Algebraic genericity of strict-order integrability

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Abstract

We provide sharp conditions on a measure μ defined on a measurable space X guaranteeing that the family of functions in the Lebesgue space $L^p(\mu, X)$ ($p \geq 1$) which are not integrable with order q for any $q > p$ (or any $q < p$) contains, except for zero, large subspaces of $L^p(\mu, X)$. This improves recent results due to Aron, García, Muñoz, Palmberg, Pérez, Puglisi and Seoane. It is also shown that many non-integrable functions of order q can be obtained even on any nonempty open subset of X, assuming that X is a topological space and μ is a Borel measure on X satisfying appropriate properties.

1 Introduction and aim of this paper

The study of the linear properties of sets of mathematical objects with *a priori* no linear structure has recently attracted the attention of a crescent number of mathematicians. This paper intends to shed some light on this topic, in the special framework of the spaces of integrable functions.

To this respect, let us recall some recent terminology introduced in [1], [3], [5] and [7]. Assume that E is a topological vector space. Then a subset A of E is called

- *lineable* if $A \cup \{0\}$ contains an infinite dimensional vector subspace,

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- *dense-lineable* or *algebraically generic* whenever $A \cup \{0\}$ contains a dense vector subspace of X,
- *spaceable* if $A \cup \{0\}$ contains some infinite dimensional closed vector subspace.

It is clear that dense-lineability implies lineability if E is infinite dimensional.

If μ is a cardinal number, then a subset A of E is said to be *μ-lineable* if $A \cup \{0\}$ contains a vector subspace of dimension μ. Note that if E is an infinite dimensional separable Baire topological vector space, then $\dim(E) = c$, the cardinality of the continuum. Hence c is the maximal dimension allowed for any vector subspace of E.

If the last notion is combined to algebraic genericity, the following concept arises naturally.

Definition 1.1. Let E be a topological vector space. Then we say that a subset $A \subset E$ is *maximal dense-lineable* if $A \cup \{0\}$ contains a dense vector subspace M such that $\dim(M) = \dim(E)$.

Note that it is not clear that maximal lineability (i.e., $\dim(E)$-lineability) plus dense-lineability must imply maximal dense-lineability.

For each $p \in [1, \infty)$ and each interval $I \subset \mathbb{R}$, let us consider the Lebesgue space $L^p(I)$ of (classes of) real measurable functions $f : I \to \mathbb{R}$ such that $|f|^p$ is integrable on I with respect to the Lebesgue measure. We also consider the space ℓ_p of real sequences (a_n) that are p-summable, that is, satisfying $\sum_{n=1}^{\infty} |a_n|^p < \infty$. In 2008, Muñoz, Palmberg, Puglisi and Seoane [9] proved the following assertions:

- If I is a bounded interval and $q > p \geq 1$ then $L^p(I) \setminus L^q(I)$ is lineable.
- If J is an unbounded interval and $p > q \geq 1$ then $L^p(J) \setminus L^q(J)$ is lineable.
- If $p > q \geq 1$ then $\ell_p \setminus \ell_q$ is lineable.

More precisely, it is proved in [9] that these sets are *c-lineable.* Aron, García, Pérez and Seoane [2] have recently completed these results by showing that, under the same conditions, the three mentioned sets are *dense-lineable* respectively in $L^p(I)$, $L^p(J)$ and ℓ_p. Note that $L^q(I) \subset L^p(I)$ if $q > p$, and $\ell_q \subset \ell_p$ if $p > q$, but none of the spaces $L^p(J), L^q(J)$ is included in the other if $p \neq q$.

Our aim in this paper is to unify, extend and improve both of the mentioned results in [2] and [9], according to the following points:

- We obtain vector subspaces exhibiting simultaneously density and maximal dimension.
- The intervals I, J endowed with the Lebesgue measure are replaced by rather general measure spaces, so containing the case of ℓ_p too. The conditions given on the measures will be sharp.
- The parameter q runs over all real numbers $> p$ (or $< p$), that is, q is not fixed.
- In the case of Borel measures, many non-integrable functions of order q are obtained even on every nonempty open subset.

The precise statements will be established in Sections 3 and 4. Section 2 is devoted to give the necessary background. In Section 5 we make some remarks and propose a number of problems.

2 Preliminaries

Let $(X, \mathcal{M}, \mu)$ be a measure space, with μ a positive measure, and let $p \in [1, \infty)$. As usual, $L^p(\mu, X)$ will denote the vector space of all (Lebesgue classes of) measurable functions $f : X \to \mathbb{R}$ such that $|f|^p$ is integrable on X. It becomes a Banach space under the norm $\|f\|_p = \big(\int_X |f|^p \, d\mu \big)^{1/p}$. If $p = \infty$, $L^\infty(\mu, X)$ represents the space of all Lebesgue classes of essentially bounded measurable functions $f : X \to \mathbb{R}$. It becomes a Banach space under the norm $\|f\| = \inf\{M > 0 : |f| \le M \ \mu\text{-almost everywhere in } X\}$.

We assume that the reader is familiar with some standard topological terminology and with a number of related properties, which can be found in any reasonable book on general topology. Anyway, we recall that a topological space X is said to be: *perfect* if it lacks isolated points; *regular at a point* $x_0 \in X$ if, given a closed set F with $x_0 \notin F$, there are open sets A, B such that $x_0 \in A$, $F \subset B$ and $A \cap B = \emptyset$; *first-countable at a point* $x_0 \in X$ if x_0 possesses a countable fundamental system of neighborhoods.

