A Seidel-Walsh theorem with linear differential operators

By

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Abstract. Assume that $\{S_n\}_1^\infty$ is a sequence of automorphisms of the open unit disk $\mathbf{D}$ and that $\{T_n\}_1^\infty$ is a sequence of linear differential operators with constant coefficients, both of them satisfying suitable conditions. We prove that for certain spaces X of holomorphic functions in the open unit disk, the set of functions $f \in X$ such that $\{(T_n f) \circ S_n : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})$ is residual in X. This extends the Seidel-Walsh theorem together with some subsequent results.

1. Introduction and terminology. In this paper, $\mathbf{N}$, $\mathbf{C}$ and $\mathbf{D}$ denote respectively the set of positive integers, the complex plane and the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$. The boundary of $\mathbf{D}$ is the unit circle $\partial\mathbf{D} = \{z \in \mathbf{C} : |z| = 1\}$. If $r > 0$ then we set $r\mathbf{D} = \{z \in \mathbf{C} : |z| \leq r\}$. For an open subset $G \subset \mathbf{C}$, $H(G)$ stands for the Fréchet space of holomorphic functions on G endowed with the topology of uniform convergence on compact subsets. A domain is a nonempty connected open subset of $\mathbf{C}$. A domain is said to be simply connected whenever its complement with respect to the extended complex plane is connected. $A(\mathbf{D})$ denotes, as usual, the Banach space of all functions which are continuous on the closure $\overline{\mathbf{D}}$ of $\mathbf{D}$ and holomorphic in $\mathbf{D}$, endowed with the maximum norm $|| \cdot ||_\infty$. Recall that every complete metrizable space is a Baire space and that a subset of a Baire space X is *residual* whenever its complement is of first category. Such a subset is "very large" in X.

If $|a| < 1 = |k|$, then we denote by $\sigma_{a,k}$ the Möbius transformation $\sigma_{a,k}(z) = k\frac{z-a}{1-\overline{a}z}$. It is well known that the group $Aut\,(\mathbf{D})$ of automorphisms of $\mathbf{D}$ is exactly the set of such transformations. Recall that a function $\sigma \in Aut\,(\mathbf{D})$ which is not the identity is said to be:

a) a *non-Euclidean rotation* if and only if σ has only one fixed point in $\mathbf{D}$,

b) a *non-Euclidean translation* if and only if σ has two fixed points on $\partial\mathbf{D}$, and finally

c) a *non-Euclidean limit rotation* if and only if σ has just one fixed point which lies

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on $\partial\mathbf{D}$.
See for instance [19, p. 231] for this classification. The mappings $\sigma_{a,1}$ ($|a| < 1$) are examples of non-Euclidean translations.

In 1941 W. Seidel and J.L. Walsh [17] established the existence of a function $f \in H(\mathbf{D})$ such that, given a simply connected domain $G \subset \mathbf{D}$ and a function $g \in H(G)$, there is a sequence $\{a_n\}_1^\infty \subset \mathbf{D}$ depending on g such that $f \circ \sigma_{a_n,1} \to g$ ($n \to \infty$) uniformly on compact subsets of G. In 1987 R.M. Gethner and J.H. Shapiro [9] and K.G. Grosse-Erdmann [11, p. 52], realized that the Seidel-Walsh theorem is a typical case of universality. In fact, with the methods in [9] and [11] (see also [13]) one can prefix a sequence $\{a_n\}_1^\infty \subset \mathbf{D}$ with $|a_n| \to 1$ ($n \to \infty$) and get a residual subset $\mathcal{U}$ of $H(\mathbf{D})$ such that the set $\{f \circ \sigma_{a_n,1} : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})$ for every $f \in \mathcal{U}$. The Seidel-Walsh theorem is a non-Euclidean version of Birkhoff's translation theorem on entire functions [4]. They have been developed and extended in several directions (see [2, 3, 6, 8, 9, 10, 11, 12, 13, 15, 16, 20] for instance). By the way, in 1995 A. Montes-Rodríguez and the first author [2] dropped the restriction that the automorphisms under consideration be of the special form $\sigma_{a,1}$. If fact, there it is proved the following.

