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Families of strongly annular functions: linear structure

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Abstract A function f holomorphic in the unit disk $\mathbb{D}$ is called strongly annular if there exists a sequence of concentric circles in $\mathbb{D}$ expanding out to the unit circle such that f goes to infinity as $|z|$ goes to 1 through these circles. The residuality of the family of strongly annular functions in the space of holomorphic functions on $\mathbb{D}$ is well known, and it is extended here to certain classes of functions. This important topological property is enriched in this paper by studying algebraic-topological properties of the mentioned family, in the modern setting of lineability. Namely, we prove that although this family is clearly nonlinear, it contains, except for the zero function, large vector subspaces as well as infinitely generated algebras. Similar results are obtained for strongly annular functions on the whole complex plane and for weighted Bergman spaces.

Keywords Strongly annular functions · entire functions · dense-lineability · algebrability · Bergman spaces.
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1 Introduction

Let us denote by $\mathbb{D}$, as usual, the open unit disk of the complex plane $\mathbb{C}$, and by $H(\mathbb{D})$ the space of holomorphic functions in $\mathbb{D}$, endowed with the

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compact-open topology τ_c. Under τ_c, this space becomes an F-space, i.e., a complete metrizable topological vector space. An interesting family in $H(\mathbb{D})$ is $\mathcal{SA}$, formed by the so-called *strongly annular functions*. By definition, a function $f \in H(\mathbb{D})$ belongs to $\mathcal{SA}$ provided that

$$\limsup_{r \to 1} \min\{|f(z)| : |z| = r\} = +\infty.$$

For the sake of convenience, we establish the next notation. Denote by Σ the set of all strictly increasing sequences $\sigma = \{r_n\}_{n \ge 1} \subset (0,1)$ with $r_n \to 1$. If σ is as before, we set $C(\sigma) := \bigcup_{n=1}^{\infty} r_n \mathbb{T}$, where $\mathbb{T} := \{z : |z| = 1\}$. Then $f \in \mathcal{SA}$ if and only if there is $\sigma = \{r_n\}_{n \ge 1} \in \Sigma$ such that $\lim_{n \to \infty} \min\{|f(z)| : |z| = r_n\} = +\infty$, or equivalently, $\lim_{\substack{|z| \to 1 \\ z \in C(\sigma)}} |f(z)| = +\infty$.

There is an extensive literature on this kind of functions, see for instance [6], [12], [13], [14], [15], [16], [18], [19], [20], [25], [26], [27], [28], [29], [30] and the references contained in them. The study of $\mathcal{SA}$ is motivated by the search of functions in $H(\mathbb{D})$ having fast radial growth. Observe that there is *not* any function $f \in H(\mathbb{D})$ such that $\lim_{|z| \to 1} |f(z)| = +\infty$. Indeed, by way of contradiction, assume that f is one of such functions. Then the set of zeros of f form a compact subset of $\mathbb{D}$. By the analytic continuation principle, this set of zeros is finite. Let P be a polynomial whose zeros are exactly those of f, counting multiplicities. It follows that $P/f \in H(\mathbb{D})$ and, since P is bounded on $\mathbb{D}$, $\lim_{|z| \to 1} |P(z)/f(z)| = 0$. By the maximum modulus principle, $P/f \equiv 0$, which is clearly impossible.

Once the existence of strongly annular functions is established, the next natural step is to study the topological nature and the size of $\mathcal{SA}$. This was carried out by Bonar and Carroll [13], who proved in 1975 that $\mathcal{SA}$ is a dense G_δ (hence residual) subset of $H(\mathbb{D})$. Therefore it can be said that $\mathcal{SA}$ is topologically large. But, what can be asserted about its algebraic structure and size? It is plain that $\mathcal{SA}$ is not even a vector space. In recent years, a plethora of papers have been published stating the existence of large algebraic structures within nonlinear sets. To this respect, the following notions have been recently introduced. Assume that X is a topological vector space and that μ is a cardinal number. Then a subset A of X is called

- *lineable* [3] if $A \cup \{0\}$ contains an infinite dimensional vector subspace,
- μ*-lineable* [3] if $A \cup \{0\}$ contains a μ-dimensional vector subspace,
- *dense-lineable* or *algebraically generic* [7] whenever $A \cup \{0\}$ contains a dense vector subspace of X,
- *maximal dense-lineable* [10] if $A \cup \{0\}$ contains a dense vector subspace M of X with $\dim(M) = \dim(X)$, and
- *algebrable* ([4] and [5]) if X is a function space and $A \cup \{0\}$ contains some infinitely generated algebra.

See also [1] and [22]. Recall that a vector space M of functions is said to be an algebra provided that $fg \in M$ if $f, g \in M$. Clearly, maximal dense-lineability implies $\dim(X)$-lineability plus dense lineability, but the converse is not true.

It follows from the definition that if a strongly annular function has a radial limit, the limit must be infinity. Consequently, by Fatou's theorem, no function in the classical Hardy spaces $H^p(\mathbb{D}) := \{f \in H(\mathbb{D}) : \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty\}$ (see e.g. Rudin [31]) can belong to $\mathcal{SA}$. Nevertheless, in 2007 Redett [30] was able to construct a strongly annular function in each weighted Bergman space $A^p_\alpha(\mathbb{D})$ $(0 < p < +\infty,\ \alpha > -1)$.

Our aim in this paper is to establish that $\mathcal{SA}$ is not only topologically large, but also algebraically large, in the sense of the above definitions. This will be accomplished in Section 4. Section 2 will be devoted to give the necessary background. In Section 3, residuality is reinforced and examined within certain subspaces of $H(\mathbb{D})$. Finally, in Sections 5 and 6 we extend our results to weighted Bergman spaces and to the space of entire functions.

2 Preliminary results

A number of preliminary assertions will be used in due course. We begin with a simple observation. If X is a separable infinite-dimensional F-space then Baire's theorem implies that $\dim(X) = c$, the cardinality of the continuum. Hence c is the maximal dimension allowed for any subspace of X. For instance, $\dim(H(\mathbb{D})) = c$.

The following statement on lineability was established in Bernal [10, Lemma 2.1], which in turn is a strengthening of Theorem 2.1 in [9]; see also Aron *et al.* [2, Theorem 2.2 and p. 152] for related results.

Lemma 1 *Assume that X is a metrizable separable topological vector space. Suppose that Γ is a family of linear subspaces of X such that $\bigcap_{S\in\Gamma} S$ is dense in X and $\bigcap_{S\in\Gamma}(X \setminus S)$ is μ-lineable, where μ is an infinite cardinal number. Then $\bigcap_{S\in\Gamma}(X \setminus S) \cup \{0\}$ contains a dense μ-dimensional vector subspace.*

In the next elementary lemma one meets the nice notion of "stronger than", coined by Aron, García, Pérez and Seoane in [2].