Now, we collect a number of special types of measures defined either on general σ-algebras or on the σ-algebra of the Borel sets of a topological space. Let $(X, \mathcal{M}, \mu)$ be a measure space. It is said that μ is *semifinite* if $\mu(A) = \sup\{\mu(B) : \ B \in \mathcal{M}, B \subset A \text{ and } \mu(B) < \infty\}$ for each set $A \in \mathcal{M}$,

while μ is called *nonatomic* if there is no atom in $\mathcal{M}$. Recall that a set $A \in \mathcal{M}$ is said to be an *atom* for μ if $\mu(A) > 0$ and if, for every $B \in \mathcal{M}$ with $B \subset A$, one has $\mu(B) = 0$ and $\mu(A \setminus B) = 0$. Assume that X is a topological space and that μ is a Borel measure of X, that is, μ is a positive measure defined on a σ-algebra $\mathcal{M}$ containing the Borel sets of X. Then it is said that μ has *full support* whenever $\mu(U) > 0$ for every nonempty open set U, and that μ is *continuous* if $\mu(\{x\}) = 0$ for all $x \in X$. Finally, μ is called *regular* if X is a Hausdorff locally compact space and the following three properties hold:
(a) $\mu(C) < \infty$ for every compact set $C \subset X$,
(b) $\mu(A) = \inf\{\mu(U) : U$ is open and $A \subset U\}$ for all $A \in \mathcal{M}$, and
(c) $\mu(U) = \sup\{\mu(C) : C$ is compact in X and $K \subset U\}$ for each open set $U \subset X$.
For instance, the Lebesgue measure on any interval of $\mathbb{R}$ is continuous, regular and has full support. It is known that each σ-finite measure is semifinite, and that every regular measure μ also satisfies (c) above if one replaces U by any set σ-finite set $A \in \mathcal{M}$ (see [10, Proposition 22.5]).

In Sections 3 and 4 the following two lemmas will be respectively needed. Lemma 2.1 is an strengthening of Theorem 2.1 in [6]. The nuance is that this time the dimension of the resulting vector subspace is specified. Lemma 2.2 has a shorter scope and replaces a topological vector space by a topological group.

Lemma 2.1. *Assume that E is a metrizable separable topological vector space. Suppose that Γ is a family of linear subspaces of E such that $\bigcap_{S\in\Gamma} S$ is dense in E and $\bigcap_{S\in\Gamma}(E \setminus S) \cup \{0\}$ contains a vector space of dimension μ, where μ is an infinite cardinal number. Then $\bigcap_{S\in\Gamma}(E \setminus S) \cup \{0\}$ contains a dense vector subspace of dimension μ.*

Proof. From the hypothesis, we can choose a dense countable set $\{z_n\}_{n\geq 1}$ in E as well as a translation-invariant distance d defining the topology of E. By denseness, we also can take, for each $n \in \mathbb{N} := \{1, 2, \cdots\}$, a vector $y_n \in \bigcap_{S\in\Gamma} S$ such that

$$d(y_n, z_n) < 1/n. \tag{1}$$

By hypothesis, there exists a linearly independent family $\{v_\alpha\}_{\alpha\in J} \subset E$ such that $\mathrm{card}\,(J) = \mu$ and

$$L \setminus \{0\} \subset \bigcap_{S\in\Gamma}(E \setminus S),$$

where we have set $L := \mathrm{span}\,(\{v_\alpha;\ \alpha \in J\})$. Since μ is infinite, we can split the set J into infinitely many mutually disjoint nonempty sets, say

$J = \bigcup_{n=1}^{\infty} J_n$. Now, the scalar multiplication is continuous on E. Therefore we derive the existence of a set $\{\varepsilon_\alpha\}_{\alpha \in J} \subset (0, \infty)$ such that

$$d(\varepsilon_\alpha v_\alpha, 0) < 1/n \quad (\alpha \in J_n,\, n \in \mathbb{N}). \tag{2}$$

Next, we define $x_{n,\alpha} := y_n + \varepsilon_\alpha v_\alpha$ ($\alpha \in J_n$, $n \in \mathbb{N}$) and

$$D := \operatorname{span}(\{x_{n,\alpha} : \alpha \in J_n,\, n \in \mathbb{N}\}),$$

the linear span generated by the vectors $x_{n,\alpha}$.

For each $n \in \mathbb{N}$, choose $\alpha_n \in J_n$ and consider the vector $u_n := x_{n,\alpha_n}$. Thanks to (1), (2), the triangle inequality and the translation-invariance of d, we get

$$\begin{aligned} d(u_n, z_n) &\le d(y_n + \varepsilon_{\alpha_n} v_{\alpha_n}, y_n) + d(y_n, z_n) \\ &= d(\varepsilon_{\alpha_n} v_{\alpha_n}, 0) + d(y_n, z_n) < 2/n \longrightarrow 0 \quad (n \to \infty). \end{aligned}$$

Hence $d(u_n, z_n) \to 0$. Since $\{z_n\}_{n \ge 1}$ is dense and E is perfect (because E is a topological vector space), we derive that $\{u_n\}_{n \ge 1}$ is dense. Consequently, D is a dense linear subspace of E.

Let us prove that $D \setminus \{0\} \subset \bigcap_{S \in \Gamma}(E \setminus S)$. To this end, fix a vector $x \in D \setminus \{0\}$. Then there exist $N \in \mathbb{N}$, scalars $c_1, \dots, c_N$ with $c_N \neq 0$ and indexes $\beta_j \in J_j$ ($j = 1, \dots, N$) satisfying $x = c_1 x_{1,\beta_1} + \cdots + c_N x_{N,\beta_N}$, that is

$$x = c_1 y_1 + \cdots + c_N y_N + c_1 \varepsilon_{\beta_1} v_{\beta_1} + \cdots + c_N \varepsilon_{\beta_N} v_{\beta_N}. \tag{3}$$

Assume, by way of contradiction, that $x \notin \bigcap_{S \in \Gamma}(E \setminus S)$. Then there would be some $S_0 \in \Gamma$ for which $x \in S_0$. But $y_1, \dots, y_N \in \bigcap_{S \in \Gamma} S \subset S_0$ and S_0 is a linear subspace, so

$$x - (c_1 y_1 + \cdots + c_N y_N) \in S_0. \tag{4}$$

Since $c_N \varepsilon_{\beta_N} \neq 0$ and the vectors v_{β_j} are linearly independent, we deduce that

$$c_1 \varepsilon_{\beta_1} v_{\beta_1} + \cdots + c_N \varepsilon_{\beta_N} v_{\beta_N} \in L \setminus \{0\} \subset \bigcap_{S \in \Gamma}(E \setminus S),$$

which contradicts (4) because of (3).