Theorem 1. *If $\{S_n\}_1^\infty \subset Aut(\mathbf{D})$, denote $\mathcal{U}(\{S_n\}) = \{f \in H(\mathbf{D}) : \{f \circ S_n : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})\}$. We have:*
a) If $\{a_n\}_1^\infty$ and $\{k_n\}_1^\infty$ are two complex sequences with $|a_n| < 1 = |k_n|$ for every $n \in \mathbf{N}$ and $S_n = \sigma_{a_n,k_n}$, then $\mathcal{U}(\{S_n\})$ is not empty if and only if it is residual if and only if $\limsup_{n\to\infty} |a_n| = 1$ if and only if the action of $\{S_n\}_1^\infty$ is properly discontinuous on $\mathbf{D}$.
b) Let $\varphi = \sigma_{a,k}$ where $k = e^{i\theta}$ and $|a| < 1$. If $\varphi_n = \varphi \circ \ldots \circ \varphi$ (n times), the $\mathcal{U}(\{\varphi_n\})$ is not empty if and only if it is residual if and only if φ is not a non-Euclidean rotation if and only if $|\sin \frac{\theta}{2}| \leq |a|$.

Recall that for a sequence $\{S_n\}_1^\infty$ of automorphisms of a complex domain G, it is said that *its action is properly discontinuous on G* if for each compact subset $K \subset G$ there exists a positive integer $m = m(K)$ such that $K \cap S_m(K) \neq \emptyset$. As we can observe, this concept is purely topological.

In [13] G. Herzog proves the following "Seidel-Walsh theorem for derivatives".

Theorem 2 *Let X be a Banach space of holomorphic functions on $\mathbf{D}$ having the following properties:*
a) Convergence in X implies compact convergence on $\mathbf{D}$.
b) $A(\mathbf{D}) \subset X$.
c) The polynomials are dense in X.
Then for every sequence $\{a_n\}_1^\infty \subset \mathbf{D}$ with $\lim_{n\to\infty} |a_n| = 1$, the set $\{f \in X : \{f' \circ \sigma_{a_n,1} : n \in \mathbf{N}\}$ is dense in $H(\mathbf{D})\}$ is a residual subset of X.

In fact, by following the proof in [13] one can easily realize that the hypothesis that X be a Banach space can be changed to the weaker one that X be a Fréchet space. Herzog also studies several examples of Banach spaces in which the latter theorem can be applied or not. We point out that, trivially, the expression $f' \circ \sigma_{a_n,1}$ in the conclusion of Theorem 2 cannot be substituted by $f \circ \sigma_{a_n,1}$; indeed, if $X = A(\mathbf{D})$ and $f \in X$, then the set $\{f \circ \sigma_{a_n,1} : n \in \mathbf{N}\}$ is uniformly bounded on $\mathbf{D}$, so it cannot be dense in $H(\mathbf{D})$. With this in mind, we propose in this paper the following question: If $\Phi(z)$ is a nonconstant polynomial and D is the differentiation operator (i.e., $Df = f'$), does the conclusion of Theorem 2 hold by changing f' to $\Phi(D)f$? Theorem 2 would be the case $\Phi(z) \equiv z$. From the above remark, the conclusion is false for $\Phi(z) =$ a constant. Our main result (Theorem 4) says that this is the only exceptional case. In fact, we obtain much more: f' can be replaced by $\Phi_n(D)f$, $\{\Phi_n\}_1^\infty$ being a sequence of polynomials whose coefficients satisfy certain boundedness restrictions.

2. Universality, linear differential operators and antiderivatives. Before stating our theorem, some definitions and assertions about universality, antiderivatives and linear differential operators generated by polynomials are in order.

If X and Y are nonempty topological spaces and $\Lambda = \{L_j\}_{j \in J}$ is a family of continuous mappings from X into Y, then an element $x \in X$ is called Λ-*universal* if the set $\{L_j(x) : j \in J\}$ is dense in Y. The proof of the main result in [13] is based upon the following theorem due to K.G. Grosse-Erdman [11, Satz 1.2.2 and Satz 1.4.2] (see also several versions in [1], [9, Section 2], [10, Section 1] and [14, Chapter 1]).

Theorem 3. *Let X, Y be metrizable topological vector spaces with X complete and Y separable, and let $\Lambda = \{L_n\}_1^\infty$ be a sequence of continuous linear operators from X to Y. Then the following statements are equivalent:*
a) The set of Λ-universal elements is a residual subset of X.
b) The set of Λ-universal elements is a dense subset of X.
c) The set $\{(x, L_n(x)) : x \in X, n \in \mathbf{N}\}$ is dense in $X \times Y$.