Lemma 2 *Suppose that the following holds:*

(a) *(X, τ_0) is a topological vector space.*
(b) *A is a dense G_δ subset of X.*
(c) *Y is a vector subspace of X and τ_1 is a topology on Y such that (Y, τ_1) is a topological vector space and τ_1 is finer than $\tau_0|_Y$.*
(d) *There is a τ_1-dense subset D of Y such that A is stronger than D, that is, $A + D \subset A$.*
(e) *$A \cap Y \neq \emptyset$.*

Then $A \cap Y$ is a dense G_δ subset of (Y, τ_1).

Proof According to (b), there are τ_0-open sets G_n $(n \ge 1)$ with $A = \bigcap_{n=1}^\infty G_n$. Then $A \cap Y = \bigcap_{n=1}^\infty (G_n \cap Y)$, which is a τ_1-G_δ subset of Y because of (c). From (e), there is $x_0 \in A \cap Y$ and, by (d), $x_0 + D \subset A \cap Y$. But $x_0 + D$ is τ_1-dense in Y. Consequently, the same holds for $A \cap Y$. $\square$

If $\varphi : \mathbb{D} \to (0, +\infty)$ is continuous and $\sigma \in \Sigma$, we define

$$\mathcal{SA}(\varphi) := \left\{ f \in H(\mathbb{D}) : \limsup_{r \to 1} \min\{ \frac{|f(z)|}{\varphi(z)} : |z| = r\} = +\infty \right\}$$

$$\text{and} \quad \mathcal{SA}(\varphi, \sigma) := \left\{ f \in H(\mathbb{D}) : \lim_{\substack{|z| \to 1 \\ z \in C(\sigma)}} \frac{|f(z)|}{\varphi(z)} = +\infty \right\}.$$

Then it is plain that $\mathcal{SA}(\varphi) = \bigcup_{\sigma \in \Sigma} \mathcal{SA}(\varphi, \sigma)$ and that $\mathcal{SA}(1) = \mathcal{SA}$.

The following assertion will be employed to study dense-lineability. Recall that if $A \subset \mathbb{C}$ then $f \in H(A)$ means that there is an open set $G = G(f) \supset A$ such that $f \in H(G) := \{\text{holomorphic functions on } G\}$.

Lemma 3 *Assume that $\varphi : \mathbb{D} \to (0, +\infty)$ is a continuous function satisfying*

$$\lim_{|z| \to 1} \frac{\log \varphi(z)}{\log \frac{1}{1-|z|}} = +\infty. \tag{1}$$

If $f \in \mathcal{SA}(\varphi)$ and $g \in H(\overline{\mathbb{D}}) \setminus \{0\}$ then $fg \in \mathcal{SA}$.

Proof Fix f, g as in the statement. Then we can choose a connected open set with $G \supset \overline{\mathbb{D}}$, $g \in H(G)$ and $g \not\equiv 0$ in G. From the analytic continuation principle one derives that there are only finitely many zeros of g on $\overline{\mathbb{D}}$. Hence we can assume that g possesses zeros $z_1, \dots, z_p$ in $\mathbb{D}$ and zeros $w_1, \dots, w_q$ on $\mathbb{T}$, with respective multiplicities $m_1, \dots, m_p$, $n_1, \dots, n_q$ (other cases are easier to handle). Then $g = PQh$, where $h \in H(G)$, h lacks zeros in $\overline{\mathbb{D}}$ and $P(z) := \prod_{k=1}^p (z - z_k)^{m_k}$, $Q(z) := \prod_{k=1}^q (z - w_k)^{n_k}$. By hypothesis, $f \in \mathcal{SA}(\varphi, \sigma)$ for some sequence $\sigma = (r_n) \in \Sigma$. Let n_0 be such that $r_n > \max\{|z_1|, \dots, |z_p|\}$ for all $n \geq n_0$, and choose α, $\beta > 0$ with $|h(z)| > \alpha$ $(z \in \mathbb{D})$ and $|z - z_k| > \beta$ $(|z| = r_n,\ n \geq n_0;\ k = 1, \dots, p)$. If $|z| = r_n$ with $n \geq n_0$ we have

$$\begin{aligned} |f(z)g(z)| &= |h(z)||P(z)||Q(z)|\varphi(z) \cdot \frac{|f(z)|}{\varphi(z)} \\ &> \alpha \prod_{k=1}^{p} |z - z_k|^{m_k} \prod_{k=1}^{q} (1 - |z|)^{n_k} \varphi(z) \cdot \frac{|f(z)|}{\varphi(z)} \\ &> \alpha \beta^{\text{degree}\,(P)} (1 - |z|)^{\text{degree}\,(Q)} \varphi(z) \cdot \frac{|f(z)|}{\varphi(z)}. \end{aligned}$$

By (1), $\lim_{|z| \to 1} (1 - |z|)^N \varphi(z) = +\infty$ for all $N \in \mathbb{N} := \{1, 2, 3, \dots\}$. But recall that $\lim_{\substack{|z| \to 1 \\ z \in C(\sigma)}} |f(z)|/\varphi(z) = +\infty$. Therefore $\lim_{\substack{|z| \to 1 \\ z \in C(\sigma)}} |f(z)g(z)| = +\infty$, that is, $fg \in \mathcal{SA}$. □

Finally, Lemma 4 will be needed to examine dense-lineability in the context of entire functions.

Lemma 4 *If f is an entire function that is not a polynomial then the family $\{f_\alpha : \alpha > 0\}$ is linearly independent, where we have set $f_\alpha(z) := f(\alpha z)$.*

Proof Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and suppose, by way of contradiction, that there is a finite linear combination $\sum_{k=1}^{N} c_k f_{\alpha_k} = 0$, where $\alpha_k > 0$, $c_k \in \mathbb{C}$ $(k = 1, \dots, N)$ and not all the c_k are zero. We can assume that $N \geq 2$, $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ and $c_N \neq 0$. Then $a_n(c_1\alpha_1^n + \cdots + c_N\alpha_N^n) = 0$ $(n \in \mathbb{N})$. Since f is not a polynomial, one can find a sequence $\{n_1 < n_2 < \cdots < n_j < \cdots\} \subset \mathbb{N}$ such that $c_1\alpha_1^{n_j} + \cdots + c_N\alpha_N^{n_j} = 0$ $(j \in \mathbb{N})$. Therefore

$$1 = -\sum_{k=1}^{N-1} c_k c_N^{-1} (\alpha_k \alpha_N^{-1})^{n_j} \longrightarrow 0 \quad (j \to \infty).$$

This is the desired contradiction. □

3 Residuality

We start with a refinement of the residuality of the family $\mathcal{SA}$. In fact, we can fix the sequence of radii supporting big values of $|f|$ as well as the rate of growth so that residuality is kept. As usual, we denote $B(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{B}(a,r) = \{z \in \mathbb{C} : |z - a| \leq r\}$ $(a \in \mathbb{C},\, r > 0)$.

