Interpolation by hypercyclic functions for differential operators

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Abstract

We prove that, given a sequence of points in a complex domain Ω without accumulation points, there are functions having prescribed values at the points of the sequence and, simultaneously, having dense orbit in the space of holomorphic functions on Ω. The orbit is taken with respect to any fixed non-scalar differential operator generated by an entire function of subexponential type, thereby extending a recent result about MacLane-hypercyclicity due to Costakis, Vlachou and Niess.

Key words and phrases: Hypercyclic function, differential operator, interpolation, mixing sequence of mappings.

1 Introduction

In 1929, Birkhoff [6] proved the existence of an entire function f which is universal in the sense that the sequence $\{f(\cdot + an) : n \geq 1\}$ of its translates $(a \in \mathbb{C} \setminus \{0\})$ is dense in the space $H(\mathbb{C})$ of entire functions. Later, in 1952, MacLane [15] demonstrated the same density property for the sequence $\{f^{(n)} : n \geq 1\}$ of derivatives of some entire function f. Formally speaking, the translation operator $\tau_a : f \mapsto f(\cdot + a)$ and the derivation operator $D : f \mapsto f'$ acting on the space $H(\mathbb{C})$ are hypercyclic, see Section 2. In 1991, Godefroy and Shapiro [10] unified both above results by establishing the hypercyclicity on $H(\mathbb{C})$ of any nonscalar differential operator $\Phi(D)$, generated by an entire function Φ with exponential type. Birkhoff's theorem and MacLane's theorem correspond, respectively, to the cases $\Phi(z) = \exp(az)$, $\Phi(z) = z$.

In a different direction, Weierstrass' interpolation theorem (see [18]) asserts the existence of holomorphic functions on a domain Ω of the complex plane $\mathbb{C}$ having prescribed values at a sequence of points without accumulation points. Recently, Costakis and Vlachou [9] and, independently, Niess ([16],[17]) have carried out the interesting task of combining both properties of hypercyclicity and interpolation. To be more specific, they have proved the existence of a holomorphic function f (in fact, of a plethora of them) that is MacLane-universal and, simultaneously, takes prescribed values at the points of a given set without accumulation points in Ω, whenever Ω is simply connected ($\Omega = \mathbb{C}$ in [16],[17]). They have also established a number of results concerning interpolation for functions enjoying either Birkhoff universality on $\mathbb{C}$ ([9],[17]), or multiplicative universality in $\mathbb{C} \setminus \{0\}$ [17], or universality in a domain with respect to some similarities $z \mapsto az + b$ [9], or universality (on multiply connected domains) of their Taylor partial sums

[9]. In [16], [17], the universality-interpolation properties are even combined with prescribed zeros with given multiplicities. It seems that Costakis [8] was the first analyst that dealt with the problem of existence of interpolating universal functions, precisely in the setting of universal Taylor series on simply connected domains.

Our aim in this paper is to extend the former result by Costakis–Vlachou and Niess. Namely, we generalize their assertion to differential operators generated by entire functions Φ with subexponential type. Section 2 is devoted to give a number of pertinent definitions and instrumental results, while Section 3 contains the exact statement of our theorem, together with its proof and a result about linear hypercyclic structure in a special case.

2 Terminology and preliminary results

Let Ω be a domain in $\mathbb{C}$, that is, Ω is a nonempty connected open subset of $\mathbb{C}$. The vector space $H(\Omega)$ of holomorphic functions on Ω becomes a complete metrizable separable space when endowed with the topology of uniform convergence on compacta. In particular, it is a second-countable Baire space. A domain Ω is said to be simply connected provided that its complement with respect to the extended complex plane is connected. If $S \subset \mathbb{C}$, then S^0 denotes the interior of S.

