Chapter 28
Discreteness, Periodicity, Holomorphy, and Factorization

V.B. Vasilyev

28.1 Introduction

The main topic of the paper is to establish some relations between the solvability of a special kind of discrete equations in certain canonical domains and holomorphy properties of their Fourier analogues. We start from the theory of pseudo-differential operators and equations [Ta81, Tr80, Sh01], and corresponding boundary value problems [Es81] and we shall try to construct a discrete analogue of this theory with forthcoming limit passage from discrete case to a continuous one.

Historically, the theory of pseudo-differential operators started from a special kind of integral operators, namely Calderon–Zygmund operators of the following type

$$(Ku)(x) = p.v. \int_{\mathbb{R}^m} K(x, x-y)u(y)dy, \qquad (28.1)$$

where the kernel $K(x, y)$ has certain specifical properties [MiPr86]. If we enlarge the class of such kernels and in particular we permit that the kernel $K(x, y)$ can be a distribution on a second variable y, then the above formula will include differential operators with variable coefficients

$$(\mathcal{D}u)(x) = \sum_{|k|=0}^{n} a_k(x) \frac{\partial^k u}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}}(x),$$

where k is a multi-index, $|k| = k_1 + \cdots k_m$.

V.B. Vasilyev (✉)
National Research Belgorod State University, Belgorod, Russia
e-mail: vbv57@inbox.ru

C. Constanda et al. (eds.), *Integral Methods in Science and Engineering, Volume 1*,
DOI 10.1007/978-3-319-59384-5_28

Indeed for this case we use the kernel

$$K(x, y) = \sum_{|k|=0}^{n} a_k(x)\delta^{k_1}(y_1)\cdots\delta^{k_m}(y_m).$$

A theory for similar operators consists in a description of functional spaces in which these operators are bounded, possible additional conditions (maybe boundary conditions) which permit to state the well-posedness of boundary value problem in a corresponding functional space and so on. Here, we would like to discuss these problems for a discrete situations and to describe some of our first results in this direction.

28.2 Discreteness

We consider functions u_d of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$, where $h > 0$ is a small parameter, and operators defined on such functions of the following type

$$(A_d u_d)(\tilde{x}) = a u_d(\tilde{x}) + \sum_{\tilde{y}\in D_d} A_d(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x} \in D_d, \tag{28.2}$$

taking partial sums of the series (28.2) over cubes

$$Q_N = \{\tilde{x} \in h\mathbb{Z}^m : \max_{1\le k\le m} |x_k| \le N\},$$

where we use following notations.

Let $D \subset \mathbb{R}^m$ be a domain, $D_d \equiv D \cap h\mathbb{Z}^m$ be a discrete set, A_d be a given function of a discrete variable defined on $h\mathbb{Z}^m$, and $a \in \mathbb{C}$. We say that the function $A_d(\tilde{x})$ is the kernel of the discrete operator A_d. This kernel may be summable, i.e. it can be generated by integrable function

$$\int_{\mathbb{R}^m} |A(x)|dx < +\infty,$$

but for this case we deal with ordinary convolution. The author has considered the more interesting and complicated case when the generating function $A(x)$ is a Calderon–Zygmund kernel [VaEtAl15-2, VaEtAl15-3]. These Calderon–Zygmund operators play an important role as the simplest model of a pseudo-differential operator [MiPr86, Es81]. Taking into account our forthcoming considerations of discrete pseudo-differential operators we shall restrict to this simplest case.

We assume that our generating function $A(x)$ is a Calderon–Zygmund kernel, i.e. it is homogeneous of order $-m$ and has vanishing mean value on unit sphere $S^{m-1} \subset \mathbb{R}^m$, also it is continuously differentiable out of the origin and by definition $A(0) = 0$.

The first question which arises in this situation is the following. Is there a certain dependence on a parameter h for a norm of the operator A_d ? Fortunately the answer is negative (see also [VaEtAl15-3] for the whole space $\mathbb{R}^m$).

Theorem 1 *Let D be a bounded domain in $\mathbb{R}^m$ with a Lipschitz boundary ∂D. Then the norm of the operator $A_d : L_2(D_d) \to L_2(D_d)$ doesn't depend on h.*

This property leaves us hope to describe the spectra of the operator A_d using methods developed for this purpose in a continuous case.