Theorem 1 *Let be prescribed a continuous function* $\varphi : \mathbb{D} \to (0, +\infty)$ *and a sequence* $\sigma \in \Sigma$. *Then the set* $\mathcal{SA}(\varphi, \sigma)$ *is residual in* $H(\mathbb{D})$. *Consequently,* $\mathcal{SA}(\varphi)$ *is also residual in* $H(\mathbb{D})$.

Proof Let $\sigma = (r_n)$, so that $0 < r_1 < r_2 < \cdots \to 1$. For every pair $m, n \in \mathbb{N}$ we denote $\mathcal{S}_{m,n} := \{f \in H(\mathbb{D}) : |f(z)| > n\varphi(z)$ for all $z \in r_m\mathbb{T}\}$. If we set $\mathcal{S}_n = \bigcup_{m \geq n} \mathcal{S}_{m,n}$ $(n \in \mathbb{N})$ then one can express

$$\mathcal{SA}(\varphi, \sigma) = \bigcap_{n=1}^{\infty} \mathcal{S}_n.$$

For each compact set $K \subset \mathbb{D}$ and each continuous function f on $\mathbb{D}$ we set $\|f\|_K := \sup\{|f(z)| : z \in K\}$ and $m(f, K) := \min\{|f(z)| : z \in K\}$. A basic open neighborhood of a function $g \in H(\mathbb{D})$ has the form $V(g, K, \varepsilon) = \{h \in H(\mathbb{D}) : \|h - g\|_K < \varepsilon\}$, where $\varepsilon > 0$ and K is a compact subset of $\mathbb{D}$.

Fix $m, n \in \mathbb{N}$. If $g \in \mathcal{S}_{m,n}$ then $\delta := m(|g| - n\varphi, r_m\mathbb{T}) > 0$. If $h \in V(g, r_m\mathbb{T}, \delta)$ then we have for all $z \in r_m\mathbb{T}$ that $-|h(z)| + |g(z)| \leq |h(z) - g(z)| < m(|g| - n\varphi, r_m\mathbb{T})$, so

$$|h(z)| > |g(z)| - m(|g| - n\varphi, r_m\mathbb{T}) \geq |g(z)| - |g(z)| + n\varphi(z) = n\varphi(z).$$

Hence $V(g, r_m\mathbb{T}, \delta) \subset \mathcal{S}_{m,n}$, which proves that $\mathcal{S}_{m,n}$ is open. Therefore every $\mathcal{S}_n$ is open. By Baire's theorem it is enough to show that each $\mathcal{S}_n$ is dense. To this end, fix a basic open set $V(g, K, \varepsilon)$. Choose $m \geq \max\{n, 3\}$ such that $K \subset B(0, r_{m-2})$. Since $r_{m-2} < r_{m-1} < r_m$, we can select $p \in \mathbb{N}$ satisfying

$$\left(\frac{r_{m-2}}{r_{m-1}}\right)^p < \frac{\varepsilon}{\|\varphi\|_{r_m\mathbb{T}}} \quad \text{and} \quad \left(\frac{r_m}{r_{m-1}}\right)^p > n + \frac{\|g\|_{r_m\mathbb{T}}}{\|\varphi\|_{r_m\mathbb{T}}}.$$

Define $f(z) := g(z)+(z/r_{m-1})^p \|\varphi\|_{r_m \mathbb{T}}$. Then $\|f-g\|_K \le (r_m/r_{m-1})^p \|\varphi\|_{r_m\mathbb{T}} < \varepsilon$, so $f \in V(g, K, \varepsilon)$. Furthermore, for all $z \in r_m\mathbb{T}$,

$$\begin{aligned}|f(z)| &> (r_m/r_{m-1})^p \|\varphi\|_{r_m\mathbb{T}} - |g(z)| \\ &> n\|\varphi\|_{r_m\mathbb{T}} + \|g\|_{r_m\mathbb{T}} - |g(z)| \ge n\varphi(z).\end{aligned}$$

Thus, $f \in V(g, K, \varepsilon) \cap \mathcal{S}_n$, which proves the density of $\mathcal{S}_n$. □

The last result will be of help in Section 4 to find algebraic genericity inside $\mathcal{SA}$.

Remark 1 With minor modifications in the proof, one can obtain the following enhancement of Theorem 1. Let $\varphi : \mathbb{D} \to (0, +\infty)$ is continuous and $0 < s_1 < r_1 < s_2 < r_2 < \cdots < s_n < r_n < \cdots \longrightarrow 1$. If we set $A := \bigcup_{n=1}^{\infty} \{z : \, s_n < |z| < r_n\}$ and

$$\mathcal{SA}(\varphi, A) := \big\{ f \in H(\mathbb{D}) : \lim_{\substack{|z|\to 1 \\ z \in A}} \frac{|f(z)|}{\varphi(z)} = +\infty \big\},$$

then $\mathcal{SA}(\varphi, A)$ is residual in $H(\mathbb{D})$. Observe that for each function $f \in \mathcal{SA}(\varphi)$ there is a set $A = A(f)$ as before such that $f \in \mathcal{SA}(\varphi, A)$; indeed, it suffices to apply the continuity of f. Note also that sequences (r_n), (s_n) can be selected so as to their corresponding set A is rather large, in the sense that its radial boundary density ω-dens(A) is maximal (that is, equal to 1). Here ω-dens$(A) := \lim_{r\to 1} \dfrac{\lambda(A \cap \{z : \, r < |z| < 1\})}{\pi(1-r^2)}$, whenever this limit exists, where λ denotes bidimensional Lebesgue measure. For related results (with different classes of functions), see Belna and Redett [8].

We finish this section by extending residuality to other spaces of holomorphic functions in $\mathbb{D}$. Among these well-behaved spaces, we find the weighted Bergman spaces $A^p_\alpha(\mathbb{D})$. For every $p \in (0, +\infty)$ and every $\alpha \in (-1, +\infty)$ the space $A^p_\alpha(\mathbb{D})$ is defined (see e.g. [23]) as the class of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{p,\alpha} := \left(\int\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha \, dxdy \right)^{\min\{1,1/p\}} < +\infty.$$

It becomes a separable F-space under the F-norm $\|\cdot\|_{p,\alpha}$. If $p \ge 1$ ($p = 2$, resp.), $\|\cdot\|_{p,\alpha}$ even makes $A^p_\alpha(\mathbb{D})$ a Banach (Hilbert, resp.) space. For $\alpha = 0$ one obtains the classical Bergman spaces $A^p(\mathbb{D}) = \{f \in H(\mathbb{D}) : \int\int_{\mathbb{D}} |f(z)|^p \, dxdy < +\infty\}$.