Let Φ be an entire function. Then Φ is said to be of exponential type if there are constant A, $B \in (0, +\infty)$ such that $|\Phi(z)| \leq A \exp(B|z|)$ $(z \in \mathbb{C})$. And Φ is called of subexponential type whenever, given $\varepsilon > 0$, there is a constant $A = A(\varepsilon) \in (0, +\infty)$ such that $|\Phi(z)| \leq A \exp(\varepsilon|z|)$ $(z \in \mathbb{C})$. In other words, Φ is of exponential (subexponential, resp.) type if and only if it has either growth order < 1 or growth order $= 1$ and type $< +\infty$

(either growth order < 1 or growth order $= 1$ and type 0, resp.). If Ω is a domain and $\Phi(z) := \sum_{n=0}^{\infty} c_n z^n$ is an entire function of subexponential type, then the expression $\Phi(D) := \sum_{n=0}^{\infty} c_n D^n$ defines a (linear, continuous) operator $\Phi(D) : f \in H(\Omega) \mapsto \sum_{n=0}^{\infty} c_n f^{(n)} \in H(\Omega)$, see [1]. Note that $\Phi(D)$ is a (generally, infinite order) linear differential operator with constant coefficients. If $\Omega = \mathbb{C}$, then the operator $\Phi(D)$ is well-defined even if Φ is of exponential type. In fact, an operator T on $H(\mathbb{C})$ has the form $T = \Phi(D)$ with Φ of exponential type if and only if T commutes will all translations τ_a ($a \in \mathbb{C}$), see [10].

For the following concepts and results on universality and other dynamical properties, the reader is referred to [14], [12], [13], [7] and [11]. Assume that $T_n : X \to Y$ ($n \in \mathbb{N} := \{1, 2, \dots\}$) is a sequence of continuous mappings between two topological spaces X, Y. Then $\{T_n\}_{n \geq 1}$ is said to be:

(a) *universal* if there exists a point $x_0 \in X$ (called *universal* for $\{T_n\}_{n \geq 1}$) whose orbit $\{T_n x_0 : n \in \mathbb{N}\}$ is dense in Y.

(b) *topologically transitive* if, given nonempty open subsets $U \subset X$, $V \subset Y$, there is $N \in \mathbb{N}$ such that $T_N(U) \cap V \neq \emptyset$.

(c) *topologically mixing* if, given nonempty open subsets $U \subset X$, $V \subset Y$, there is $N \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$ for $n \geq N$.

If $T : X \to X$ is a continuous selfmapping, then T is called *universal* (*topologically transitive, topologically mixing*, resp.) provided that the sequence $\{T^n\}_{n \geq 1}$ of its iterates is universal (topologically transitive, topologically mixing, resp.). If X, Y are topological vector spaces and the T_n (or T, if we are dealing with selfmappings) are linear, then it is customary to say *hypercyclic* instead of *universal*. We denote $\mathcal{U}(\{T_n\}_{n \geq 1}) = \{x \in X : x$ is universal for $\{T_n\}_{n\geq 1}$ and, if $X = Y$, $\mathcal{U}(T) = \{x \in X : x$ is universal for $T\} = \mathcal{U}(\{T^n\}_{n\geq 1})$.

Recall that a subset A of a Baire topological space X is residual when it contains a dense G_δ subset. It can be said that a residual subset is "topologically large" in X. The connection between universality and transitivity is given by the following well-known result.

Birkhoff's transitivity theorem. *Assume that $T_n : X \to Y$ $(n \in \mathbb{N})$ is a sequence of continuous mappings between two topological spaces X, Y. Suppose that X is Baire and that Y is second-countable. Then the following are equivalent:*
(i) $\{T_n\}_{n\geq 1}$ *is topologically transitive.*
(ii) $\{T_n\}_{n\geq 1}$ *is densely universal, that is,* $\mathcal{U}(\{T_n\}_{n\geq 1})$ *is dense in* X.
(iii) $\mathcal{U}(\{T_n\}_{n\geq 1})$ *is residual in* X.

One easily derives that, under the same hypotheses on X and Y, the sequence $\{T_n\}_{n\geq 1}$ is topologically mixing if and only if $\{T_{n_k}\}_{k\geq 1}$ is densely universal for every subsequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$.

