

(b) This follows from Lemma 2.3(b,c) together with the fact $U(G) \subset HC((T_{a,n}))$ proved in the preceding paragraph (with $M = \mathbb{N}_0$). □

For instance, if Π is one of the two open half-planes determined by a straight line passing through the origin and G is any simply connected domain contained in Π, then $G \cap (-G) = \varnothing$, and so the sequence $(T_{-2,n})$ is universal on $H(G)$.

Remark 2.6. Contrary to the case $a = -1$ (Theorem 2.10), we do not know whether or not the condition $G \cap (a+1)G = \varnothing$ in Theorem 2.5 is necessary for the universality of $(T_{a,n})$.

For any meromorphic function R we will consider the set $\mathcal{P}_R$ of its poles in the extended plane, that is, $\mathcal{P}_R = \{z \in \mathbb{C}_\infty : R(z) = \infty\}$. The following three lemmas will be used in the proof of our main result, with which we conclude this section.

Lemma 2.7. *Let $G \subset \mathbb{C}$ be a simply connected domain such that $0 \notin G$. Then the family $\mathcal{R}_0$ of rational functions R with $\mathcal{P}_R \subset \{0\}$ is a dense subset of $H(G)$.*

Proof. As a consequence of the Runge approximation theorem, if A is a subset of $\mathbb{C}_\infty$ containing exactly one point in each connected component of $\mathbb{C}_\infty \setminus G$, then the family of rational functions R with $\mathcal{P}_R \subset A$ is a dense subset of $H(G)$ (see, e.g., [37, Chapter 13]). In our case, the set $\mathbb{C}_\infty \setminus G$ is connected and $0 \in \mathbb{C}_\infty \setminus G$, so it is enough to choose $A = \{0\}$. □

Lemma 2.8. *Assume that X and Y are separable F-spaces. Let $(T_n) \subset L(X,Y)$ be a mixing sequence. Then $HC((T_n))$ is dense-lineable.*

Proof. In [10] it is proved that, if X and Y are metrizable separable topological vector spaces and (T_n) is a sequence in $L(X,Y)$ such that $HC((T_{n_k}))$ is dense for every sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$, then $HC((T_n))$ contains, except for 0, a dense vector subspace of X. The conclusion of this lemma follows from the fact that being mixing implies transitivity of each subsequence (T_{n_k}), and this in turn is equivalent to the denseness of each set $HC((T_{n_k}))$ (in fact, all that is needed is X to be, in addition, a Baire space). □

Lemma 2.9. *Let $G \subset \mathbb{C}$ be a simply connected domain with $0 \notin G$, and M be an infinite subset of N_0. Then the set*

$$\mathcal{S}_{t,M}(G) := \big\{ f \in H(G) : \overline{\{\widetilde{T_n} f\}_{n \in M}} \supset \{constants\} \big\}$$

is dense in $H(G)$.

Proof. In [40, Theorem 4.7], the statement of the lemma is proved for the case $M = \mathbb{N}_0$ by showing that $U(G) \subset \mathcal{S}_t(G) = \mathcal{S}_{t,\mathbb{N}_0}(G)$. With the same approach it can be seen that $U(G, M) \subset \mathcal{S}_{t,M}(G)$. But, by Lemma 2.3, the set $U(G, M)$ is dense in $H(G)$. Thus, $\mathcal{S}_{t,M}(G)$ is dense too. □

Theorem 2.10. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $\widetilde{T_n} : H(G) \to H(G)$ $(n \in \mathbb{N})$ defined in* (1)*. Then the following properties are equivalent:*

(a) $0 \notin G$.
(b) *The sequence $(\widetilde{T_n})$ is universal, that is, $\mathcal{S}(G) \neq \varnothing$.*
(c) *The sequence $(\widetilde{T_n})$ is mixing.*
(d) *The set $\mathcal{S}(G)$ is residual in $H(G)$.*
(e) *The set $\mathcal{S}(G)$ is dense-lineable in $H(G)$.*

Proof. Recall that $\mathcal{S}(G) = HC((\widetilde{T_n})_{n\geq 0})$, where

$$\widetilde{T_n} f(z) = \sum_{j=0}^{n} \frac{f^{(j)}(z)}{j!}(-z)^j.$$

The implication (b) $\Rightarrow$ (a) has been already proved in [40] (alternatively, see Proposition 2.1), while (c) $\Rightarrow$ (b) is trivial because any mixing sequence of operators on a separable F-space is universal. On the other hand, the implications (d) $\Rightarrow$ (b) and (e) $\Rightarrow$ (b) are also evident because if a set is dense then it is, trivially, nonempty. That (c) $\Rightarrow$ (d) is a consequence of the fact that mixing implies transitive. And (c) $\Rightarrow$ (e) follows from Lemma 2.8 as applied to our sequence $(\widetilde{T_n})$ and $X = H(G) = Y$.