If, in addition, there is a dense subset C of X such that $\lim_{n\to\infty} L_n(x)$ exists for all $x \in C$, then a), b) and c) are equivalent to
d) The set of Λ-universal elements is not empty.

We will also use Theorem 3 in the proof of our main result. Next, let us consider a polynomial $\Phi(z) = \sum_{j=0}^m a_j z^j$. We can associate to it a linear differential operator $L = \Phi(D)$, that is, $L = \sum_{j=0}^m a_j D^j$ with $D^0 = I =$ the identity operator. We adopt the following terminology. We say that a sequence $\{\Phi_n(z) = \sum_{j=0}^J a_j^{(n)} z^j\}_{n=1}^\infty$ of polynomials with the same degree $J \in \mathbf{N}$ is *C-bounded* whenever the following two conditions are satisfied:
A) Each sequence $\{a_j^{(n)} : n \in \mathbf{N}\}$ $(j = 0, 1, 2, ..., J)$ is bounded.
B) There exists a positive constant α such that $|a_J^{(n)}| \geq \alpha$ for all $n \in \mathbf{N}$.

It is obvious that a subsequence of a C-bounded sequence is also C-bounded. The following elementary two lemmas will be employed in the next section.

Lemma 1. *Let* $\{\Phi_n(z) = \sum_{j=0}^{J} a_j^{(n)} z^j\}_{n=1}^{\infty}$ *be a sequence of polynomials with the same degree* $J \in \mathbf{N}$ *such that every sequence* $\{a_j^{(n)} : n \in \mathbf{N}\}$ $(j = 0, 1, 2, ..., J)$ *is bounded. Then there is a subsequence* $\{\Phi_{n_k} : k \in \mathbf{N}\}$ *and a polynomial* P *satisfying that* $\Phi_{n_k}(D)\varphi \to P(D)\varphi$ $(k \to \infty)$ *in* $H(\mathbf{C})$ *for every entire function* φ.

P r o o f. Since the sequence $\{a_0^{(n)} : n \in \mathbf{N}\}$ is bounded, there is a subsequence $\{m_{0k} : k \in \mathbf{N}\}$ of positive integers such that $a_0^{(m_{0k})}$ tends to a complex number a_0 as $k \to \infty$. But the sequence $a_1^{(m_{0k})}$ is also bounded, so there is a point $a_1 \in \mathbf{C}$ and a subsequence $\{m_{1k} : k \in \mathbf{N}\}$ of $\{m_{0k} : k \in \mathbf{N}\}$ such that $a_1^{(m_{1k})} \to a_1$ $(k \to \infty)$. Continuing this process gives after finitely many steps a sequence $\{n_k = m_{Jk} : k \in \mathbf{N}\}$ of positive integers and a finite complex sequence $\{a_j\}_0^J$ such that $a_j^{(n_k)} \to a_j$ $(k \to \infty)$ for every $j \in \{0, 1, 2, ..., J\}$. If we now define

$$P(z) = \sum_{j=0}^{J} a_j z^j,$$

then the conclusion of the lemma is evident. ////

Lemma 2. *Assume that* G *and* Ω *are two nonempty open subsets of* $\mathbf{C}$ *and that* $H : \Omega \to \mathbf{C}$, $H_k : \Omega \to \mathbf{C}$ $(k \in \mathbf{N})$, $\Psi : G \to \mathbf{C}$, $\Psi_k : G \to \mathbf{C}$ $(k \in \mathbf{N})$ *are functions satisfying the following properties:*
i) H_k *tends to* H $(k \to \infty)$ *uniformly on compact sets in* Ω.
ii) Ψ_k *tends to* Ψ $(k \to \infty)$ *uniformly on compact sets in* G.
iii) $\Psi(G) \subset \Omega$.
iv) Ψ *is continuous on* G *and* H *is continuous on* Ω.
Then $H_k \circ \Psi_k \to H \circ \Psi$ $(k \to \infty)$ *uniformly on compact sets in* G.