Godefroy and Shapiro [10] proved the hypercyclicity of any operator $\Phi(D)$ on $H(\mathbb{C})$, with Φ a nonconstant entire function of exponential type. In fact, they worked in $\mathbb{C}^N$, and their result can be extended to a Runge domain Ω in $\mathbb{C}^N$ (if $N = 1$, Runge equals simply connected) whenever Φ is of subexponential type (see [2] and [5], where sequences of operators $\Phi_n(D)$ are also considered). The approach followed by Godefroy and Shapiro shows that, in fact, the operators $\Phi(D)$ are mixing.

In Section 3, we will need the next four lemmas. Lemma 2.1 is an analogue to an auxiliary bounded interpolation result given in [9], except that in ours exponentials are employed instead of polynomials. Lemma 2.2 follows from a well-known denseness assertion whose proof can be found in [10]. Lemma 2.3 says that the differential operators induced by entire functions of subexponential type are "internally controlled". Its easy proof is sketched in [4]. Finally, Lemma 2.4 furnishes a statement about the strong linear structure of a topologically mixing sequence of linear mappings. By e_a ($a \in \mathbb{C}$) we denote the function $e_a(z) := \exp(az)$, while $\mathrm{span} X_0$ will stand for the linear span of a subset X_0 of a vector space.

Lemma 2.1. *Assume that L is a compact subset of $\mathbb{C}$, that $a_1, \ldots, a_m$ are different points in L, and that A is a nonempty open subset of $\mathbb{C}$. Then there are a constant $M = M(L, a_1, \ldots, a_m, A) \in (0, +\infty)$ and a finite set of functions $\{\alpha_1, \ldots, \alpha_m\} \subset \mathrm{span}\{e_a : a \in A\}$, depending only on A and the points $a_1, \ldots, a_m$, satisfying the following property: Given a pair f, h of complex-valued functions defined on L, there exists a function $F : L \to \mathbb{C}$ satisfying:*

(a) $\sup_{z \in L} |F(z) - h(z)| \leq M \sup_{z \in L} |f(z) - h(z)|$.

(b) $F(a_j) = h(a_j)$ $(j = 1, \ldots, m)$.

(c) $F - f = \sum_{j=1}^{m} (h(a_j) - f(a_j)) \alpha_j$.

Proof. We can assume that $A \neq \mathbb{C}$. Let us choose a point $c \in A$ and select a positive number d satisfying

$$d < \frac{1}{2m} \inf\{|z - c| : z \in \mathbb{C} \setminus A\} \tag{1}$$

and

$$d < \min_{\substack{j,l \in \{1,\ldots,m\} \\ j \neq l}} \frac{1}{|a_j - a_l|}. \tag{2}$$

We define

$$\Pi_j(z) := \prod_{l \in \{1,\ldots,m\} \setminus \{j\}} (e_d(z - a_l) - 1) \quad (j = 1, \ldots, m).$$

From (2), it follows that

$$0 < d|a_j - a_l| < 1 < 2\pi$$

for all $j, l \in \{1, \ldots, m\}$ with $j \neq l$, so $\Pi_j(a_j) \neq 0$ for all $j \in \{1, \ldots, m\}$. In addition, $\Pi_j(a_l) = 0$ whenever $j \neq l$. Now, we set

$$\alpha_j(z) := e_c(z - a_j)\frac{\Pi_j(z)}{\Pi_j(a_j)} \quad (j = 1, \ldots, m)$$

and

$$M := 1 + \sum_{j=1}^{m} \sup_{z \in L} |\alpha_j(z)|.$$

Observe that each function α_j is a finite linear combination of functions of the form e_{c+nd} $(0 \leq n \leq m)$. But each point $c + nd$ is in A because of (1). Hence the functions α_j are in $\mathrm{span}\{e_a : a \in A\}$.

