U-OPERATORS

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Abstract

Inspired by a statement of W. Luh asserting the existence of entire functions having together with all their derivatives and antiderivatives some kind of additive universality or multiplicative universality on certain compact subsets of the complex plane or of, respectively, the punctured complex plane, we introduce in this paper the new concept of U-operators, which are defined on the space of entire functions. Concrete examples, including differential and antidifferential operators, composition, multiplication and shift operators, are studied. A result due to Luh, Martirosian and Müller about the existence of universal entire functions with gap power series is also strengthened.

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1 Introduction

Let us denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Z}$ the set of all integers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, by $\mathbb{C}$ the complex plane, by $H(G)$ the Fréchet space of all complex holomorphic functions on a domain $G \subset \mathbb{C}$, endowed with the compact-open topology, and by $A(K)$ the set of all functions which are continuous on K and holomorphic in its interior K^0, where $K \subset \mathbb{C}$ is a compact set. Introducing the maximum norm $||f||_K := \max_{z \in K} |f(z)|$, the space $A(K)$ becomes a Banach space.

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Since Birkhoff proved in 1929 [12] the existence of an entire function f which is universal in the sense that the sequence of its additive translates $\{f(z+n) : n \in \mathbb{N}\}$ is dense in the space of entire functions $\mathcal{E} := H(\mathbb{C})$, a great number of papers have been written dealing with this topic or similar ones, yielding in many cases unexpected results. An excellent survey (updated till 1998) for the concepts, history and results about the subject of universality and the related one of hypercyclicity is [20].

In 1941 Seidel and Walsh [38] extended Birkhoff's theorem to non-Euclidean translates on the unit disk $\mathbb{D} = \{|z| < 1\}$. In 1988 Zappa [40] also established an analogous result to that of Birkhoff, this time for the punctured complex plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. He proved the existence of a holomorphic function f on $\mathbb{C}^*$ with the property that for any compact set $K \subset \mathbb{C}^*$, whose complement is connected, the set of the multiplicative translates $\{f(cz) : c \in \mathbb{C}^*\}$ is dense in $A(K)$. In this line of research, Montes and the first author [11] (compare also [25]) have characterized the sequences $(\varphi_n) \subset \mathrm{Aut(G)} := \{\text{automorphisms of } G\}$ –where $G \subset \mathbb{C}$ is a domain– for which there exist functions $f \in H(G)$ such that the sequence $(f \circ \varphi_n)$ has the analogous universal property for compact subsets of G. We recall this characterization in Theorem 1.1 below, but some terminology is first needed. By $\mathcal{K}(G)$ we denote the family of all compact subsets of G, while $\mathcal{M}(G)$ will stand for the family $\{K \in \mathcal{K}(G) : \mathbb{C} \setminus K \text{ is connected}\} = \{K \in \mathcal{K}(G) : G \setminus K \text{ is connected}\}$. A sequence $(\varphi_n) \subset \mathrm{Aut}(G)$ is said to be *run-away* whenever it acts properly discontinuously on G, that is, given $K \in \mathcal{K}(G)$, there is $n \in \mathbb{N}$ such that $K \cap \varphi_n(K) = \emptyset$.

Theorem 1.1. *Let $(\varphi_n) \subset \mathrm{Aut}(G)$. Then the following conditions are equivalent:*

(a) *The sequence (φ_n) is run-away.*

(b) *There exists a function $f \in H(G)$ such that $(f \circ \varphi_n)$ is dense in $A(K)$ for all $K \in \mathcal{M}(G)$.*

(c) *There exists a residual set of functions $f \in H(G)$ such that $(f \circ \varphi_n)$ is dense in $A(K)$ for all $K \in \mathcal{M}(G)$.*

We point out that in parts (b)–(c) the density of $(f \circ \varphi_n)$ can occur in $H(G)$ itself whenever G is not isomorphic to $\mathbb{C}^*$. Taking into account that $\{z \mapsto z+n : n \in \mathbb{N}\}$, $\{z \mapsto nz : n \in \mathbb{N}\}$ and $\{z \mapsto (n-1-nz)/((n-1)z-n) :$

$n \in \mathbb{N}\}$ are run-away sequences of automorphisms of $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{D}$ respectively, Theorem 1.1 extends and unifies Birkhoff–Seidel–Walsh–Zappa's theorems. It should be pointed out that several authors, including Luh, Duyos-Ruis, Blair, Rubel, Grosse-Erdmann, Gethner, Shapiro and Godefroy, had earlier extended Birkhoff's theorem in some direction, see [20] for a complete list of references.

We now focus our attention on a recent result of Luh that improves Birkhoff–Zappa's theorems, but this time following another point of view. In it $f^{(j)}$ denotes as usual the derivative of f of order j if $j \in \mathbb{N}_0$, and if $j \in \mathbb{N}$ the symbol $f^{(-j)}$ denotes the unique antiderivative F of order j satisfying $F^{(k)}(0) = 0$ for all $k \in \{0, 1, ..., j-1\}$. His statement (see [26, Theorem]) is proved constructively and, after adapting notations, reads as follows.

Theorem 1.2. *Let be $(a_n) \subset \mathbb{C}$ a given sequence with $a_n \to \infty$. Then there exists an entire function f with the following properties:*

(a) *For any fixed $j \in \mathbb{Z}$, the sequence of 'additive translates' $\{f^{(j)}(z + a_n) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C})$.*

(b) *For any fixed $j \in \mathbb{Z}$, the sequence of 'multiplicative translates' $\{f^{(j)}(a_n z) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C}^*)$.*

As a matter of fact, in [26, Theorem] the sequence (a_n) is just assumed to be unbounded, but the formulation is equivalent because a subsequence tending to infinity can be taken out. Luh's theorem also asserted a further property for f, namely, the sequence of derivatives $\{f^{([|a_n|])} : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C})$ ($[x]$ denotes the integer part of x). We will not consider this property because it is of a different nature and, in addition, it can be derived by using Baire-category methods together with the fact that the differentiation operator on $\mathcal{E}$ is densely hereditarily hypercyclic –see [20] for concepts, results and references– which in turn is a strong generalization of MacLane's theorem [32] about the existence of an entire function whose sequence of derivatives is dense in $\mathcal{E}$. Theorem 1.2 provides two novelties if it is compared to Birkhoff–Zappa's theorem: First, the function f can be replaced by the result of the action on f of the *operators* of differentiation and antidifferentiation, and secondly, the universal function f can be chosen to be *entire*, even in the case in which the domain ($\mathbb{C}^*$, this time) is not the whole plane $\mathbb{C}$. In [39, Kapitel 4] some extensions of Theorem 1.2 are shown by replacing $(z+a_n), (a_n z)$ to certain sequences $(S_n(z))$, not necessarily holomorphic, defined on some subsets of $\mathbb{C}$.

The two novelties described in the last paragraph motivate the introduction of the new concept of 'U–operators', that will be developed in the subsequent sections of this paper. Concrete examples of this new kind of operators as well as sufficient conditions will be given, and Theorem 1.2 will be strongly improved. It should be pointed out that, by following a different point of view, several other kinds of operators have been recently introduced regarding the 'wild' behavior near the boundary that they produce when acting on certain holomorphic functions in a domain of $\mathbb{C}$. The starting point of this related theory is, in turn, a strong result also due to Luh [24] about the existence of holomorphic 'monsters', see [24, 1, 2, 27, 37, 28, 6, 7, 14, 9, 10, 8, 30, 31].

Finally, in the last part of Section 6 we will strengthen a recent deep result due to Luh, Martirosian and Müller [29, Theorem 1], who proved constructively the existence of an entire function with lacunary power series expansion having dense additive and multiplicative translates. An improved version of their result is established in [30, Theorem 2]. Such a version reads as follows.

Theorem 1.3. *Let $Q \subset \mathbb{N}_0$ with upper density $\overline{\Delta}(Q) = 1$ and let (a_n) be a complex sequence with $a_n \to \infty$ as $n \to \infty$. Then there exists an entire function f with lacunary power series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \textit{with} \quad c_n = 0 \qquad \textit{for} \quad n \notin Q$$

satisfying the following properties:

(a) *The sequence $\{f(z + a_n) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C})$.*

(b) *The sequence $\{f(a_n z) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C}^*)$.*

As in Theorem 1.2, the sequence (a_n) can be just assumed to be unbounded. Several notions of density of a subset of $\mathbb{N}_0$ will be recalled in Section 6.

2 U–operators: sufficient criteria and first examples

Observe first that in Theorem 1.2 both sequences $(z + a_n)$ and $(a_n z)$ tend to infinity uniformly on compact subsets, respectively in $\mathbb{C}$ and $\mathbb{C}^*$. Hence, in order that everything works well with the new kind of operators to be introduced, the domains G to be considered must be *unbounded*, because otherwise every entire function would be bounded on G, which would prevent the desired density of any sequence of 'G-translates' of it. Specifically, we assume that the set

$$\omega(G) := \{(\varphi_n) \subset \mathrm{Aut}(G) : \varphi_n \to \infty\, (n \to \infty) \text{ uniformly on compacta in } G\}$$

be nonempty, in which case we say that G is an *ω-domain.* It is clear that if $(\varphi_n) \in \omega(G)$, then (φ_n) is run-away. Note that the sequences given at the beginning of this section are respectively in $\omega(\mathbb{C})$ and $\omega(\mathbb{C}^*)$, so $\mathbb{C}$ and $\mathbb{C}^*$ are ω-domains. In fact, it is not difficult to see that $\omega(\mathbb{C}) = \{(a_n + b_n z) : b_n \neq 0 \text{ for all } n \in \mathbb{N} \text{ and } a_n \to \infty, a_n/b_n \to \infty \text{ as } n \to \infty\}$ and $\omega(\mathbb{C}^*) = \{(a_n z) : a_n \neq 0 \text{ for all } n \in \mathbb{N} \text{ and } a_n \to \infty \text{ as } n \to \infty\}$. As for an essentially different example, the upper halfplane $\{\mathrm{Im}\ z > 0\}$ is also an ω-domain; indeed, take $\psi(z) = (2z-1)/(2-z)$ $(\in \mathrm{Aut}(\mathbb{D}))$, $\psi_n = \psi\circ\cdots\circ\psi$ (n-fold), $h(z) = (z-i)/(z+i)$ and $\varphi_n = h^{-1}\circ\psi_n\circ h$ $(n \in \mathbb{N})$; then $(\varphi_n) \in \omega(\{\mathrm{Im}\ z > 0\})$. It should be warned that not every unbounded domain is an ω-domain. For instance, if G has finite connectivity ≥ 3 then by Heins' theorem [22] the group $\mathrm{Aut}(G)$ is finite, hence no equence in $\mathrm{Aut}(G)$ can be run-away and, consequently, $\omega(G) = \emptyset$. Finally, an unbounded infinite-connected domain may not be an ω-domain: just take $G = \mathbb{C}\setminus[\{1/n : n \in \mathbb{N}\}\cup\{0\}]$; a simple application of the Casorati–Weierstrass theorem and of the Open Mapping Theorem for holomorphic functions shows that $\mathrm{Aut}\,(G)$ reduces to the identity on G.

Next, we give the definition of U–operators. Observe that in it the condition on G of being an ω-domain is in fact not strictly necessary, but we keep it because otherwise the property would become vacuous. By *operator* we mean a (*not* necessarily linear) continuous selfmapping on some space, mainly on $\mathcal{E}$.

Definition 2.1. We say that an operator $T : \mathcal{E} \to \mathcal{E}$ is a *U–operator* whenever the following property is satisfied:

Given an ω-domain $G \subset \mathbb{C}$ and a sequence $(\varphi_n) \in \omega(G)$, there exists a dense subset of entire functions f such that the sequence $\{((Tf) \circ \varphi_n)|_K : n \in \mathbb{N}\}$ is dense in $A(K)$ for every $K \in \mathcal{M}(G)$.

For the sake of convenience, we rewrite the last definition in the language of universality. Recall that if X, Y are topological spaces then a sequence $T_n : X \to Y$ $(n \in \mathbb{N})$ of continuous selfmappings is said to be *universal* whenever there is some element $x \in X$, also called universal (for (T_n)), whose orbit $\{T_n x : n \in \mathbb{N}\}$ is dense in Y. And (T_n) is said to be *densely universal* if the set $\mathcal{U}((T_n))$ of universal elements for (T_n) is dense in X. If X, Y are linear topological spaces and the mappings T_n are also linear then the word 'universal' is frequently replaced by 'hypercyclic'. The condition given in Definition 2.1 tells us that, for given G, $K \in \mathcal{M}(G)$ and (φ_n), the sequence

$$T_n : f \in \mathcal{E} \mapsto ((Tf) \circ \varphi_n)|_K \in A(K) \qquad (n \in \mathbb{N}) \tag{1}$$

is densely universal.