P r o o f. Fix $\varepsilon > 0$ and a compact set $L \subset G$. Then $\Psi(L)$ is a compact subset of Ω by iii) and iv). Set $\alpha = (1/2)\,\mathrm{dist}\,(\Psi(L), \mathbf{C} \setminus \Omega)$ (or any positive number if $\Omega = \mathbf{C}$). Then $\alpha > 0$ and there exists $k_1 \in \mathbf{N}$ such that $|\Psi_k(z) - \Psi(z)| \leq \alpha$ for all $k \geq k_1$ and all $z \in L$. In particular $\Psi_k(z) \in \Omega$ for all $k \geq k_1$ and all $z \in L$, so $H_k \circ \Psi_k(z)$ makes sense for these k, z. Denote $L_1 = \{w \in \mathbf{C} : \mathrm{dist}\,(w, \Psi(L)) \leq \alpha\}$. Then L_1 is compact and $\Psi(L) \subset L_1 \subset \Omega$. Moreover $\Psi_k(L) \subset L_1$ for all $k \geq k_1$. From i) there exists $k_2 \in \mathbf{N}$ (with $k_2 \geq k_1$) such that $|(H_k - H)(w)| < \varepsilon/2$ for all $k \geq k_2$ and all $w \in L_1$. In particular,

$$|(H_k - H)(\Psi_k(z))| < \varepsilon/2 \quad (\forall k \geq k_2, \forall z \in L). \tag{A}$$

By iv), H is uniformly continuous on the compact set L_1, so there exists $\delta \in (0, \alpha)$ satisfying $|H(w) - H(w')| < \varepsilon/2$ whenever $w, w' \in L_1$ and $|w - w'| < \delta$. From ii), there is $k_0 \in \mathbf{N}$ (with $k_0 \geq k_2$) such that $|\Psi_k(z) - \Psi(z)| < \delta$ $(\forall z \in L)$, so

$$|H \circ \Psi_k(z) - H \circ \Psi(z)| < \varepsilon/2 \quad (\forall z \in L, \forall k \geq k_0). \tag{B}$$

Finally, if we combine (A) and (B) and apply the triangle inequality, we obtain

$$|H_k \circ \Psi_k(z) - H \circ \Psi(z)| < \varepsilon \quad (\forall z \in L, \, \forall k \geq k_0),$$

as desired. ////

We conclude this section with the following elementary well-known statement about derivatives. We establish it as a lemma for future references.

Lemma 3. *Let $G \subset \mathbf{C}$ be a simply connected domain, $a \in G$, $F \in H(G)$ and, for each $k \in \mathbf{N}$,*

$$(I_k F)(z) = \int_a^z \frac{(z-\xi)^{k-1}}{(k-1)!} \cdot F(\xi)\, d\xi \quad (z \in G),$$

where the integration is taken along any rectifiable curve in G joining a to z. If we set $I_0 F = F$, then $I_k F$ is well-defined for every $k \in \{0, 1, 2, ...\}$, $I_k F \in H(G)$ and $(I_k F)^{(j)} = I_{k-j} F$ for $j \in \{0, 1, ..., k\}$.

3. The main result. We are now ready to state the main theorem of this paper. The unique condition on the sequence of automorphisms is the proper discontinuity of its action, mentioned in Theorem 1.

Theorem 4. *Let X be a Fréchet space of holomorphic functions on $\mathbf{D}$ having the following properties:*
a) Convergence in X implies compact convergence on $\mathbf{D}$.
b) $A(\mathbf{D}) \subset X$.
c) The polynomials are dense in X.

Assume that $\{S_n\}_1^\infty$ is a sequence of automorphisms of $\mathbf{D}$ and that $\{\Phi_n\}_1^\infty$ is a C-bounded sequence of polynomials. Denote $T_n = \Phi_n(D)$ $(n \in \mathbf{N})$ and consider the set

$$\mathcal{U} = \{f \in X : \{(T_n f) \circ S_n : n \in \mathbf{N}\} \text{ is dense in } H(\mathbf{D})\}.$$

Then $\mathcal{U}$ is a residual set of X if and only if $\mathcal{U}$ is not empty if and only if the action of $\{S_n\}_1^\infty$ is properly discontinuous on $\mathbf{D}$.

