



Frequently hypercyclic weighted backward shifts on spaces of real analytic functions

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Abstract: We study frequent hypercyclicity in the case of weighted backward shift operators acting on locally convex spaces of real analytic functions. We obtain certain conditions on frequent hypercyclicity and linear chaoticity of these operators using dynamical transference principles and the frequent hypercyclicity criterion.

Key words: Frequent hypercyclicity, linear chaos, spaces of real analytic functions, weighted backward shifts

1. Introduction

Let $A(\Omega)$ denote the space of real analytic functions on a nonempty open subset Ω of \mathbb{R} , equipped with its natural topology of inductive limit of the spaces $H(U)$ of holomorphic functions on U , where U runs through all complex open neighborhoods of Ω .

Although the spaces $H(U)$ are Fréchet with the topology of uniform convergence on compact subsets, and monomials form a (Schauder) basis for $H(U)$, the space $A(\Omega)$ is not metrizable and does not admit a Schauder basis [9]. This means that we neither have the Baire category theorem nor a sequential representation for $A(\Omega)$ at our disposal, which are important tools in linear dynamics. Due to these obstacles, we define weighted backward shifts on $A(\Omega)$ as in [8] by considering their action on monomials using the density of polynomials in $A(\Omega)$. For more information on the topology of $A(\Omega)$, we refer the reader to the survey of [7]. For the functional analytic tools that we use, we refer the reader to the monograph [12].

Definition 1.1 A continuous linear operator $B_w : A(\Omega) \rightarrow A(\Omega)$ is called a weighted backward shift if

$$B_w(\mathbf{1}) = 0, \quad B_w(x^n) = w_{n-1}x^{n-1}, \quad n \geq 1,$$

where $\mathbf{1}$ denotes the constant unit function, and $w = (w_n)_n$ is a sequence of nonzero scalars, called a weight sequence.

In this setting, an immediate question is for which sequences w are the corresponding weighted backward shifts B_w well defined and continuous. In the case of Ω containing 0, we have the following characterization [8, Theorem 2.2].

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Theorem 1.2 *Let Ω be an open set in \mathbb{R} containing 0, and let $w = (w_n)$ be a sequence of scalars. For any linear map $B : A(\Omega) \rightarrow A(\Omega)$, the following are equivalent:*

(a) *B is the weighted backward shift B_w with weight sequence w .*

(b) *There is an analytic functional T with support contained in*

$$V(\Omega) := \{x \in \mathbb{R} \mid xy \in \Omega \text{ for every } y \in \Omega\}$$

such that $\langle x^n, T \rangle = w_n$ for every $n \in \mathbb{N}$, and

$$B(f)(x) = B_T(f)(x) := \left\langle \frac{f(xy) - f(0)}{xy}, T_y \right\rangle \text{ for every } f \in A(\Omega).$$

(c) *There is a function τ holomorphic on the Riemann sphere outside some compact set $K \subset V(\Omega) \subset \mathbb{R}$ with Laurent series representation $\tau(z) = \sum_{n=0}^{\infty} w_n/z^{n+1}$ and*

$$B(f)(x) = B_\tau(f)(x) := \frac{1}{2\pi i} \int_\gamma \frac{\tau(w)(f(xw) - f(0))}{xw} dw$$

for a suitably chosen curve γ .

(d) *B maps a function $f(z) = \sum_{n=0}^{\infty} f_n z^n$ around zero into a real analytic function on Ω represented around zero by the series $\sum_{n=1}^{\infty} f_n w_{n-1} z^{n-1}$.*

We recall that a continuous linear operator T on a topological vector space X is *hypercyclic* if there is a vector x in X such that the set $\{T^n x : n \in \mathbb{N}\}$, called the *orbit* of x under T , is dense in X . If X is a Fréchet space, then the hypercyclicity of T is equivalent to T being *topologically transitive*; that is, for any pair U, V of nonempty open subsets of X , there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$.

In the case of general topological vector spaces, any hypercyclic operator is topologically transitive; however, the converse may no longer hold true. From this viewpoint, an operator T on a topological vector space X is defined to be *chaotic* if T is topologically transitive, and has a dense set of periodic points [5]. For more on the dynamics of linear operators we refer the reader to the monographs [3] and [10].