Consequently, all we need to prove is that (a) implies (c). So, we assume $0 \notin G$. Our goal is to show that $(\widetilde{T_n})_{n\in\mathbb{N}_0}$ is mixing. This is equivalent to show that $(\widetilde{T_n})_{n\in M}$ is transitive for every infinite subset $M \subset \mathbb{N}_0$. With this aim, fix such a subset M as well as two nonempty open sets U, W of $H(G)$. We should find $n_0 \in M$ such that $\widetilde{T_{n_0}}(U) \cap W \neq \varnothing$. Recall that the family of all sets of the form

$$V(f, K, \varepsilon) = \{g \in H(G) : |g(z) - f(z)| < \varepsilon \text{ for all } z \in K\}$$

($f \in H(G)$, $\varepsilon > 0$, K a compact subset of G) is an open basis for the topology of $H(G)$. Now, recall that since G is simply connected, the set $\mathcal{P}$ of all polynomials and the set $\mathcal{R}_0$ (Lemma 2.7) are dense in $H(G)$. Moreover, we have $V(f, K, \varepsilon) \subset V(f, L, \alpha)$ if $K \supset L$ and $\varepsilon < \alpha$. Then there are $\varepsilon > 0$, $P \in \mathcal{P}$, $R \in \mathcal{R}_0$ and a compact subset $K \subset G$ such that $U \supset V(P, K, \varepsilon)$ and $W \supset V(R, K, \varepsilon)$.

Thus, we should search for an $m \in M$ enjoying the property that there is a function $f \in H(G)$ such that $f \in V(P, K, \varepsilon)$ and $\widetilde{T_m} f \in V(R, K, \varepsilon)$ or, equivalently, such that

$$|f(z) - P(z)| < \varepsilon \text{ and } |(\widetilde{T_m} f)(z) - R(z)| < \varepsilon \text{ for all } z \in K. \qquad (3)$$

Let $p := \text{degree}(P)$. On the one hand, if $n \geq p$ and $z \in \mathbb{C}$, we obtain from the Taylor expansion that

$$\begin{aligned}(\widetilde{T_n}P)(z) &= \sum_{j=0}^{n} \frac{P^{(j)}(z)}{j!}(-z)^j = \sum_{j=0}^{p} \frac{P^{(j)}(z)}{j!}(-z)^j \\ &= \sum_{j=0}^{p} \frac{P^{(j)}(z)}{j!}(0-z)^j = P(0). \qquad (4)\end{aligned}$$

On the other hand, there are $b_0, b_1, \ldots, b_q \in \mathbb{C}$ such that

$$R(z) = b_0 + \frac{b_1}{z} + \cdots + \frac{b_q}{z^q} =: b_0 + R_0(z).$$

According to Lemma 2.9, we can find a function $\varphi \in H(G)$ and an infinite subset $M_0 \subset M$ such that

$$|\varphi(z)| < \frac{\varepsilon}{2} \quad \text{and} \quad |(\widetilde{T_n}\varphi)(z) - (-P(0) + b_0)| < \varepsilon \quad (z \in K,\, n \in M_0). \qquad (5)$$

Now, since $K \subset G$ is compact and $0 \notin G$, we can find $C_K \in (0,1)$ such that

$$|z| > C_K \quad \text{for all} \quad z \in K. \qquad (6)$$

Since M_0 is infinite, we can choose $m \in M_0$ (hence $m \in M$) satisfying

$$m > p \quad \text{and} \quad m > \frac{2q \cdot \max_{1 \leq k \leq q} |b_k|}{\varepsilon \cdot C_K^q}. \qquad (7)$$