We need to reformulate Definition 2.1 in a more comfortable way. This will be done in Theorem 2.2, but for this the following topological lemma is necessary. Its content can be found in [11, Lemma 2.9] (see [26, Lemma 3] for the special case $G = \mathbb{C}^*$).

Lemma 2.1. *For every domain $G \subset \mathbb{C}$ there exists a sequence $(K_m) \subset \mathcal{M}(G)$ such that for every $K \in \mathcal{M}(G)$ there is a positive integer m_0 with $K \subset \overset{0}{K}_{m_0}$.*

We remark that while in Definition 2.1 the universal function f does not depend on the compact set K, in part (b) of the next result it is allowed to *depend* on K. As usual, $B(a, r)$ $(\overline{B}(a, r))$ will stand for the open (closed, respectively) ball with center a and radius r $(a \in \mathbb{C}, r > 0)$.

Theorem 2.2. *Assume that T is an operator on $\mathcal{E}$. Then the following properties are equivalent:*

(a) *T is a U-operator.*

(b) *Given an ω-domain $G \subset \mathbb{C}$, a sequence $(\varphi_n) \in \omega(G)$ and a compact set $K \in \mathcal{M}(G)$, the sequence (T_n) defined by* (1) *is densely universal.*

(c) *Given an ω-domain $G \subset \mathbb{C}$, $\sigma = (\varphi_n) \in \omega(G)$, $K \in \mathcal{M}(G)$, $\varepsilon > 0$ and $g \in A(K)$, the set*

$$(2)\quad A(T, G, K, \sigma, \varepsilon, g) := \{f \in \mathcal{E} : \text{ there exists } n \in \mathbb{N} \text{ with } \|(Tf)\circ\varphi_n - g\|_K < \varepsilon\}$$

is dense in $\mathcal{E}$.

(d) *Given an ω-domain $G \subset \mathbb{C}$, $\sigma = (\varphi_n) \in \omega(G)$, $K \in \mathcal{M}(G)$, $\varepsilon > 0$, $r > 0$, $g \in A(K)$ and $h \in \mathcal{E}$, the set*

$$U(T, G, K, \sigma, \varepsilon, r, g, h) := \{f \in \mathcal{E} : \|f - h\|_{\overline{B}(0,r)} < \varepsilon$$

$$(3)\qquad \text{and there exists } n \in \mathbb{N} \text{ such that } \|(Tf) \circ \varphi_n - g\|_K < \varepsilon\}$$

is nonempty.

Proof. It is straightforward that (c) and (d) are equivalent because the family of sets $D(h, \varepsilon, r)$ $(h \in \mathcal{E}, \varepsilon > 0, r > 0)$ given by

$$D(h, \varepsilon, r) = \{f \in \mathcal{E} : \|f - h\|_{\overline{B}(0,r)} < \varepsilon\}$$

is a basis for the topology of $\mathcal{E}$, and

$$U(T, G, K, \sigma, \varepsilon, r, g, h) = A(T, G, K, \sigma, \varepsilon, g) \cap D(h, \varepsilon, r).$$

On the other hand, it is trivial that (a) implies (b). Assume now that (b) holds. Then (c) is satisfied as soon as one realizes that

$$\mathcal{U}((T_n)) = \bigcap \{A(T, G, K, \sigma, \varepsilon, g) : \varepsilon > 0, g \in A(K)\}.$$

Finally, our goal is to prove that T is a U–operator by starting from (c). Observe first that each set defined by (2) can be written as

$$A(T, G, K, \sigma, \varepsilon, g) = \bigcup_{n \in \mathbb{N}} T_n{}^{-1}(B_K(g, \varepsilon)),$$

where $B_K(g, \varepsilon)$ is the open ball $\{h \in A(K) : \|h - g\|_K < \varepsilon\}$ in $A(K)$. Therefore the continuity of each T_n shows that $A(T, G, K, \sigma, \varepsilon, g)$ is an open subset of $\mathcal{E}$. But note that if (g_j) is any fixed denumerable dense subset of $A(K)$ (for instance, (g_j) may be the set of restrictions to K of polynomials whose coefficients have rational real and imaginary parts) then

$$\mathcal{U}((T_n)) = \bigcap_{j,k\in\mathbb{N}} A(T,G,K,\sigma,\frac{1}{k},g_j).$$

Hence $\mathcal{U}((T_n))$ is a countable intersection of dense subsets in the Baire space $\mathcal{E}$. At this point it is convenient to write $T_n = T_n^{(K)}$, with the emphasis in the fact that for given G, σ the sequence (T_n) depends on K. In order to see that T is a U–operator it must be shown that the set

$$\mathcal{L}(T,G,\sigma) := \bigcap \{\mathcal{U}((T_n^{(K)})) : K \in \mathcal{M}(G)\}$$

is dense in $\mathcal{E}$. But if (K_m) is the sequence of compact sets furnished by Lemma 2.1 then

$$\mathcal{L}(T,G,\sigma) = \bigcap_{m\in\mathbb{N}} \mathcal{U}((T_n^{(K_m)})). \tag{4}$$

Indeed, given $K \in \mathcal{M}(G)$ there exists $m_0 \in \mathbb{N}$ with $K \subset K_{m_0}$. If $f \in \mathcal{U}((T_n^{(K_{m_0})}))$ then for each fixed polynomial $P(z)$ there is a sequence $\{n_1 < n_2 < ... < n_j < ...\} \subset \mathbb{N}$ such that

$$(Tf)(\varphi_{n_j}(z)) \to P(z) \quad (j \to \infty)$$

uniformly on K_{m_0}, hence on K. Now Mergelyan's theorem [36, Chapter 20] implies that the set of polynomials is dense in $A(K)$, therefore the sequence $\{((Tf) \circ \varphi_n)|_K \ : \ n \in \mathbb{N}\}$ is also dense in $A(K)$, which proves (4). Thus, $\mathcal{L}(T,G,\sigma)$ is a countable intersection of residual subsets in $\mathcal{E}$. Then $\mathcal{L}(T,G,\sigma)$ is residual itself, so dense, and this finishes the proof. □

From the proof it is clear that in parts (c)–(d) it can be supposed that g is just a polynomial.

Our next task should be, obviously, to identify some U–operator. It happens that the simplest operator does the job.

Theorem 2.3. *The identity operator I on $\mathcal{E}$ is a U–operator.*

Proof. Let us try to apply condition (d) in Theorem 2.2. Fix G, $\sigma = (\varphi_n), K$, ε, r, g as in that theorem and consider the set $U := U(T = I, G, K, \sigma, \varepsilon, r, g, h)$ given by (3). We want to show that $U \neq \emptyset$, that is, there is an entire function f and some $n \in \mathbb{N}$ with $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and $\|f \circ \varphi_n - g\|_K < \varepsilon$. Since $\varphi_n(z) \to \infty$ $(n \to \infty)$ uniformly on K, there exists n with $|\varphi_n(z)| > r$ for all $z \in K$. Then $\overline{B}(0,r) \cap \varphi_n(K) = \emptyset$. In addition, $\varphi_n(K)$ is a compact subset of G with connected complement because φ_n is an isomorphism on G. Therefore the set $L := \overline{B}(0,r) \cup \varphi_n(K)$ is a compact subset of $\mathbb{C}$ with connected complement. Consider the function $F : L \to \mathbb{C}$ defined by

$$F(z) = \begin{cases} h(z) & \text{if } |z| \leq r \\ g(\varphi_n^{-1}(z)) & \text{if } z \in \varphi_n(K). \end{cases}$$

We have that $F \in A(L)$, so by Mergelyan's theorem there exists a polynomial f with $\|f - F\|_L < \varepsilon$. This implies that $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and $\|f - g \circ \varphi_n^{-1}\|_{\varphi_n(K)} < \varepsilon$. But the last inequality is the same as $\|f \circ \varphi_n - g\|_K < \varepsilon$, which finishes the proof. □

We can now produce a big family of U–operators via composition of operators.

Theorem 2.4. *Suppose that T, S are operators on $\mathcal{E}$ in such a way that T is a U–operator and S is linear and onto. Then TS is a U–operator.*

Proof. If we follow the notations in the proof of Theorem 2.2 one must demonstrate that for fixed ω-domain G and sequence $\sigma \in \omega(G)$ the set $\mathcal{L}(TS, G, \sigma)$ is dense in $\mathcal{E}$. For this, observe that

$$\mathcal{L}(TS, G, \sigma) = S^{-1}(\mathcal{L}(T, G, \sigma)),$$

hence $\mathcal{L}(TS, G, \sigma)$ is dense because $\mathcal{L}(T, G, \sigma)$ is dense and the Open Mapping Theorem (recall that $\mathcal{E}$ is an F-space) guarantees that if $V \subset \mathcal{E}$ is a nonempty open set then $S(V)$ is also a nonempty open set. □

The last theorem carries an important consequence, namely, every differentiation operator D^j $(j \geq 0)$ is a U–operator. Here, as usual, $D^0 = I$ and

$D^j f = f^{(j)}$ ($j \in \mathbb{N}$). But much more can be obtained. Recall that an entire function $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ is called of *exponential type* whenever there exist positive constants A, B such that

$$|\Phi(z)| \le A \exp(B|z|) \qquad (z \in \mathbb{C}).$$

In such a case the series $\Phi(D) = \sum_{j=0}^{\infty} a_j D^j$ defines an operator on $\mathcal{E}$. Hence, it is a kind of infinite order differentiation operator with constant coefficients. Consider the translation operators τ_a ($a \in \mathbb{C}$) defined as $\tau_a f(z) = f(z+a)$ ($z \in \mathbb{C}$, $f \in \mathcal{E}$). It happens that a linear operator S on $\mathcal{E}$ commutes with the translation operators τ_a if and only if it commutes with the differentiation operator D if and only if $S = \Phi(D)$ for some $\Phi \in \mathcal{E}$ with exponential type if and only if there is a complex Borel measure μ on $\mathbb{C}$ with compact support such that $Sf(z) = \int f(z+w)\, d\mu(w)$ for all $z \in \mathbb{C}$ and all $f \in \mathcal{E}$, see for instance [18, Section 5].

Theorem 2.5. (a) *If S is an onto linear operator on $\mathcal{E}$ then S is a U–operator.*

(b) *If S is a linear operator on $\mathcal{E}$ that commutes with translations then S is a U–operator.*

Proof. As for part (a), combine Theorems 2.3–2.4. Now part (b) is a consequence of the Malgrange–Ehrenpreis theorem that asserts that every differentiation operator $\Phi(D)$ is surjective on $\mathcal{E}$, see [16] and [33]. □

One might believe that having dense range and being a U–operator are equivalent. Nevertheless, this is *false*. Indeed, each antidifferentiation operator D^{-N} ($N \in \mathbb{N}$) given by $D^{-N}(f) = f^{(-N)}$ is a U–operator (see Section 4) but evidently it has not dense range. We want to pose here the following problem, whose answer is unknown to us up to date, compare with Theorem 2.5(a):

Is a U–operator any operator on $\mathcal{E}$ with dense range?

We now focus our attention on the search of workable conditions under which an operator T on $\mathcal{E}$ is a U–operator. For this, let us introduce two new concepts. We say that T has *ω–dense range* whenever there is $R > 0$ such that the restriction mapping

$$T_M : f \in \mathcal{E} \to (Tf)|_M \in A(M)$$

has dense range for any $M \in \mathcal{M}(\{|z| > R\})$. Any operator on $\mathcal{E}$ with dense range has, obviously, ω–dense range. We say that T is *ω-stable* whenever the following property is satisfied: For every $r > 0$ there is $R > 0$ such that for each $f \in \mathcal{E}$, each $\varepsilon > 0$ and each $M \in \mathcal{M}(\{|z| > R\})$ there exists $\delta > 0$ and $S \in \mathcal{M}(\{|z| > r\})$ such that if $g \in \mathcal{E}$ and $\|f - g\|_S < \delta$ then $\|Tf - Tg\|_M < \varepsilon$. This property has obviously an easier formulation if T is linear.