Finally, we have to demonstrate that $\dim(D) = \mu$. Since $\operatorname{card}\{(n, \alpha) : \alpha \in J_n,\, n \in \mathbb{N}\}) = \operatorname{card}\left(\bigcup_{n=1}^{\infty} J_n\right) = \mu$, it is enough to show that the family $\{x_{n,\alpha}\}_{\alpha \in J_n,\, n \in \mathbb{N}}$ is linearly independent. This is easy: assume that a finite linear combination such as the right hand side of (3) is 0. Fix any $S_0 \in \Gamma$. Then $c_1 y_1 + \cdots + c_N y_N \in S_0$. Therefore

$$-(c_1 \varepsilon_{\beta_1} v_{\beta_1} + \cdots + c_N \varepsilon_{\beta_N} v_{\beta_N}) \in S_0 \cap \big[\cap_{S \in \Gamma} (E \setminus S) \cup \{0\} \big] = \{0\}.$$

Thus $c_1 \varepsilon_{\beta_1} v_{\beta_1} + \cdots + c_N \varepsilon_{\beta_N} v_{\beta_N} = 0$. But the vectors v_α are linear independent, so $c_j \varepsilon_{\beta_j} = 0$ for all j and, consequently, $c_1 = c_2 = \cdots = c_N = 0$. This concludes the proof. □

Lemma 2.2. *Assume that $(E, *)$ is a topological group. Suppose that Γ is a family of subgroups of E such that $\bigcap_{S \in \Gamma} S$ is dense in E. Then the set $\bigcap_{S \in \Gamma} (E \setminus S)$ is either empty or dense in E.*

Proof. Suppose that $\bigcap_{S \in \Gamma} (E \setminus S) \neq \emptyset$, so that it has at least one element, say x_0. Since the "translation" $x \in E \mapsto x * x_0 \in E$ is a homeomorphism from E onto itself, we get that the set $A := \{x * x_0 : x \in \bigcap_{S \in \Gamma} S\}$ is dense in E. Therefore, it is enough to show that $A \subset \bigcap_{S \in \Gamma} (E \setminus S)$. To see this, let us argue by contradiction and assume that there is an element $y = x * x_0 \in A \setminus \bigcap_{S \in \Gamma} (E \setminus S)$, where $x \in \bigcap_{S \in \Gamma} S$. Then there exists $S_0 \in \Gamma$ such that $x * x_0 \in S_0$. Hence $x_0 = x^{-1} * (x * x_0)$ also belongs to S_0 because S_0 is a semigroup. This is the desired contradiction. □

Next, we state two propositions whose content is surely well known. But, since we have not been able to find a reference, a proof will be furnished. We want to isolate a property that will be used later. A measure space $(X, \mathcal{M}, \mu)$ is said to satisfy *property* (σ) provided that there is a countable set $\mathcal{S} \subset \mathcal{M}$ satisfying the following: given $M \in \mathcal{M}$ with $\mu(M) < \infty$ and $\varepsilon > 0$, there exists a set $A \in \mathcal{S}$ such that $\mu(M \bigtriangleup A) < \varepsilon$. By $A \bigtriangleup B$ we have denoted the symmetric difference $(A \setminus B) \cup (B \setminus A)$ between two sets A and B. As usual, $\mathbb{Q}$ will stand for the set of rational numbers and χ_A will denote the indicator function of a subset A of X.

Proposition 2.3. Let $p \in [1, \infty)$ and $(X, \mathcal{M}, \mu)$ be a measure space.

(a) If (σ) is satisfied, then the space $L^p(\mu, X)$ is separable.

(b) If X is a Hausdorff second-countable locally compact topological space and μ is a regular measure on X, then $L^p(\mu, X)$ is separable.

Proof. According to [12, Chapter 3], the set S_t of step functions, i.e., the set of all measurable functions $f : X \to \mathbb{R}$ such that $f(X)$ is finite and $\mu(\{x \in X : f(x) \neq 0\}) < \infty$, is dense in $L^p(\mu, X)$, for any $p \in [1, \infty)$ and any positive measure on an arbitrary measurable space X. Now, assume that μ is as in (a), and consider the countable set $\mathcal{S} \subset \mathcal{M}$ furnished by (σ). Since the set $\mathcal{A} := \left\{ \sum_{i=1}^N q_i \chi_{A_i} : q_i \in \mathbb{Q},\, A_i \in \mathcal{S} \text{ for } i \in \{1, \dots, N\},\, N \in \mathbb{N} \right\}$ is countable and S_t is dense in $L^p(\mu, X)$, it is enough to prove that, given $\varepsilon > 0$ and $f \in S_t$, there exists a function $g \in \mathcal{A}$ such that $\|f - g\|_p < \varepsilon$. This is easy: just take into account the denseness of $\mathbb{Q}$ in $\mathbb{R}$, the identities $\chi_A = \chi_A^p$, $|\chi_A - \chi_B| = \chi_{A \triangle B}$ and the elementary inequality $(\sum_{i=1}^N a_i)^p \le N^p (\sum_{i=1}^N a_i^p)$ for $a_i \in [0, \infty)$. This demonstrates (a).