For each $k \in \{1, \ldots, q\}$, let us define the numbers d_k and a_k by

$$d_k := \sum_{j=0}^{m} \frac{k(k+1)\cdots(k+j-1)}{j!} \quad \text{and} \quad a_k := \frac{b_k}{d_k}, \qquad (8)$$

with the convention $\frac{k(k+1)\cdots(k+j-1)}{j!} := 1$ if $j = 0$. Observe that $d_k \geq m+1$ for all $k \in \{1, \ldots, q\}$. We also define the function

$$f := P + \varphi + S, \quad \text{where} \quad S(z) := \frac{a_1}{z} + \cdots + \frac{a_q}{z^q}. \qquad (9)$$

Obviously, $f \in H(G)$. Let $\psi_k(z) := z^{-k}$ for $k \in \mathbb{N}$. An easy computation gives $\widetilde{T_m}\psi_k = d_k \psi_k$. Hence, by linearity, $\widetilde{T_m}S = \sum_{k=1}^{q} a_k d_k \psi_k = \sum_{k=1}^{q} b_k \psi_k = R_0$. On the one hand, we have by (5), (6), (7), (8), (9) and the triangle inequality that, for all $z \in K$,

$$\begin{aligned}|f(z) - P(z)| &\leq |\varphi(z)| + |S(z)| \leq \frac{\varepsilon}{2} + \sum_{k=1}^{q} \left|\frac{a_k}{z^k}\right| \\ &= \frac{\varepsilon}{2} + \sum_{k=1}^{q} \frac{|b_k|}{|d_k z^k|} \leq \frac{\varepsilon}{2} + \sum_{k=1}^{q} \frac{|b_k|}{m\, C_K^k} \\ &< \frac{\varepsilon}{2} + \frac{q \cdot \max_{1 \leq k \leq q} |b_k|}{m\, C_K^q} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

On the other hand, from (4), (5), (7), (9), the triangle inequality and the linearity of $\widetilde{T_m}$ we get for all $z \in K$ that

$$|(\widetilde{T_m} f)(z) - R(z)| = \Big|(\widetilde{T_m} P)(z) + (\widetilde{T_m}\varphi)(z) + (\widetilde{T_m} S)(z) - b_0 - \sum_{k=1}^{q} \frac{b_k}{z^k}\Big|$$
$$\leq |P(0) + (\widetilde{T_m}\varphi)(z) - b_0| + \Big|(\widetilde{T_m} S)(z) - R_0(z)\Big| < \varepsilon + 0 = \varepsilon.$$

Consequently, (3) holds for the chosen function f, and we are done. □

Question 2.11. *Let $G \subset \mathbb{C}$ be a simply connected domain with $0 \notin G$. Is $\mathcal{S}(G)$ spaceable?*

3. Differential polynomials associated to power series

Let $G \subset \mathbb{C}$ be a domain. We can associate to each polynomial $P(z) = \sum_{k=0}^{N} a_k z^k$ with complex coefficients a_k a differential operator $P(D) = \sum_{k=0}^{N} a_k D^k \in L(H(G))$, where $D^k f = f^{(k)}$ for $k \in \mathbb{N}_0$. Then $P(D)f = \sum_{k=0}^{N} a_k f^{(k)}$. Therefore, any (formal) power series $\sum_{n=0}^{\infty} c_n z^n$ (or, that is the same, any sequence $\mathbf{c} = (c_n) \in \mathbb{C}^{\mathbb{N}_0}$) defines, in a natural way, a sequence $\{T_{\mathbf{c},n}\}_{n \geq 0}$ of operators on $H(G)$ given by $T_{\mathbf{c},n} = \sum_{j=0}^{n} c_j D^j$, that is,

$$(T_{\mathbf{c},n} f)(z) = \sum_{j=0}^{n} c_j f^{(j)}(z) \quad (f \in H(G)). \tag{10}$$

Then it is natural to ask for the universality of such a sequence.