For instance, by using Malgrange–Ehrenpreis' theorem together with Mergelyan's theorem, it is easy to see that every nonzero differential operator $\Phi(D)$ has ω-dense range. Also the antidifferential operator D^{-N} has ω-dense range; indeed, an adequate application of Mergelyan's theorem yields that the polynomials with a zero of order $\geq N$ at the origin are dense in $A(M)$ whenever $M \in \mathcal{M}(\mathbb{C})$ with $0 \notin M$, and these polynomials are clearly in the range of D^{-N}. On the other hand, from Cauchy's integral formula for derivatives, it is not difficult to realize that $\Phi(D)$ is ω-stable whenever Φ is of *subexponential type*. Recall that Φ is of subexponential type whenever given $\varepsilon > 0$ there is a constant $K = K(\varepsilon) > 0$ such that $|\Phi(z)| \leq Ke^{\varepsilon|z|}$ for all $z \in \mathbb{C}$; equivalently, $n|a_n|^{1/n} \to 0$ $(n \to \infty)$ if $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$. Every entire function of subexponential type is, trivially, of exponential type.

A combination of ω-denseness and ω-stability will give a positive result.

Theorem 2.6. *Assume that T is an operator on $\mathcal{E}$ such that for every $r > 0$ there is $R > 0$ satisfying that for each $M \in \mathcal{M}(\{|z| > R\})$ the following properties hold:*

(i) *The restriction mapping T_M has dense range.*

(ii) *For every $f \in \mathcal{E}$ and every $\varepsilon > 0$ there exist $\delta > 0$ and $S \in \mathcal{M}(\{|z| > r\})$ such that if $\varphi \in \mathcal{E}$ and $\|f - \varphi\|_S < \delta$ then $\|Tf - T\varphi\|_M < \varepsilon$.*

Then T is a U–operator.

Proof. Fix an ω-domain $G \subset \mathbb{C}$, $\sigma = (\varphi_n) \in \omega(G)$, $K \in \mathcal{M}(G)$, $\varepsilon > 0$, $r > 0$, $g \in A(K)$, $h \in \mathcal{E}$, and the corresponding set $U(T, G, K, \sigma, \varepsilon, r, g, h) =: U$ given by (3). Our goal is to prove that $U \neq \emptyset$.

Since $\sigma \in \omega(G)$ there exists $m \in \mathbb{N}$ with $\varphi_m(K) \subset \{|z| > R\}$, where $R > 0$ is the number associated to r given by hypothesis. Observe that $\varphi_m(K) \in \mathcal{M}(G)$ $(\subset \mathcal{M}(\mathbb{C}))$ because φ_n is a homeomorphism from G into itself. Therefore, by (i) and the fact that $g \circ \varphi_m^{-1} \in A(\varphi_m(K))$, there exists an entire function f_1 such that

$$(5) \qquad \|Tf_1 - g \circ \varphi_m^{-1}\|_{\varphi_m(K)} < \frac{\varepsilon}{2}.$$

Now, by (ii) there exist $\delta > 0$ and $S \in \mathcal{M}(\mathbb{C})$ with $S \subset \{|z| > r\}$ such that for all $\varphi \in \mathcal{E}$

$$(6) \qquad \|\varphi - f_1\|_S < \delta \quad \text{implies that} \quad \|T\varphi - Tf_1\|_{\varphi_m(K)} < \frac{\varepsilon}{2}.$$

Note that the complement of the compact set $L := \overline{B}(0,r) \cup S$ is connected because S and $\overline{B}(0,r)$ share this property and they are disjoint. Hence Runge's approximation theorem together with the fact that F is holomorphic on an open subset containing L allows to select a polynomial f (so $f \in \mathcal{E}$) satisfying

$$\|f - F\|_L < \min\{\delta, \varepsilon\},$$

where $F : L \to \mathbb{C}$ is the function belonging to $A(L)$ given by

$$F(z) = \begin{cases} h(z) & \text{if } z \in \overline{B}(0,r) \\ f_1(z) & \text{if } z \in S. \end{cases}$$

Thus, we obtain $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and, in addition, $\|f - f_1\|_S < \delta$. Due to (6), the last inequality yields $\|Tf - Tf_1\|_{\varphi_m(K)} < \frac{\varepsilon}{2}$. Now, this together with (5) and the triangle inequality gives $\|Tf - g \circ \varphi_m^{-1}\|_{\varphi_m(K)} < \varepsilon$, which is clearly equivalent to $\|(Tf) \circ \varphi_m - g\|_K < \varepsilon$. Summarizing, f is an entire function satisfying $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and $\|(Tf) \circ \varphi_m - g\|_K < \varepsilon$ for some $m \in \mathbb{N}$. In other words, $U \neq \emptyset$. □

Corollary 2.7. *Suppose that T is an operator on $\mathcal{E}$ which is ω-stable and has ω-dense range. Then T is a U–operator.*

The remarks about $\Phi(D)$ just before Theorem 2.6 together with Corollary 2.7 yield again that, at least for entire functions Φ of subexponential type, $\Phi(D)$ is a U–operator. Observe that this time the proof does not depend on the fact that I is a U–operator, compare the proof of Theorem 2.5.

3 Composition and multiplication operators

In this section, conditions in order that composition and multiplication operators be U–operators are investigated. Recall that if $\varphi \in \mathcal{E}$ then its associated right-composition operator C_φ, left-composition (or superposition) operator L_φ and multiplication operator M_φ are defined on $\mathcal{E}$ as $C_\varphi(f) = f \circ \varphi$, $L_\varphi(f) = \varphi \circ f$, $M_\varphi(f) = f\varphi$. Observe that C_φ and M_φ are linear but L_φ is not, except for trivial cases.

As for right-composition operators we suspect that only the similarities on the plane, that is, the polynomials $\varphi(z) = az + b$ of degree one or, equivalently, the automorphisms of $\mathbb{C}$ (which in turn are the unique one-to-one entire functions), generate U–operators. Although we have not been able to give a complete characterization, we have obtained the following result.

Theorem 3.1. *Assume that φ is an entire function. We have:*

(a) *If C_φ is a U–operator then φ is a polynomial.*

(b) *If φ is a similarity then C_φ is a U–operator.*

(c) *If $\varphi(z) = P((z - \alpha)^N)$ for some $\alpha \in \mathbb{C}$, some positive integer $N \geq 2$ and some polynomial P then C_φ is not a U–operator.*

(d) *If φ is a polynomial with degree$\,(\varphi) = 2$ then C_φ is not a U–operator.*

Proof. Fix any value $a \in \mathbb{C}$. If φ were not a polynomial then the point of infinity would be an essential singularity for φ, whence by Casorati–Weierstrass' theorem a sequence $(z_n) \subset \mathbb{C}$ with $z_n \to \infty$ $(n \to \infty)$ could be found in such a way that $\varphi(z_n) \to a$ $(n \to \infty)$. Consider the ω-domain $G := \mathbb{C}$, the sequence $\sigma := (\varphi_n(z) = z + z_n) \in \omega(\mathbb{C})$ and the compact set $K := \{0\} \in \mathcal{M}(\mathbb{C})$. Assume that f satisfies the property of Definition 2.1 for $T := C_\varphi$. Then for $g \equiv 0$ we would get an increasing sequence $(n_j) \subset \mathbb{N}$ with $f(\varphi(\varphi_{n_j}(z))) \to g(z)$ $(j \to \infty)$ on $A(K)$, that is, $f(\varphi(z_{nj})) \to 0$ $(j \to \infty)$. But $(f(\varphi(z_{nj})))$ tends to $f(a)$, hence $f(a) = 0$ for all $a \in \mathbb{C}$, i.e. $f \equiv 0$, which is clearly impossible. This proves (a). On the other hand, if φ is a similarity then, clearly, C_φ is linear, onto (so it has dense range) and ω-stable. Therefore part (b) is a consequence of either Theorem 2.5(a) or Corollary 2.7. As for (d), observe that any polynomial $\varphi(z) = az^2 + bz + c$ of degree two can be written in the form $\varphi(z) = P((z-\alpha)^2)$, where $\alpha = -b/2a$ and $P(z) = az - c - (b^2/4a)$. Hence (d) follows from (c).

Finally, let us prove (c). Assume that $\varphi(z) = P((z-\alpha)^N)$ with α, N, P as in the hypothesis, and consider the ω-domain $G := \mathbb{C} \setminus \{\alpha\}$, the sequence $(\varphi_n(z) := \alpha + n(z-\alpha)) \in \omega(\mathbb{C} \setminus \{\alpha\})$ and the circle arc $K := \{\alpha + \exp(it) : 0 \le t \le 2\pi/N\}$, which is in $\mathcal{M}(\mathbb{C} \setminus \{\alpha\})$ because $N \ge 2$. Suppose, by the way of contradiction, that C_φ is a U–operator. Then we would obtain an entire function f such that one can associate to the function $g(z) := 1/(z-\alpha) \in A(K)$ an adequate increasing sequence $(n_j) \subset \mathbb{N}$ satisfying $(C_\varphi f)(\varphi_{n_j}(z)) \to g(z)$ $(j \to \infty)$ uniformly on K, that is, $f(P(n_j^N(z-\alpha)^N)) \to 1/(z-\alpha)$ $(j \to \infty)$ uniformly on K. Therefore, after taking N–powers,

$$\lim_{j\to\infty} \sup_{z\in K} \left| f(P(n_j^N(z-\alpha)^N))^N - \frac{1}{(z-\alpha)^N} \right| = 0.$$

Consider the circle arcs $K_\nu = \alpha + \omega_\nu(K - \alpha)$ $(\nu \in \{0, 1, ..., N-1\})$, where $\omega_\nu = \exp(2\pi\nu/N)$. Of course, $K_0 = K$. Denote by S the circle with center α and radius 1. Then $S = K_0 \cup K_1 \cup ... \cup K_{N-1}$. Given $z \in S$ there is $\nu \in \{0, 1, .., N-1\}$ with $z \in K_\nu$, so $\alpha + \omega_\nu^{-1}(z-\alpha) \in K$. But also

$$\left| f\left(P(n_j^N(\alpha + \omega_\nu^{-1}(z-\alpha) - \alpha)^N)\right)^N - \frac{1}{(\alpha + \omega_\nu^{-1}(z-\alpha) - \alpha)^N} \right| =$$

$$= \left| f\left(P(n_j^N(z-\alpha)^N)\right)^N - \frac{1}{(z-\alpha)^N} \right|$$

because $\omega_\nu^N = 1$. Hence the $\lim_{j\to\infty} \sup_{z\in S}$ of the last expression equals zero; in other words,

$$f\left(P(n_j^N(z-\alpha)^N)\right)^N \to \frac{1}{(z-\alpha)^N} \quad (j \to \infty)$$

uniformly on S. Then there exists $j_0 \in \mathbb{N}$ such that

$$\left| f\left(P(n_{j_0}^N(z-\alpha)^N)\right)^N - \frac{1}{(z-\alpha)^N} \right| < 1 \qquad (z \in S),$$

so $|[(z-\alpha) f\left(P(n_{j_0}^N(z-\alpha)^N)\right)]^N - 1| < 1$ for all $z \in S$. But, due to the Maximum Modulus Principle, the last inequality holds for all z in the open ball of center α and radius 1, in particular for $z = \alpha$, that is, $1 < 1$. This is absurd, so the theorem is proved. $\square$

Next, we show a characterization of the property of U–operator for L_φ in terms of existence of an 'approximate right inverse' for φ, see [9, Section 3]. The characterization in terms *only* of φ remains as an open question.