Now, assume that X and μ are as in (b). Select a countable open basis $\mathcal{U}$ for X and define $\mathcal{S}$ to be the family of all finite unions of members of $\mathcal{U}$. Then $\mathcal{S}$ is a countable subfamily of $\mathcal{M}$. Therefore, it is sufficient to prove that (σ) is satisfied with $\mathcal{S}$. For this, fix $\varepsilon > 0$ and $M \in \mathcal{M}$ with $\mu(M) < \infty$. Since μ is regular, we can find a compact set K and an open set U satisfying

$$K \subset M \subset U \quad \text{and} \quad \mu(U) - \frac{\varepsilon}{2} < \mu(M) < \mu(K) + \frac{\varepsilon}{2}. \tag{5}$$

Fix $x \in K$. Then $x \in U$, so there is $B_x \in \mathcal{U}$ with $x \in B_x \subset U$, because $\mathcal{U}$ is an open basis. Therefore the family $\{B_x : x \in K\}$ is an open covering of K. From the compactness of K, it follows the existence of a finite subfamily $\{B_{x_j} : j = 1, \dots, J\}$ such that $K \subset \bigcup_{j=1}^J B_{x_j} =: A$. Then $A \in \mathcal{S}$ and $K \subset A \subset U$. From this and (5), one derives that $M \triangle A \subset U \setminus K$ and $\mu(M \triangle A) \le \mu(U \setminus K) = \mu(U) - \mu(K) < \varepsilon$. Consequently, $\mathcal{S}$ satisfies the property of (a) and the proof is finished. □

The last proposition can be applied, of course, to the spaces $L^p(I)$ and ℓ_p, where $p \in [0, \infty)$ and I is any interval of $\mathbb{R}$. Indeed, the Lebesgue measure and the cardinal measure on $\mathbb{N}$ are special instances of (b). It is well known that, on the contrary, $L^\infty(\mu, X)$ is seldom separable.

Note that the sufficient condition for non-separability furnished in the proposition below is rather common, and works, again, for the Lebesgue measure as well as for the cardinal measure on $\mathbb{N}$.

Proposition 2.4. Let $(X, \mathcal{M}, \mu)$ be a measure space such that there is an infinite family $\mathcal{N} \subset \mathcal{M}$ whose members are pairwise disjoint such that $0 < \mu(A) < \infty$ for all $A \in \mathcal{N}$. Then the space $L^\infty(\mu, X)$ is not separable.

Proof. One can select a sequence $\{A_n\}_{n \ge 1}$ of mutually disjoint sets in $\mathcal{N}$ with positive measure. Consider the family $\mathcal{F} := \{B_S = \bigcup_{n \in S} A_n : S \subset \mathbb{N}\}$, and let $B_S, B_{\tilde{S}}$ be two distinct members of $\mathcal{F}$. Then $S \triangle \tilde{S} \ne \emptyset$ and $|\chi_{B_S} - \chi_{B_{\tilde{S}}}| = \chi_{B_S \triangle B_{\tilde{S}}}$. Hence $|\chi_{B_S} - \chi_{B_{\tilde{S}}}| = 1$ on the set $B_S \triangle B_{\tilde{S}}$, and this set has positive measure. Consequently, the open norm-balls $B(\chi_F, 1/2)$ $(F \in \mathcal{F})$ are mutually disjoint. The non-separability of our space follows from the fact that $\mathcal{F}$ is uncountable. □

To finish this section, we show a purely topological preliminary result, to be used in Section 3. If $A \subset X$, where X is a topological space, then $\overline{A}$ will stand for the closure of A.

Lemma 2.5. *Let X be a T_1 topological space. Assume that x_0 is a non-isolated point of X such that X is regular and first-countable at it. Then, for each open neighborhood U of x_0, there exists a fundamental system $\{U_n\}_{n\geq 1}$ of open neighborhoods of x_0 such that*

$$\overline{U_{n+1}} \subset U_n \subset U \quad \text{and} \quad U_n \setminus \overline{U_{n+1}} \neq \emptyset \quad \text{for all } n \in \mathbb{N}.$$

Proof. Fix an open neighborhood U of x_0. Since X is first-countable at x_0, there exists a fundamental system $\{V_n\}_{n\geq 1}$ of open neighborhoods of x_0. By replacing, if necessary, each V_n by $U \cap V_1 \cap V_2 \cap \cdots \cap V_n$, we can assume that $V_{n+1} \subset V_n \subset U$ for every $n \in \mathbb{N}$. Pick any point $x_1 \in V_1 \setminus \{x_0\}$, which is possible because x_0 is not isolated. Now, we can apply regularity at x_0 to find open sets A_1, B_1 satisfying $x_0 \in A_1$, $(X \setminus V_1) \cup \{x_1\} \subset B_1$ and $A_1 \cap B_1 = \emptyset$ (note that $\{x_1\}$ is closed because X is T_1). Therefore $V_1 \setminus \{x_1\} \supset X \setminus B_1 \supset A_1$. But $X \setminus B_1$ is closed, whence $V_1 \setminus \{x_1\} \supset \overline{A_1}$. Define $U_1 := V_1$ and $U_2 := V_2 \cap A_1$. On the one hand, U_1, U_2 are open sets containing x_0 with $U_1 \subset V_1, U_2 \subset V_2$. On the other hand, $\overline{U_2} \subset \overline{A_1} \subset V_1 = U_1$ and $x_1 \in U_1 \setminus \overline{A_1} \subset U_1 \setminus \overline{U_2}$. Hence $U_1 \setminus \overline{U_2} \neq \emptyset$. By starting with U_2, can pick a point $x_2 \in U_2 \setminus \{x_0\}$, and a similar process drives us to an open set U_3 satisfying $x_0 \in U_3$, $\overline{U_3} \subset U_2$, $U_3 \subset V_3$ and $x_2 \in U_2 \setminus \overline{U_3}$ (so $U_2 \setminus \overline{U_3} \neq \emptyset$). It is plain that this procedure generates the desired sequence $\{U_n\}_{n\geq 1}$ of open neighborhoods of x_0. It is still a fundamental system for x_0 due to the fact that $U_n \subset V_n$ for all $n \geq 1$. □

3 Algebraic genericity

We start by labeling two properties that will be often used. These properties are related to a measure space $(X, \mathcal{M}, \mu)$:

(α) $\inf\{\mu(A) : A \in \mathcal{M}, \mu(A) > 0\} = 0$.