Starting with the work of Rolewicz [14], backward shifts have been important examples in the theory of dynamics of linear operators. The dynamical properties of weighted backward shifts on Fréchet sequence spaces have been studied extensively (see [10, Section 4.1]). Certain conditions on hypercyclicity and chaoticity of weighted backward shifts on spaces of real analytic functions were given in [8].

In this paper we focus on the linear dynamical property called *frequent hypercyclicity*, introduced by Bayart and Grivaux [1], for which an orbit meets every nonempty open set frequently in terms of positive lower density. We recall that the *lower density* of a subset A of \mathbb{N}_0 is defined as

$$\underline{\text{dens}}(A) = \liminf_{N \rightarrow \infty} \frac{\text{card}\{n \in A : 0 \leq n \leq N\}}{N + 1}.$$

Definition 1.3 *An operator T on a topological vector space X is called frequently hypercyclic if there is a vector $x \in X$ such that for every nonempty open subset U of X ,*

$$\underline{\text{dens}}\{n \in \mathbb{N}_0 : T^n x \in U\} > 0.$$

In Section 2 we obtain some necessary and some sufficient conditions for frequent hypercyclicity of weighted backward shifts acting on $A(\Omega)$ by using dynamical transference principles, and a construction due to Bonnet. In Section 3 we adapt the so-called frequent hypercyclicity criterion to the nonmetrizable case to obtain alternative sufficient conditions for frequent hypercyclicity of weighted backward shifts on $A(\Omega)$. In Section 4 we consider questions about relations between frequent hypercyclicity and linear chaos in the case of weighted backward shifts on $A(\Omega)$.

2. Conditions on frequent hypercyclicity by transference principles

As a linear dynamical property, frequent hypercyclicity is *preserved under quasiconjugacy*, as stated in the next proposition [6, Theorem 3.4].

Proposition 2.1 *If X and Y are topological vector spaces, $S : Y \rightarrow Y$, $T : X \rightarrow X$ are linear continuous operators, and there is a continuous map $\phi : Y \rightarrow X$ with dense range such that $T \circ \phi = \phi \circ S$, (i.e. T is quasiconjugate to S), then the frequent hypercyclicity of S implies the frequent hypercyclicity of T .*

For weighted backward shifts B_w acting on $A(\Omega)$, we obtain two useful quasiconjugacies due to the following proposition [8, Proposition 2.4].

Proposition 2.2 *Let Ω be an open subset of \mathbb{R} containing 0, and $B_w : A(\Omega) \rightarrow A(\Omega)$ be a weighted backward shift. Then B_w acts as a weighted backward shift continuously on the space $H(\mathbb{C})$ of entire functions, and on the space $H(\{0\})$ of germs of holomorphic functions at zero.*

Since $H(\mathbb{C})$ is dense in $A(\Omega)$, and $A(\Omega)$ is dense in $H(\{0\})$, Propositions 2.1 and 2.2 together imply the following.

Proposition 2.3 *Let B_w be a weighted backward shift on $A(\Omega)$, where Ω is an open set in \mathbb{R} containing 0.*

- (i) *If B_w acting on $H(\mathbb{C})$ is frequently hypercyclic, then B_w is frequently hypercyclic.*
- (ii) *If B_w is frequently hypercyclic, then B_w acting on $H(\{0\})$ is frequently hypercyclic.*

Using the characterization for frequent hypercyclicity of weighted backward shifts on $H(\mathbb{C})$ [10, Example 9.16(a)], we immediately obtain the following sufficient condition for frequent hypercyclicity in the case of weighted backward shifts acting on $A(\Omega)$. Note that this condition also characterizes the chaoticity of weighted backward shifts acting on $H(\mathbb{C})$.

Theorem 2.4 *Let Ω be an open subset of \mathbb{R} containing 0, and $B_w : A(\Omega) \rightarrow A(\Omega)$ be a weighted backward shift with a weight sequence $w = (w_n)_n$ of nonzero scalars. If*

$$\lim_{n \rightarrow \infty} \left(\prod_{\nu=1}^n |w_{\nu-1}| \right)^{1/n} = \infty, \quad (2.1)$$

then B_w is frequently hypercyclic.