Theorem 3.2. *Assume that φ is an entire function. Then the following properties are equivalent:*

(a) *The superposition operator L_φ is a U–operator.*

(b) *There is a sequence $(f_n) \subset \mathcal{E}$ such that $(\varphi \circ f_n)$ tends to the identity function locally uniformly in $\mathbb{C}$.*

Proof. Let us suppose that (a) holds. Then by taking $T = L_\varphi$, $G = \mathbb{C}$, $\varphi_n(z) = z + n$ $(n \in \mathbb{N})$ in Definition 2.1 one obtains the existence of at least one entire function f such that, for every closed ball B, $(L_\varphi f)(\varphi_n(z)) \to z$ $(n \to \infty)$ in $A(B)$. Equivalently, $\varphi(f(z+n)) \to z$ as $n \to \infty$ uniformly on B. Therefore (b) is satisfied if we just take $f_n(z) = f(z+n)$ $(n \in \mathbb{N})$. Conversely, assume that (b) holds. From the continuity of φ it is easy to see that L_φ is always ω-stable. On the other hand, if we fix a set $M \in \mathcal{M}(\mathbb{C})$ and $g \in \mathcal{E}$ then we have that $g(M)$ is compact, whence $\sup_{z \in g(M)} |\varphi(f_n(z)) - z| \to 0$ $(n \to \infty)$ or, that is the same, $\sup_{z \in M} |\varphi(f_n(g(z))) - g(z)| \to 0$ $(n \to \infty)$. This tells us that

$$L_\varphi(f_n \circ g) \to g \quad (n \to \infty) \ \text{ in } \ A(M),$$

hence the restriction mapping $(L_\varphi)_M : f \in \mathcal{E} \to (L_\varphi f)|_M \in A(M)$ has dense range due to Mergelyan's theorem. Consequently, L_φ has ω-dense range and the result is completely proved after an application of Corollary 2.7. $\square$

We point out here that, in order that (b) is satisfied, the injectivity of φ is sufficient but not necessary (in fact, any entire universal function in the sense of Birkhoff –see Section 1– satisfies (b)), and its surjectivity is necessary but not sufficient, see [9, Section 3].

We finish this section by characterizing the multiplication U–operators.

Theorem 3.3. *Let be given an entire function φ. Then the following properties are equivalent:*

(a) *For all operator T on $\mathcal{E}$ that is ω-stable and has ω-dense range, $M_\varphi T$ is a U–operator.*

(b) *The multiplication operator M_φ is a U–operator.*

(c) *There exists an operator T on $\mathcal{E}$ such that $M_\varphi T$ is a U–operator.*

(d) *The set $Z(\varphi)$ of zeros of φ is finite.*

Proof. Because the identity operator is ω-stable and has ω-dense range (so it is a U-operator), it is trivial that (a) implies (b) and that (b) implies (c).

Assume now that (c) holds, that is, $M_\varphi T$ is a U–operator for some operator T on $\mathcal{E}$. Let us suppose, by a way of contradiction, that (d) is false, so there are points z_n ($n \in \mathbb{N}$) tending to infinity with $\varphi(z_n) = 0$ for all n. If $G = \mathbb{C}$ and $(\varphi_n(z) := z + z_n) \in \omega(G)$ then there must be an entire function f such that the sequence $(\varphi \circ \varphi_n)((Tf) \circ \varphi_n)$ is dense in $A(K := \{0\}) = \{\text{the constants}\}$, which is absurd because $\varphi(\varphi_n(0))(Tf)(\varphi_n(0)) = 0$ for all n. Therefore the zero set of φ is finite. Finally, we start from the fact that $Z(\varphi)$ is finite. Our aim is to prove (a), hence let us fix an ω-stable operator T on $\mathcal{E}$ with ω-dense range. From the continuity of φ, it is immediate that $M_\varphi T$ is also ω-stable. On the other hand, there is $R > 0$ such that the restriction mapping $T_M : f \in \mathcal{E} \to (Tf)|_M \in A(M)$ has dense range for any $M \in \mathcal{M}(\{|z| > R\})$. We can suppose without loss of generality that $R > \max\{|z| : z \in Z(\varphi)\}$. Let us fix $\varepsilon > 0$, $M \in \mathcal{M}(\{|z| > R\})$ and $g \in A(M)$. Then $g/\varphi \in A(M)$, therefore there exists $f \in \mathcal{E}$ with $\|Tf - (g/\varphi)\|_M < \varepsilon/\|\varphi\|_M$. Hence $\|(M_\varphi T)f - g\|_M < \varepsilon$ and $M_\varphi T$ also as ω-dense range. Now Corollary 2.7 anew finishes the proof. $\square$

4 Integral operators

In this section we are going to discover some classes of integral operators defined on the space $\mathcal{E}$, including the antidifferentiation operator D^{-N}, which are U–operators.

Along this section the symbol φ will denote an entire function $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of two complex variables. The Volterra operator of the first kind associated to φ is defined by

$$V_\varphi : f \in \mathcal{E} \mapsto V_\varphi f \in \mathcal{E}, \quad (V_\varphi f)(z) = \int_0^z f(t)\varphi(z,t)\,dt \quad (z \in \mathbb{C}),$$

where the integral is taken along any rectifiable arc joining the origin to z. We will prove in due course that, under adequate conditions on the kernel φ, the Volterra operator V_φ with or without a perturbation by a differential operator is a U–operator. In particular, our results also include Volterra operators of the second class $\lambda I + V_\varphi$. Now, we recall the notion of (generally infinite order) antidifferential operators with constant coefficients, see [4, Section 2]. Let $\Psi(z) = \sum_{j=0}^\infty c_j z^j$ be a formal complex power series. By taking into account that

$$D^{-j} f(z) = \int_0^z f(t)\,\frac{(z-t)^{j-1}}{(j-1)!}\,dt \qquad (j \in \mathbb{N},\ f \in \mathcal{E},\ z \in \mathbb{C})$$

it is not difficult to see that if we assume in addition that $\lim_{j\to\infty} |c_j|^{1/j}/j = 0$ then the series $\Psi(D^{-1}) = \sum_{j=0}^\infty c_j D^{-j}$ defines an operator on $\mathcal{E}$. Indeed, if we set

$$\varphi(z,t) = \sum_{j=1}^\infty c_j \frac{(z-t)^{j-1}}{(j-1)!}$$

then φ is entire in both variables and $\Psi(D^{-1}) = c_0 I + V_\varphi$. Of course, Volterra operators and operators $\Psi(D^{-1})$ include the operators D^{-N} $(N \in \mathbb{N})$.

The following lemmas will reveal useful in order to find integral U–operators. But a little further notation is needed. By ∂A we mean the boundary of any set $A \subset \mathbb{C}$. If K is a compact set and $a \in K$ then $A_a(K)$ will denote the subspace of all functions of $A(K)$ with a zero at a, endowed with the same norm $\|\cdot\|_K$. In order to avoid troubles with integration along arcs we will consider the class Π of closed Jordan regions L whose boundary ∂L is a polygonal closed curve which consists of finitely many segments that are parallel to the axes. Observe that each integral $\int_a^b F(t)\,dt$ makes sense and is unambiguously defined for each $F \in A(L)$ and each pair of points $a, b \in L$ whenever $L \in \Pi$. Indeed, the complement of L is connected and a, b can be joined by a piecewise continuously differentiable arc yielding in L.

Lemma 4.1. *Let S be an operator on $\mathcal{E}$ and $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ an entire function of two variables. Assume that there exists an $R > 0$ such that for each $r > R$*

and each $M \in \mathcal{M}(\{|z| > r\})$ there are $L \in \mathcal{M}(\{|z| > r\}) \cap \Pi$ with $M \subset L$ and a point $a \in \partial L \setminus M$ such that

(a) *the operator S extends continuously to a mapping $S_1 : A(L) \to A(M)$,*

(b) *the mapping $Q : A_a(L) \to A(M)$ defined by*

$$Qf(z) = S_1 f(z) + \int_a^z f(t)\varphi(z,t)\,dt \quad (z \in M)$$

has dense range.

Then $S + V_\varphi$ is a U–operator.

Proof. Fix a set $U = U(T = S + V_\varphi, G, K, \sigma = (\varphi_n), \varepsilon, r, g, h)$ as in (3). According Theorem 2.2, we should show that $U \neq \emptyset$. We may suppose $r > R$ without loss of generality. Since K is a compact subset of G and $\sigma \in \omega(G)$, there is $n \in \mathbb{N}$ such that $\varphi_n(K) \cap \overline{B}(0,r) = \emptyset$. Then $M := \varphi_n(K) \in \mathcal{M}(\{|z| > r\})$ because φ_n is automorphism of G. By hypothesis, there exists $L \in \mathcal{M}(\{|z| > r\}) \cap \Pi$ with $M \subset L$ and a point $a \in \partial L$ such that (a) and (b) are satisfied. It is clear that we can find a rectifiable Jordan arc γ joining the origin to a with $\gamma \cap L = \{a\}$ and such that the compact set $\overline{B}(0,r) \cup \gamma \cup L$ has connected complement. By using a suitable parametrization of the arc γ, it is not difficult to construct a function f_1 that is continuous on $\overline{B}(0,r) \cup \gamma$, agrees with h on $\overline{B}(0,r)$ and satisfies $f_1(a) = 0$. Consider the mapping $S_2 : A(\gamma) \to A(M)$ given by

$$S_2 f(z) = g(\varphi_n^{-1}(z)) - \int_\gamma f(t)\varphi(z,t)\,dt \quad (z \in M). \tag{7}$$

It is well defined because $g \in A(K)$, $K \subset G$, $M = \varphi_n(K)$ and $\varphi_n^{-1} \in H(G)$. It follows from (b) that there exists a function $f_2 \in A_a(L)$ such that

$$|Qf_2(z) - S_2 f_1(z)| < \varepsilon \quad (z \in M). \tag{8}$$

On the other hand, the mapping $S_1 : A(L) \to A(M)$ is continuous (by (a)). Also the mappings S_2 and

$$S_3 : A(L) \to A(M), \quad S_3 f(z) = \int_a^z f(t)\varphi(z,t)\,dt \quad (z \in M)$$

are obviously continuous. Therefore, by (7) and (8), there exists $\delta > 0$ such that if $f \in \mathcal{E}$ satisfies

$$(9) \qquad |f(z) - f_1(z)| < \delta \ (z \in \gamma) \ \text{ and } \ |f(z) - f_2(z)| < \delta \ (z \in L)$$

then

$$(10) \qquad |S_1 f(z) + S_3 f(z) - S_2 f(z)| < \varepsilon \quad (z \in M).$$

Consider the function $f_3 : L_0 \to \mathbb{C}$ defined as

$$f_3(z) = \begin{cases} f_1(z) & \text{if } z \in \overline{B}(0,r) \cup \gamma \\ f_2(z) & \text{if } z \in L, \end{cases}$$

where $L_0 := \overline{B}(0,r) \cup \gamma \cup L$. From the fact $f_1(a) = 0 = f_2(a)$ one obtains that $f_3 \in A(L_0)$. But the compact set L_0 has connected complement. Consequently, it follows from Mergelyan's theorem that there exists a polynomial f satisfying $\|f - f_3\|_{L_0} < \min\{\varepsilon, \delta\}$. Hence, $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and (9) holds. Then f also satisfies (10), which can be rewritten as $|Tf(z) - g(\varphi_n^{-1}(z))| < \varepsilon \ (z \in M)$. But this is equivalent to $||(Tf) \circ \varphi_n - g||_K < \varepsilon$. Summarizing, $f \in U$, so $U \neq \emptyset$. $\square$

We remark that if the operator S is linear then, due to the density of $\mathcal{E}$ in $A(L)$, condition (a) is equivalent to the following: For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $f \in \mathcal{E}$ and $\|f\|_L < \delta$ then $\|Sf\|_M < \varepsilon$.

Lemma 4.2. *For every $L \in \Pi$ and every $a \in L$, there exists a finite positive constant $\beta = \beta(L,a)$ satisfying the following property: To each $z \in L$ we can associate a piecewise continuously differentiable arc $\gamma_z : [0,1] \to L$ joining a to z and a finite subset $F_z \subset [0,1]$ such that $|\gamma_z'(u)| \le \beta|z-a|$ for all $u \in [0,1] \setminus F_z$.*

Proof. Let us fix L, a as in the statement. From the shape of L it is evident that a number $R \in (0,+\infty)$ can be chosen in such a way that $B(a,R) \cap L$ is starlike with respect to a. If $z \in B(a,R) \cap L$ then we simply define γ_z as the segment joining a to z, i.e. $\gamma_z(u) = a + (z-a)u \quad (0 \le u \le 1)$, hence $|\gamma_z'(u)| = |z-a|$ for all $u \in (0,1)$. Assume now that $z \in L \setminus B(a,R)$. Let N be the number of segments of ∂L. Then it is clear that one can pick a polygonal arc $\gamma_z \subset L$ joining a to z consisting of $m = m(z)$ segments which are parallel to the axes, with $m \le N$. Now if we parametrize such segments in the obvious way on $[0,1/m],[1/m,2/m],\dots,[(m-1)/m,1]$ then $|\gamma_z'(u)|$ is not greater than $m\,\mathrm{diam}(L)$ in the interior of each one. Therefore $|\gamma_z'(u)| \le N\mathrm{diam}(L)$ for all $u \in [0,1]\setminus F_z$, where $F_z = \{0,1/m,2/m,\dots,1\}$. Hence $|\gamma_z'(u)| \le N\mathrm{diam}(L)|z-a|/R$ for such values of u whenever $z \in L \setminus B(a,R)$. Thus, the constant

$$\beta := \max\left\{1, \frac{N\mathrm{diam}(L)}{R}\right\}$$

does the job. $\square$

Lemma 4.3. *If $L \in \Pi$, $a \in L$, φ is an entire function of two variables and α is an entire function with $\alpha(z) \neq 0$ for all $z \in L$, then the operator $Q_{\alpha,\varphi} : A_a(L) \to A_a(L)$ given by*

$$Q_{\alpha,\varphi}f(z) = \alpha(z)f(z) + \int_a^z f(t)\varphi(z,t)\,dt \quad (z \in L)$$

is onto.

