Automatic Semigroups vs Automaton Semigroups

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Abstract

We develop an effective and natural approach to interpret any semigroup admitting a special language of greedy normal forms as an automaton semigroup, namely the semigroup generated by a Mealy automaton encoding the behaviour of such a language of greedy normal forms under one-sided multiplication. The framework embraces many of the well-known classes of (automatic) semigroups: free semigroups, free commutative semigroups, trace or divisibility monoids, braid or Artin–Tits or Krammer or Garside monoids, Baumslag–Solitar semigroups, etc. Like plactic monoids or Chinese monoids, some neither left- nor right-cancellative automatic semigroups are also investigated, as well as some residually finite variations of the bicyclic monoid. It provides what appears to be the first known connection from a class of automatic semigroups to a class of automaton semigroups. It is worthwhile noting that, "being an automatic semigroup" and "being an automaton semigroup" become dual properties in a very automata-theoretical sense. Quadratic rewriting systems and associated tilings appear as the cornerstone of our construction.

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1 Introduction

The half century long history of the interactions between (semi)group theory and automata theory went through a pivotal decade from the mid-eighties to the mid-nineties. Contemporaneously but independently, two new theories truly started to develop and thrive: automaton (semi)groups on the one hand with the works of Aleshin [2, 3] and Grigorchuk [27, 28] and the book [47], and automatic (semi)groups on the other hand with the work of Cannon and Thurston and the book [24]. We refer to [55] for a clear and short survey on the known interactions between groups and automata. A deeper and more extended survey by Bartholdi and Silva can be found out in two chapters [7, 8] of the forthcoming AutoMathA handbook. We can refer to [14, 45] for automaton semigroups and to [17, 33] for automatic semigroups.

Remote siblings. As their very name indicates, automaton (semi)groups and automatic (semi)groups share a same defining object: the *automaton* or the letter-to-letter transducer in this case. Beyond this common origin, these two topics until now happened to remain largely distant both in terms of community and in terms of tools or results. Typically, any paper on one or the other topic used to contain a sentence like "it should be emphasised that, despite their similar names, the notions of automaton (semi)groups are entirely separate from

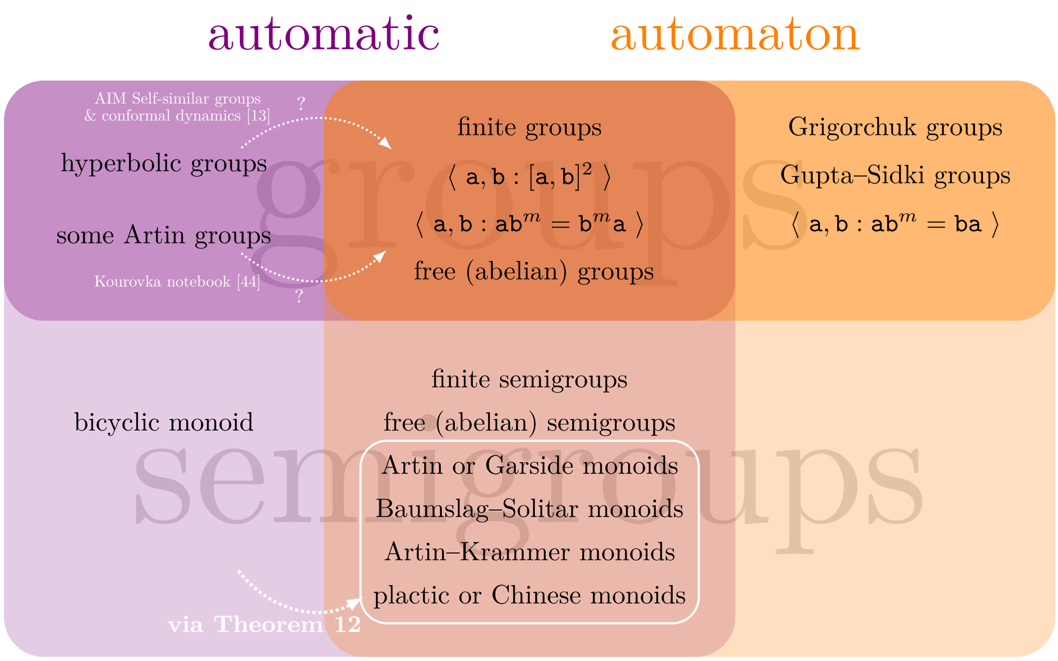


Figure 1 The big picture: comparing the classes of automatic *vs* automaton (semi)groups.

the notions of automatic (semi)groups". This was best evidenced by the above-mentioned valuable handbook chapter [7] which splits into exactly two sections (automatic groups and automaton groups) without any reference between one and the other appearing explicitly.

