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# Optimal domain for the Hardy operator*

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## Introduction

In the 1920's, Hardy proved what today is one of the classical inequalities on integration:

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(s)\,ds\right)^p dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^\infty f(x)^p dx\right)^{1/p}$$

for $1<p<\infty$ and $f$ any positive measurable real function. This inequality has been studied by a lot of mathematicians producing a great variety of Hardy type inequalities which apply in different areas of mathematics.
Considering the average of functions as an operator $S$, i.e.

$$Sf(x) := \frac{1}{x}\int_0^x f(s)\,ds, \quad x>0,$$

for $f\in L^1_{loc}\big([0,\infty)\big)$, the Hardy's inequality establishes that $S\colon L^p\big([0,\infty)\big)\to L^p\big([0,\infty)\big)$ is well defined and continuous. This $S$ is called the ***Hardy operator***.

- Is there any space $Y$ larger than $L^p\big([0,\infty)\big)$ such that $S\colon Y\to L^p\big([0,\infty)\big)$?
- Which is the largest space $Y$ such that $S\colon Y\to L^p\big([0,\infty)\big)$?

## The general case

Let $X$ be a *Banach function space* (B.f.s.), i.e. a Banach space of measurable real functions on $[0,\infty)$, satisfying

$$|f|\le|g| \text{ a.e. and } g\in X \;\Rightarrow\; f\in X \text{ and } \|f\|_X\le\|g\|_X\,.$$

- Is there any B.f.s. $Y$ such that $S\colon Y\to X$?
- Which is the largest B.f.s. $Y$ such that $S\colon Y\to X$?

Note that if $S\colon Y\to X$ is well defined is automatically continuous, since it is a positive operator between Banach lattices.

## Optimal domain for $S$

The space

$$[S,X] := \{\, f\colon[0,\infty)\to\mathbb{R} \text{ measurable: } S|f|\in X\,\}$$

is a B.f.s. with the norm $\|f\|_{[S,X]}=\|S|f|\|_X$. Moreover,

- $S\colon[S,X]\to X$,
- $[S,X]$ is the largest B.f.s. $Y$ satisfying $S\colon Y\to X$.

So, $[S,X]$ is the ***optimal domain*** for $S$ considered with values in $X$.

Is it possible to give a precise description for $[S,X]$?

**Remark.** For a positive kernel operator $T$ satisfying some suitable conditions, it is possible to describe $[T,X]$ in terms of interpolation spaces (see [1] and [2]). Unfortunately, the Hardy operator $S$ does not satisfies these conditions. However, we can describe $[S,X]$ for some particulars $X$.

[1] G. P. Curbera and W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.
[2] O. Delgado, *Optimal domains for kernel operators on* $[0,\infty)\times[0,\infty)$, Studia Math. **174** (2006), 131–145.

## Particular cases of $[S,X]$

- $\big[S,L^1\big([0,\infty)\big)\big]=\{0\}$.
- $\big[S,L^{1,\infty}\big([0,\infty)\big)\big]=L^1\big([0,\infty)\big)$, for the quasi-B.f.s.

$$L^{1,\infty}\big([0,\infty)\big)=\Big\{f\colon[0,\infty)\to\mathbb{R} \text{ measurable}: \sup_{t\ge0}\big\{tf^*(t)\big\}<\infty\Big\},$$

where $f^*$ is the decreasing rearrangement of $f$, i.e.

$$f^*(t)=\inf\{c>0:\ m\big(\{x\in[0,\infty]:\ |f(x)|>c\}\big)\le t\}$$

for $m$ being the Lebesgue measure.

- Let $\varphi\colon[0,\infty)\to[0,\infty)$ be an increasing concave function with $\varphi(0)=0$ and consider the Lorentz space

$$\Lambda_\varphi=\Big\{f\colon[0,\infty)\to\mathbb{R} \text{ measurable}: \int_0^\infty f^*(t)\,d\varphi(t)<\infty\Big\}.$$

**Theorem.** Suppose that $\varphi$ satisfies
✓ $\varphi(0^+)=0$,
✓ $\theta_\varphi(t):=\int_t^\infty\frac{\varphi'(s)}{s}\,ds<\infty$ for all $t>0$,
✓ $\exists\, C>0$ such that $\frac{\varphi(t)}{t}\le C\,\theta_\varphi(t)$ for all $t>0$.
Then,

$$[S,\Lambda_\varphi]=L^1_{\theta_\varphi},$$

where $L^1_{\theta_\varphi}$ denotes the space of integrable functions with respect to the Lebesgue measure with density $\theta_\varphi$.