Proof. Observe first that $Q_{\alpha,\varphi}f$ is well defined because $Q_{\alpha,\varphi}f(a) = 0$ for all $f \in A_a(L)$. Since $\alpha(z) \neq 0$ for all $z \in L$, the statement is derived from the fact that the operator $I - K : A_a(L) \to A_a(L)$ is invertible (so onto), where K is the operator

$$Kf(z) = \int_a^z f(t)\varphi_1(z,t)\,dt \quad (z \in L)$$

and $\varphi_1(z,t) = -\varphi(z,t)/\alpha(z)$. If the spectrum $\sigma(K)$ reduces to $\{0\}$ one would have $\sigma(I-K) = \{1\}$, hence $0 \notin \sigma(I-K)$, so obtaining the invertibility of $I-K$. Therefore, according to Gelfand's formula for the spectral radius, it must be shown that $\lim_{n\to\infty} \|K^n\|^{1/n} = 0$, where $\|K\| = \sup\{\|Kf\|_L : \|f\|_L \le 1\}$, the norm in the space $L(A_a(L))$ of linear operators on $A_a(L)$. Take a constant $\beta \in (0,+\infty)$ and the family of arcs $\{\gamma_z : z \in L\}$ joining a to z as Lemma 4.2 asserts. Therefore, the length of each partial arc $\gamma_z|_{[0,u]}$ from a up to $\gamma(u)$ is not greater that $\beta u|z-a|$ and, in particular, $|\gamma_z(u)-a| \le \beta u|z-a|$ $(u \in [0,1])$.

Let us fix $f \in A_a(L)$ with $\|f\|_L \le 1$ and denote $C = \sup\{|\varphi_1(z,t)| : \ z,t \in L\}$. From the definition of the operator K we obtain, for all $z \in L$,

$$|Kf(z)| = \left| \int_0^1 f(\gamma_z(u))\varphi_1(z,\gamma_z(t))\gamma_z'(u)\,du \right| \le$$

$$\le \int_0^1 |f(\gamma_z(u))||\varphi_1(z,\gamma_z(t))||\gamma_z'(u)|\,du \le C\beta|z-a|.$$

Then

$$\left|K^2 f(z)\right| = \left| \int_{\gamma_z} (Kf)(t)\varphi_1(z,t)\,dt \right| = \left| \int_0^1 (Kf)(\gamma_z(u))\varphi_1(z,\gamma_z(u))\gamma_z'(u)\,du \right| \le$$

$$\le \int_0^1 |Kf(\gamma_z(u))|C|\gamma_z'(u)|\,du \le C^2\beta \int_0^1 |\gamma_z(u)-a|\beta|z-a|\,du \le$$

$$\le C^2\beta^3|z-a|^2 \int_0^1 u\,du = \frac{C^2\beta^3|z-a|^2}{2!}.$$

By induction we are led to the following inequality, which holds for every $n \in \mathbb{N}$:

$$|K^n f(z)| \le \frac{C^n\beta^{n+1}|z-a|^n}{n!} \le \frac{C^n\beta^{n+1}\mathrm{diam}(L)^n}{n!} \qquad (z \in L).$$

Whence

$$\|K^n\|^{1/n} \le C\beta \frac{\mathrm{diam}(L)\beta^{1/n}}{(n!)^{1/n}} \to 0 \quad (n \to \infty)$$

and we are done. □

Recall that $Z(f)$ denotes the subset of G consisting of the zeros of a function $f : G \to \mathbb{C}$. We are now ready to establish our theorem.

Theorem 4.4. *Assume that $N \in \mathbb{N}_0$ and that $a_n(z)$ $(n = 0,\ldots,N)$ are entire functions, in such a way that $a_N(z)$ has finitely many zeros. Assume also that P is a polynomial and that Φ is an entire function of subexponential type. Let $\Psi(z) = \sum_{j=0}^\infty c_j z^j$ be a formal power series with $\lim_{j\to\infty}(|c_j|^{1/j}/j) = 0$. We have:*

(A) *The operator T on $\mathcal{E}$ defined by*

$$Tf(z) = \sum_{j=0}^{N} a_j(z)f^{(j)}(z) + V_\varphi(z) \quad (f \in \mathcal{E}, \ z \in G)$$

is a U–operator.

(B) *If P is non-zero then $P(D) + V_\varphi$ is a U–operator. If P is nonconstant then $P(D) + \Psi(D^{-1})$ is a U–operator. If $\lambda \in \mathbb{C} \setminus \{0\}$ then the Volterra operator of the second kind $\lambda I + V_\varphi$ is a U–operator.*

(C) *If for some $N \in \mathbb{N}_0$ the entire function $w \mapsto (\partial^N \varphi / \partial z^N)(w, w)$ has finitely many zeros and each function $w \mapsto (\partial^n \varphi / \partial z^n)(w, w)$ ($n = 0, ..., N-1$) vanishes identically then V_φ is a U–operator.*

(D) *If Ψ is non-zero then $\Psi(D^{-1})$ is a U–operator. In particular, if P is non-zero then $P(D^{-1})$ is a U–operator.*

(E) *If Φ is nonconstant then $\Phi(D) + P(D^{-1})$ is a U–operator.*

Proof. It is evident that (B) is a consequence of (A). Furthermore, (D) is derived from (C). Indeed, for the case $c_0 = 0$, let $N = \min\{j \in \mathbb{N}_0 : c_{j+1} \neq 0\}$. Then $\Psi(D^{-1}) = V_\varphi$ with $\varphi(z, t) = \sum_{j=N}^{\infty} c_{j+1}(z - t)^j / j!$, hence $(\partial^N \varphi / \partial z^N)(w, w) = c_{N+1} \neq 0 = (\partial^n \varphi / \partial z^n)(w, w)$ ($n = 0, ..., N-1$) for all $w \in \mathbb{C}$ and (C) applies. The case $c_0 \neq 0$ follows in a similar way from (B).

Thus, our goal is to prove (A), (C) and (E). As for (A), let us check that the hypotheses (a)–(b) of Lemma 4.1 are fulfilled when S is defined as $Sf = \sum_{j=0}^{N} a_j(\cdot) D^j f$.

Clearly, (a) holds for every pair of sets $M, L \in \mathcal{M}(\mathbb{C})$ with $M \subset L$. On the other hand, choose $R = 1 + \max\{|z| : z \in Z(a_N)\}$ and fix $r > R$ and $M \in \mathcal{M}(\{|z| > r\})$. It is not difficult to realize that a connected compact set $L \subset \{|z| > r\}$ can be constructed in such a way that $M \subset L^0$, $\mathbb{C} \setminus L$ is connected, and ∂L consists of finitely many segments which are parallel to the axes, that is, $L \in \mathcal{M}(\{|z| > r\}) \cap \Pi$. Hence, condition (b) of Lemma 4.1 will be satisfied as soon as we show that the operator $Q : A_a^N(L) \to A(M)$ defined by

$$Qf(z) = \sum_{j=0}^{N} a_j(z) f^{(j)}(z) + \int_a^z f(t)\varphi(z, t)\, dt \quad (z \in M)$$

has dense range, where a is any fixed point in ∂L (so $a \in L \setminus M$) and $A_a^N(L)$ is the subspace of $A_a(L)$ consisting of all functions $f \in A(L)$ that are N-times continuously differentiable in L with $f^{(n)}(a) = 0$ for $n = 0, ..., N$.

For this, consider any entire function $\psi(z,t)$ of two complex variables such that for each $z \in \mathbb{C}$ the function $t \in \mathbb{C} \mapsto \psi(z,t) \in \mathbb{C}$ is an N-antiderivative of $\varphi(z,\cdot)$ (of course, $\psi = \varphi$ if $N = 0$) in such a way that $(\partial^j \psi/\partial t^j)(z,a) = 0$ for $j = 0,...,N-1$. After integration by parts (N times) we obtain, for $f \in A_a^N(L)$,

$$\int_a^z f(u)\varphi(z,u)\,du = \int_a^z f(u)\frac{\partial^N \psi}{\partial t^N}(z,u)\,du =$$

$$= \sum_{n=0}^{N-1}(-1)^n \left[f^{(n)}(z)\frac{\partial^{N-n-1}\psi}{\partial t^{N-n-1}}(z,z) - f^{(n)}(a)\frac{\partial^{N-n-1}\psi}{\partial t^{N-n-1}}(z,a)\right] +$$

$$+(-1)^N \int_a^z f^{(N)}(u)\psi(z,u)\,du =$$

$$= \sum_{n=0}^{N-1}(-1)^n f^{(n)}(z)\frac{\partial^{N-n-1}\psi}{\partial t^{N-n-1}}(z,z) + (-1)^N \int_a^z f^{(N)}(u)\psi(z,u)\,du.$$

Hence

$$Qf(z) = a_N(z)f^{(N)}(z) + \sum_{n=0}^{N-1} b_n(z)f^{(n)}(z) + (-1)^N \int_a^z f^{(N)}(t)\psi(z,t)\,dt$$

for certain entire functions b_n $(n = 0,...,N-1)$.

But $f^{(n)} = D_a^{-N+n} f^{(N)}$ $(n = 0,...,N-1)$ for $f \in A_a^N(L)$, where $D_a^{-j}h$ denotes the unique j-antiderivative H of order j of h such that $H^{(k)}(a) = 0$ $(k = 0,...,j-1)$. Therefore

$$\sum_{n=0}^{N-1} b_n(z)f^{(n)}(z) = \sum_{n=0}^{N-1} b_n(z)D_a^{-N+n}f^{(N)}.$$

Then our mapping Q can be written as

$$Qf(z) = a_N(z)D^N f(z) + \int_a^z (D^N f(t))\psi_1(z,t)\,dt,$$

where ψ_1 is an entire function of two variables; specifically,

$$\psi_1(z,t) = (-1)^N \psi(z,t) + \sum_{n=0}^{N-1} b_n(z)\frac{(z-t)^{N-n-1}}{(N-n-1)!}.$$

Next, let us consider the operator $Q_{a_N,\psi_1} : A_a(L) \to A_a(L)$, where Q_{a_N,ψ_1} is defined as in Lemma 4.3; it should be observed that $a_N(z) \neq 0$ for all $z \in L$

because $L \subset \{|z| > r\}$. Then, by Lemma 4.3, $Q_{a_N,\psi_1} : A_a(L) \to A_a(L)$ is onto. But $A_a(L)$ is dense in $A(M)$; indeed, if $g \in A(M)$ then the function $g(z)/(z-a)$ also belongs to $A(M)$ because $a \notin M$, so given $\varepsilon > 0$ Mergelyan's theorem furnishes a polynomial P with $|P(z) - (g(z)/(z-a))| < \varepsilon/\text{diam}(L)$ $(z \in M)$, hence the function $P_1(z) := (z-a)P(z)$ is in $A_a(L)$ and satisfies $\|P_1 - g\|_M < \varepsilon$. Consequently, $Q_{a_N,\psi_1} : A_a(L) \to A(M)$ has dense range. Hence Q has also dense range because $Q = Q_{a_N,\psi_1} \circ D^N$ and the mapping $D^N : A_a^N(L) \to A_a(L)$ is, trivially, onto. This completes the proof of (A).