Related open problems. A significant problem is to recognise whether a given (semi)group is self-similar, that is, an automaton (semi)group, see Figure 1. Amongst the thirty-odd listed problems from [13], we can pick the one with the number 1.1:

> **Problem A.** It seems quite difficult to show whether a given group is self-similar. Are Gromov hyperbolic groups self-similar? Find obstructions to self-similarity.

Amongst the unsolved problems in group theory from the Kourovka Notebook [44], the one (with number 16.84) asked by Sushchanskii (see also [40, 51]) can be formulated as follows:

> **Problem B.** Is the n-strand braid group $\mathbf{B}_n$ a subgroup of some automaton group?

All these questions can be meaningfully rephrased in terms of semigroups or monoids.

Our contributions. The aim here is to establish a possible connection between being an automatic semigroup and being an automaton semigroup. Preliminary observations are that these classes intersect non trivially and that neither is included in the other (see Figure 1). Like the Grigorchuk group for instance, many automaton groups are infinite torsion groups, hence cannot be automatic groups. By contrast, it is an open question whether every automatic group is an automaton group. The latter is related to the question whether every automatic group is residually finite, which remains open despite the works by Wise [58] and Elder [23]. Like the bicyclic monoid, some automatic semigroups are not residually finite, hence cannot be automaton semigroups (see [14] for instance). As for the intersection, we know that at least finite semigroups, free semigroups (of rank at least 2, see [11, 12]), free abelian semigroups happen to be both automatic semigroups and automaton semigroups.

We propose here a new and natural way to interpret algorithmically each semigroup from a wide class of automatic semigroups – encompassing all the above-mentioned classes – as an automaton semigroup (Theorem 12). Furthermore, it is worthwhile noting that, in all these cases, "being an automatic semigroup" and "being an automaton semigroup" become dual properties in a very automata-theoretical sense (Corollary 15).

Occurring as the very first bridge between two hitherto irreconcilable research areas, Theorem 12 allows us to also provide a (more than) positive answer to the monoidal version of Problem **B**. While the n-strand braid monoid $\mathbf{B}_{n+}^{1}$ is the paradigmatic example of an automatic monoid from [24, Chapter 9], Theorem 12 implies that $\mathbf{B}_{n+}^{1}$ is (not only a submonoid of) an automaton monoid as well, see Example 23. From this significant milestone arise various new questions and perspectives, that will hopefully pervade both research fields.

Organisation. The structure of the paper is as follows. As a simple preliminary, Section 2 illustrates in a deliberately informal manner how a single Mealy automaton can be used in order to define both self-similar structures and automatic structures (via a principle of duality). In Section 3, we set up the notations for Mealy automata and recall necessary notions of dual automata, cross-diagrams, and self-similar structures. In Section 4, we recall basics about normal forms and automatic structures, and we give necessary notions of quadratic normalisations and square-diagrams. Section 5 is devoted to our main results (Theorem 12 and Corollary 15), while Section 6 finally gathers several carefully selected examples, counterexamples, and open problems.

2 A preliminary example

As their very name indicates, automaton (semi)groups and automatic (semi)groups share a same defining object. In both cases, a *Mealy automaton* (see Definition 1) basically transforms words into words.

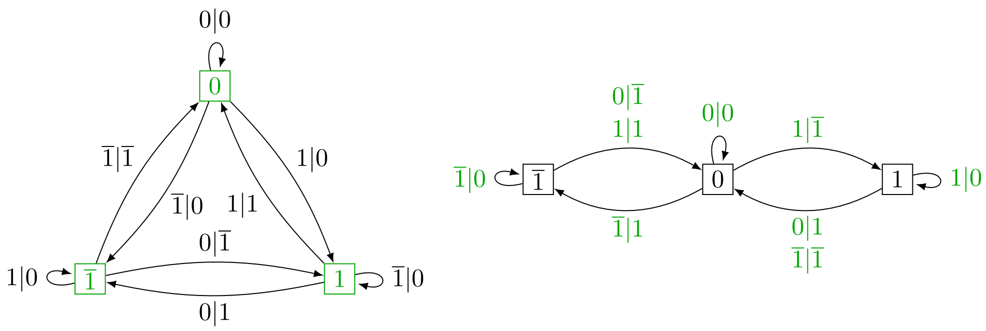


Figure 2 Two (dual) Mealy automata: the left-hand one computing the division by 3 in base 2 (most significant digit first) *vs* the right-hand one computing the multiplication by 2 in base 3 (least significant digit first).