P r o o f. Firstly, observe that each function S_n has the form $S_n = \sigma_{a_n,k_n}$, where $|a_n| < 1 = |k_n|$ for all $n \in \mathbf{N}$. We start with the case $\limsup_{n\to\infty} |a_n| < 1$, i.e., we are now assuming that the action of $\{S_n\}_1^\infty$ is not properly discontinuous on $\mathbf{D}$ (see Theorem 1). There is a constant $\mu \in (0,1)$ such that $|a_n| < \mu$ for all $n \in \mathbf{N}$. Fix $r \in (0,1)$. Then, for $|z| \leq r$,

$$|S_n(z)| = |k_n \cdot \frac{z - a_n}{1 - \overline{a_n} z}| \leq \frac{r + \mu}{1 + r\mu} < 1.$$

We have used the elementary inequality $|\frac{a+b}{1+\overline{a}b}| \leq \frac{A+B}{1+AB}$, which holds whenever $|a| \leq A < 1$, $|b| \leq B < 1$. By hypothesis, every funcion Φ_n has the form

$$\Phi_n(z) = \sum_{j=0}^{J} b_j^{(n)} z^j \quad (z \in \mathbf{C}),$$

$J \in \mathbf{N}$ fixed, in such a way that there are positive real constants B_j $(j = 0, 1, ..., J)$ satisfying that $|b_j^{(n)}| \leq B_j$ $(n \in \mathbf{N})$.

Assume that $f \in \mathcal{U}$. If $L = \{z : |z| \leq \frac{r+\mu}{1+r\mu}\}$, then L is a compact subset of $\mathbf{D}$ and so the set

$$\bigcup_{n \in \mathbf{N}} [(\Phi_n(D)f) \circ S_n](|z| \leq r)$$

is bounded, because it is contained in the disk $\{|z| \leq s\}$, where $s := \sum_{j=0}^{J} B_j \cdot \sup_{w \in L} |f^{(j)}(w)| < +\infty$. Thus, the set $\{(\Phi_n(D)f) \circ S_n : n \in \mathbf{N}\}$ cannot be dense in $H(\mathbf{D})$, which contradicts the fact that $f \in \mathcal{U}$. Consequently, $\mathcal{U}$ is empty if the action of $\{S_n\}_1^\infty$ is not properly discontinuous on $\mathbf{D}$.

Since a residual subset is trivially not empty, the only property to be proved is that $\mathcal{U}$ is residual whenever $\limsup_{n\to\infty} |a_n| = 1$.

As in [13], we first prove the latter property in the case that $X = A(\mathbf{D})$. Define the mappings

$$L_n : A(\mathbf{D}) \to H(\mathbf{D}) \quad (n \in \mathbf{N})$$

by $L_n(f) = (T_n f) \circ S_n$. Since convergence in $A(\mathbf{D})$ implies compact convergence on $\mathbf{D}$, these mappings are linear and continuous. Our goal is to prove that the set

$$\{(f, L_n(f)) : f \in A(\mathbf{D}), n \in \mathbf{N}\} \tag{1}$$

is dense in $A(\mathbf{D}) \times H(\mathbf{D})$. An application of Theorem 3 would yield the conclusion in this case.

Since $\limsup_{n\to\infty} |a_n| = 1$ and $|k_n| = 1$ for every $n \in \mathbf{N}$, we can suppose with no loss of generality, by taking a subsequence if necessary, that there are points γ_1, γ_2 on $\partial\mathbf{D}$ such that $a_n \to \gamma_1$ and $k_n \to \gamma_2$ as $n \to \infty$. If $\gamma = -\gamma_1\gamma_2$, then the chain of inequalities

$$\begin{aligned}
|S_n(z) - \gamma| =& |k_n \cdot \frac{z - a_n}{1 - \overline{a_n} z} + \gamma_1\gamma_2| \\
\leq& |k_n \cdot \frac{z - a_n}{1 - \overline{a_n} z} + \gamma_1 k_n| + |\gamma_1\gamma_2 - \gamma_1 k_n| \\
=& |\frac{z - a_n}{1 - \overline{a_n} z} + \gamma_1| + |k_n - \gamma_2| \\
=& |\frac{\gamma_1 - a_n + z(1 - \overline{a_n}\gamma_1)}{1 - \overline{a_n} z}| + |k_n - \gamma_2| \\
\leq& \frac{|a_n - \gamma_1|}{1 - |z|} + |1 - \overline{a_n}\gamma_1| \cdot \frac{|z|}{1 - |z|} + |k_n - \gamma_2|
\end{aligned}$$

shows that $S_n(z) \to \gamma$ $(n \to \infty)$ uniformly on compact subsets of $\mathbf{D}$.