(β) $\sup\{\mu(A) : A \in \mathcal{M}, \mu(A) < \infty\} = \infty$.

The exact conditions under which the inclusions among the Lebesgue spaces $L^p(\mu, X)$ hold are well known. The assertions of the following theorem can be found in the paper [11] by J.L. Romero and in [10, Section 14.8].

Theorem 3.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space and p and q be extended real numbers satisfying $1 \le p < q \le \infty$. We have:*

(a) $L^p(\mu, X) \subset L^q(\mu, X)$ *if and only if* $\inf\{\mu(A) : A \in \mathcal{M},\ \mu(A) > 0\} > 0$.

(b) $L^q(\mu, X) \subset L^p(\mu, X)$ *if and only if* $\sup\{\mu(A) : A \in \mathcal{M},\ \mu(A) < \infty\}$ *is finite.*

For instance, if μ is a finite measure on $(X, \mathcal{M})$ and ν is the cardinal measure on an infinite set Y, then $L^q(\mu, X) \subset L^p(\mu, X)$ and $L^p(\mu, Y) \subset L^q(\mu, Y)$ whenever $p < q$. In particular, we recover the inclusion relations $L^q(I) \subset L^p(I)$ (I = a bounded interval of $\mathbb{R}$) and $\ell_p \subset \ell_q$ as well as the non-inclusion relation $L^r(J) \not\subset L^s(J)$ ($r, s \in [1, \infty]$ with $r \neq s$, J = an unbounded interval of $\mathbb{R}$).

Remarks 3.2. 1. Note that the last theorem can be reformulated as follows:

- Let $1 \le p < q \le \infty$. We have that $L^p(\mu, X) \setminus L^q(\mu, X) \neq \emptyset$ if and only if (α) holds.
- Let $1 \le q < p \le \infty$. We have that $L^p(\mu, X) \setminus L^q(\mu, X) \neq \emptyset$ if and only if (β) holds.

2. In [10, pp. 233–235] it is also proved that (α) is true if and only if there exists a sequence (A_n) of pairwise disjoint measurable sets with $0 < \mu(A_n) < 1/2^n$ ($n \in \mathbb{N}$), while (β) holds if and only if there exists a sequence (A_n) of pairwise disjoint measurable sets with $1 < \mu(A_n) < \infty$ ($n \in \mathbb{N}$).

As a matter of fact, conditions (α) and (β) will turn also to be sharp conditions for much finer properties than the mere non-vacuousness of the set $L^p \setminus L^q$. This question will be studied in the present section, so reaching more general conclusions than the ones by Aron-García-Muñoz-Palmberg-Pérez-Puglisi-Seoane given in the Introduction. We present the following concepts. After them, we will state our main result.

Definition 3.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $p \in [1, \infty)$. The members of the set $L^p_{l\text{-}strict} := L^p(\mu, X) \setminus \bigcup_{q \in [1,p)} L^q(\mu, X)$ will be called *left-strict p-order integrable functions*. The members of the set $L^p_{r\text{-}strict} := L^p(\mu, X) \setminus \bigcup_{q \in (p,\infty]} L^q(\mu, X)$ will be called *right-strict p-order integrable functions*. Finally, the members of the set $L^p_{strict} := L^p(\mu, X) \setminus \bigcup_{q \in [1,\infty] \setminus \{p\}} L^q(\mu, X)$ are said to be *strict p-order integrable functions*.

We notice that $L^p_{strict} = L^p_{l\text{-}strict} \cap L^p_{r\text{-}strict}$, $L^1_{l\text{-}strict} = L^1(\mu, X)$ and $L^1_{r\text{-}strict} = L^1_{strict}$.

Theorem 3.4. *Assume that $p \in [1, \infty)$, that $(X, \mathcal{M}, \mu)$ is a measure space and that $L^p(\mu, X)$ is separable. We have:*

(a) *The set $L^p_{r\text{-}strict}$ is maximal dense-lineable if and only if (α) holds.*

(b) *If $p > 1$, then $L^p_{l\text{-}strict}$ is maximal dense-lineable if and only if (β) holds.*

(c) *If $p > 1$, then L^p_{strict} is maximal dense-lineable if and only if both (α) and (β) hold.*

Proof. From Theorem 3.1 and Remark 3.2.1, it follows that conditions (α), (β) and $(\alpha) + (\beta)$ are respectively necessary in (a), (b) and (c). It should be proved that they are also sufficient.