An alternative way to state condition (2.1) in Theorem 2.4, as in [8, Proposition 3.2], is that for every $R > 0$,

$$\lim_{n \rightarrow \infty} \left(\prod_{\nu=1}^n w_{\nu-1} \right)^{-1} R^n = 0.$$

An immediate example of a frequently hypercyclic weighted backward shift acting on $A(\Omega)$ is the differentiation operator that has the weight sequence $w = (n + 1)_n$.

In order to obtain a necessary condition for frequent hypercyclicity, we need to understand the frequent hypercyclicity of weighted backward shifts acting on $H(\{0\})$. In the case of Fréchet sequence spaces with unconditional basis, a necessary condition is given as follows [6, Proposition 4.7].

Proposition 2.5 *Let B_w be a weighted backward shift on a Fréchet sequence space X in which the canonical unit sequences $\{e_n : n \in \mathbb{N}\}$ are an unconditional basis. If B_w is frequently hypercyclic, then there exists a subset A of \mathbb{N}_0 with positive lower density such that*

$$\sum_{n \in A} \left(\prod_{\nu=1}^n w_{\nu} \right)^{-1} e_n$$

converges.

However, equipped with its natural inductive limit topology, $H(\{0\})$ is an (LF)-space [12, Example 24.37(2)]. By a construction due to Bonet (Bonet J, Dynamics of weighted backward shifts on coechelon spaces, unpublished manuscript, 2015), we may use a quasiconjugacy in a Fréchet space setting.

Theorem 2.6 *Let Ω be an open subset of \mathbb{R} containing 0, and $B_w : A(\Omega) \rightarrow A(\Omega)$ be a weighted backward shift with a weight sequence $w = (w_n)_n$ of nonzero scalars. If B_w is frequently hypercyclic, then there exists a subset A of \mathbb{N} with positive lower density and there exists $R > 0$ such that the series*

$$\sum_{n \in A} \left(\prod_{\nu=1}^n w_{\nu-1} \right)^{-1} R^n \tag{2.2}$$

converges.

Proof If we assume that $B_w : A(\Omega) \rightarrow A(\Omega)$ is frequently hypercyclic, then B_w acting on $H(\{0\})$ is also frequently hypercyclic by Proposition 2.3. By using the idea of Bonet (also see [8, Proposition 3.2]), we obtain a Köthe sequence space $X = \lambda^1((v^{(j)})_{j \in \mathbb{N}_0})$ with the sequence of seminorms $(\|\cdot\|_{v^{(j)}})_{j \in \mathbb{N}_0}$ chosen inductively starting from $v^{(0)}$, where

$$\|x\|_{v^{(j)}} = \sum_{n=1}^{\infty} |x_n v_n^{(j)}|$$

for all $j \in \mathbb{N}_0$ and $x = (x_n)_n$ in X , with each $v^{(j)} = (v_n^{(j)})_n$ satisfying

$$\sup_n \frac{v_n^{(j)}}{e^{-kn}} < \infty$$

for all $k \in \mathbb{N}$. By this construction, X is a Fréchet sequence space with a continuous dense map from $H(\{0\})$ to X , and B_w acts on X as a weighted backward shift. Thus, by Proposition 2.1, B_w acting on X is also frequently hypercyclic.

Considering the canonical unit sequences $(e_n)_{n \in \mathbb{N}}$, which form an absolute basis in X , Proposition 2.5 implies that there is a subset A of \mathbb{N} with positive lower density such that

$$\sum_{n \in A} \left(\prod_{\nu=1}^n w_\nu \right)^{-1} e_n \tag{2.3}$$

converges in X .

If we assume contrarily that the series (2.2) diverges for every $R > 0$ and for every $A \subset \mathbb{N}$ with positive lower density, then taking the set A as above so that the series (2.3) converges in X , the series

$$\sum_{n \in A} \left(\prod_{\nu=1}^n |w_{\nu-1}| \right)^{-1} R^n$$

diverges to infinity for every $R > 0$.