28.3 Periodicity

Roughly speaking the Fourier image of the lattice $h\mathbb{Z}^m$ is a periodic structure with basic cube of periods $\hbar\mathbb{T}^m$, where $\hbar = \frac{h^{-1}}{2\pi}$. More precisely if we introduce a discrete Fourier transform by the formula

$$\widetilde{u}_d(\xi) \equiv (F_d u_d)(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m \tag{28.3}$$

taking partial sums of the series (28.3) over cubes Q_N, we can use this construction to give a definition of a discrete pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \int_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi} \widetilde{A}_d(\xi) \widetilde{u}_d(\xi) d\xi,$$

where the function $\widetilde{A}_d(\xi)$ is called the symbol of the operator A_d.

Let us note that this discrete Fourier transform preserves all basic properties of standard Fourier transform. Only one principal distinction is periodicity of Fourier images.

Definition 1 The symbol $\widetilde{A}_d(\xi)$ is called an elliptic symbol (and operator A_d is called an elliptic one) if $\operatorname{ess}\inf_{\xi \in \hbar\mathbb{T}^m} |\widetilde{A}_d(\xi)| > 0$.

Proposition 1 *The operator $A_d : L_2(h\mathbb{Z}^m) \to L_2(h\mathbb{Z}^m)$ is invertible iff it is an elliptic operator.*

Many interesting properties of the operator A_d related to a comparison between continuous and discrete cases can be found in [VaEtAl15-2].

28.4 Holomorphy

The property of holomorphy arises if we try to obtain a Fourier image for a so-called paired equation

$$(A_d P_+ + B_d P_-)U_d = V_d, \tag{28.4}$$

where $P_\pm$ are projectors on some canonical domains (see below), A_d, B_d are discrete operators similar to (28.2).

If, for example, $P_\pm$ are projectors on discrete half-spaces $h\mathbb{Z}^m_\pm = \{\tilde{x} \in h\mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \pm\tilde{x}_m > 0\}$, and we want to use standard properties of the Fourier transform related to a convolution then, in order to find a Fourier image of the product $\chi_+(\tilde{x})U(\tilde{x})$ where χ_+ is an indicator of the $h\mathbb{Z}^m_+$, we need to go out in a complex domain [VaEtAl15-1, VaEtAl15-3].

We introduce for fixed $\xi' = (\xi_1, \cdots, \xi_{m-1})$

$$\Pi_\pm = \{\xi_m \pm i\tau \in \mathbb{C} : \xi_m \in h^{-1}[-\pi, \pi], \tau > 0\}.$$

Theorem 2 *Let $H_\pm$ be subspaces of the space $L_2(\hbar\mathbb{T}^m)$ consisting of functions which admit holomorphic extensions into upper and lower complex half-strips $\Pi_\pm$ on a last variable ξ_m under almost all fixed $\xi' = (\xi_1, \cdots, \xi_{m-1})$. Then we have the following decomposition*

$$L_2(\hbar\mathbb{T}^m) = H_+ \oplus H_-.$$

Indeed the decomposition is given by the following operators

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi_m) = \frac{1}{2\pi i} p.v. \int\limits_{-\pi h^{-1}}^{\pi h^{-1}} \tilde{u}_d(\xi', t) \cot\frac{h(t-\xi_m)}{2} dt,$$

$$P^{per}_{\xi'} = 1/2(I + H^{per}_{\xi'}),\ Q^{per}_{\xi'} = 1/2(I - H^{per}_{\xi'}),$$

so that

$$F_d P_+ u_d = P^{per}_{\xi'}\widetilde{u}_d, \quad F_d P_- u_d = Q^{per}_{\xi'}\widetilde{u}_d.$$

28.5 Factorization

The concept of factorization is needed if we consider an original equation in a non-whole lattice, i.e. $D \neq \mathbb{R}^m$. We extract from $\mathbb{R}^m$ some so-called canonical domains. The fact that to obtain Fredholm conditions for an elliptic operator (or equation) on a manifold, and in particular in a domain of m-dimensional space, we need to obtain an invertibility conditions for a local representative of original operator, is called a local principle [MiPr86, Va00]. Roughly speaking such local representatives are simple model operators in canonical domains. If for example we are interested in studying a Calderon–Zygmund operator on a manifold with a boundary of the type (28.1), we need to describe invertibility conditions for the following model operators in the following canonical domains:

- for inner points x_0 of a manifold

$$u(x) \longmapsto p.v. \int\limits_{\mathbb{R}^m} K(x_0, x-y)u(y)dy,$$

- for boundary points x_0 on smooth parts of a boundary

$$u(x) \longmapsto p.v. \int\limits_{\mathbb{R}^m_+} K(x_0, x-y)u(y)dy,$$

  where $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}$,
- for boundary points x_0 for which their neighborhood is diffeomorphic to $C^a_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x' = (x_1, \cdots, x_{m-1}), x_m > a|x'|, a > 0\}$

$$u(x) \longmapsto p.v. \int\limits_{C^a_+} K(x_0, x-y)u(y)dy.$$

It is natural to expect similar properties for general discrete operators. That's why we consider here the simplest model operators in cones. So we have the following canonical domains: $\mathbb{R}^m$, $\mathbb{R}^m_+$, C^a_+. It is essential that all these domains are cones but the first two include a whole straight line.

The case $D = \mathbb{R}^m$ is very simple (from modern point of view; there was a lot of mathematicians whose papers have helped us to clarify this situation). If a symbol $\widetilde{A}_d(\xi)$ of the operator A_d from (28.2) is elliptic, then such operator A_d is invertible at least in the space $L_2(h\mathbb{Z}^m)$. We apply the discrete Fourier transform (28.3) and obtain immediately that the operator A_d is unitary equivalent to a multiplication operator on its symbol.

We proceed to describe the half-space case. First we recall the following definition.

Definition 2 Factorization of an elliptic symbol $\widetilde{A}_d(\xi)$ is called its representation in the form

$$\widetilde{A}_d(\xi) = \widetilde{A}^+_d(\xi) \cdot \widetilde{A}^-_d(\xi),$$

where the factors $\widetilde{A}^\pm_d(\xi)$ admit a bounded holomorphic continuation into upper and lower complex half-strips $\varPi_\pm$ for almost all $\xi' = (\xi_1, \cdots, \xi_{m-1}) \in \mathbb{T}^{m-1}$.

Now we come back to the equation (28.4), we apply the discrete Fourier transform F_d and we obtain the following equation

$$\frac{\widetilde{A}_d(\xi',\xi_m)+\widetilde{B}_d(\xi',\xi_m)}{2}\tilde{U}_d(\xi)+$$

$$\frac{\widetilde{A}_d(\xi',\xi_m)-\widetilde{B}_d(\xi',\xi_m)}{4\pi i}p.v.\int\limits_{-\pi h^{-1}}^{\pi h^{-1}}\tilde{U}_d(\xi',\eta)\cot\frac{h(\eta-\xi_m)}{2}d\eta=\tilde{V}_d(\xi). \qquad (28.5)$$

where $\widetilde{A}_d(\xi),\widetilde{B}_d(\xi)$ are symbols of discrete operators A_d, B_d. Of course, equation (28.5) is related to the corresponding Riemann boundary value problem [Ga81, Mu76, VaEtAl13, VaEtAl15-1, VaEtAl15-3], so the following result is valid.

Theorem 3 *For $m \geq 3$ the equation (28.5) is uniquely solvable in the space $L_2(h\mathbb{Z}^m)$ iff operators A_d, B_d are elliptic and*

$$Ind\,\widetilde{A}_d(\cdot,\xi_m)\widetilde{B}_d^{-1}(\cdot,\xi_m)=0.$$

The key role for a proof of the theorem is played by the concept of factorization for an elliptic symbol, and it can be constructed exactly by means of operator $H_{\xi'}^{per}$ [VaEtAl15-1, VaEtAl15-3].

28.5.1 Conical Case

This section is devoted to the last and most complicated case. Let $\chi_+(\tilde{x})$ be a characteristic function of the discrete cone D_d and $S_d(z)$ be the following function

$$S_d(z)=\sum_{\tilde{x}\in D_d}\chi_+(\tilde{x})e^{i\tilde{x}\cdot z}, \quad z\in T(\overset{*}{D}), \quad z=\xi+i\tau,$$

where $\overset{*}{D}=\{x\in\mathbb{R}^m : x\cdot y>0, \forall y\in D\}$, $T(D)$ is a specific domain in a multidimensional complex space $\mathbb{C}^m$ so that $T(D)=\hbar\mathbb{T}^m+iD$.