The Mealy automaton displayed on Figure 2 (left) is some signed-digit version of one of the most classical examples of a transducer (see [53, Prologue] for a delightfully alternate history). Signed-digit numeration systems [4, 18] are not the topic, now they provide a special opportunity to illustrate our purpose. When starting from the state 0 and reading any binary word or any $\{\bar{1}, 0, 1\}$-word $\mathbf{u}$ (most significant digit first), it computes the division by 3 in base 2 by outputting the (quotient) $\{\bar{1}, 0, 1\}$-word $\mathbf{v}$ (most significant digit first) satisfying

$$(\mathbf{u})_2 = 3 \times (\mathbf{v})_2 + f \qquad (\%)$$

where the (remainder) $f \in \{\bar{1}, 0, 1\}$ corresponds to the arrival state of the run, and where $(\mathbf{w})_b$ denotes by convention the number that is represented by $\mathbf{w}$ in base b.

For the current preliminary section, let us now focus on this basic example and consider the two different viewpoints described as follows.

On the one hand, it seems natural to consider the set of those functions (from $\{\bar{1}, 0, 1\}$-words to $\{\bar{1}, 0, 1\}$-words) thus associated with each state, then to compose them with each other, and finally to study the (semi)group which is generated by such functions.

For instance, the function $\mathbf{u} \mapsto \mathbf{v}$ associated with the state 0 (satisfying Equation (%)) can be squared, cubed, and so on, to obtain functions, which can be again interpreted as the division by $9, 27, \ldots$ (in base 2 with most significant digit first), or can be composed with the functions induced by the other two states. The generated semigroup happens to be the rank 3 free semigroup $\{\bar{1}, 0, 1\}^+$ (provided that the three states and their induced functions are identified). This simple idea coincides with the notion of automaton (semi)groups or self-similar structures (see Definition 2). With this crucial standpoint, we can compute (semi)group operations by manipulating the corresponding Mealy automaton (see [5, 7, 35, 46]), and hopefully foresee some combinatorial and dynamical properties by examining its shape (see [6, 9, 10, 19, 25, 26, 34, 36, 37, 38, 51, 56] for instance).

On the other hand, it may be also natural to simply iterate the runs. The starting language is again over the (input/output) alphabet, now the images of the transformations are some languages over the stateset.

For instance, restarting again from the state 0, the previously output word $\mathbf{v}$ (satisfying Equation (%)) can be read in turn, and so on. The successive arrival states can be then collected and concatenated in order to obtain here the decomposition of $(\mathbf{u})_2$ in base 3 (least significant digit first). The whole process has thus inherently a quadratic time complexity.

This second idea coincides with the fundamental notion of automatic (semi)groups (see Definition 3 and [7, 17, 24, 33, 51]), for which Mealy automata can compute normal forms.

To conclude this preliminary section, let us mention that states and letters of any Mealy automaton play a symmetric role, and that several properties can be beneficially derived from the so-called *dual (Mealy) automaton*, obtained by exchanging the stateset and the alphabet (see [36] for an overview).

For instance, Figure 2 displays a pair of dual automata. While the left-hand automaton allows to compute the division by 3 in base 2 (most significant digit first) as we have seen just above, its dual automaton (right) essentially computes the multiplication by 2 in base 3 (least significant digit first). More precisely, its state 0 induces the function $x \mapsto 2 \times x$ on $\mathbb{Z}$, while its states $\bar{1}$ and 1 induce the functions $x \mapsto 2 \times x - 1$ and $x \mapsto 2 \times x + 1$ respectively: they together generate the semigroup $\langle\ \bar{1}, 0, 1 : 01 = 1\bar{1}, 0\bar{1} = \bar{1}1\ \rangle_+$. Let us also mention that the induced functions happen to be invertible and to generate a group which is isomorphic with the so-called Baumslag–Solitar group $\mathbf{BS}(2,1) = \langle\ \mathtt{a}, \mathtt{b} : \mathtt{ab}^2 = \mathtt{ba}\ \rangle$ (see Figure 1, Example 20, and [51]).