Fix two polynomials p, q and a number $\varepsilon \in (0,1)$. Let $\beta = 1 + ||q||_\infty + \sum_{j=0}^{J} B_j ||p^{(j)}||_\infty$ (recall that $||\cdot||_\infty$ is the maximum norm on $\overline{\mathbf{D}}$) and choose $m \in \mathbf{N}$ such that

$$m > \frac{2\beta \cdot (1 + \sum_{j=0}^{J} B_j)}{\alpha\varepsilon}, \tag{2}$$

where α is the positive constant furnished by the definition of C-boundedness for $\{\Phi_n\}_1^\infty$, i.e., $|b_J^{(n)}| \geq \alpha$ for all $n \in \mathbf{N}$.

Since $S_n \to \gamma$ $(n \to \infty)$ in $H(\mathbf{D})$, there exists $n_0 \in \mathbf{N}$ satisfying

$$\sup_{|z| \leq 1-\varepsilon} |1 - \gamma^{-m} S_{n_0}(z)^m| \leq \frac{\varepsilon}{2\beta}. \tag{3}$$

Take the function

$$F(z) = z^m \cdot (q(S_{n_0}^{-1}(z)) - (\Phi_{n_0}(D)p)(z)),$$

which is in $H(|a_{n_0}|^{-1}\mathbf{D})$. By hypothesis,

$$|b_J^{(n_0)}| \geq \alpha. \tag{4}$$

With the notation of Lemma 3, take $a = 0$ and define the function

$$h = \frac{1}{\gamma^m b_J^{(n_0)}} I_J F$$

on the domain $G = |a_{n_0}|^{-1}\mathbf{D}$. Then $h \in H(|a_{n_0}|^{-1}\mathbf{D})$ and so $h \in A(\mathbf{D})$. If we choose the segment $[0,z]$ as integration curve, we get

$$|h^{(j)}(z)| = |\frac{1}{b_J^{(n_0)}} I_{J-j} F(z)|$$

$$= |\frac{1}{b_J^{(n_0)}} \int_0^z \frac{(z-\xi)^{J-1-j}}{(J-1-j)!} \cdot \xi^m \cdot (q(S_{n_0}^{-1}(\xi)) - (\Phi_{n_0}(D)p)(\xi))\, d\xi|$$

$$= |\frac{1}{b_J^{(n_0)}} \int_0^1 \frac{z^{J-1-j}(1-t)^{J-1-j} \cdot z}{(J-1-j)!} \cdot (zt)^m \cdot (q(S_{n_0}^{-1}(zt)) - (\Phi_{n_0}(D)p)(zt))\, dt|$$

$$\leq \frac{1}{|b_J^{(n_0)}|} \int_0^1 t^m \cdot \beta dt = \frac{\beta}{(m+1)|b_J^{(n_0)}|} < \frac{\beta}{m \cdot \alpha} < \frac{\varepsilon}{2(1 + \sum_{j=0}^{J} B_j)}$$

for all $z \in \overline{\mathbf{D}}$ and all $j \in \{0, 1, ..., J-1\}$, because of (2), (4) and the facts

$$|\frac{z^{J+m-j}(1-t)^{J-1-j}}{(J-1-j)!}| \leq 1 \quad (z \in \overline{\mathbf{D}},\, t \in [0,1])$$

and

$$|(\Phi_{n_0}(D)p)(\xi)| = |\sum_{j=0}^{J} b_j^{(n_0)} p^{(j)}(\xi)| \leq \sum_{j=0}^{J} B_j ||p^{(j)}||_\infty$$

for every $\xi \in \overline{\mathbf{D}}$. Hence

$$||h^{(j)}||_\infty < \frac{\varepsilon}{2(1 + \sum_{j=0}^{J} B_j)} \quad (j \in \{0, 1, ..., J-1\}). \tag{5}$$

Define $g = p + h$. Then $g \in A(\mathbf{D})$ and $||p - g||_\infty = ||h||_\infty < \varepsilon$. Moreover,

$$q(z) - (L_{n_0}g)(z) = q(z) - (L_{n_0}p)(z) - (L_{n_0}h)(z)$$

$$= q(z) - (\Phi_{n_0}(D)p)(S_{n_0}(z)) - b_J^{(n_0)} h^{(J)}(S_{n_0}(z)) - \sum_{j=0}^{J-1} b_j^{(n_0)} h^{(j)}(S_{n_0}(z))$$

$$= q(z) - (\Phi_{n_0}(D)p)(S_{n_0}(z)) - \gamma^{-m} S_{n_0}(z)^m \cdot (q(z) - (\Phi_{n_0}(D)p)(S_{n_0}(z))) - \sum_{j=0}^{J-1} b_j^{(n_0)} h^{(j)}(S_{n_0}(z))$$

$$= (1 - \gamma^{-m} S_{n_0}(z)^m) \cdot (q(z) - (\Phi_{n_0}(D)p)(S_{n_0}(z))) - \sum_{j=0}^{J-1} b_j^{(n_0)} h^{(j)}(S_{n_0}(z))$$

for all $z \in |a_{n_0}|^{-1}\mathbf{D}$.