Theorem 2 *Assume that Y is a Baire topological vector space with $Y \subset H(\mathbb{D})$ such that Y is endowed with a topology τ which is finer that $\tau_c|_Y$. Let $\varphi : \mathbb{D} \to (0, +\infty)$ be continuous, and $\sigma \in \Sigma$. We have:*

(a) *If $\mathcal{SA} \cap Y \neq \emptyset$ and there is a dense subset $\mathcal{D}$ of Y such that each function $f \in \mathcal{D}$ is bounded on $\mathbb{D}$, then $\mathcal{SA} \cap Y$ is residual in Y.*
(b) *If $\mathcal{SA}(\varphi) \cap Y \neq \emptyset$ and there is a dense subset $\mathcal{D}$ of Y such that f/φ is bounded on $\mathbb{D}$ for each $f \in \mathcal{D}$, then $\mathcal{SA}(\varphi) \cap Y$ is residual in Y.*

(c) *If* $\mathcal{SA}(\varphi,\sigma)\cap Y\neq\emptyset$ *and there is a dense subset* $\mathcal{D}$ *of* Y *such that* f/φ *is bounded on* $\mathbb{D}$ *for each* $f\in\mathcal{D}$, *then* $\mathcal{SA}(\varphi,\sigma)\cap Y$ *is residual in* Y.

Proof Apply Lemma 2 with $X:=H(\mathbb{D})$, $\tau_0:=\tau_c$, $\tau_1:=\tau$ and $D:=\mathcal{D}$. In the situation of (c), the proof of Theorem 1 reveals that $A:=\mathcal{SA}(\varphi,\sigma)$ is a G_δ subset of X. Clearly $A+D\subset A$. By Lemma 2, $\mathcal{SA}(\varphi,\sigma)\cap Y$ is a dense G_δ-subset of Y. Since Y is Baire, $\mathcal{SA}(\varphi,\sigma)\cap Y$ is residual in Y. Hence (c) is proved. Under the hypotheses of (b), there must exist $s\in\Sigma$ such that $\mathcal{SA}(\varphi,s)\cap Y\neq\emptyset$. From (c), $\mathcal{SA}(\varphi,s)\cap Y$ is residual in Y. Since $\mathcal{SA}(\varphi,s)\subset\mathcal{SA}(\varphi)$, this larger set is also residual, which proves (b). Part (a) is the special case of (b) when one takes $\varphi\equiv 1$. □

Corollary 1 *If* $p\in(0,+\infty)$ *and* $\alpha\in(-1,+\infty)$ *then* $\mathcal{SA}\cap A^p_\alpha(\mathbb{D})$ *is residual in* $A^p_\alpha(\mathbb{D})$.

Proof By Redett's result [30], $\mathcal{SA}\cap A^p_\alpha(\mathbb{D})\neq\emptyset$. Just apply Theorem 2(a) with $Y:=A^p_\alpha(\mathbb{D})$ and $\mathcal{D}:=\{\text{polynomials}\}$, and take into account that convergence in $A^p_\alpha(\mathbb{D})$ implies convergence in each compact subset of $\mathbb{D}$ [23, Prop. 1.1] and that the polynomials form a dense subset of $A^p_\alpha(\mathbb{D})$ [23, Prop. 1.3]. □

4 Lineability of $\mathcal{SA}$

We proceed to study the lineability of $\mathcal{SA}$ and of subfamilies of it. By $\text{span}(Y)$ we denote the linear span of a family Y of functions, while $\langle f\rangle$ will stand for the span of $\{f\}$, that is, the set $\{\lambda f:\,\lambda\in\mathbb{C}\}$.

Theorem 3 $\mathcal{SA}$ *is maximal dense-lineable in* $H(\mathbb{D})$.

Proof Consider the function $\varphi(z):=\exp\frac{1}{1-|z|}$ $(z\in\mathbb{D})$. According to Theorem 1, we can select a function $f_0\in\mathcal{SA}(\varphi)$.

Consider the functions $e_\alpha(z):=\exp(\alpha z)$ $(\alpha>0)$ and the set

$$M:=\text{span}\{e_\alpha f_0:\,\alpha>0\}.$$

It is clear that M is a vector subspace of $H(\mathbb{D})$. Moreover, $\dim(M)=c$. Indeed, since the cardinality of $(0,+\infty)$ is c, it is enough to prove the linear independence of the functions $e_\alpha f_0$ $(\alpha>0)$. For this, consider a nontrivial linear combination

$$\sum_{j=1}^{N} a_j e_{\alpha_j} f_0 = 0$$

where, without loss of generality, we can assume that $0<\alpha_1<\alpha_2<\cdots<\alpha_N$ and $a_N\neq 0$. Since $f_0\not\equiv 0$, the analytic continuation principle guarantees the existence of an open interval $I\subset(-1,1)$ such that $f_0(x)\neq 0$ for all $x\in I$. Then, after dividing by f_0 and transposing terms, we get $a_N=-\sum_{j=1}^{N-1}a_j e^{(\alpha_j-\alpha_N)x}$ $(x\in I)$. Now, the analytic continuation principle comes again in our help, yielding that the last equality holds for all $x \in \mathbb{R}$. Letting $x \to +\infty$, we have $e^{(\alpha_j - \alpha_N)x} \to 0$ $(j = 1, \ldots, N-1)$, hence $a_N = 0$, a contradiction. Therefore the functions $e_\alpha f_0$ $(\alpha > 0)$ are independent.

Fix $f \in M \setminus \{0\}$. Then $f = f_0 g$, where g is a nonzero linear combination $\sum_{j=1}^N a_j e_{\alpha_j}$ as before. Since, obviously, φ satisfies (1) and $g \in H(\overline{\mathbb{D}}) \setminus \{0\}$, it follows from Lemma 3 that $M \setminus \{0\} \subset \mathcal{SA}$, whence $\mathcal{SA}$ is c-lineable.

To conclude, take $X := H(\mathbb{D})$ and $\Gamma := \{\langle f \rangle + \mathcal{P} : f \in H(\mathbb{D}) \setminus \mathcal{SA}\}$, where $\mathcal{P} := \{\text{polynomials}\}$. Since $0 \notin \mathcal{SA}$ and the sum of a polynomial and of a function in $H(\mathbb{D}) \setminus \mathcal{SA}$ stays in $H(\mathbb{D}) \setminus \mathcal{SA}$ (i.e. $H(\mathbb{D}) \setminus \mathcal{SA}$ is stronger than $\mathcal{P}$), we have on one hand that $\bigcap_{S \in \Gamma} S = \mathcal{P}$, which is dense in X, and on the other hand that $\bigcap_{S \in \Gamma} (X \setminus S) = \mathcal{SA}$, which is c-lineable. According to Lemma 1, $\mathcal{SA} \cup \{0\}$ contains a dense c-dimensional vector subspace or, that is the same, $\mathcal{SA}$ is maximal dense-lineable. □

In the next assertion, we settle algebrability.