Finally, if we fix functions $f, h : L \to \mathbb{C}$ and define $F := f + \sum_{j=1}^{m}(h(a_j) - f(a_j))\alpha_j$, then it is straightforward that (a), (b) and (c) are fulfilled. $\square$

Lemma 2.2. *Let $A \subset \mathbb{C}$ be a subset with some finite accumulation point, and $\Omega \subset \mathbb{C}$ be a simply connected domain. Then the set* $\mathrm{span}\{e_a : a \in A\}$ *is dense in the space $H(\Omega)$. In particular, this holds if A is a nonempty open subset of $\mathbb{C}$.*

Lemma 2.3. *Let $\Omega \subset \mathbb{C}$ be a domain and Φ be an entire function of subexponential type. Assume that K, L are compact sets in Ω with $L \subset K^0$. Then there exists a constant $C = C(K, L) \in (0, +\infty)$ such that*

$$\sup_{z \in L} |(\Phi(D)f)(z)| \leq C \sup_{z \in K} |f(z)| \quad \textit{for all} \quad f \in H(\Omega).$$

Lemma 2.4. *Let X, Y be two metrizable separable topological vector spaces, such that X is Baire. If $T_n : X \to Y$ $(n \in \mathbb{N})$ is a topologically mixing sequence of continuous linear mappings, then there exists a dense linear manifold $\mathcal{D} \subset X$ such that $\mathcal{D} \setminus \{0\} \subset \mathcal{U}(\{T_n\}_{n\geq 1})$.*

Proof. Note that X is Baire, Y is second-countable and, by Birkhoff's transitivity theorem, each sequence $\{T_{n_k}\}_{k\geq 1}$ with $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ is densely hypercyclic. Then the result follows from [3, Theorem 2]. □

3 $\Phi(D)$-hypercyclicity with interpolation

In this section, we establish our main result. For this, a bit of additional notation is needed. Let $\Omega \subset \mathbb{C}$ be a domain. By $\mathbb{C}^{\mathbb{N}}$ we denote, as usual, the class of sequences of complex numbers, while $\omega(\Omega)$ will stand for the set of sequences of pairwise different points in Ω without accumulation points in Ω. If $\{a_n\}_{n\geq 1} \in \omega(\Omega)$ and $\{b_n\}_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$, then we denote (as in [9])

$$\Gamma = \Gamma(\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}) := \{f \in H(\Omega) : \ f(a_n) = b_n \ \text{ for all } \ n \in \mathbb{N}\}.$$

Observe that Γ is nonempty (by Weierstrass' interpolation theorem). In addition, Γ is closed in $H(\Omega)$, hence completely metrizable, so a Baire space.

Costakis and Vlachou [9] have proved that, if Ω is simply connected, the set $\mathcal{U}(D) \cap \Gamma$ is residual in Γ. This admits an extension as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and Φ be a nonconstant entire function of subexponential type. Assume that $\{a_n\}_{n\geq 1} \in \omega(\Omega)$ and $\{b_n\}_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$. Consider the corresponding set $\Gamma = \Gamma(\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1})$. Then the sequence of mappings*

$$T_n := \Phi(D)^n|_{\Gamma} : \Gamma \longrightarrow H(\Omega) \quad (n \geq 1)$$

is topologically mixing. In particular, there is a residual subset of functions in Γ that are $\Phi(D)$-hypercyclic.

Proof. The second part is a consequence of Birkhoff's transitivity theorem. Then our task is to prove that $\{T_n\}_{n\geq 1}$ is topologically mixing.

Fix two nonempty open subsets $U \subset \Gamma$, $V \subset H(\Omega)$. We should exhibit a number $N \in \mathbb{N}$ satisfying the following: For every $n \geq N$, there is a function $f \in H(\Omega)$ such that

$$f \in U \quad \text{and} \quad T_n f \in V. \tag{3}$$

For prescribed open sets U, V as before, we can find functions $h \in \Gamma, \varphi \in H(\Omega)$, a number $\varepsilon \in (0,1)$ and a compact set $L \subset \Omega$ such that

$$U \supset \Gamma \cap \{f \in H(\Omega) : \, |f(z) - h(z)| < \varepsilon \ \text{ for all } \ z \in L\}$$

and

$$V \supset \{f \in H(\Omega) : \, |f(z) - \varphi(z)| < \varepsilon \ \text{ for all } \ z \in L\}.$$