Suppose that (α) holds. Firstly, we are going to demonstrate that $L^p_{r\text{-}strict}$ is c-lineable. According to Remark 3.2.2, we can select a sequence $(A_n) \subset \mathcal{M}$ of pairwise disjoint sets with $0 < \mu(A_n) < 1/2^n$ $(n \geq 1)$. For each $a \in (1, \infty)$, consider the function $f_a : X \to [0, \infty)$ given by

$$f_a = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}(\log(n+1))^{a/p}\mu(A_n)^{1/p}} \cdot \chi_{A_n}.$$

It is clear that f_a is measurable. From the disjointness of the sets A_n, it follows that $f_a^p = \sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^a \mu(A_n)} \cdot \chi_{A_n}$. Moreover, we have that

$$\|f_a\|_p = \left[\int_X f_a^p \, d\mu\right]^{1/p} = \left[\sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^a}\right]^{1/p} =: \delta.$$

Now, δ is finite: use, for instance, Cauchy's condensation principle. Then each f_a belongs to $L^p(\mu, X)$. At this point, we define the vector subspace M of $L^p(\mu, X)$ by

$$M := \operatorname{span}(\{f_a : \ a \in (1, \infty)\}).$$

The functions f_a are linearly independent. Indeed, by way of contradiction, consider a finite linear combination $c_1 f_{\alpha_1} + \cdots + c_N f_{\alpha_N} = 0$ such that the α_j are pairwise distinct and not all the scalars c_j are null. Without loss of generality, we can assume that $N \geq 2$ and $\alpha_1 < \cdots < \alpha_N$ and $c_1 \neq 0$. Since $(\log(n+1))^{\alpha_2 - \alpha_1} \to \infty$ as $n \to \infty$, we can choose $n_0 \geq 2$ such that

$$(\log(n+1))^{\frac{\alpha_2 - \alpha_1}{p}} > 2 \sum_{j=2}^{N} |c_j / c_1| \quad (n \geq n_0). \tag{6}$$

In particular we have that

$$\frac{|c_1|}{n_0^{1/p}(\log(n_0+1))^{\alpha_1/p}\mu(A_{n_0})^{1/p}} > \sum_{j=2}^{N} \frac{|c_j|}{n_0^{1/p}(\log(n_0+1))^{\alpha_j/p}\mu(A_{n_0})^{1/p}}.$$

Therefore $|c_1 f_{\alpha_1}| > |-(c_2 f_{\alpha_2} + \cdots + c_N f_{\alpha_N})|$ on A_{n_0}, which is absurd.

Furthermore, each $f \in M \setminus \{0\}$ happens to not belong to $L^q(\mu, X)$, for any $q > p$. Indeed, such an f has the form $f = c_1 f_{\alpha_1} + \cdots + c_N f_{\alpha_N}$ with the c_j, the α_j and N as in the last paragraph. Select an $n_0 \in \mathbb{N}$ such that (6) works. Let $n \geq n_0$. Then on A_n we have:

$$\begin{aligned} |f| &\geq |c_1 f_{\alpha_1}| - (|c_2 f_{\alpha_2}| + \cdots + |c_N f_{\alpha_N}|) \\ &= \frac{|c_1|}{n^{1/p}(\log(n+1))^{\alpha_1/p}\mu(A_n)^{1/p}} - \sum_{j=2}^{N} \frac{|c_j|}{n^{1/p}(\log(n+1))^{\alpha_j/p}\mu(A_n)^{1/p}} \\ &\geq \frac{|c_1|}{n^{1/p}(\log(n+1))^{\alpha_1/p}\mu(A_n)^{1/p}} - \frac{\sum_{j=2}^{N}|c_j|}{n^{1/p}(\log(n+1))^{\alpha_2/p}\mu(A_n)^{1/p}} \\ &\geq \frac{1}{2} \cdot \frac{|c_1|}{n^{1/p}(\log(n+1))^{\alpha_1/p}\mu(A_n)^{1/p}}. \end{aligned}$$

Now, fix $q > p$. Then it follows that

$$\begin{aligned} \|f\|_q^q &\geq \sum_{n=n_0}^{\infty} \frac{1}{2^q} \cdot \frac{|c_1|^q}{n^{q/p}(\log(n+1))^{\alpha_1 q/p}\mu(A_n)^{q/p}} \mu(A_n) \\ &\geq \sum_{n=n_0}^{\infty} \frac{1}{2^q} \cdot \frac{(2^{\frac{q}{p}-1})^n |c_1|^q}{n^{q/p}(\log(n+1))^{\alpha_1 q/p}} = \infty \end{aligned}$$

because the general term of this series is unbounded since $2^{\frac{q}{p}-1} > 1$. Consequently, $f \notin L^q(\mu, X)$ whenever $q \in (p, \infty)$. The facts that $\mu(A_n) > 0$ and $d_n := \frac{1}{2} \cdot \frac{|c_1|}{n^{1/p}(\log(n+1))^{\alpha_1/p}\mu(A_n)^{1/p}} \to \infty$ as $n \to \infty$ (because $\mu(A_n) < 1/2^n$), together with the inequality $|f| \geq d_n$ on A_n ($n \geq n_0$), show that $f \notin L^\infty(\mu, X)$. Hence $M \setminus \{0\} \subset L^p_{r\text{-}strict}$. So far, we have proved that the last set is $\mathfrak{c}$-lineable. In order to see that it is in fact maximal dense-lineable, it is enough to observe that $L^p_{r\text{-}strict} = \bigcap_{S \in \Gamma}(E \setminus S)$, where $E = L^p(\mu, X)$ and Γ is the family of all subspaces $L^p(\mu, X) \cap L^q(\mu, X)$ ($q > p$). Note that the subspace $\bigcap_{S \in \Gamma} S$ is dense in E because it contains the class S_t of step functions. By applying Lemma 2.1, the proof of (a) is concluded.

Part (b) is proved in a similar way, only by taking into account that this time the sets A_n can be chosen such that $1 < \mu(A_n)$ for all n (see Remark 3.2). Then the above subspace M works for our goal, and a further use of Lemma 2.1 provides the desired maximal dense-lineability. The unique change is the business of proving that $f \notin L^q(\mu, X)$ whenever $q < p$, where f is as before. This is easy, because we have now that

$$\begin{aligned} \|f\|_q^q &\geq \sum_{n=n_0}^{\infty} \frac{1}{2^q} \cdot \frac{|c_1|^q}{n^{q/p}(\log(n+1))^{\alpha_1 q/p}\mu(A_n)^{q/p}} \cdot \mu(A_n) \\ &\geq \sum_{n=n_0}^{\infty} \frac{1}{2^q} \cdot \frac{|c_1|^q}{n^{q/p}(\log(n+1))^{\alpha_1 q/p}}. \end{aligned}$$

This series diverges, for $q/p < 1$. Therefore $\|f\|_q = \infty$, as required.