The infinite sum exists for $\tau\neq 0$ but does not exist for $\tau=0$ because it is formally the discrete Fourier transform (28.3) of the nonsummable indicator χ_+. If we fix a certain function $u_d\in L_2(h\mathbb{Z}^m)$, then we have $\chi_+\circ u_d\in L_2(D_d)$, and therefore the discrete Fourier transform $\widetilde{\chi_+\circ u_d}$ is defined and belongs to $L_2(\hbar\mathbb{T}^m)$. So, according to properties of the discrete Fourier transform (28.3), we have

$$(F_d(\chi_+\circ u))(\xi)=\lim_{\tau\to 0+}\int\limits_{\hbar\mathbb{T}^m}S_d(z-y)\tilde{u}_d(y)dy,$$

and the last integral exists at least in L_2-sense.

Thus we study a corresponding analogue of the equation (28.4) and we also need complex variables and relevant analogue of Riemann boundary value problem [BoMa48, Vl07, Va00].

Let $A(\hbar\mathbb{T}^m)$ be a subspace of $L_2(\hbar\mathbb{T}^m)$ consisting of functions which admit an analytical extension into $T(\overset{*}{D})$, and $B(\hbar\mathbb{T}^m)$ is an orthogonal complementation of the subspace $A(\hbar\mathbb{T}^m)$ in $L_2(\hbar\mathbb{T}^m)$ so that

$$L_2(\hbar\mathbb{T}^m) = A(\hbar\mathbb{T}^m) \oplus B(\hbar\mathbb{T}^m).$$

First of all we deal with a jump problem formulated in the following way: finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\hbar\mathbb{T}^m), \Phi^{-} \in B(\hbar\mathbb{T}^m)$, such that

$$\Phi^{+}(\xi) - \Phi^{-}(\xi) = g(\xi), \quad \xi \in \hbar\mathbb{T}^m, \tag{28.6}$$

where $g(\xi) \in L_2(\hbar\mathbb{T}^m)$ is given.

Proposition 2 *The operator $S_d : L_2(\hbar\mathbb{T}^m) \to A(\hbar\mathbb{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\hbar\mathbb{T}^m)$.*

Proof According to standard properties of the discrete Fourier transform F_d we have

$$F_d(\chi_+(\tilde{x})u_d(\tilde{x})) = \lim_{\tau\to 0} \int\limits_{\hbar\mathbb{T}^m} S_d(z-\eta)\widetilde{u}_d(\eta)d\eta,$$

where $\chi_+(\tilde{x})$ is the indicator of the set D_d. It implies the boundedness of the operator B_d. The second assertion follows from holomorphic properties of the kernel $S_d(z)$. In other words for arbitrary function $v \in A(\hbar\mathbb{T}^m)$ we have

$$v(z) = \int\limits_{\hbar\mathbb{T}^m} S_d(z-\eta)v(\eta)d\eta, \quad z \in T(\overset{*}{D}).$$

It is an analogue of the Cauchy integral formula. □

Theorem 4 *The jump problem has unique solution for arbitrary right-hand side from $L_2(\hbar\mathbb{T}^m)$.*

Proof Indeed there is an equivalent unique representation of the space $L_2(D_d)$ as a direct sum of two subspaces. If we denote $\chi_+(x), \chi_-(x)$ the indicators of the discrete sets $D_d, h\mathbb{Z}^m \setminus D_d$, respectively, then the following representation

$$u_d(\tilde{x}) = \chi_+(\tilde{x})u_d(\tilde{x}) + \chi_-(\tilde{x})u_d(\tilde{x})$$

is unique and holds for an arbitrary function $u_d \in L_2(h\mathbb{Z}^m)$. After applying the discrete Fourier transform we have

$$F_d u_d = F_d(\chi_+ u_d) + F_d(\chi_- u_d),$$

where $F_d(\chi_+ u_d) \in A(\hbar\mathbb{T}^m)$ according to the proposition 2, and thus $F_d(\chi_- u_d) = F_d u_d - F_d(\chi_+ u_d) \in B(\hbar\mathbb{T}^m)$ because $F_d u_d \in L_2(h\mathbb{T}^m)$.