Assume that $|z| \leq 1 - \varepsilon$. If we apply inequalities (3) and (5) then we get

$$\begin{aligned}
|q(z) - (L_{n_0}g)(z)| \leq & |1 - \gamma^{-m} S_{n_0}(z)^m| \cdot |q(z) - (\Phi_{n_0}(D)p)(S_{n_0}(z))| \\
& + \sum_{j=0}^{J-1} B_j ||h^{(j)}||_\infty \\
< & \frac{\varepsilon}{2\beta} \cdot (||q||_\infty + \sum_{j=0}^{J} B_j ||p^{(j)}||_\infty) + \frac{\varepsilon}{2(1 + \sum_{j=0}^{J} B_j)} \cdot \sum_{j=0}^{J-1} B_j \\
< & \frac{\varepsilon}{2\beta} \cdot \beta + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus the closure of the set given in (1) contains the set of all (p, q) (p and q polynomials), but this set is dense in $A(\mathbf{D}) \times H(\mathbf{D})$, so the first subset is dense in $A(\mathbf{D}) \times H(\mathbf{D})$, as required.

Now, if X is a Fréchet space as in the hypothesis, then the mappings $L_n : X \to H(\mathbf{D})$ ($n \in \mathbf{N}$) defined as $L_n(f) = (T_n f) \circ S_n$ are continuous since X satisfies a). As before, we can assume by taking a subsequence if necessary that $S_n(z)$ tends in the topology of $H(\mathbf{D})$ as $n \to \infty$ to a point $\gamma \in \partial\mathbf{D}$. By Lemma 1, we may suppose with no loss of generality that there is a polynomial P such that $T_n\varphi \to P(D)\varphi$ $(n \to \infty)$ in $H(\mathbf{C})$ for every entire function φ. Fix a polynomial $\varphi(z)$ and apply Lemma 2 on $G = \mathbf{D}$, $\Omega = \mathbf{C}$, $\Psi_n = S_n$, $\Psi =$ the constant γ, $H_n = T_n\varphi$ and $H = P(D)\varphi$. We obtain $L_n\varphi = (T_n\varphi) \circ S_n \to (P(D)\varphi)(\gamma)$ $(n \to \infty)$ in $H(\mathbf{D})$. Thus, $\lim_{n\to\infty} L_n\varphi$ exists in $H(\mathbf{D})$ for every $\varphi \in \{\text{polynomials}\}$. Since X satisfies b), the set $\mathcal{U}$ is not empty. Finally, since X satisfies c), Theorem 3 yields (for $Y = H(\mathbf{D})$ and $C = \{\text{polynomials}\}$) that $\mathcal{U}$ is a residual subset of X. ////

4. Final remarks. 1. Assume that φ is an automorphism of the open unit disk which is not a non-Euclidean rotation or, equivalently, with no fixed point in $\mathbf{D}$. Then by Theorem 1 the action of the sequence of iterates $S_n = \varphi \circ \ldots \circ \varphi$ (n times, $n \in \mathbf{N}$) is properly discontinuous. Hence the statement of Theorem 4 is true for this sequence. It is well known that if φ is a general holomorphic self-mapping of the open unit disk with no fixed point, then the Denjoy-Wolff theorem (see for instance [7] or [18, p. 78]) asserts that its corresponding sequence of iterates tends uniformly on compact subsets to a constant of modulus one (the “Denjoy-Wolff point” of φ). This constant would be the point γ found in the proof of Theorem 4 in the case that S_n be the sequence of iterates of an automorphism as above.
2. The statement of Theorem 4 holds trivially for a single nonconstant polynomial P (i.e., for $\Phi_n = P$ for all $n \in \mathbf{N}$) or, equivalently, for a finite linear differential operator $P(D)$ with constant coefficients which is not a multiple of the identity.
3. The authors do not know whether the conclusion of the main result holds for a sequence of polynomials which is not necessarily C-bounded.

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