Since Ω is simply connected and the set $\{a_n\}_{n\geq 1}$ has no accumulation points in Ω, it is not difficult to find a sequence of compact sets $L_k \subset \Omega$ $(k \in \mathbb{N})$ and a sequence $\{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ satisfying, for each k, the following properties:

- The set $\mathbb{C} \setminus L_k$ is connected,
- $L \subset L_1^0$, $L_k \subset L_{k+1}^0$ and $\Omega = \bigcup_{k\geq 1} L_k$, and
- $L_k \cap \{a_n\}_{n\geq 1} = \{a_1, \dots, a_{m_k}\}$.

By hypothesis, the entire function Φ is not constant, so the open sets $A := \{z \in \mathbb{C} : \, |\Phi(z)| < 1\}$, $B := \{z \in \mathbb{C} : \, |\Phi(z)| > 1\}$ are nonempty. It follows from Lemma 2.2 that the set $\text{span}\{e_a : \, a \in A\}$ is dense in $H(\Omega)$.

In particular, we derive the existence of a function $h_1 \in \operatorname{span}\{e_a : a \in A\}$ satisfying

$$|h(z) - h_1(z)| < \frac{\varepsilon}{4M_1} \quad (z \in L_1), \tag{4}$$

where $M_1 = M(L_1, a_1, \dots, a_{m_1}, A)$ is the positive constant provided by Lemma 2.1.

Observe that if $a \in A$ then $|\Phi(a)| < 1$. Since $\Phi(D)e_c = \Phi(c)e_c$ for all $c \in \mathbb{C}$, we have $\Phi(D)^n e_c = \Phi(c)^n e_c$ for every $n \in \mathbb{N}$. Hence $\Phi(D)^n e_a \to 0$ $(n \to \infty)$ compactly in $\mathbb{C}$ whenever $a \in A$ and, consequently,

$$\Phi(D)^n \alpha \to 0 \;\; (n \to \infty) \;\; \text{compactly for every } \alpha \in \operatorname{span}\{e_a : a \in A\}. \tag{5}$$

In particular, there is $N_1 \in \mathbb{N}$ such that

$$|(\Phi(D)^n h_1)(z)| < \frac{\varepsilon}{6} \quad (z \in L_1,\, n \geq N_1). \tag{6}$$

Again by Lemma 2.2, we can find a function $\psi \in \operatorname{span}\{e_b : b \in B\}$ such that

$$|\psi(z) - \varphi(z)| < \frac{\varepsilon}{6} \quad (z \in L_1). \tag{7}$$

The function ψ has the form $\psi = \sum_{i=1}^{p} c_i e_{b_i}$, where $p \in \mathbb{N}$, $c_1, \dots, c_p$ are constants and $\{b_1, \dots, b_p\} \subset B$. Then $|\Phi(b_i)| > 1$ $(i = 1, \dots, p)$. Therefore the functions

$$\varphi_n := \sum_{i=1}^{p} \frac{c_i}{(\Phi(b_i))^n} e_{b_i} \quad (n \in \mathbb{N})$$

satisfy

$$\Phi(D)^n \varphi_n = \psi \quad (n \in \mathbb{N}) \tag{8}$$

and

$$\varphi_n \longrightarrow 0 \quad (n \to \infty) \quad \text{compactly in } \mathbb{C}. \tag{9}$$

Property (9) yields specially the existence of a positive integer N_2 such that

$$|\varphi_n(z)| < \frac{\varepsilon}{4M_1} \quad (z \in L_1,\, n \geq N_2). \tag{10}$$

From (4), (10) and the triangle inequality we get

$$|h(z) - (\varphi_n(z) + h_1(z))| < \frac{\varepsilon}{2M_1} \quad (z \in L_1,\, n \geq N_2). \tag{11}$$