Example 1 If $m = 2$ and C^2_+ is the first quadrant of $\mathbb{R}^2$, then a solution of a jump problem is given by formulas

$$\Phi^+(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2$$

$$\Phi^-(\xi) = \Phi^+(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in C^2_+.$$

The last decomposition will help us to formulate the periodic Riemann boundary value problem which is very distinct for one-dimensional case and multidimensional one. The principal non-correspondence is that the subspace $B(\hbar\mathbb{T}^m)$ consists of boundary values of certain analytical functions in one-dimensional case, but this set has an unknown nature for a multidimensional case.

A multidimensional periodic variant of Riemann boundary value problem can be formulated as follows: finding two functions $\Phi^{\pm}(\xi)$ such that $\Phi^+(\xi) \in A(\hbar\mathbb{T}^m)$, $\Phi^-(\xi) \in B(\hbar\mathbb{T}^m)$ and the following linear relation holds

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \tag{28.7}$$

where $G(\xi), g(\xi)$ are given functions on $\hbar\mathbb{T}^m$. We assume here that $G(\xi) \in C(\hbar\mathbb{T}^m)$, $G(\xi) \neq 0, \forall \xi \in \hbar\mathbb{T}^m$.

Definition 3 Periodic wave factorization of a function $G(\xi)$ is called its representation in the form

$$G(\xi) = G_{\neq}(\xi) G_{=}(\xi),$$

where factors $G^{\pm 1}_{\neq}(\xi), G^{\pm 1}_{=}(\xi)$ admit a bounded analytical continuation into complex domains $T(\overset{*}{D}), T(-\overset{*}{D})$, respectively.

Theorem 5 *If $G(\xi)$ admits periodic wave factorization, then multidimensional Riemann boundary value problem has a unique solution for arbitrary right-hand side $g(\xi) \in L_2(\hbar\mathbb{T}^m)$.*

Proof We rewrite a multidimensional Riemann boundary value problem in the form

$$G^{-1}_{\neq}(\xi)\Phi^+(\xi) - G_{=}(\xi)\Phi^-(\xi) = G^{-1}_{\neq}(\xi) g(\xi)$$

and obtain a jump problem (28.6).

Indeed for arbitrary two functions $f, g \in L_2(h\mathbb{Z}^m)$ such that *supp* $f \subset h\mathbb{Z}^m \setminus (-D_d)$, *supp* $g \subset (-D_d)$ according to properties of discrete Fourier transform F_d we have

$$(F_d^{-1}(f \circ g))(\tilde{x}) = ((F_d^{-1}f) * (F_d^{-1}g))(\tilde{x}) \equiv$$

$$\sum_{\tilde{y} \in h\mathbb{Z}^m} f_1(\tilde{x} - \tilde{y}) g_1(\tilde{y}) = \sum_{\tilde{y} \in -D_d} f_1(\tilde{x} - \tilde{y}) g_1(\tilde{y}),$$

where $f_1 = F_d^{-1}f, g_1 = F_d^{-1}g$ and according to the proposition 2 *supp* $g_1 \subset -D_d$.

Further since we have *supp* $f_1 \subset h\mathbb{Z}^m \setminus (-D_d)$ then for $\tilde{x} \in D_d, \tilde{y} \in -D_d$ we have $\tilde{x} - \tilde{y} \in D_d$ so that $f_1(\tilde{x} - \tilde{y}) = 0$ for such $\tilde{x}, \tilde{y}$. Thus *supp* $(f_1 * g_1) \subset h\mathbb{Z}^m \setminus D_d$. □

This solution can be constructed by means of the kernel $S_d(z)$.

Remark 1 If $m = 1$ the required factorization exists and can be constructed by the periodic analogue of Hilbert transform (see above). If $m \geq 2$ there is no an effective algorithm for constructing the required periodic wave factorization. One can give some sufficient conditions, for example, *supp* $F_d^{-1}(\ln G(\xi)) \subset D_d \cup (-D_d)$.

Now we consider the elliptic equation (28.4) with $\widetilde{A}_d(\xi), \widetilde{B}_d(\xi) \in C(\hbar\mathbf{T}^m)$. As above, one can establish the needed relationship between periodic multidimensional Riemann boundary value problem (28.7) and the corresponding integral equation in Fourier images similar to one-dimensional case [Ga81, Mu76, VaEtAl15-1] and can obtain the following result.