Denoting the set A as an increasing sequence $A = (n_l)_l$, for every $k \in \mathbb{N}$ we can find $n_{l_k} \in A$ such that for every $m \in A$ with $m \geq n_{l_k}$ we have

$$\sum_{l=1}^m \left(\prod_{\nu=1}^{n_l} |w_{\nu-1}| \right)^{-1} e^{-kn_l} \geq k.$$

Taking $v_n = e^{-kn}$ for $n_{l_k} \leq n \leq n_{l_{k+1}}$, we obtain a sequence $v = (v_n)_n$ with $\sup_n v_n / e^{-kn} < \infty$ for all $k \in \mathbb{N}_0$, which we may choose as $v^{(0)}$ in our construction of $X = \lambda^1((v^{(j)})_{j \in \mathbb{N}_0})$.

Since the canonical unit sequences $(e_n)_n$ form an absolute basis in X , for the continuous seminorm $\|\cdot\|_{v^{(0)}}$ on X there is a continuous seminorm q on X and $C > 0$ such that

$$\begin{aligned} \sum_{n \in A} \left(\prod_{\nu=1}^n |w_{\nu-1}| \right)^{-1} \|e_n\|_{v^{(0)}} &= \sum_{n \in A} \left(\prod_{\nu=1}^n |w_{\nu-1}| \right)^{-1} v_n \\ &\leq Cq \left(\sum_{n \in A} \left(\prod_{\nu=1}^n w_{\nu-1} \right)^{-1} e_n \right), \end{aligned}$$

which is a contradiction since the series $\sum_{n \in A} \left(\prod_{\nu=1}^n |w_{\nu-1}| \right)^{-1} v_n$ diverges to infinity. □

3. Frequent hypercyclicity criterion

A sufficient condition for frequent hypercyclicity of operators on separable Fréchet spaces is provided by the *frequent hypercyclicity criterion* [1, 6], which we state in the next proposition. We recall that a series $\sum_{n=1}^\infty x_n$ in a topological vector space is *unconditionally convergent* if for any bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ the series $\sum_{n=1}^\infty x_{\sigma(n)}$ converges.

Proposition 3.1 *Let T be an operator on a separable Fréchet space X . If there is a dense subset X_0 of X and a map $S : X_0 \rightarrow X_0$ such that for any $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S^n x$ converges unconditionally,

(iii) $TSx = x$,

then T is frequently hypercyclic.

Since $A(\Omega)$ is not metrizable, this criterion cannot be applied directly to operators acting on $A(\Omega)$. However, the frequent hypercyclicity criterion also works for operators on separable sequentially complete locally convex spaces due to the following characterization of unconditional convergence, as given in [11].

Proposition 3.2 *Let E be a sequentially complete locally convex space. Then the following conditions are equivalent:*

(i) *The series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent.*

(ii) *For any increasing sequence (n_k) of \mathbb{N} , $\sum_{k=1}^{\infty} x_{n_k}$ converges in E .*

(iii) *For every neighborhood V of 0 in E , there exists $N \in \mathbb{N}$ such that $\sum_{n \in F} x_n \in V$ for any finite set $F \subset \{N, N + 1, N + 2, \dots\}$.*

Using this characterization of unconditional convergence, we can adapt the proof of the frequent hypercyclicity criterion [10, Theorem 9.9] to obtain the following.

Theorem 3.3 *Let T be an operator on a separable sequentially complete locally convex space X . If there is a dense subset X_0 of X and a map $S : X_0 \rightarrow X_0$ such that for any $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S^n x$ converges unconditionally,

(iii) $TSx = x$,

then T is frequently hypercyclic.

Since $A(\Omega)$ is a complete separable locally convex space, Theorem 3.3 provides alternative sufficient conditions for frequent hypercyclicity of weighted backward shifts acting on $A(\Omega)$. The following is an example of a frequently hypercyclic backward shift on $A(\Omega)$ that does not satisfy the conditions of Theorem 2.4.

Example 3.4 *Let Ω be the interval $(-1, 1)$, and B be the unweighted backward shift acting on $A(\Omega)$ with the weight sequence $w = (w_n)$, where $w_n = 1$ for all n . We apply Theorem 3.3 by taking X_0 to be the set of all polynomials, which is dense in $A(\Omega)$, and $S : X_0 \rightarrow X_0$ to be the linear map with $S(x^n) = x^{n+1}$ for all $n \in \mathbb{N}_0$.*

For any polynomial $p(x)$, condition (i) holds since $\sum_{n=0}^{\infty} B^n p(x)$ is again a polynomial, and therefore unconditionally convergent. By computation, we can show that

$$\sum_{n=0}^{\infty} S^n p(x) = p(1) \left(\sum_{n=0}^{\infty} x^n \right) + q(x)$$

where $q(x)$ is a polynomial with degree less than the degree of $p(x)$. Hence, $\sum_{n=0}^{\infty} S^n p(x)$ converges unconditionally in $A(\Omega)$, and condition (ii) holds. Condition (iii) also holds by the definitions of B and S .