However, before going on, it is worth mentioning that there are some restrictions on the desired universality. For instance, if the series $\sum_{n=0}^{\infty} c_n z^n$ is "very convergent", we should not get our hopes up too much. To be more explicit, assume that $\Phi(z) = \sum_{n=0}^{\infty} c_n z^n$ is an entire function of subexponential type, that is, given $\varepsilon > 0$, there is a constant $K = K(\varepsilon) \in (0, +\infty)$ such that $|\Phi(z)| \leq K\, e^{\varepsilon |z|}$ for all $z \in \mathbb{C}$. Then the infinite order differential operator $\Phi(D) = \sum_{n=0}^{\infty} c_n D^n$ is well defined on $H(G)$; see, e.g., [8] (in fact, it makes sense on $H(\mathbb{C})$ if Φ is just of exponential type, that is, if there are constants $A, B \in (0, +\infty)$ satisfying $|\Phi(z)| \leq A\, e^{B|z|}$ for all $z \in \mathbb{C}$). The corresponding sequence $\{T_{\mathbf{c},n}\}_{n \geq 0}$ of operators satisfies

$$T_{\mathbf{c},n} f \longrightarrow \Phi(D) f = \sum_{k=0}^{\infty} c_k f^{(k)} \quad (n \to \infty)$$

uniformly on compacta in G, so we have a kind of "anti-hypercyclicity" in this case.

With this in mind, we have got a partial positive result (Theorem 3.2) by assuming that $\mathbf{c}$ is *not* the sequence of Taylor coefficients of an entire function (i.e., $\limsup_{n\to\infty} |c_n|^{1/n} > 0$) as well as some "angular" behavior of these coefficients. The remaining cases in which the series $\sum_{n=0}^{\infty} c_n z^n$ does not define an entire function of subexponential type stay –as far as we know–

as an open problem. For the proof, we need the following lemma, which is in the line of the eigenvalue criteria given in [11, 14, 23]. However, the lemma cannot be deduced from those criteria. Moreover, its content might be of some interest by itself. By $\mathrm{span}(A)$ we represent the linear span of a subset A of a vector space.

Lemma 3.1. *Assume that X is a separable F-space and that $(T_n)_{n\geq 0} \subset L(X)$. Suppose that there are subsets $D, E \subset X$ satisfying the following conditions:*

(a) *D and $\mathrm{span}(E)$ are dense in X.*
(b) *For each $d \in D$, the sequence $\{T_n\, d\}_{n\geq 0}$ converges in X.*
(c) *Each $e \in E$ is an eigenvector of every T_n $(n \geq 0)$, with eigenvalue $\lambda(T_n, e)$, say.*
(d) *$\lim_{n\to\infty} \lambda(T_n, e) = \infty$ for all $e \in E$.*

Then $(T_n)_n$ is mixing and the set $HC((T_n)_n)$ is dense-lineable in X.

Proof. The second conclusion follows from Lemma 2.8. As for the first conclusion, we want to prove that every subsequence (T_{n_k}) of (T_n) is transitive. Let us denote $R_k := T_{n_k}$ for $k \in \mathbb{N}$.

In order to show that (R_k) is transitive, fix two nonempty open sets $U, V \subset X$. Our goal is to exhibit an $m \in \mathbb{N}$ such that $R_m(U) \cap V \neq \varnothing$. By the denseness of D assumed in (a), there is $d \in D \cap U$. It follows from (b) the existence of a vector $f \in X$ such that $R_k\, d \to f$ as $k \to \infty$. Now, by the denseness of $\mathrm{span}(E)$ this time, there is $e \in \mathrm{span}(E) \cap (V - f)$, because the translate $V - f$ of V is also open and nonempty. Since $e \in \mathrm{span}(E)$, we can find finitely many scalars μ_j and vectors $e_j \in E$ $(j = 1, \ldots, q)$ such that $e = \sum_{j=1}^{q} \mu_j e_j$. Thanks to (c) and (d), we have $R_k e_j = \lambda(T_{n_k}, e_j) e_j$ and $\lim_{k\to\infty} \lambda(T_{n_k}, e_j) = \infty$ for all $j \in \{1, \ldots, q\}$. In particular, there is $k_1 \in \mathbb{N}$ such that $\lambda(T_{n_k}, e_j) \neq 0$ for all $k \geq k_1$ and all $j \in \{1, \ldots, q\}$. Next, for any $k \geq k_1$, we define

$$x_k := d + \sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} e_j.$$

Since $\frac{\mu_j}{\lambda(T_{n_k}, e_j)} \to 0$ $(k \to \infty)$ for $j \in \{1, \ldots, q\}$, it follows from the continuity of the multiplication by scalars in a topological vector space that $x_k \to d + 0 = d$ as $k \to \infty$. As $d \in U$ and U is open, there exists $k_2 \geq k_1$ such that $x_k \in U$ for all $k \geq k_2$. Finally, we get

$$\begin{aligned} R_k x_k &= R_k\, d + R_k\Big(\sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} e_j \Big) = R_k\, d + \sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} R_k e_j \\ &= R_k\, d + \sum_{j=1}^{q} \mu_j e_j = R_k\, d + e \longrightarrow f + e \quad \text{as} \quad k \to \infty. \end{aligned}$$