It remains to demonstrate (c) under the assumptions $p > 1$, (α) and (β). Again by Remark 3.2.2 and (α), there are infinitely many pairwise disjoint measurable sets A_n with $0 < \mu(A_n) < 1/2^n$ $(n \in \mathbb{N})$. But observe that the set $A := \bigcup_{n=1}^{\infty} A_n$ has finite measure. It follows that the property (β) is also satisfied by the measure subspace $(X \setminus A, \mathcal{M}_{X\setminus A}, \mu|_{X\setminus A})$. This entails the existence of infinitely many mutually disjoint measurable sets B_n with $1 < \mu(B_n) < \infty$ and $A_k \cap B_n = \emptyset$ $(n, k \in \mathbb{N})$. Let $C_1 := A_1, C_2 := B_1, C_3 := A_2, C_4 := B_2, \ldots$. Then we define the vector subspace $M := \operatorname{span}(\{f_\alpha : \alpha \in (1,\infty)\})$, where this time $f_\alpha = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}(\log(n+1))^{\alpha/p}\mu(C_n)^{1/p}} \cdot \chi_{C_n}$. From here, the proof is an appropriate combination of the approaches of (a) and (b). The details (cumbersome, but easy) are left to the reader. □

Remarks 3.5. 1. The separability of $L^p(\mu, X)$ is a general hypothesis in the last theorem. According to Proposition 2.3, the condition is fulfilled if the property (σ) is satisfied.

2. In the case $p = 1$ we have, trivially, that $L^1_{l\text{-}strict}$ is always maximal dense-lineable, while (by (a)) L^1_{strict} is maximal dense lineable if and only if (α) holds.

3. The last theorem is no longer valid if $p = \infty$. Indeed, neither (α) nor (β) is compatible with the separability of $L^\infty(\mu, X)$: see Proposition 2.4.

We conclude this section with the following corollary, which provides two examples of when the main theorem applies.

Corollary 3.6. Assume that $p \in [1, \infty)$, that $(X, \mathcal{M}, \mu)$ is a measure space and that $L^p(\mu, X)$ is separable. Then $L^p_{r\text{-}strict}$ is maximal dense-lineable if at least one of the following properties is true:

(a) The measure μ is semifinite and nonatomic.

(b) X is a T_1 topological space and there is a non-isolated point $x_0 \in X$ such that X is regular and first-countable at it, $\mu(\{x_0\}) = 0$ and there is an open neighborhood U of x_0 with $\mu(U) < \infty$ and $\mu(V) > 0$ for any nonempty open set $V \subset U$.

Proof. If (a) is true, then $[0, \mu(M)] = \{\mu(A) : A \in \mathcal{M} \text{ and } A \subset M\}$ for every set $M \in \mathcal{M}$ (see for instance [10, Theorem 11.27]). Hence (α) is satisfied and Theorem 3.3 applies. If we start from (b), it follows from Lemma 2.5 that there exists a fundamental system $\{U_n\}_{n\geq 1}$ of open neighborhoods of x_0 such that $\overline{U_{n+1}} \subset U_n \subset U$ and $U_n \setminus \overline{U_{n+1}} \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore the the "annuli" $A_n := U_n \setminus \overline{U_{n+1}}$ satisfy $\mu(A_n) > 0$ $(n \geq 1)$. Since (U_n) is a decreasing sequence with intersection $\{x_0\}$ (because X is T_1) and the U_n's have finite measure, one derives that $\lim_{n\to\infty} \mu(U_n) = \mu(\{x_0\}) = 0$ (see [12, Chap. 1]). Hence $\mu(A_n) \to 0$, which shows that (α) is again fulfilled. □

According to Proposition 2.3(b), the separability of $L^p(\mu, X)$ is guaranteed by a set of conditions, some of which are finer than those given in part (b) of the last corollary.

4 Non-integrability on any open set

This short section is devoted to show that under appropriate, rather mild, conditions on a regular measure, right-strictness can be reinforced to involve every nonempty open set.

Theorem 4.1. *Let X be a Hausdorff first-countable separable locally compact perfect topological space. Assume that μ is a Borel measure on X such that μ is continuous, regular and has full support. Let $p \in [1, \infty)$ and $\mathcal{C}$ be the set of all functions $f \in L^p(\mu, X)$ such that, for every nonempty open set U of X and every $q > p$, $f \notin L^q(\mu, U)$. Then $\mathcal{C}$ is a dense subset of $L^p(\mu, X)$.*

Proof. We can consider the space $E := L^p(\mu, X)$ as a topological group under the operation $+$. If τ is the topology of X, our set $\mathcal{C}$ can be written as $\mathcal{C} = \bigcap_{S\in\Gamma}(E \setminus S)$, where $\Gamma := \{\{f \in L^p(\mu, X) : f|_U \in L^q(\mu, U)\} : q > p$ and $U \in \tau \setminus \{\emptyset\}\}$. Observe that each member of Γ is a semigroup of E. Moreover, the set $\bigcap_{S\in\Gamma} S$ is dense in E, for it contains the set S_t of step functions. According to Lemma 2.2, it is enough to show that $\mathcal{C} \neq \emptyset$.