Let us prove (C). We will again try to apply Lemma 4.1. Condition (a) is trivially satisfied for $S = 0$. Let $R = \max\{|z| : z \in Z(f)\}$ and fix $r > R$ and $M \in \mathcal{M}(\{|z| > r\})$. As before, choose any compact set $L \in \Pi$ with $L \subset \{|z| > r\}$ and $M \subset L^0$. Fix any $a \in \partial L$, so $a \in L \setminus M$. We should verify condition (b) of Lemma 4.1.

By hypothesis

$$\text{(11)} \qquad \frac{\partial^N \varphi}{\partial z^N}(w,w) \neq 0 = \frac{\partial^n \varphi}{\partial z^n}(w,w) \qquad (w \in L,\ n = 0, \dots, N-1).$$

Consider the mapping $Q : A_a(L) \to A(M)$ given by $Qf(z) = \int_a^z f(t)\varphi(z,t)\,dt$. Our goal is to show that it has dense range. By using an application of Mergelyan's theorem which is similar to that used in the proof of part (A) we obtain that the linear combinations of $(z-a)^m$ $(m \geq N+2)$ are dense in $A(M)$. Hence Q will have dense range as soon as we find for a fixed $m \geq N+2$ a function $f \in A_a(L)$ such that

$$\text{(12)} \qquad f(z) = (z-a)^m \quad (z \in L).$$

Due to (11) and to Leibniz's rule, the function Qf is $(N+1)$-times continuously differentiable on L with

$$\text{(13)} \qquad D^n(Qf)(w) = \int_a^w f(t)\frac{\partial^n \varphi}{\partial z^n}(w,t)\,dt \quad (n = 0, ..., N)$$

and

$$D^{N+1}(Qf)(w) = f(w)\frac{\partial^N \varphi}{\partial z^N}(w,w) + \int_a^w f(t)\frac{\partial^{N+1}\varphi}{\partial z^{N+1}}(w,t)\,dt$$

for all $w \in L$. Now, the not-equal part of (11) and Lemma 4.3 for $\alpha(w) := (\partial^N \varphi/\partial z^N)(w,w)$ and φ changed to $\partial^{N+1}\varphi/\partial z^{N+1}$ imply that $D^{N+1} \circ Q : A_a(L) \to A_a(L)$ is onto, whence there exists a function $f \in A_a(L)$ with $D^{N+1}(Qf)(w) = m!(w-a)^{m-N-1}/(m-N-1)!$ for all $w \in L$. Then $D^{N+1}[Qf - h] = 0$ on L, where $h(z) := (z-a)^m$. But $D^n[Qf-h](a) = 0$ $(n = 0, ..., N)$ by (13), hence $Qf - h = 0$ on L, which proves (12) and (C).

Finally, we prove (E). Let $\Phi(D) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of subexponential type, $M \in \mathcal{M}(\mathbb{C})$, $L \in \Pi$ with $L^0 \supset M$ and $a \in \partial L$. Since $(n!|a_n|)^{1/n} \to 0$ $(n \to \infty)$ we get $|a_n| \leq (\mathrm{dist}(M, \partial L)/2)^n/n!$ for n large enough. From this and from Cauchy's inequalities one obtains easily that given $\varepsilon > 0$ there is a $\delta > 0$ such that if $f \in \mathcal{E}$ and $\|f\|_L < \delta$ then $\|\Phi(D)f\|_M < \varepsilon$. In other words, the condition given just after Lemma 4.1 is satisfied for the linear operator $S = \Phi(D)$, hence condition (a) in that lemma is fulfilled. The extension of $\Phi(D)$ to a continuous mapping $A(L) \to A(M)$ will be also denoted by $\Phi(D)$, and similarly for related operators. Therefore, our final task is to verify condition (b) of Lemma 4.1, that is, we should check that the mapping $Q : A_a(L) \to A(M)$ given by $Qf = \Phi(D)f + P(D^{-1})f$ has dense range. By Mergelyan's theorem it suffices to show that given an $\varepsilon > 0$ and a polynomial g there exists $f \in A_a(L)$ such that $\|Qf - g\|_M < \varepsilon$.

For this, assume that $P(z) = p_0 z^N + p_1 z^{N-1} + ... + p_N$ and define the new entire function Φ_1 of subexponential type by $\Phi_1(z) = z^N \Phi(z) + \sum_{n=0}^{N} p_n z^n$. Then $Q = \Phi_1(D) \circ J \circ D_a^{-N}$, where $D_a^{-N} : A_a(L) \to A_a^N(L)$, $J : A_a^N(L) \to A(K)$ and $\Phi_1(D) : A(K) \to A(M)$. Here K is a member in Π that has been selected to satisfy $M \subset K^0 \subset K \subset L^0$ (so $a \notin K$), and J is the inclusion $J(f) = f$. Note that $\Phi_1(D) : A(K) \to A(M)$ is well defined by the same reasoning as that in the beginning of the proof of this part. Since $\Phi_1 \neq 0$ (because Φ is nonconstant) Malgrange–Ehrenpreis' theorem guarantees that $\Phi_1(D) : \mathcal{E} \to \mathcal{E}$ is onto, hence $\Phi_1(D) : A(K) \to A(M)$ has dense range because $\mathcal{E}$ is dense in $A(M)$ due to Mergelyan's theorem. Again by an adequate application of Mergelyan's theorem (the fact $a \notin K$ is crucial) we have that J has dense range. But D_a^{-N} is clearly onto, so it has dense range. Consequently, Q also has dense range and we are done. $\square$

We stress here that not every Volterra operator is a U–operator. For instance, set $\varphi(z,t) := \sin(\pi z)$, $G := \mathbb{C}$, $(\varphi_n(z) := z+n) \in \omega(G)$ and $K := \{0\}$, and fix $f \in \mathcal{E}$. Then

$$((V_\varphi f) \circ \varphi_n)(z) = 0 \to 0 \quad (n \to \infty) \text{ for all } \ z \in K,$$

so $((V_\varphi f) \circ \varphi_n)$ is not dense in $A(K) = \{\text{constants}\}$ and V_φ cannot be a U–operator.

5 Large linear manifolds of entire functions with universal translates

Before continuing our research of further classes of U–operators we take a break in this section in order to establish the promised improvement of Theorem 1.2, see Theorem 5.2 below. It will be shown that the family of entire functions which are universal in the sense of the former theorem is very large in both topological and algebraic senses.

The following statement can be found in [8] and it will be needed in Theorem 5.2. It furnishes a sufficient condition for the existence of large linear manifolds of vectors which are simultaneously hypercyclic with respect to each member of a countable family of sequences of linear mappings. It is in turn an extension of an assertion due to the first author, see [5, Theorem 2]. It should be noted that in [8] the final spaces Y_k were all the same, but a glance to the proof reveals that they can be different.

Lemma 5.1. *Let X and Y_k ($k \in \mathbb{N}$) be metrizable topological vector spaces such that X is Baire and separable. Assume that, for each $k \in \mathbb{N}$, $T_n^{(k)} : X \to Y_k$ ($n \in \mathbb{N}$) is a sequence of continuous linear mappings satisfying that $(T_{n_j}^{(k)})$ is densely hypercyclic for every sequence $\{n_1 < n_2 < ... < n_j < ...\} \subset \mathbb{N}$. Then there is a dense linear submanifold $M \subset X$ such that*

$$M \setminus \{0\} \subset \bigcap_{k \in \mathbb{N}} \mathcal{U}((T_n^{(k)})).$$

Theorem 5.2. *Suppose that (S_j) is a countable family of U–operators on $\mathcal{E}$ and that (G_k) is a countable family of ω-domains in $\mathbb{C}$. For each k, assume that $\{\varphi_{n,k} : n \in \mathbb{N}\} \in \omega(G_k)$. Then we have:*

(a) *There exists a residual subset of entire functions f such that each sequence $\{((S_j f) \circ \varphi_{k,n})|_K : n \in \mathbb{N}\}$ is dense in $A(K)$ for every $K \in \mathcal{M}(G_k)$, every k and every j.*

(b) *If every S_j is linear then there exists a dense linear manifold $M \subset \mathcal{E}$ such that each non–zero function $f \in M$ satisfies the same density property given in* (a).

Proof. With the notation of Section 2 we have that for each j, each k and each $K \in \mathcal{M}(G_k)$ the sequence of mappings

$$S_{j,k,n}^{(K)} : f \in \mathcal{E} \mapsto ((S_j f) \circ \varphi_{k,n})|_K \in A(K) \qquad (n \in \mathbb{N})$$

is densely hypercyclic. Since $\mathcal{E}$ is a Baire space and $A(K)$ is second-countable the set $\mathcal{U}((S_{j,k,n}^{(K)}))$ of hypercyclic vectors for that sequence is a dense G_δ-subset of $\mathcal{E}$, see [20, Theorem 1]. Now for given k let us select a sequence $(K_{k,m}) \subset \mathcal{M}(G_k)$ as that given in Lemma 2.1. Denote by A the subset of functions $f \in \mathcal{E}$ satisfying the property stated in (a). Then

$$A = \bigcap_{j,k,K} \mathcal{U}((S_{j,k,n}^{(K)})) = \bigcap_{j,k,m} \mathcal{U}((S_{j,k,n}^{(K_{k,m})})),$$

where the second equality is derived as in the proof of Theorem 2.2. Then A is a countable intersection of dense G_δ-subsets of $\mathcal{E}$, therefore A is itself a dense G_δ-subset, hence a residual subset of $\mathcal{E}$. This proves (a).

As for (b) choose $X := \mathcal{E}$, $Y_{k,m} := A(K_{k,m})$ and $T_n^{(j,k,m)} := S_{j,k,n}^{(K_{k,m})}$ in Lemma 5.1 (a trivial variant of it has been used by employing double and triple indexes) and take into account that each subsequence $(T_{n_p}^{(j,k,m)})$ of $(T_n^{(j,k,m)})$ is densely hypercyclic because each S_j is a U–operator and a subsequence of a member of $\omega(G_k)$ also belongs to $\omega(G_k)$. This concludes the proof. $\square$

Corollary 5.3. *Let be given a countable family (G_k) of ω-domains in $\mathbb{C}$ and, for each $k \in \mathbb{N}$, a sequence $\{\varphi_{k,n} : n \in \mathbb{N}\} \in \omega(G_k)$. Then there exists a residual set $A \subset \mathcal{E}$ and a dense linear manifold $M \subset \mathcal{E}$ satisfying the following:*

(a) *For any fixed $f \in A$, $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ the sequence of 'G_k-translates' $\{f^{(j)}(\varphi_{k,n}(z)) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(G_k)$.*

(b) *The inclusion $M \setminus \{0\} \subset A$ holds.*

Proof. Differentiation and antidifferentiation operators D^j ($j \in \mathbb{Z}$) are U–operators. $\square$

6 Taylor shifts and gap series

In this final section a kind of operators is considered on $\mathcal{E}$ when it is regarded as the space of complex sequences (a_n) with $|a_n|^{1/n} \to 0$ $(n \to \infty)$. In this setting and in connection with universality, the weighted backward shifts have been studied in [18, 34, 3, 21]. Recall if $w = \{w_n : n \in \mathbb{N}_0\}$ is a complex sequence then the weighted backward shift associated to w is the mapping defined on $\mathcal{E}$ as

$$B_w : f(z) = \sum_{n=0}^{\infty} a_n z^n \mapsto (B_w f)(z) = \sum_{n=0}^{\infty} w_n a_{n+1} z^n.$$

It is easy to see that if the sequence $\{|w_n|^{1/n} : n \in \mathbb{N}\}$ is bounded then B_w defines actually an operator on $\mathcal{E}$. Observe that the differentiation operator D is the special case $D = B_w$ with weight sequence $w_n = n+1$. In [3] the first author introduced a more general notion which is closed under composition, namely, the Taylor shifts (in [21] they are called 'pseudo-shifts', and they are considered in a more general setting). An operator $T : \mathcal{E} \to \mathcal{E}$ it said to be a *Taylor shift* if and only if there are a complex sequence $w = \{w_n : n \in \mathbb{N}_0\}$ and a one-to-one selfmapping $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ such that $Tf(z) = \sum_{n=0}^{\infty} w_n a_{\varphi(n)} z^n$ whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($f \in \mathcal{E}$, $z \in \mathbb{C}$). Equivalently, T is linear and, for every $n \in \mathbb{N}_0$,

$$T(z^n) = \begin{cases} w_m z^m & \text{if} \quad n = \varphi(m) \\ 0 & \text{if} \quad n \notin \varphi(\mathbb{N}_0). \end{cases}$$

Then we will denote $T = T_{w,\varphi}$. We remark that $B_w = T_{w,\varphi}$ with $\varphi(n) = n+1$. Clearly, $T_{w,\varphi}$ is not one-to-one if φ is not onto.