Besides, such a Mealy automaton (right) can be used to compute the base 2 from the base 3 representation of the fractional part of any rational number, by iterating runs as explained above. For instance, finitely iterated runs from the state 0 and the initial word $0001\bar{1}$ produce the infinite word $(0\bar{1}0010)^\omega$, both words representing (the fractional part of) the rational $-\frac{2}{9}$ in base 3 (least significant digit first) and in base 2 (most significant digit first) respectively.

This innocuous example allows to illustrate the quite simple machineries associated both with automaton semigroups and with automatic semigroups. It also aims to give an informal glimpse of their behaviours through the duality principle: for instance, division *vs* multiplication, factor *vs* base, least *vs* most significant digit first, integer part *vs* fractional part.

3 Mealy automata and self-similar structures

We first recall the formal definition of an automaton. Possible references are [7, 14, 45, 47].

▶ **Definition 1.** *A* (finite, deterministic, and complete) automaton *is a triple* $\big(Q, \Sigma, \tau = (\tau_i \colon Q \to Q)_{i \in \Sigma}\big)$*, where the* stateset Q *and the* alphabet Σ *are non-empty finite sets, and where the* τ_i*'s are functions.*

A Mealy automaton *is a quadruple* $\big(Q, \Sigma, \tau = (\tau_i \colon Q \to Q)_{i \in \Sigma}, \sigma = (\sigma_x \colon \Sigma \to \Sigma)_{x \in Q}\big)$ *such that both* (Q, Σ, τ) *and* (Σ, Q, σ) *are automata.*

In other terms, a Mealy automaton is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet.

The graphical representation of a Mealy automaton is standard, see Figures 2 and 7.

In a Mealy automaton $\mathcal{A} = (Q, \Sigma, \tau, \sigma)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}(\mathcal{A}) = (\Sigma, Q, \sigma, \tau)$:

$$x \xrightarrow{\;i\,|\,j\;} y \;\in \mathcal{A} \qquad \Longleftrightarrow \qquad i \xrightarrow{\;x\,|\,y\;} j \;\in \mathfrak{d}(\mathcal{A}).$$

We view $\mathcal{A} = (Q, \Sigma, \tau, \sigma)$ as an automaton with an input and an output tape, thus defining mappings from input words over Σ to output words over Σ. Formally, for $x \in Q$, the map $\sigma_x \colon \Sigma^* \to \Sigma^*$, extending $\sigma_x \colon \Sigma \to \Sigma$, is defined recursively by:

$$\forall i \in \Sigma,\ \forall \mathbf{s} \in \Sigma^*, \qquad \sigma_x(i\mathbf{s}) = \sigma_x(i)\sigma_{\tau_i(x)}(\mathbf{s}) \,.$$

The above equation can be easier to understood when depicted by a *cross-diagram* (see [1]):

$$\begin{tikzcd} & i \arrow[dd] & & \mathbf{s} \arrow[dd] & \\ x \arrow[rr] & & \tau_i(x) \arrow[rr] & & \tau_{\mathbf{s}}(\tau_i(x)) \\ & \sigma_x(i) & & \sigma_{\tau_i(x)}(\mathbf{s}) & \end{tikzcd}$$

By convention, the image of the empty word is itself. The mapping σ_x for each $x \in Q$ is length-preserving and prefix-preserving. We say that σ_x is the *production function* associated with $(\mathcal{A}, x)$. For $\mathbf{x} = x_1 \cdots x_n \in Q^n$ with $n > 0$, set $\sigma_{\mathbf{x}} \colon \Sigma^* \to \Sigma^*, \sigma_{\mathbf{x}} = \sigma_{x_n} \circ \cdots \circ \sigma_{x_1}$. Denote dually by $\tau_i \colon Q^* \to Q^*, i \in \Sigma$, the production functions associated with the dual automaton $\mathfrak{d}(\mathcal{A})$. For $\mathbf{s} = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, set $\tau_{\mathbf{s}} \colon Q^* \to Q^*,\ \tau_{\mathbf{s}} = \tau_{s_n} \circ \cdots \circ \tau_{s_1}$.

▶ **Definition 2.** *The semigroup of mappings from* Σ^* *to* Σ^* *generated by* $\{\sigma_x, x \in Q\}$ *is called the* semigroup generated by $\mathcal{A}$ *and is denoted by* $\langle\, \mathcal{A} \,\rangle_+$*. When* $\mathcal{A}$ *is invertible, its production functions are permutations on words of the same length and thus we may consider the corresponding group instead; this group is the* group generated by $\mathcal{A}$ *and is denoted by* $\langle\, \mathcal{A} \,\rangle$*. A (semi)group is called an* automaton (semi)group *whenever it can be generated by some Mealy automaton. The term* self-similar *is used as a synonym.*

4 Quadratic normalisations and automatic structures

This section gathers the definitions of some classical notions like normal form or automatic structure (see [24, 17, 33]), together with the slighly more specific notion of a quadratic normalisation (see [20, 22]).