Moreover, (10) implies that there is a constant $C \in (0, +\infty)$ for which

$$|\varphi_n(z)| \leq C \quad (z \in L_1,\, n \in \mathbb{N}). \tag{12}$$

Consider the function $\alpha_j \in \operatorname{span}\{e_a : a \in A\}$ $(j = 1, \ldots, m_1)$ furnished by Lemma 2.1 and associated to $A, a_1, \ldots, a_{m_1}$. By (5), there exists a number $N_3 \in \mathbb{N}$ satisfying for each $j \in \{1, \ldots, m_1\}$ that

$$|(\Phi(D)^n \alpha_j)(z)| < \frac{\varepsilon}{6(C + \sup_{L_1} |h| + \sup_{L_1} |h_1|)} \quad (z \in L_1,\, n \geq N_3). \tag{13}$$

Define $N := \max\{N_1, N_1, N_3\} \in \mathbb{N}$ and fix a number $n \geq N$. According to Lemma 2.1 and (11), the function

$$g_1 = g_{1,n} := \varphi_n + h_1 + \sum_{j=1}^{m_1} [h(a_j) - (\varphi_n(a_j) + h_1(a_j))]\alpha_j \tag{14}$$

enjoys the following properties:

$$\sup_{z \in L_1} |g_1(z) - h(z)| < \frac{\varepsilon}{2} \tag{15}$$

and

$$g_1(a_j) = h(a_j) = b_j \quad (j = 1, \ldots, m_1). \tag{16}$$

In addition, we obtain from (6), (7), (8), (13), (14), the linearity of $\Phi(D)^n$ and the triangle inequality that

$$\sup_{z \in L_1} |(\Phi(D)^n g_1)(z) - \varphi(z)| < \frac{\varepsilon}{2}. \tag{17}$$

Now, we mimic the approach in [9] and follow on by constructing adequate functions $g_2 = g_{2,n}$, $g_3 = g_{3,n}, \ldots$. Since n has been fixed and we do not need the help of the functions φ_n in the remainder of the proof, we may dispense with the subindex n from now on.

Let $k \geq 1$ and suppose that g_k has been constructed. Recall that $L \subset L_1^0$. According to Lemma 2.3, there exists a constant $C_0 = C_0(L_1, L) \in (0, +\infty)$ such that

$$\sup_{z \in L} |(\Phi(D)^n f)(z)| \leq C_0 \sup_{z \in L_1} |f(z)| \quad (f \in H(\Omega)). \tag{18}$$

Consider the constant $M_{k+1} = M(K_k, a_{m_k+1}, \dots, a_{m_{k+1}}, A) \in (0, +\infty)$ provided by Lemma 2.1 (at this point, the corresponding simpler lemma in [9] may also be invoked), where $K_k := L_k \cup \{a_{m_k+1}, \dots, a_{m_{k+1}}\}$. Since the complement of this compact set is connected and the function $F_k : K_k \to \mathbb{C}$ given by

$$F_k(z) = \begin{cases} g_k(z) & \text{if} \quad z \in L_k \\ h(z) & \text{if} \quad z \in \{a_{m_k+1}, \dots, a_{m_{k+1}}\} \end{cases}$$

is holomorphic in a neighborhood of K_k, an application of Runge's approximation theorem (see [18]) provides a polynomial h_{k+1} such that

$$|h_{k+1}(z) - F_k(z)| < \frac{\varepsilon}{2^{k+1} M_{k+1}(1 + C_0)} \quad (z \in K_k).$$

It follows from Lemma 2.1 the existence of a function $g_{k+1} \in H(\Omega)$ satisfying

$$|g_{k+1}(z) - g_k(z)| < \frac{\varepsilon}{2^{k+1}(1 + C_0)} < \frac{\varepsilon}{2^{k+1}} < \frac{1}{2^{k+1}} \quad (z \in L_k) \tag{19}$$

and

$$g_{k+1}(a_j) = F_k(a_j) = h(a_j) = b_j \quad (j = 1, \dots, m_{k+1}). \tag{20}$$

To conclude (20), observe that we have inductively (from (16)) that $g_k(a_j) = b_j$ $(j = 1, \dots, m_k)$.