The following theorem provides with a sufficient criterium for a Taylor shift to be a U–operator. It covers the case of differentiation operators D^N ($N \in \mathbb{N}_0$), which of course are already known to be U–operators as particular instances of operators $\Phi(D)$.

Theorem 6.1. *Let be given a complex sequence $\{w_n : n \in \mathbb{N}_0\}$ and a one-to-one selfmapping $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the following properties:*

(a) $0 < \inf_{n\in\mathbb{N}} |w_n|^{1/n} \le \sup_{n\in\mathbb{N}} |w_n|^{1/n} < +\infty$ *and* $w_0 \neq 0$,

(b) $0 < \liminf_{n\to\infty} \varphi(n)/n \le \sup_{n\in\mathbb{N}} \varphi(n)/n < +\infty$.

Then the Taylor shift $T_{w,\varphi}$ *is a U–operator.*

Proof. As seen in [3, Theorem 3.2], the last inequality in (a) together with the first inequality in (b) guarantees that $T := T_{w,\varphi}$ is a well-defined operator on $\mathcal{E}$. Recall that T is linear. According to Corollary 2.5(a), it is enough to show that T is onto. For this, fix an entire function $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Let us define

$$a_n = \begin{cases} \frac{b_{\varphi^{-1}(n)}}{w_{\varphi^{-1}(n)}} & \text{if } n \in \varphi(\mathbb{N}_0) \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $w_j \neq 0$ for all j. Consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. It is clear that, formally, $Tf = g$. Hence it suffices to check that $f \in \mathcal{E}$, that is, $\lim_{n\to\infty} |a_n|^{1/n} = 0$. We have

$$\left|\frac{b_{\varphi^{-1}(n)}}{w_{\varphi^{-1}(n)}}\right|^{1/n} = \left(\left|b_{\varphi^{-1}(n)}\right|^{\frac{1}{\varphi^{-1}(n)}}\right)^{\varphi^{-1}(n)/n} \cdot \left(\frac{1}{\left|w_{\varphi^{-1}(n)}\right|^{\frac{1}{\varphi^{-1}(n)}}}\right)^{\varphi^{-1}(n)/n}.$$

Now observe that $\left|w_{\varphi^{-1}(n)}\right|^{1/\varphi^{-1}(n)}$ is bounded away from zero by the first inequality of (a), that $\varphi^{-1}(n)/n$ is asymptotically bounded away from zero by the last inequality of (b) and that $\left|b_{\varphi^{-1}(n)}\right|^{1/\varphi^{-1}(n)} \to 0$ as $n \to \infty$ because g is entire. Therefore,

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{b_{\varphi^{-1}(n)}}{w_{\varphi^{-1}(n)}}\right|^{1/n} = 0,$$

as required. □

It is natural to ask whether non-onto Taylor shifts U–operators can exist. They exist indeed, even with $\varphi =$ the identity on $\mathbb{N}_0$. Specifically, we next study the Euler differential operator, see its definition below. It is related to certain lacunary power series, which will be also dealt with in the final part of this section.

Assume that $\Phi(z) = \sum_{n=0}^\infty c_n z^n$ is an entire function of subexponencial type. Consider the operator $E : \mathcal{E} \to \mathcal{E}$ given by $Ef(z) = zf'(z)$. Then the *Euler differential operator* $\Phi(E)$ associated to Φ is defined as

$$\Phi(E) : f \in \mathcal{E} \to \Phi(E)f = \sum_{n=0}^{\infty} c_n E^n f \in \mathcal{E}.$$

It happens that $\Phi(E)$ is in fact a linear well-defined operator on $\mathcal{E}$, and that $\Phi(E)f(z) = \sum_{n=0}^\infty \Phi(n) a_n z^n$ whenever $f(z) = \sum_{n=0}^\infty a_n z^n$, see [23, pages 46–54]. Hence $\Phi(E) = T_{w,\varphi}$ with $w_n = \Phi(n)$, $\varphi(n) = n$ $(n \in \mathbb{N}_0)$.

In order to establish the desired property for $\Phi(E)$ we need two auxiliary lemmas. The first one is classic and can be found in [13, Theorem 9.1.4]. The second one is a recent lacunary result and may be seen in [28, Lemma] and [30, Lemma], see also [29, Lemma 2]. A little further terminology is in order. Recall that if $Q \subset \mathbb{N}_0$ and $\nu(A)$ denotes the number of elements of a finite set A then the *upper* (*lower*, respectively) *density* $\overline{\Delta}(Q)$ ($\underline{\Delta}(Q)$, respectively) of Q and the *maximal* (*minimal*, respectively) *density* $\Delta_{\max}(Q)$ ($\Delta_{\min}(Q)$, respectively) of Q in the sense of Pólya [35] are defined as

$$\overline{\Delta}(Q) = \limsup_{n\to\infty} \frac{\nu(Q \cap [0,n])}{n},$$

$$\underline{\Delta}(Q) = \liminf_{n\to\infty} \frac{\nu(Q \cap [0,n])}{n},$$

$$\Delta_{\max}(Q) = \lim_{\alpha\to 1^-} \left(\limsup_{r\to\infty} \frac{\nu(Q\cap[0,r]) - \nu(Q\cap[0,\alpha r])}{(1-\alpha)r} \right),$$

$$\Delta_{\min}(Q) = \lim_{\alpha\to 1^-} \left(\liminf_{r\to\infty} \frac{\nu(Q\cap[0,r]) - \nu(Q\cap[0,\alpha r])}{(1-\alpha)r} \right).$$

The *density* $\Delta(Q)$ of Q is defined as

$$\Delta(Q) = \lim_{n\to\infty} \frac{\nu(Q \cap [0,n])}{n},$$

if such a limit exists, that is, if $\overline{\Delta}(Q) = \underline{\Delta}(Q)$. In addition, we denote by $\mathcal{E}_Q$ the subspace of $\mathcal{E}$ consisting of all entire functions with null Taylor nth-coefficient at the origin for every $n \notin Q$. Therefore $\mathcal{E}_Q$ is a space of gap series. Note that $\Phi(E)f \in \mathcal{E}_Q$ if $Q = \mathbb{N}_0 \setminus \Phi^{-1}(0)$. Moreover, for $A \subset \mathbb{C}$ and for $\alpha \in [0,\pi)$ we set

$$A_\alpha := \{ze^{i\theta} : \; z \in A, \, |\theta| \le \alpha\}.$$

Lemma 6.2. *If Φ is a non-zero entire function of subexponential type then* $\Delta\left(\mathbb{N}_0 \setminus \Phi^{-1}(0)\right) = 1$.

Lemma 6.3. *Let $K \in \mathcal{M}(\mathbb{C})$ with $0 \in K^0$ and assume that Q is a subset of $\mathbb{N}_0$ satisfying at least one of the following two conditions:*

(a) *The component of K containing the origin is starlike with respect to 0 and $\overline{\Delta}(Q) = 1$.*

(b) *The minimal density satisfies $\Delta_{\min}(Q) = \delta \in (0,1]$ and there exists a Jordan arc γ connecting ∞ with the boundary of the maximal disk with center 0 which is contained in K^0 and having the property $\gamma_{\pi(1-\delta)} \cap K = \emptyset$.*

Suppose that $\varepsilon > 0$ and that f is holomorphic on some open set containing K such that f has a power series representation around the origin of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with } a_n = 0 \ \text{ for } n \notin Q.$$

Then there exists a polynomial $P \in \mathcal{E}_Q$ such that $|f(z) - P(z)| < \varepsilon$ for all $z \in K$.

The proof of the following result is inspired by that of [15, Theorem 4.6].

Theorem 6.4. *If Φ is a non-zero entire function of subexponential type then the Euler differential operator $\Phi(E)$ is a U–operator.*

Proof. According to Corollary 2.7, we would be done as soon as we prove that $\Phi(E)$ has ω-dense range and is ω-stable.

Fix any $R > 0$, any $M \in \mathcal{M}(\{|z| > R\})$ and any $g \in A(M)$. By Mergelyan's theorem, there exists a polynomial P_1 such that

$$|g(z) - P_1(z)| < \frac{\varepsilon}{2} \quad (z \in M). \tag{14}$$

Consider $\Omega := B(0,R) \cup \{|z| > R\}$, $K := \overline{B}(0,R/2) \cup M$ and $Q := \mathbb{N}_0 \setminus \Phi^{-1}(0)$. Then, by Lemma 6.2, $\Delta(Q) = 1$. But $K \in \mathcal{M}(\mathbb{C})$, $0 \in K^0$ and Ω is an open set containing K, therefore from Lemma 6.3 (under condition (a)) there is some polynomial $P \in \mathcal{E}_Q$ such that

$$|f(z) - P(z)| < \frac{\varepsilon}{2} \quad (z \in K), \tag{15}$$

where $f : \Omega \to \mathbb{C}$ is defined as

$$f(z) := \begin{cases} P_1(z) & \text{if } |z| > R \\ 0 & \text{if } |z| < R. \end{cases} \tag{16}$$

By (14), (15) and (16) we get

$$|g(z) - P(z)| < \varepsilon \quad (z \in M). \tag{17}$$

Now, we define the polynomial h as follows. Assume that $P(z) = \sum_{n \in Q \cap \{0,1,\ldots,N\}} a_n z^n$. Then $h(z) := \sum_{n \in Q \cap \{0,1,\ldots,N\}} (a_n/\Phi(n)) z^n$. Trivially, $h \in \mathcal{E}$ and $\Phi(E)h = P$. Thus, by (17),

$$|(\Phi(E)h)(z) - g(z)| < \varepsilon \quad \text{for all } z \in M.$$

This shows that the restriction mapping $\Phi(E)_M : \mathcal{E} \to A(M)$ has dense range, so $\Phi(E)$ has ω-dense range. As for ω-stability, fix $r > 0$ and select $R := r$. Given $\varepsilon > 0$ and $M \in \mathcal{M}(\{|z| > r\})$ we have to find $\delta > 0$ and $S \in \mathcal{M}(\{|z| > r\})$ in such way that $||\Phi(E)f||_M < \varepsilon$ whenever f is an entire function with $\|f\|_S < \delta$. We can choose a compact set $S \in \Pi$ (see the notation just before Lemma 4.1) such that $M \subset S^0 \subset S \subset \{|z| > r\}$, so $S \in \mathcal{M}(\{|z| > r\})$. Set $\alpha := \inf\{|t - z| : t \in \Gamma,\ z \in M\} > 0$, where $\Gamma = \partial S$. Let us denote $\beta := \max\{|t| : t \in \Gamma\}$, hence $\beta \in (0, +\infty)$. Since $\Phi(z) := \sum_{n=0}^{\infty} c_n z^n$ has subexponential type, there exists a constant $C \in (0, +\infty)$ such that

$$|c_n| \le \frac{C}{n!} \left(\frac{\alpha}{2\beta} \right)^n \qquad (n \in \mathbb{N}_0).$$

Define $\delta := \varepsilon\pi\alpha/(C \,\text{length}\,(\Gamma))$ and fix $f \in \mathcal{E}$ with $\|f\|_S < \delta$. According to [23, pages 46–54], we have

$$E^n f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{P_n(z,t) f(t)}{(t - z)^{n+1}}\, dt \quad (n \in \mathbb{N}_0,\ z \in M),$$

where $P_n(z,t)$ is a polynomial of two variables z, t satisfying $|P_n(z,t)| < n!\beta^n$ for all $z \in M$ and $t \in \Gamma$. In fact, P_n does not depend on f. Finally, for every $z \in M$ we obtain

$$|\Phi(E)f(z)| = \left|\sum_{n=0}^{\infty} c_n (E^n f)(z)\right| \le \frac{1}{2\pi}\sum_{n=0}^{\infty} |c_n| \left|\oint_\Gamma \frac{P_n(z,t)f(t)}{(t-z)^{n+1}}\,dt\right| \le$$

$$\le \frac{1}{2\pi}\sum_{n=0}^{\infty} \frac{C}{n!}\left(\frac{\alpha}{2\beta}\right)^n \cdot \frac{n!\beta^n\|f\|_S}{\alpha^{n+1}} \cdot \text{ length }(\Gamma) < \frac{C\delta \text{ length }(\Gamma)}{2\pi\alpha}\sum_{n=0}^{\infty}\frac{1}{2^n} = \varepsilon,$$

as required. □

There are other non-onto Taylor shift U–operators $T_{w,\varphi}$ with $\varphi(n) = n$ ($n \in \mathbb{N}$) which are essentially different from Euler differential operators, but also related to gap Taylor series. Our result is contained in Theorem 6.5 (see below) and strengthens Theorem 1.3. On the other hand, the condition $\overline{\Delta}(Q) = 1$ is 'essentially' necessary in order that the property of density in $A(K)$ ($K \in \mathcal{M}(\mathbb{C}^*)$) can be satisfied for some $f \in \mathcal{E}_Q$; indeed, it is shown in [29, Theorem 2] that $\Delta_{\max}(Q) = 1$ in such case.