Therefore, Theorem 3.3 implies that $B : A(\Omega) \rightarrow A(\Omega)$ is frequently hypercyclic.

4. Frequent hypercyclicity and linear chaos

We note that any operator T on a separable sequentially complete locally convex space X that satisfies the frequent hypercyclicity criterion (Theorem 3.3) is also chaotic and mixing (that is, for any pair U, V of nonempty open subsets of X , there exists some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for every $n \geq N$), which can be shown by adapting the proof of [10, Proposition 9.11]. For example, the unweighted backward shift in Example 3.4 is also chaotic since it satisfies the frequent hypercyclicity criterion, and also due to [8, Corollary 4.11].

Bayart and Grivaux showed that there is a frequently hypercyclic operator on c_0 that is not chaotic [2]. Also, a weighted backward shift on ℓ_p is chaotic if and only if it is frequently hypercyclic, due to Bayart and Ruzsa [4]. A natural question thus arises in our setting.

Problem 4.1 *Is there a frequently hypercyclic weighted backward shift on $A(\Omega)$ that is not chaotic?*

Menet was able to construct a chaotic operator that is not frequently hypercyclic [13]. This leads to the following question.

Problem 4.2 *Is there a chaotic operator (weighted backward shift) on $A(\Omega)$ that is not frequently hypercyclic?*

These questions are also related to obtaining a complete characterization of frequent hypercyclicity for weighted backward shifts on $A(\Omega)$. Thus, we conclude with the following question.

Problem 4.3 *Is it possible to completely characterize frequently hypercyclic (or chaotic) weighted backward shifts on $A(\Omega)$?*

References

- [1] Bayart F, Grivaux S. Frequently hypercyclic operators. T Am Math Soc 2006; 358: 5083-5117.
- [2] Bayart F, Grivaux S. Invariant Gaussian measures for operators on Banach spaces and linear dynamics. P Lond Math Soc 2007; 94: 181-210.
- [3] Bayart F, Matheron E. Dynamics of Linear Operators. Cambridge, UK: Cambridge University Press, 2009.
- [4] Bayart F, Ruzsa IZ. Difference sets and frequently hypercyclic weighted shifts. Ergodic Theory Dynam Systems 2015; 35: 691-709.
- [5] Bonnet J. Hypercyclic and chaotic convolution operators. J Lond Math Soc 2000; 62: 253-262.

- [6] Bonilla A, Grosse-Erdmann KG. Frequent hypercyclic operators and vectors. *Ergodic Theory Dynam Systems* 2007; 27: 383-404.
- [7] Domański P. Notes on real analytic functions and classical operators. In: *Topics in Complex Analysis and Operator Theory, Winter School in Complex Analysis and Operator Theory*; February 2010; Valencia, Spain. Providence, RI, USA: AMS, 2012, pp. 3-47.
- [8] Domański P, Kariksız CD. Eigenvalues and dynamical properties of weighted backward shifts on the space of real analytic functions. *Studia Math* 2018; 242: 57-78.
- [9] Domański P, Vogt D. The space of real analytic functions has no basis. *Studia Math* 2000; 142: 187-200.
- [10] Grosse-Erdmann KG, Peris A. *Linear Chaos*. Berlin, Germany: Springer, 2011.
- [11] Junde W, Ronglu L. Unconditional convergent series on locally convex spaces. *Taiwanese J Math* 2000; 4: 253-259.
- [12] Meise R, Vogt D. *Introduction to Functional Analysis*. Oxford, UK: Clarendon Press, 1997.
- [13] Menet Q. Linear chaos and frequent hypercyclicity. *Trans Amer Math Soc* 2017; 369: 4977-4994.
- [14] Rolewicz S. On orbits of elements. *Studia Math* 1969; 32: 17-22.