To this end, observe that our hypotheses on X and μ imply that the conditions given in the part (b) of Corollary 3.5 are satisfied at all points $x_0 \in$

X. As in the proof of the mentioned corollary, we can consider an appropriate decreasing countable basis $U_n = U_{n,x_0}$ $(n \geq 1)$ of open neighborhoods for x_0, and then select mutually disjoint measurable subsets $A_n = A_{n,x_0}$ $(n \in \mathbb{N})$ having positive measure and such that $\mu(A_{n,x_0}) \to 0$. By passing to a subsequence, if necessary, one may suppose that $0 < \mu(A_{n,x_0}) < 1/2^n$ for every n. Similarly to the proof of Theorem 3.3, we consider the measurable function $f_{x_0} : X \to [0,\infty)$ given by

$$f_{x_0} = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}(\log(n+1))^{2/p}\mu(A_{n,x_0})^{1/p}} \cdot \chi_{A_{n,x_0}}.$$

As in the mentioned theorem (with $\alpha = 2$), we get that $\|f_{x_0}\|_p = \big[\sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^2}\big]^{1/p} =: \beta < \infty$. Observe that β does not depend on x_0. At this point, the separability of X comes to our help, so providing us with a dense countable subset $\{x_k : k \geq 1\} \subset X$. Define the measurable function $f : X \to [0,\infty)$ as

$$f = \sum_{k=1}^{\infty} 2^{-k} f_{x_k} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{2^{-k}}{n^{1/p}(\log(n+1))^{2/p}\mu(A_{n,x_k})^{1/p}} \cdot \chi_{A_{n,x_k}}. \qquad (7)$$

Our final task is to show that $f \in \mathcal{C}$. From the Minkowsky inequality, we obtain

$$\|f\|_p \leq \sum_{k=1}^{\infty} \|2^{-k} f_{x_k}\|_p = \sum_{k=1}^{\infty} 2^{-k}\beta = \beta < \infty.$$

Hence $f \in L^p(\mu, X)$.

Our proof will be concluded as soon as we show that, for every $q > p$ and every $U \in \tau \setminus \{\emptyset\}$, $f \notin L^q(\mu, U)$. For this, fix q and U as before. From the density of (x_k) it follows that there is $k_0 \in \mathbb{N}$ with $y := x_{k_0} \in U$. Since the $U_{n,y}$'s form a decreasing basis of open neighborhoods for y, we can find an $n_0 \in \mathbb{N}$ such that $U_{n,y} \subset U$ for all $n \geq n_0$. Hence $A_{n,y} \subset U$ for such integers n. Since all terms in (7) are nonnegative, we derive that

$$|f|^q \geq \sum_{n=n_0}^{\infty} \frac{2^{-k_0 q}}{n^{q/p}(\log(n+1))^{2q/p}\mu(A_{n,y})^{q/p}} \cdot \chi_{A_{n,y}},$$

from which one deduces that

$$\begin{aligned} \|f\|_q^q &\geq 2^{-k_0 q} \cdot \sum_{n=n_0}^{\infty} \frac{1}{n^{q/p}(\log(n+1))^{2q/p}\mu(A_{n,y})^{(q/p)-1}} \\ &\geq 2^{-k_0 q} \cdot \sum_{n=n_0}^{\infty} \frac{(2^{\frac{q}{p}-1})^n}{n^{q/p}(\log(n+1))^{2q/p}} = \infty, \end{aligned}$$

because $(q/p) - 1 > 0$. This proves the desired conclusion if q is finite. If $q = \infty$, suffice it to observe that $\inf_{A_{n,y}} |f| \to \infty$ $(n \to \infty)$ and that $\mu(A_{n,y}) > 0$ for all $n \in \mathbb{N}$. □

Observe that, from the proof of Lemma 2.2, we obtain a dense affine linear subspace of functions with the property described in the last theorem. On the contrary, note also that, like in Corollary 3.5, we cannot expect left-strictness in the conclusion of Theorem 4.1 because our measure may well be finite.

5 Final remarks

1. The nice notion of "A stronger than B" introduced in [2] can also be used to face the problem of dense-lineability.
2. If the separability of $L^p(\mu, X)$ is not guaranteed, we at least keep c-lineability in the conclusions of Theorem 3.3 and Corollary 3.5. By extending in the trivial way the strictness to the case $p = \infty$, we have also c-lineability here. We pose as an open problem to study the dense-lineability and the maximal dense-lineability in this case.
3. We also want to pose the question of whether the special sets studied in this paper are residual/spaceable under appropriate conditions. Moreover, we do not know whether the set $\mathcal{C}$ considered in Theorem 4.1 is lineable/c-lineable/dense-lineable/maximal-dense lineable.
4. Regarding this theorem, it is worth warning the reader that the expression "$f \notin L^q(\mu, U)$ for every nonempty open set U" cannot be replaced by "$f \notin L^q(\mu, A)$ for every measurable set A with $\mu(A) > 0$". Indeed, since $f \in L^p(\mu, X)$, we have that f is finite μ-almost everywhere. Thus there is a set Z with $\mu(Z) = 0$ such that $X = Z \cup \bigcup_{n=1}^{\infty} \{x \in X : |f(x)| \leq n\}$. Hence some set $\{x \in X : |f(x)| \leq n_0\} =: A$ has positive measure. But $|f|^q \leq n_0^{q-p}|f|^p$ on A, so $f \in L^q(\mu, A)$.
5. In [4] the following concept is introduced. A subset A of a topological vector space of scalar functions is called *algebrable* if $A \cup \{0\}$ contains an infinitely generated algebra. We cannot expect algebrability in our setting. For instance, if $q > p \geq 1$ and μ is finite, then the set $A := L^p(\mu, X) \setminus L^q(\mu, X)$ is *not* algebrable. Indeed, if A were algebrable, there would exist $f \in A$ with $f^N \in A$ for all $N \in \mathbb{N}$. Choose N with $Np > q$. Then $f \in L^{Np}(\mu, X) \subset L^q(\mu, X)$, which is absurd.
6. Lineability properties of families of functions that are either Riemann-integrable or non-Riemann integrable are studied in [8].

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