Since $f+e \in f+(V-f)=V$ and V is open, one can find $k_3 \geq k_2$ such that $R_k x_k \in V$ for all $k \geq k_3$. Consequently, we obtain $R_m(U) \cap V \neq \varnothing$ as soon as we choose $m := k_3$. This had to be shown. □

We are now ready to state our final theorem.

Theorem 3.2. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $T_{\mathbf{c},n} : H(G) \to H(G)$ $(n \in \mathbb{N}_0)$ defined in* (10)*, where $\mathbf{c} = (c_n)_{n\geq 0}$ satisfies the following conditions:*

(i) $\limsup_{n\to\infty} |c_n|^{1/n} > 0$.
(ii) *There exist $\alpha \in \mathbb{R}$ and a sequence $(\theta_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}_0}$ with*

$$\min\Big\{\limsup_{n\to\infty} |\theta_n|, \limsup_{n\to\infty} \big|\theta_n - \frac{\pi}{2}\big|, \limsup_{n\to\infty} |\theta_n - \pi|, \limsup_{n\to\infty} \big|\theta_n - \frac{3\pi}{2}\big|\Big\} < \frac{\pi}{2}$$

such that $\arg c_n = n\,\alpha + \theta_n$ whenever $c_n \neq 0$.

Then $(T_{\mathbf{c},n})$ is mixing and, in particular, universal. Moreover, the set $HC((T_{\mathbf{c},n}))$ is dense-lineable in $H(G)$.

Proof. The second part of the conclusion follows from the first one and Lemma 2.8. Hence, our goal is to prove that $(T_{\mathbf{c},n})$ is mixing. We will use Lemma 3.1 with $X := H(G)$, $T_n := T_{\mathbf{c},n}$ $(n \geq 0)$, $D := \mathcal{P} = \{\text{polynomials}\}$ and $E := \{e_\lambda : \lambda \in \{t\,e^{-i\alpha} : t > R\}\}$, where $e_a(z) := e^{az}$ $(a \in \mathbb{C})$ and R is the radius of convergence of the power series $\sum_{n=0}^\infty c_n z^n$, that is, $R = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1}$. Observe that $0 \leq R < +\infty$ by (i), which yields $E \neq \varnothing$.

On the one hand, the denseness of D in X follows from the simple connectedness of G. On the other hand, it is known (see, e.g., [24, Lemma 2.34]) that if $\Lambda \subset \mathbb{C}$ is a set with an accumulation point, then $\text{span}(\{e_\lambda : \lambda \in \Lambda\})$ is dense in $H(\mathbb{C})$, and hence in $H(G)$ due to Runge's approximation theorem and the simple connectedness of G. Consequently, $\text{span}(E)$ is dense in X and condition (a) of Lemma 3.1 is fulfilled. Now, if $P \in \mathcal{P}$ and $N = \text{degree}(P)$ then $P^{(n)} = 0$ for all $n > N$, and so $T_n P = \sum_{j=0}^N c_j P^{(j)} := Q$ for all $n \geq N$. Hence $T_n P \to Q$ as $n \to \infty$, which tells us that condition (b) in Lemma 3.1 is also satisfied. As for condition (c), notice that $e_\lambda^{(n)} = \lambda^n e_\lambda$ for all $\lambda \in \mathbb{C}$ and all $n \in \mathbb{N}_0$, which entails $T_n e_\lambda = \lambda(T_n, e_\lambda) e_\lambda$, where $\lambda(T_n, e_\lambda) = \sum_{j=0}^n c_j \lambda^j$, that is, each $e_\lambda \in E$ is in fact an eigenvector for all T_n. Let us verify, finally, condition (d) in Lemma 3.1.