Theorem 4 *$\mathcal{SA}$ is algebrable.*

Proof For $f \in H(\mathbb{D})$ the standard notation $M(f, r) := \max\{|f(z)| : |z| = r\}$ $(0 < r < 1)$ will be used. We start with a function $f_1 \in \mathcal{SA}$. Then there is a sequence of radii $\sigma = (r_n) \in \Sigma$ such that $\lim_{n\to\infty} \min\{|f_1(z)| : z \in r_n\mathbb{T}\} = +\infty$. Hence $f_1 \in \mathcal{SA}(\varphi_0, \sigma)$, where $\varphi_0 \equiv 1$.

Let $\varphi_1(z) := \exp M(f_1, |z|)$. According to Theorem 1, we can select a function $f_2 \in \mathcal{SA}(\varphi_1, \sigma)$. By induction, assume that for some $N \geq 2$ the functions $f_1, \ldots, f_{N-1}, \varphi_0, \ldots, \varphi_{N-2}$ have been already determined. Then we define $\varphi_{N-1}(z) := \exp M(f_{N-1}, |z|)$ and, again by Theorem 1, one can choose a function $f_N \in \mathcal{SA}(\varphi_{N-1}, \sigma)$. Therefore we obtain a sequence of functions $(f_n) \subset H(\mathbb{D})$ such that $f_n \in \mathcal{SA}(\varphi_{n-1}, \sigma)$ $(n \geq 1)$, where $\varphi_0 \equiv 1$ and $\varphi_j(z) \equiv \exp M(f_j, |z|)$ $(j \geq 1)$. Define M as the algebra generated by the functions f_n $(n \in \mathbb{N})$. Our task is to show that (f_n) is a minimal system of generators of M and that each nonzero member of M belongs to $\mathcal{SA}$.

In order to achieve the first part of the task, it is enough to prove that for each $N \geq 2$ the function f_N is not algebraically generated by $f_1, \ldots, f_{N-1}$. To do this, assume by way of contradiction that for some $N \geq 2$ there exists a polynomial $P(z_1, \ldots, z_{N-1})$ in $N-1$ variables without constant term such that $f_N = P(f_1, \ldots, f_{N-1})$. Denote by m the number of monomials forming P, by α the maximum of the moduli of the coefficients of P, and by p the degree of P. Then

$$|f_N(z)| = |P(f_1(z), \ldots, f_{N-1}(z))| \leq m\alpha \Big[\prod_{j=1}^{N-1} (1 + M(f_j, |z|)) \Big]^p \quad (z \in \mathbb{D}).$$

From the construction of $f_1, \ldots, f_N$ we can choose $n_0 \in \mathbb{N}$ satisfying

$$M(f_{N-1}, |z|) \geq M(f_j, |z|) \quad (j = 1, \ldots, N-1) \quad \text{and}$$

$$|f_N(z)| \geq \exp M(f_{N-1}, |z|) \geq 2\, m\alpha (1 + M(f_{N-1}, |z|))^{pN}$$

for all $z \in A := \bigcup_{n \geq n_0} r_n \mathbb{T}$. It follows that, if $z \in A$,

$$1 \leq m\alpha \frac{(1 + M(f_{N-1}, |z|))^{pN}}{\exp M(f_{N-1}, |z|)} \leq \frac{1}{2},$$

so providing the desired contradiction.

Finally, fix $f \in M \setminus \{0\}$. Then there is $N \in \mathbb{N}$ and a nonzero polynomial $P(z_1, \dots, z_N)$ (without constant term, but this is inmaterial) such that $f = P(f_1, \dots, f_N)$. We proceed by induction on N. If $N = 1$ then $f = P(f_1) = \sum_{k=0}^{m} a_k f_1^k$, say, where $m \in \mathbb{N}$ and $a_m \in \mathbb{C} \setminus \{0\}$. If $m = 1$ then it is trivial that $f \in \mathcal{SA}$. If $m \geq 2$ then, for $z \in C(\sigma)$, we have

$$|f(z)| \geq |a_m||f_1(z)|^m \big(1 - \sum_{k=0}^{m-1} |a_k||f_1(z)|^{k-m}\big) \longrightarrow +\infty \quad (|z| \to 1)$$

because $|f_1(z)| \to +\infty$ as $z \in C(\sigma)$, $|z| \to 1$. Therefore $f \in \mathcal{SA}(1, \sigma)$. Assume now that, for some $N \geq 2$, any nonconstant polynomial Q of $N - 1$ variables satisfies $Q(f_1, \dots, f_{N-1}) \in \mathcal{SA}(1, \sigma)$. Let $f = P(f_1, \dots, f_N)$, where P is as in the beginning of this paragraph. Then there are $m \in \mathbb{N}$ and polynomials $Q_1, \dots, Q_m$ of $N - 1$ variables with $Q_m \not\equiv 0$ such that $f = \sum_{k=0}^{m} Q_k(f_1, \dots, f_{N-1}) f_N^k$. By the induction hypothesis, either Q_m is constant or $Q_m(f_1, \dots, f_{N-1}) \in \mathcal{SA}(1, \sigma)$. Choose $n_0 \in \mathbb{N}$ so large that $f_N(z) \neq 0$ if $|z| = r_n$ and $n \geq n_0$. Then

$$|f(z)| = |Q_m(f_1(z), \dots, f_{N-1}(z)) f_N(z)^m| \cdot \big(1 - \sum_{k=0}^{m-1} \frac{Q_k(f_1, \dots, f_{N-1})}{f_N(z)^{m-k}}\big)$$

for such points z. But, in view of the exponential growth of f_N with respect to $f_1, \dots, f_{N-1}$ on $\bigcup_{n \geq n_0} r_n \mathbb{T}$, one gets that the last sum tends to zero as $|z| \to 1$ ($z \in C(\sigma)$). Moreover, it is plain that $|Q_m(f_1, \dots, f_{N-1}) f_N^m|$ tends to $+\infty$ along $C(\sigma)$. Hence $f \in \mathcal{SA}(1, \sigma)$. This completes induction and shows that $M \setminus \{0\} \subset \mathcal{SA}(1, \sigma) \subset \mathcal{SA}$, which had to be proved. □

Remark 2 With slight modifications of the proofs of Theorems 3–4, it is not difficult to demonstrate the following improvement: Assume that $\varphi : \mathbb{D} \to (0, +\infty)$ is continuous and that $\sigma \in \Sigma$. Then $\mathcal{SA}(\varphi, \sigma)$ (and so $\mathcal{SA}(\varphi)$) is maximal dense-lineable and algebrable.