Since the sequence $\{L_k\}_{k \geq 1}$ of compact sets of Ω is exhaustive, each compact set $S \subset \Omega$ is contained in all L_k $(k \geq k_0)$ for some $k_0 \in \mathbb{N}$. But, by (19), the series

$$g_{k_0} + \sum_{k=k_0}^{\infty} (g_{k+1} - g_k)$$

converges uniformly on S. Therefore, the sequence $\{g_k\}_{k\geq 1}$ converges compactly to a function $g : \Omega \to \mathbb{C}$. By the Weierstrass convergence theorem, $g \in H(\Omega)$. Let us prove that g is the desired function.

Firstly, fix $j \in \mathbb{N}$. Then there is $k_0 \in \mathbb{N}$ such that $m_k > j$ for every $k \geq k_0$. According to (20), $g_k(a_j) = b_j$ $(k \geq k_0)$, so $g(a_j) = \lim_{k\to\infty} g_k(a_j) = b_j$. Hence $g \in \Gamma$.

Secondly, note that, in particular,

$$g = g_1 + \sum_{k=1}^{\infty} (g_{k+1} - g_k). \tag{21}$$

Thanks to (15) and (19), we get for $z \in L$ ($\subset L_k$ for all k) that

$$|g(z) - h(z)| \leq |g_1(z) - h(z)| + \sum_{k=1}^{\infty} |g_{k+1}(z) - g_k(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which together with the fact $g \in \Gamma$ yields

$$g \in U.$$

Finally, by joining (18) to the first inequality in (19) we are driven to

$$|(\Phi(D)^n (g_{k+1} - g_k))(z)| < \frac{\varepsilon}{2^{k+1}} \quad (z \in L), \tag{22}$$

where the fact $L \subset L_k$ has been used once more. Now, by (17), (21), (22), the linearity of $\Phi(D)^n$ and the triangle inequality, we derive

$$|(\Phi(D)^n g)(z) - \varphi(z)| < \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon \quad (z \in L).$$

This allows to conclude

$$T_n g = \Phi(D)^n g \in V.$$

Summarizing, the function g fulfils (3) and the proof is finished. □

Remark 3.2. Observe that the result by Costakis–Vlachou and Niess is the special case $\Phi(z) = z$. Moreover, Theorem 3.1 includes, in particular, all nonscalar differential polynomials $\Phi(D) = c_0 + c_1 D + \cdots + c_N D^N$. Note also that our theorem is *sharp*, at least in terms of order and type of growth. Indeed, if we allow Φ to be of exponential type then the conclusion of Theorem 3.1 is no longer necessarily true, even though $\Phi(D)$ is well-defined. For instance, take $\Omega = \mathbb{C}$, $\Phi(z) = \exp z$, $a_n = n$, $b_n = 0$ $(n \in \mathbb{N})$. Then $\Phi(D)$ is the 1-translation operator τ_1 and there is no interpolating τ_1-hypercyclic entire function f, because, if this were the case, the set $\{f(n)\}_{n\geq 1}$ would have to be dense in $\mathbb{C}$, which is clearly false. This is the reason why some condition must be imposed on the points a_n in order to guarantee Birkhoff-hypercyclicity, namely, for every $N \in \mathbb{N}$, there exist infinitely many numbers $n \in \mathbb{N}$ with $\{a_n\}_{n\geq 1} \cap \{z : \, |z - n| \leq N\} = \emptyset$ (see [9] and [17]).

If the interpolation values b_n are 0 $(n \in \mathbb{N})$, we evidently have that the set $\Gamma_0 = \Gamma_0(\{a_n\}_{n\geq 1}) := \{f \in H(\Omega) : \, f(a_n) = 0 \text{ for all } n \in \mathbb{N}\}$ is a vector subspace of $H(\Omega)$. We will consider in Γ_0 the topology inherited from $H(\Omega)$. Hence Theorem 3.1 yields the following consequence, that puts the end to this paper.