For this, take any $n \in \mathbb{N}_0$ and any $\lambda = t\,e^{-i\alpha}$ with $t > R$. From (ii), at least one of the following inequalities is true: $\limsup_{n\to\infty} |\theta_n| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \frac{\pi}{2}| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \pi| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \frac{3\pi}{2}| < \frac{\pi}{2}$. Suppose that the first inequality holds. Then there is $N \in \mathbb{N}$ such that $\sup_{n>N} |\theta_n| < \frac{\pi}{2}$. Let $\gamma := \inf_{n>N} \cos\theta_n$. Note that $\gamma > 0$. Let $n > N$.

Also by (ii) and the triangle inequality, we can estimate:

$$\begin{aligned}|\lambda(T_n, e_\lambda)| &= \Big|\sum_{j=0}^{n} c_j\,\lambda^j\Big| = \Big|\sum_{j=0}^{n} |c_j|\, e^{i(j\alpha+\theta_j)}(t\,e^{-i\alpha})^j\Big| = \Big|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}\Big| \\ &\geq \mathrm{Re}\Big(\sum_{j=N+1}^{n} |c_j|\, t^j\, e^{i\,\theta_j}\Big) - \sum_{j=0}^{N} |c_j|\, t^j \\ &= \sum_{j=0}^{n} |c_j|\, t^j\, \cos\theta_j - \sum_{j=0}^{N} |c_j|\, t^j \\ &\geq \gamma \cdot \sum_{j=0}^{n} |c_j|\, t^j - \sum_{j=0}^{N} |c_j|\, t^j \longrightarrow +\infty \quad \text{as} \quad n \to \infty\end{aligned}$$

because the series with positive terms $\sum_{n=0}^{\infty} |c_n| t^n$ diverges: indeed, $t > R$, the radius of convergence. If $\limsup_{n\to\infty} |\theta_n - \frac{\pi}{2}| < \frac{\pi}{2}$ holds, the reasoning is similar by considering $\gamma := \inf_{n>N} \sin\theta_n$ and taking imaginary parts instead of real parts. The remaining third and four cases $\limsup_{n\to\infty} |\theta_n - \pi| < \frac{\pi}{2}$ and $\limsup_{n\to\infty} |\theta_n - \frac{3\pi}{2}| < \frac{\pi}{2}$ are analogous, just by considering the inequalities $|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}| \geq \mathrm{Re}\,(\sum_{j=N+1}^{n} -|c_j|\, t^j\, e^{i\,\theta_j}) - \sum_{j=0}^{N} |c_j|\, t^j$ $|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}| \geq \mathrm{Im}\,(\sum_{j=N+1}^{n} -|c_j|\, t^j\, e^{i\,\theta_j}) - \sum_{j=0}^{N} |c_j|\, t^j$ and letting $\gamma := \inf_{n>N} |\cos\theta_n|$, $\gamma := \inf_{n>N} |\sin\theta_n|$, respectively. Thus, (d) is satisfied and the proof is concluded. □

Corollary 3.3. *Let $G \subset \mathbb{C}$ be a simply connected domain, and assume that $\mathbf{c} = (c_n)_{n\geq 0}$ is a sequence satisfying $c_n \geq 0$ for all $n \geq 0$ and $\limsup_{n\to\infty} c_n^{1/n} > 0$. Then $(T_{\mathbf{c},n})$ is mixing on $H(G)$.*

Remarks 3.4. 1. For instance, the sequence of operators on $H(G)$ given by $\{\sum_{k=0}^{n}(k+i)(1+i)^k D^k\}_{n\in\mathbb{N}_0}$ is universal, for any simply connected domain $G \subset \mathbb{C}$.

2. In [23] the hypercyclicity of a nonscalar operator $\Phi(D)$ on $H(\mathbb{C})$ is established, which in particular yields Birkhoff's theorem [15] and MacLane's theorem [27] on hypercyclicity of the translation operator and the derivative operator, respectively. Note that this is equivalent to the universality of the sequence $(\Phi^n(D))$. Concerning universality of sequences of differential operators not being the iterates of a single one, the reader can find a number of results in [9, 11, 14, 36], but none of them covers Theorem 3.2. Moreover, the set $HC(\Phi(D))$ is spaceable, as proved by Petersson, Shkarin and Menet [30, 35, 39] (see also [24, Section 10.1]). This fact together with the results of this section motivates the next and final question.

Question 3.5. *Let $G \subset \mathbb{C}$ be a simply connected domain. Under what conditions is $HC((T_{\mathbf{c},n}))$ spaceable in $H(G)$?*

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