Remark 3 This paper deals with special unbounded analytic (hence continuous) functions on $\mathbb{D}$ or $\mathbb{C}$ under the focus of lineability. To this respect, families of unbounded continuous functions on more general topological spaces have been already studied from this point of view. Namely, García, Martín and Seoane [21, Theorem 4.1] have recently proved that, in every non-compact metric space Ω, the set of all continuous unbounded real functions defined on it is algebrable.

5 Bergman spaces

The purpose of this section is to analyze lineability of $\mathcal{SA}$ in the context of weighted Bergman spaces. The next lemma, which is due to Buckley, Koskela and Vukotic [17], will be employed (as in Redett [30]) to construct strongly annular functions in this setting. Functions involved in it are associated to lacunary series.

Lemma 5 *Let* $p \in (0,+\infty)$*,* $\alpha \in (-1,+\infty)$*,* $(p_n) \subset \mathbb{N}$ *with* $p_{n+1} > 2p_n$ $(n \geq 1)$*, and* $f(z) = \sum_{n=0}^{\infty} a_n z^{p_n} \in H(\mathbb{D})$*. Then* $f \in A_\alpha^p(\mathbb{D})$ *if and only if* $\sum_{n=1}^{\infty} |a_n|^p p_n^{-\alpha-1} < +\infty$*.*

We establish our result in the next theorem.

Theorem 5 *The set* $\mathcal{SA} \cap A_\alpha^p(\mathbb{D})$ *is dense-lineable in* $A_\alpha^p(\mathbb{D})$*.*

Proof Suppose that we have already proved that $\mathcal{SA} \cap A_\alpha^p(\mathbb{D})$ is lineable. Then we can argue as in the final part of the proof of Theorem 3, by using Lemma 1 with $\mu = \mathrm{card}(\mathbb{N})$, $X := A_\alpha^p(\mathbb{D})$ and $\Gamma := \{\langle f \rangle + \mathcal{P} : f \in A_\alpha^p(\mathbb{D}) \setminus \mathcal{SA}\}$; again, we use the fact that polynomials form a dense subset of $A_\alpha^p(\mathbb{D})$ [23, Prop. 1.3]. Then we conclude the dense-lineability of $\mathcal{SA} \cap A_\alpha^p(\mathbb{D})$. Alternatively, [2, Theorem 2.2 for F-spaces] can also be used to prove dense-lineability.

Therefore we have only to show the lineability of $\mathcal{SA} \cap A_\alpha^p(\mathbb{D})$. In some steps of the proof we will follow (and modify) the nice construction given by Redett in [30].

We proceed by induction. Choose $p_1 = a_1 = 2$ and $0 < r_1 < 1$ so that $a_1 r_1^{p_1} > 1$. Since the series $\sum_{n=1}^{\infty} n r_1^n$ converges, one can select $m_1 \in \mathbb{N}$ so that $\sum_{n=m_1}^{\infty} n r_1^n < a_1 r_1^{p_1} - 1$. Then

$$a_1 r_1^{p_1} > 1 + \sum_{n=m_1}^{\infty} n r_1^n.$$

Observe that $a_1^p p_1^{-\alpha-1} < a_1^p = 2^p$. Next, we choose a positive integer $a_2 > \max\{2(1+a_1), m_1\}$. With a_2 fixed, select $p_2 \in \mathbb{N}$ such that $p_2 > \max\{2p_1, a_2\}$ and $a_2^p p_2^{-\alpha-1} < 2^p - a_1 p_1^{-\alpha-1}$. Therefore $\sum_{n=1}^{2} a_n^p p_n^{-\alpha-1} < 2^p$. Since $a_2 > 2(1+a_1)$ and $a_2 x^{p_2} \to a_2$ as $x \to 1$, we can pick $r_2 \in (r_1, 1)$ with $a_2 r_2^{p_2} > 2(1+a_1)$. As before, the series $\sum_{n=1}^{\infty} n r_2^n$ converges. Then one can select $m_2 \in \mathbb{N}$ such that $\sum_{n=m_2}^{\infty} n r_2^n < (1/2) a_2 r_2^{p_2} - (1+a_1)$, whence

$$a_2 r_2^{p_2} > 2\big(1 + a_1 + \sum_{n=m_2}^{\infty} n r_2^n\big).$$

By induction, assume that $a_1, \ldots, a_n, p_1, \ldots, p_n, r_1, \ldots, r_n, m_1, \ldots, m_n$ have already been selected in the above manner.

At step $n+1$, we choose a positive integer $a_{n+1} > \max\{(n+1)(1 + \sum_{k=1}^{n} a_k), m_n\}$. Since $\sum_{k=1}^{n} a_k^p p_k^{-\alpha-1} < 2^p$, one can select a positive integer

$p_{n+1} > \max\{2p_n, a_{n+1}\}$ such that $a_{n+1}p_{n+1}^{-\alpha-1} < 2^p - \sum_{k=1}^n a_k^p p_k^{-\alpha-1}$. Therefore

$$\sum_{k=1}^{n+1} a_k^p p_k^{-\alpha-1} < 2^p. \tag{3}$$

In addition, since $a_{n+1}x^{p_{n+1}} \to a_{n+1}$ as $x \to 1$, we may choose $r_{n+1} \in (r_n, 1)$ such that $a_{n+1}r_{n+1}^{p_{n+1}} > (n+1)(1 + \sum_{k=1}^n a_k)$. Once more, the convergence of $\sum_{k=1}^\infty kr_{n+1}^k$ ensures the existence of a number $m_{n+1} \in \mathbb{N}$ with $\sum_{k=m_{n+1}}^\infty kr_{n+1}^k < \frac{a_{n+1}r_{n+1}^{p_{n+1}}}{n+1} - (1 + \sum_{k=1}^n a_k)$, whence

$$a_{n+1}r_{n+1}^{p_{n+1}} > (n+1)\big(1 + \sum_{k=1}^n a_k + \sum_{k=m_{n+1}}^\infty kr_{n+1}^k\big). \tag{4}$$

Next, we fix a countable collection $\{i(j,k)\}_{k\geq 1}$ $(j \in \mathbb{N})$ of pairwise disjoint strictly increasing sequences of natural numbers and define

$$f_j(z) := \sum_{k=1}^\infty a_{i(j,k)} z^{p_{i(j,k)}} \quad (j \in \mathbb{N}). \tag{5}$$

Since $p_{n+1} > 2p_n$ and, thanks to (3), the series $\sum_{n=1}^\infty a_n^p p_n^{-\alpha-1}$ converges, it follows from Lemma 5 that each $f_j \in A_\alpha^p(\mathbb{D})$. Define

$$M := \mathrm{span}\{f_j : j \in \mathbb{N}\}.$$

Since $p_1 < p_2 < p_3 < \cdots$ and the mapping $i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is injective, the family $\{f_j : j \in \mathbb{N}\}$ is linearly independent. Thus M is an infinite-dimensional vector subspace of $A_\alpha^p(\mathbb{D})$. Our task is to show that $M \setminus \{0\} \subset \mathcal{SA}$.