Corollary 3.3. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, $\{a_n\}_{n\geq 1} \in \omega(\Omega)$ and Φ be a nonconstant entire function with subexponential type. Then there exists an infinite-dimensional linear submanifold $\mathcal{D}$ of $H(\Omega)$ such that each function $f \in \mathcal{D} \setminus \{0\}$ is $\Phi(D)$-hypercyclic and satisfies $f(a_n) = 0$ for all $n \in \mathbb{N}$.*

Proof. Let $Y := H(\Omega)$. Then Y is metrizable and separable, so $X := \Gamma_0$ is also a metrizable separable topological vector space. Since Y is complete and X is closed in Y, we obtain that X is complete, so Baire. By Theorem

3.1, the sequence $T_n : \Gamma_0 \to H(\Omega)$ ($n \in \mathbb{N}$) defined as $T_n = \Phi(D)^n|_{\Gamma_0}$ is topologically mixing. Then an application of Lemma 2.4 yields the existence of a dense linear submanifold $\mathcal{D}$ of Γ_0 such that $\mathcal{D} \setminus \{0\} \subset \mathcal{U}(\{T_n\}_{n \geq 1})$. This is the desired submanifold. Indeed, Γ_0 is infinite-dimensional (here is an easy argument: if $f \in \Gamma_0$, then all functions $z \mapsto z^n f(z)$, $n \in \mathbb{N}$, belong to Γ_0 and are linearly independent), so its dense subspace $\mathcal{D}$ must also be infinite-dimensional. $\square$

References

[1] C.A. Berenstein, R. Gay, Complex Analysis and Special Topics in harmonic Analysis, Springer, New York, 1995.

[2] L. Bernal-González, Hypercyclic sequences of differential and antidifferential operators, J. Approx. Theory 96 (1999) 323–337.

[3] L. Bernal-González, Densely hereditarily hypercyclic sequences and large hypercyclic manifolds, Proc. Amer. Math. Soc. 127 (1999) 3279–3285.

[4] L. Bernal-González, A. Bonilla, M.C. Calderón, J.A. Prado-Bassas, Maximal cluster sets of L-analytic functions on arbitrary curves, Constr. Approx. 25 (2007) 211–219.

[5] L. Bernal-González, J.A. Prado-Tendero, Sequences of differential operators: exponentials, hypercyclicity and equicontinuity, Ann. Polon. Math. 77 (2001) 169–187.

[6] G.D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris 189 (1929) 473–475.

[7] J. Bonet, F. Martínez-Giménez, A. Peris, Linear chaos on Fréchet spaces, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (2003) 1649–1655.

[8] G. Costakis, Zeros and interpolation by universal Taylor series on simply connected domains, Math. Proc. Camb. Phil. Soc. 139 (2005) 149–160.

[9] G. Costakis, V. Vlachou, Interpolation by universal-hypercyclic functions, Preprint.

[10] G. Godefroy, J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991) 229–269.

[11] S. Grivaux, Hypercyclic operators, mixing operators, and the Bounded Steps Problem, J. Oper. Theory 54 (2005) 147–168.

[12] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (1999) 345–381.

[13] K.G. Grosse-Erdmann, Recent developments in hypercyclicity, Rev. R. Acad. Cien. Serie A. Mat. 97 (2003) 273–286.

[14] S. Kolyada, L. Snoha, Some aspects of topological transitivity – A survey, Grazer Math. Ber., Bericht Nr. 334 (1997) 3–35.

[15] G.R. MacLane, Sequences of derivatives and normal families, J. Anal. Math. 2 (1952) 72–87.

[16] M. Niess, MacLane functions with prescribed zeros and interpolation properties, Analysis 27 (2007), 323–332.

[17] M. Niess, Generic approach to multiply universal functions, Complex Var. Elliptic Equ., to appear.

[18] W. Rudin, Real and Complex Analysis, 3rd edition, McGraw-Hill, New York-St. Louis-San Francisco, 1987.