Theorem 6 *If* $\widetilde{A}_d(\xi)\widetilde{B}_d^{-1}(\xi)$ *admit the periodic wave factorization, then the equation* (28.4) *has a unique solution in the space* $L_2(h\mathbf{Z}^m)$.

Proof Applying the discrete Fourier transform to the equation (28.4), we obtain the following integral equation with operator S_d

$$\widetilde{A}_d(\xi)(S_d\widetilde{U}_d)(\xi) + \widetilde{B}_d(\xi)(I - S_d\widetilde{U}_d)(\xi) = \widetilde{V}_d$$

which is equivalent to certain periodic Riemann boundary value problem similar to (28.7). It was done in [Va00] for non-periodic case, and it looks the same for a periodic case. Then, according to Theorem 5, we obtain the required assertion. □

Conclusion

The author hopes these consideration will be useful for constructing basic elements of discrete theory of elliptic pseudo-differential equations and boundary value problems on manifolds with a boundary (possibly non-smooth) taking into account latest author's results [Va11, Va13, Va15].

References

[BoMa48] Bochner, S., Martin, W.T.: Several Complex Variables. Princeton University Press, Princeton, NY (1948)

[Es81] Eskin, G.: Boundary Value Problems for Elliptic Pseudodifferential Equations. AMS, Providence, RI (1981)

[Ga81] Gakhov, F.D.: Boundary Value Problems. Dover, Mineola, NY (1981)

[MiPr86] Mikhlin, S.G., Prössdorf, S.: Singular Integral Operators. Akademie, Berlin (1986)

[Mu76] Muskhelishvili, N.I.: Singular Integral Equations. North Holland, Amsterdam (1976)

[Sh01] Shubin, M.A.: Pseudodifferential Operators and Spectral Theory. Springer, Berlin/Heidelberg (2001)

[Ta81] Taylor, M.E.: Pseudodifferential Operators. Princeton University Press, Princeton, NJ (1981)

[Tr80] Treves, F.: Introduction to Pseudodifferential Operators and Fourier Integral Operators. Springer, New York, NY (1980)

[VaEtAl13] Vasilyev, A.V., Vasilyev, V.B.: Discrete singular operators and equations in a half-space. Azerb. J. Math. **3**, 84–93 (2013)

[VaEtAl15-1] Vasil'ev, A.V., Vasil'ev, V.B.: Periodic Riemann problem and discrete convolution equations. Differ. Equ. **51**, 652–660 (2015)

[VaEtAl15-2] Vasilyev, A.V., Vasilyev, V.B.: Discrete singular integrals in a half-space. In: Mityushev, V., Ruzhansky, M. (eds.) Current Trends in Analysis and Its Applications. Research Perspectives, pp. 663–670. Birkhäuser, Basel (2015)

[VaEtAl15-3] Vasil'ev, A.V., Vasil'ev, V.B.: On the solvability of certain discrete equations and related estimates of discrete operators. Dokl. Math. **92**, 585–589 (2015)

[Va00] Vasil'ev, V.B.: Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Elliptic Boundary Value Problems in Non-Smooth Domains. Kluwer Academic Publishers, Dordrecht/Boston/London (2000)

[Va11] Vasilyev, V.B.: Asymptotical analysis of singularities for pseudo differential equations in canonical non-smooth domains. In: Constanda, C., Harris, P. (eds.) Integral Methods in Science and Engineering, pp. 379–390. Birkhäuser, Boston, MA (2011)

[Va13] Vasilyev, V.B.: Pseudo differential equations on manifolds with non-smooth boundaries. In: Pinelas, S., Došlá, Z., Došlý, O., Kloeden, P.E. (eds.) Differential and Difference Equations and Applications. Springer Proceedings in Mathematics & Statistics, vol. 47, pp. 625–637. Birkhäuser, Basel (2013)

[Va15] Vasilyev, V.B.: New constructions in the theory of elliptic boundary value problems. In: Constanda, C., Kirsch, A. (eds.) Integral Methods in Science and Engineering, pp. 629–641. Birkhäuser, New York, NY (2015)

[Vl07] Vladimirov, V.S.: Methods of the Theory of Functions of Many Complex Variables. Dover, Mineola, NY (2007)