We now consider the 'gap operator' $I_Q : \mathcal{E} \to \mathcal{E}$ given by

$$(I_Q f)(z) = \sum_{n\in Q} a_n z^n, \text{ where } f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $Q \subset \mathbb{N}_0$ is fixed. Observe that $I_Q = T_{w,\varphi}$ with $\varphi(n) = n$ for all n and

$$w_n = \begin{cases} 1 & \text{if } n \in Q \\ 0 & \text{if } n \notin Q. \end{cases}$$

Note that the next theorem is *not* contained in Theorem 6.4 because, given $Q \subset \mathbb{N}_0$ with $\overline{\Delta}(Q) = 1$ and $Q \ne \mathbb{N}_0$, there exists no entire function Φ of subexponential type satisfying $\Phi(n) = 1$ for $n \in Q$ and $\Phi(n) = 0$ for $n \notin Q$. Indeed, if such a function exists then $\Phi_1(z) := \Phi(z) - 1$ would also be of subexponential type; but $\Phi_1^{-1}(0) = Q$, so $\underline{\Delta}(\mathbb{N}_0 \setminus \Phi_1^{-1}(0)) = \underline{\Delta}(\mathbb{N}_0 \setminus Q) = 0 \ne 1$, hence $\Phi_1 \equiv 0$ by Lemma 6.2. Therefore $\Phi \equiv 1$, which is absurd.

Theorem 6.5. *Suppose that Q is a subset of $\mathbb{N}_0$ with $\overline{\Delta}(Q) = 1$. We have:*

(a) *The gap operator I_Q is a U–operator.*

(b) *Let be given a countable family* (G_k) *of* ω*-domains in* $\mathbb{C}$*. For each* k*, assume that* $\{\varphi_{k,n} : n \in \mathbb{N}\} \in \omega(G_k)$*. Then there exists an infinite-dimensional linear manifold* $M \subset \mathcal{E}_Q$ *such that for every* $F \in M \setminus \{0\}$ *the sequence* $\{(F \circ \varphi_{k,n})|_K : n \in \mathbb{N}\}$ *is dense in* $A(K)$ *for every* $K \in \mathcal{M}(G_k)$ *and every* k*.*

Proof. (a) Assume that a set $U := U(T = I_Q, , G, K, \sigma = (\varphi_n), \varepsilon, r, g, h)$ as in part (d) of Theorem 2.2 is fixed. As remarked after Theorem 2.2, it can be supposed without loss of generality that g is a polynomial. It has to be shown that U is nonempty. Since $\sigma \in \omega(G)$, there exists $n \in \mathbb{N}$ with $\overline{B}(0, r) \cap \varphi_n(K) = \emptyset$. Consider the set $L := \overline{B}(0, r) \cup \varphi_n(K)$. Then $L \in \mathcal{M}(\mathbb{C})$ because $K \in \mathcal{M}(\mathbb{C})$ and φ_n is a homeomorphism from G into itself. In addition, $0 \in L^0$ and the component of L containing 0 ($= \overline{B}(0, r)$) is starlike with respect to 0. Let us consider the function

$$F(z) = \begin{cases} (I_Q h)(z) & \text{if } z \in \overline{B}(0, r) \\ g(\varphi_n^{-1}(z)) & \text{if } z \in \varphi_n(K). \end{cases}$$

Observe that F is holomorphic on some open set containing L; indeed, $I_Q h$ is entire and $g \circ \varphi_n^{-1} \in H(G)$. On the other hand, F has, obviously, a power series representation around the origin with gaps at the indexes belonging to $\mathbb{N}_0 \setminus Q$. By Lemma 6.3(a), there is a polynomial $P \in \mathcal{E}_Q$ such that

$$|F(z) - P(z)| < \varepsilon \quad (z \in L).$$

In particular,

$$\|I_Q h - P\|_{\overline{B}(0,r)} < \varepsilon$$

and

$$\|P \circ \varphi_n - g\|_K < \varepsilon.$$

Now define

$$f := P + I_{\mathbb{N}_0 \setminus Q} h.$$

It is clear that $f \in \mathcal{E}$ and $I_Q f = P$. Hence

$$\|f - h\|_{\overline{B}(0,r)} = \|P + I_{\mathbb{N}_0 \setminus Q} h - I_Q h - I_{\mathbb{N}_0 \setminus Q} h\|_{\overline{B}(0,r)} < \varepsilon$$

and

$$\|(I_Q f) \circ \varphi_n - g\|_K < \varepsilon.$$

Consequently, $f \in U$ and we are done.

(b) Let us suppose that (G_k) and $\{\varphi_{k,n} : n \in \mathbb{N}\}$ $(k \in \mathbb{N})$ are as in the hypothesis. If we apply part (b) of Theorem 5.2 on the constant sequence $S_j = I_Q$ then we obtain a dense linear manifold $\widetilde{M} \subset \mathcal{E}$ such that, for all $f \in \widetilde{M} \setminus \{0\}$, each sequence $\{((I_Q f) \circ \varphi_{k,n})|_K : n \in \mathbb{N}\}$ is dense in $A(K)$ for every $K \in \mathcal{M}(G_k)$ and every $k \in \mathbb{N}$. Define

$$M := I_Q(\widetilde{M}).$$

Then M is a linear manifold in $\mathcal{E}_Q$. Moreover, if $F \in M \setminus \{0\}$, then $F = I_Q f$ for some $f \in \widetilde{M} \setminus \{0\}$, so the approximation property of the statement holds. Finally, M is dense in $I_Q(\mathcal{E}) = \mathcal{E}_Q$, hence M must be infinite-dimensional. □

To finish, we would like to say something in the case of the weaker condition $\Delta_{\min}(Q) > 0$ for the subset $Q \subset \mathbb{N}_0$. In such a case, Luh, Martirosian and Müller were able to prove (see [28, Theorem 1]) that for a given sequence $(a_n) \subset \mathbb{C}$ tending to ∞ (again, the statement is equivalent to '(a_n) is unbounded') there exists a function $f \in \mathcal{E}_Q$ such that the sequence of translates $\{f(z + a_n) : n \in \mathbb{N}\}$ is dense in $A(K)$ for all $K \in \mathcal{M}(\mathbb{C})$. In our next (and final) theorem we obtain a strong improvement with a different proof. We remark that by Mergelyan's theorem density in $\mathcal{E}$ implies density in every $A(K)$ with $K \in \mathcal{M}(\mathbb{C})$.

Theorem 6.6. *Let be given a subset $Q \subset \mathbb{N}_0$ with $\Delta_{\min}(Q) > 0$ and a sequence $(\varphi_n) \in \omega(\mathbb{C})$. Then there exists an infinite-dimensional linear manifold $M \subset \mathcal{E}_Q$ such that for every $F \in M \setminus \{0\}$ the sequence $\{F \circ \varphi_n : n \in \mathbb{N}\}$ is dense in the space $\mathcal{E}$.*

Proof. We have that $\varphi_n(z) = a_n + b_n z$ $(n \in \mathbb{N})$ for some complex sequences (a_n), (b_n) with $b_n \neq 0$ for all n and $a_n \to \infty$, $a_n/b_n \to \infty$ as $n \to \infty$. For given $\varepsilon > 0$, $r > 0$, $R > 0$ and polynomials g, h we can select as in the proof of Theorem 6.5 a positive integer n with $\overline{B}(0,r) \cap \varphi_n(\overline{B}(0,R)) = \emptyset$. Consider also the corresponding function F defined on $L := \overline{B}(0,r) \cup \varphi_n(\overline{B}(0,R))$ given by

$$F(z) = \begin{cases} (I_Q h)(z) & \text{if } z \in \overline{B}(0,r) \\ g(\varphi_n^{-1}(z)) & \text{if } z \in \varphi_n(\overline{B}(0,R)). \end{cases}$$

Now, n can be choosen in such a way that there exists a Jordan arc γ connecting ∞ with the boundary of $\overline{B}(0,r)$ such that

$$\gamma_{\pi(1-\delta)} \cap L = \emptyset, \tag{18}$$

where $\delta := \Delta_{\min}(Q)$ (this will be shown at the end of the proof). Therefore Lemma 6.3(b) applies, yielding a polynomial $P \in \mathcal{E}_Q$ such that $||P - F||_L < \varepsilon$. Then as in the proof of Theorem 6.5 we obtain a function $f \in \mathcal{E}$ with $\|f - h\|_{\overline{B}(0,r)} < \varepsilon$ and $\|(I_Q f) \circ \varphi_n - g\|_{\overline{B}(0,R)} < \varepsilon$. Let us define

$$G(g,R,\varepsilon) := \{f \in \mathcal{E} : \|(C_{\varphi_n} I_Q) f - g\|_{\overline{B}(0,R)} < \varepsilon \text{ for some } n \in \mathbb{N}\}.$$

Then we have just proved that each $G(g,R,\varepsilon)$ is a dense subset of $\mathcal{E}$. On the other hand, it is not difficult to realize that every $G(g,R,\varepsilon)$ is open and that

$$\mathcal{U}((C_{\varphi_n} I_Q)) = \bigcap_{j,k,l \in \mathbb{N}} G(g_j, k, 1/l),$$

where (g_j) is an enumeration of polynomials whose coefficients have rational real and imaginary parts. By Baire's theorem, $\mathcal{U}((C_{\varphi_n} I_Q))$ is dense. In other words, the sequence $C_{\varphi_n} I_Q : \mathcal{E} \to \mathcal{E}$ $(n \in \mathbb{N})$ is densely hypercyclic. But the same holds for every subsequence $(C_{\varphi_{n_j}} I_Q)$ $(n_1 < n_2 < n_3 < \ldots)$ because, trivially, (φ_{n_j}) also belongs to $\omega(\mathbb{C})$. From Lemma 5.1 as applied on $X := \mathcal{E} =: Y_k$ for all k (or from [5, Theorem 2]), there is a dense linear manifold $\widetilde{M} \subset \mathcal{E}$ with $\widetilde{M} \setminus \{0\} \subset \mathcal{U}((C_{\varphi_n} I_Q))$. If now we define $M := I_Q(\widetilde{M})$ then we can conclude as in the proof of part (b) of Theorem 6.5.

Thus, we would be done if (18) is obtained for some suitable Jordan arc γ. Recall that $L = \overline{B}(0,r) \cup \varphi_n(\overline{B}(0,R))$ where n is such that the union is disjoint. Observe that $\varphi_n(\overline{B}(0,R)) = \overline{B}(a_n, R|b_n|)$. If $a_n = |a_n| e^{i\theta_n}$, consider the angle $S(n) = \{z \in \mathbb{C} \setminus \{0\} : \theta_n - \pi\delta < \arg z < \theta_n + \pi\delta\}$. Since $\lim_{n\to\infty} b_n/a_n = 0$ we can choose our integer n in such a way that $|b_n/a_n| < \sin(\pi\delta)$, so $\overline{B}(a_n, R|b_n|) \subset$

$S(n)$. Let us define the Jordan arc $\gamma := \{-te^{i\theta_n} : t > r\}$. Then γ connects ∞ with the point $-r$ of the boundary of $\overline{B}(0,r)$. In addition, $\gamma_{\pi(1-\delta)} \cap \overline{B}(0,r) = \emptyset$ and $\gamma_{\pi(1-\delta)} \subset \mathbb{C} \setminus S(n)$, whence (18) holds. $\square$

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