Fix $f \in M \setminus \{0\}$. Then there are $N \in \mathbb{N}$ and scalars $c_1, \ldots, c_N$ such that $c_N \neq 0$ and $f = \sum_{j=1}^N c_j f_j$. We can assume without loss of generality that $N \geq 2$ and $c_N = 1$. Let $\alpha := \max\{|c_j| : 1 \leq j \leq N-1\}$. It follows from (5) that for all z with $|z| = r_{i(N,k)}$ we have

$$\begin{aligned}
|f(z)| &\geq |c_N f_N(z)| - \sum_{j=1}^{N-1} |c_j||f_j(z)| \geq |f_N(z)| - \alpha \sum_{j=1}^{N-1} |f_j(z)| \\
&\geq a_{i(N,k)}|z|^{p_{i(N,k)}} - \sum_{\substack{\nu=1\\ \nu\neq k}}^\infty a_{i(N,\nu)}|z|^{p_{i(N,\nu)}} - \alpha \sum_{j=1}^{N-1} |f_j(z)| \\
&= a_{i(N,k)} r_{i(N,k)}^{p_{i(N,k)}} - \sum_{\substack{\nu=1\\ \nu\neq k}}^\infty a_{i(N,\nu)} r_{i(N,k)}^{p_{i(N,\nu)}} - \alpha \sum_{j=1}^{N-1} |f_j(z)| \\
&\geq a_{i(N,k)} r_{i(N,k)}^{p_{i(N,k)}} - (1+\alpha) \sum_{\substack{n=1\\ n\neq i(N,k)}}^\infty a_n r_{i(N,k)}^{p_n}.
\end{aligned}$$

Fix $k_0 \in \mathbb{N}$ so large that $i(N,k) \ge 1+\alpha$ for all $k \ge k_0$. Then (4) drives us to

$$\begin{aligned} |f(z)| &\ge i(N,k) + i(N,k) \sum_{n=1}^{i(N,k)-1} a_n + i(N,k) \sum_{n=m_{i(N,k)}}^{\infty} n r_{i(N,k)}^n \\ &\quad - (1+\alpha) \sum_{n=1}^{i(N,k)-1} a_n - (1+\alpha) \sum_{n=i(N,k)+1}^{\infty} a_n r_{i(N,k)}^{p_n} \\ &\ge i(N,k) + i(N,k) \sum_{n=m_{i(N,k)}}^{\infty} n r_{i(N,k)}^n - (1+\alpha) \sum_{n=i(N,k)+1}^{\infty} a_n r_{i(N,k)}^{a_n} \\ &\ge i(N,k) + (i(N,k) - 1 - \alpha) \sum_{n=m_{i(N,k)}}^{\infty} n r_{i(N,k)}^n, \end{aligned}$$

where we have used that $p_n \ge a_n$, $a_n \in \mathbb{N}$ for all $n \ge 1$ and that $a_n \ge a_{i(N,k)+1} \ge m_{i(N,k)}$ for all $n \ge i(N,k)+1$. Consequently, $|f(z)| \ge i(N,k)$ whenever $|z| = r_{i(N,k)}$ and $k \ge k_0$. This ensures $\lim_{k\to\infty} \min\{|f(z)| : |z| = r_{i(N,k)}\} = +\infty$, which implies $f \in \mathcal{SA}$. □

Remark 4 The same approach of the proof of Theorem 5 shows that the vector space $\widetilde{M} := \{\sum_{j=1}^{\infty} c_j f_j : (c_j) \subset \mathbb{C}$ is bounded$\}$, which is larger than M, also satisfies $\widetilde{M} \setminus \{0\} \subset \mathcal{SA} \cap A_\alpha^p(\mathbb{D})$.

6 Entire functions

The results of sections 3–4 can be extended to the space $H(\mathbb{C})$ of entire functions (again, $H(\mathbb{C})$ is $\mathfrak{c}$-dimensional and becomes an F-space when endowed with the compact-open topology), but there is an important different nuance. Namely, entire functions f with $\lim_{z\to\infty} |f(z)| = +\infty$ *do* exist. These functions are exactly the nonconstant polynomials. Nevertheless, we can define analogously the family $\mathcal{SA}_e$ of *strongly annular entire functions* as

$$\mathcal{SA}_e := \{f \in H(\mathbb{C}) : \limsup_{r\to\infty} \min\{|f(z)| : |z| = r\} = +\infty\}.$$

Hence $\mathcal{SA}_e \supset \{$nonconstant polynomials$\}$, so $\mathcal{SA}_e$ is at least dense in $H(\mathbb{C})$. Incidentally, we obtain that $\mathcal{SA}_e$ is lineable because $\mathcal{SA}_e \supset \{$polynomials P with $P \not\equiv 0$ and $P(0) = 0\}$. But much more is true: see Theorem 7 below.

A specially intriguing result is the one given in Proposition 1 below (see Boas [11, p. 39]) that is due to Wiman. Recall that, if $\rho \in (0,+\infty)$ and f is entire, the *ρ-type* of f is defined as

$$\tau_\rho(f) = \limsup_{r\to\infty} \frac{\log M(f,r)}{r^\rho},$$

where $M(f,r) := \max\{|f(z)| : |z| = r\}$ $(r > 0)$. In our terminology, Wiman's theorem reads as follows.

Proposition 1 *If f is a nonconstant entire function with $\tau_{1/2}(f) = 0$ then $f \in \mathcal{SA}_e$.*

Let us settle appropriate notation as passing from $\mathbb{D}$ to $\mathbb{C}$. Assume that $\varphi : \mathbb{C} \to (0, +\infty)$ is a continuous function. Let Σ_∞ be the family of all strictly increasing unbounded sequences $\sigma = (r_n) \subset (0, +\infty)$. For each $\sigma \in \Sigma_\infty$ we set $D(\sigma) := \bigcup_{n=1}^{\infty} r_n \mathbb{T}$. Denote

$$\mathcal{SA}_e(\varphi) := \{f \in H(\mathbb{C}) : \limsup_{r\to\infty} \min\{\frac{|f(z)|}{\varphi(z)} : |z| = r\} = +\infty\} \quad \text{and}$$

$$\mathcal{SA}_e(\varphi, \sigma) := \{f \in H(\mathbb{C}) : \lim_{\substack{z\to\infty \\ z\in D(\sigma)}} \frac{|f(z)|}{\varphi(z)} = +\infty\} \quad (\sigma \in \Sigma_\infty).$$

It is evident that $\mathcal{SA}_e(1) = \mathcal{SA}_e$ and $\mathcal{SA}_e(\varphi) = \bigcup_{\sigma\in\Sigma_\infty} \mathcal{SA}_e(\varphi, \sigma)$.