In particular

$$\big[S,L^{p,1}\big([0,\infty)\big)\big]=L^1_{t^{1/p-1}}, \quad \text{for } 1<p<\infty,$$

since $L^{p,1}\big([0,\infty)\big)=\Lambda_\varphi$ for $\varphi(t)=t^{1/p}$ and $\theta_\varphi(t)=\frac{1}{(p-1)}\frac{1}{t^{1-1/p}}$.

## $[S,X]$ may not be r.i.

Recall that a B.f.s. $X$ is *rearrangement invariant* (r.i.), if whenever $f$ and $g$ have the same distribution function and $f\in X$ it follows that $g\in X$.

**Theorem.** If $X$ is an r.i. B.f.s. and $S\colon X\to X$, then

$$[S,X] \text{ is not r.i.}$$

In particular $X\subsetneq[S,X]$.

Note that every r.i. B.f.s. $X$ satisfies $L^1\cap L^\infty\subset X\subset L^1+L^\infty$. Then, the claim of the above theorem follows, since $[S,X]\not\subseteq L^1+L^\infty$. A function $g\in L^1+L^\infty\backslash[S,X]$ is given by

$$g(t)=\begin{cases} h'\big(t-(t_{k+1}-1)\big) & \text{if } t\in(t_{k+1}-1,t_{k+1})\\ 0 & \text{in other case,}\end{cases}$$

where $t_{k+1}=e^k$, $h(t)=1-\sqrt{1-t}$ for $t\in[0,1]$ and $h'$ is the derivative of $h$.

From the Hardy inequality it follows that $S\colon L^p\big([0,\infty)\big)\to L^p\big([0,\infty)\big)$ for $1<p<\infty$. Then $\big[S,L^p\big([0,\infty)\big)\big]$ is not r.i. and $L^p\big([0,\infty)\big)\subsetneq\big[S,L^p\big([0,\infty)\big)\big]$. This holds also for $p=\infty$.

## R.I. optimal domain for $S$

Since $[S,X]$ may not be r.i., now a natural question is:

Which is the largest r.i. B.f.s. contained in $[S,X]$?

The space

$$[S,X]_{r.i.}:=\{f\colon[0,\infty)\to\mathbb{R} \text{ measurable}:\ Sf^*\in X\}$$

is an r.i. B.f.s. with the norm $\|f\|_{[S,X]_{r.i.}}=\|Sf^*\|_X$. Moreover,

- $[S,X]_{r.i.}\subset[S,X]$, since $S|f|\le Sf^*$,
- $[S,X]_{r.i.}$ is the largest r.i. B.f.s. contained in $[S,X]$.

So, $[S,X]$ is the ***r.i. optimal domain*** for $S$ considered with values in $X$.

Is it possible to give a precise description for $[S,X]_{r.i.}$?

## Particular cases of $[S,X]_{r.i.}$

- $\big[S,L^1\big([0,\infty)\big)\big]_{r.i.}=\{0\}$.
- $\big[S,L^{1,\infty}\big([0,\infty)\big)\big]_{r.i.}=L^1\big([0,\infty)\big)$.

Consider the Lorentz space $X=\Lambda_\varphi$ and $\theta_\varphi(s)=\int_s^\infty\frac{\varphi'(y)}{y}\,dy$. Then,

- $[S,\Lambda_\varphi]_{r.i.}=\Lambda_{\int_0^t\theta_\varphi(s)\,ds}$, whenever $\varphi(0^+)=0$,
- $[S,\Lambda_\varphi]_{r.i.}=L^\infty\cap\Lambda_{\int_0^t\theta_\varphi(s)\,ds}$, whenever $\varphi(0^+)>0$.

## When does $X$ coincide with $[S,X]_{r.i.}$?

For every r.i. B.f.s. $X$, since $f^*\le Sf^*$, it follows that $[S,X]_{r.i.}\subset X$. If $X$ also satisfies $S\colon X\to X$, then

$$[S,X]_{r.i.}\subset X\subset[S,X].$$

So, since $[S,X]_{r.i.}$ is the largest r.i. B.f.s. contained in $[S,X]$, we have that $[S,X]_{r.i.}=X$.

Conversely, suppose $[S,X]_{r.i.}=X$. Then $X$ is obviously r.i. and, since $|Sf|\le S|f|\le Sf^*$, it follows that $S\colon X\to X$. Therefore, the following proposition holds.

**Proposition.** $X$ r.i. and $S\colon X\to X \;\Leftrightarrow\; [S,X]_{r.i.}=X$

In particular,

$$\big[S,L^p\big([0,\infty)\big)\big]_{r.i.}=L^p\big([0,\infty)\big)$$

for all $1<p\le\infty$.

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