A closer look at the proofs of Theorems 1 and 4 together with Remark 2 reveals that they also work for entire functions, after easy modifications. Consequently, we only state the results, leaving the proof to the interested reader.

Theorem 6 *Let $\sigma \in \Sigma_\infty$ and $\varphi : \mathbb{C} \to (0, +\infty)$ be continuous. Then $\mathcal{SA}(\varphi, \sigma)$ (hence $\mathcal{SA}(\varphi)$ and so $\mathcal{SA}$) is residual in $H(\mathbb{C})$ and algebrable.*

Nevertheless, the study of maximal dense-lineability for $\mathcal{SA}_e$ requires additional work, because the proof of Theorem 3 cannot be adapted so easily to $H(\mathbb{C})$: indeed, the functions e_α $(\alpha > 0)$ satisfy $\lim_{r\to\infty} \min\{|e_\alpha(z)| : |z| = r\} = \lim_{r\to\infty} e^{-\alpha r} = 0$, which is far from being useful to obtain vector spaces inside $\mathcal{SA}_e \cup \{0\}$. Moreover, there is no analogue to Lemma 3 at our disposal. This forces us to proceed under a different focus. Recall that the growth order and the lower growth order of an entire function f are respectively defined as $\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log r}$ and $\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(f,r)}{\log r}$. In this section we set $m(f, r) := \min\{|f(z)| : |z| = r\}$ $(r > 0)$. The following assertion, which is due to Pólya (see [24, pp. 137–144]) and is more precise than Proposition 1, will be needed to prove our statement.

Lemma 6 *If f is an entire function with $\rho(f) \in (0, 1)$ and $\omega < \cos(\pi\rho)$, then there is a sequence $(r_n) \in \Sigma_\infty$ such that $m(f, r_n) > M(f, r_n)^\omega$ for all $n \geq 1$.*

In fact, we obtain a result stronger than the expected one. For each $\gamma > 0$, we denote $\varphi_\gamma(z) := \exp(|z|^\gamma)$ $(z \in \mathbb{C})$.

Theorem 7 *For each $\gamma \in (0, 1/2)$, the family $\mathcal{SA}_e(\varphi_\gamma)$ is maximal dense-lineable in $H(\mathbb{C})$. Hence $\mathcal{SA}_e$ is maximal dense-lineable in $H(\mathbb{C})$.*

Proof Fix $\gamma \in (0, 1/2)$ and select $\gamma' \in (\gamma, 1/2)$. Observe that $\mathcal{SA}_e(\varphi_\gamma) = \{f \in H(\mathbb{C}) : \limsup_{r\to\infty} \frac{m(f,r)}{e^{r^\gamma}} = +\infty\}$. For any function $g(z) = \sum_{n=1}^{\infty} a_n z^n \in H(\mathbb{C})$, the quantities $\rho(g)$ and $\lambda(g)$ satisfy

$$\liminf_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \leq \lambda(g) \leq \rho(g) = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)}, \tag{2}$$

see Boas [11, Chap. 2].

Now, we take

$$g_0(z) := \sum_{n=1}^{\infty} n^{-n/\gamma'} z^n.$$

For every $\alpha > 0$, denote $f_\alpha(z) := g_0(\alpha z)$. By Lemma 4, the set

$$M := \text{span}\,\{f_\alpha \,:\, \alpha > 0\}$$

is a vector subspace of $H(\mathbb{C})$ satisfying $\dim(M) = c$. Fix $f \in M \setminus \{0\}$. Then there are $N \in \mathbb{N}$, scalars $c_1, \dots, c_N$ with $c_N \neq 0$ and numbers $0 < \alpha_1 < \cdots < \alpha_N$ satisfying $f = c_1 f_{\alpha_1} + \cdots + c_N f_{\alpha_N}$. Then $f(z) = \sum_{n=1}^{\infty} b_n z^n$, where $b_n = n^{-n/\gamma'} c_N \alpha_N^n (1 + \sum_{k=1}^{N-1} (c_k/c_N)(\alpha_k/\alpha_N)^n)$. Therefore $\log(1/|b_n|) \sim (n/\gamma') \log n$ $(n \to \infty)$, so $\lim_{n\to\infty} \frac{n \log n}{\log(1/|b_n|)} = \gamma'$. According to (2), $\rho(f) = \gamma' = \lambda(f)$, from which one derives that $\lim_{r\to\infty} \frac{\log\log M(f,r)}{\log r} = \gamma'$. In particular, if we fix $\gamma'' \in (\gamma, \gamma')$, we get $M(f,r) > e^{r^{\gamma''}}$ asymptotically. Since $\rho(f) = \gamma' \in (0, 1/2) \subset (0,1)$, an application of Lemma 6 (with $\omega := (1/2)\cos(\pi\gamma') > 0$) yields that, for some sequence $(r_n) \in \Sigma_\infty$, $m(f, r_n) > M(f, r_n)^\omega > e^{\omega r_n^{\gamma''}}$. But $e^{\omega r^{\gamma''}} > r e^{r^\gamma}$ for r large enough. Thus, there is $n_0 \in \mathbb{N}$ such that $m(f, r_n)/e^{r_n^\gamma} > r_n$ if $n \geq n_0$. Hence $\limsup_{r\to\infty} m(f,r)/e^{r^\gamma} = +\infty$, i.e. $f \in \mathcal{SA}_e(\varphi_\gamma)$. Therefore $M \setminus \{0\} \subset \mathcal{SA}_e(\varphi_\gamma)$. Consequently, $\mathcal{SA}_e$ is c-lineable.

Finally, we can conclude as in the proof of Theorem 3. Let us take $X := H(\mathbb{C})$ and $\Gamma := \{\langle f \rangle + \mathcal{P} \,:\, f \in H(\mathbb{C}) \setminus \mathcal{SA}_e(\varphi_\gamma)\}$, where $\mathcal{P}$ is the set of polynomials. Since $0 \in H(\mathbb{C}) \setminus \mathcal{SA}_e(\varphi_\gamma)$, we derive that $\bigcap_{S\in\Gamma} S = \mathcal{P}$, which is dense in X. Furthermore, $\bigcap_{S\in\Gamma}(X \setminus S) = \mathcal{SA}_e(\varphi_\gamma)$, because the sum of a polynomial and a member of $\mathcal{SA}_e(\varphi_\gamma)$ stays in $\mathcal{SA}_e(\varphi_\gamma)$ (again we meet the notion of "stronger than" given in [2]). From Lemma 1, we deduce the existence of a dense c-dimensional vector subspace contained, except for zero, in $\mathcal{SA}_e(\varphi_\gamma)$. This finishes the proof. □

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