For any set $\mathcal{Q}$, we denote by $\mathcal{Q}^+$ the free semigroup over $\mathcal{Q}$ (*resp.* by $\mathcal{Q}^*$ the free monoid and by $\mathtt{1}$ its unit element) and call its elements *$\mathcal{Q}$-words*. We write $|\mathbf{w}|$ for the length of a $\mathcal{Q}$-word $\mathbf{w}$, and $\mathbf{w}\mathbf{w}'$ for the product of two $\mathcal{Q}$-words $\mathbf{w}$ and $\mathbf{w}'$.

▶ **Definition 3.** *Let S be a semigroup with a generating set $\mathcal{Q}$. A* normal form *for $(S, \mathcal{Q})$ is a (set-theoretic) section of the canonical projection* EV *from the language of $\mathcal{Q}$-words onto S, that is, a map* NF *that assigns to each element of S a distinguished representative $\mathcal{Q}$-word with $\mathrm{EV} \circ \mathrm{NF} = \mathrm{Id}_S$:*

$$\begin{tikzcd} \mathrm{EV} : \mathcal{Q}^+ \arrow[r] & S \arrow[l, bend left, "\mathrm{NF}"] \end{tikzcd}$$

Whenever $\mathrm{NF}(S)$ is regular, it provides a right-automatic structure *for S if the language $\mathcal{L}_q = \{ \ (\mathrm{NF}(a)\#^{|\mathrm{NF}(aq)|}, \ \mathrm{NF}(aq)\#^{|\mathrm{NF}(a)|} \ ) : a \in S \ \}$ over the alphabet $(\mathcal{Q} \sqcup \{\#\})^2$ is regular for each $q \in \mathcal{Q}$, where the normal forms of a pair are right-padded with an extra symbol $\# \notin \mathcal{Q}$ to equalise the lengths. The semigroup S can then be called a* (right-)automatic semigroup.

We mention here the thorough and precious study in [32] of the different notions (right- or left-reading-padding *vs* right- or left-multiplication) of automaticity for semigroups.

▶ Remark 4. In his seminal work [24, Chapter 9], Thurston shows how the whole set of these different automata recognizing the multiplication – that is, recognizing the languages $\mathcal{L}_q$ – in Definition 3 can be replaced with advantage by a single letter-to-letter transducer over the alphabet $\mathcal{Q}$ (see Definition 14) that computes the normal forms via iterated runs: each run both provides one symbol of the final normal form and outputs a word still to be normalised.

One will often consider the associated normalisation $\mathrm{N} = \mathrm{NF} \circ \mathrm{EV}$ over $\mathcal{Q}$.

▶ **Definition 5.** *A* normalisation *is a pair $(\mathcal{Q}, \mathrm{N})$, where $\mathcal{Q}$ is a set and N is a map from $\mathcal{Q}^+$ to itself satisfying, for all $\mathcal{Q}$-words $\mathbf{u}, \mathbf{v}, \mathbf{w}$:*
- $|\mathrm{N}(\mathbf{w})| = |\mathbf{w}|$,
- $|\mathbf{w}| = 1 \Rightarrow \mathrm{N}(\mathbf{w}) = \mathbf{w}$,
- $\mathrm{N}(\mathbf{u}\,\mathrm{N}(\mathbf{w})\,\mathbf{v}) = \mathrm{N}(\mathbf{u}\mathbf{w}\mathbf{v})$.

A $\mathcal{Q}$-word $\mathbf{w}$ satisfying $\mathrm{N}(\mathbf{w}) = \mathbf{w}$ is called N-normal*. If S is a semigroup, we say that $(\mathcal{Q}, \mathrm{N})$ is a* normalisation for S *if S admits the presentation*

$$\langle \ \mathcal{Q} : \{ \mathbf{w} = \mathrm{N}(\mathbf{w}) \mid \mathbf{w} \in \mathcal{Q}^+ \} \ \rangle_+ .$$

We associate with every element $q \in \mathcal{Q}$ a q-labeled edge and with a product the concatenation of the associated edges, and represent equalities in the ambient semigroup using commutative diagrams, that we shall often organise as tilings and that we call here *square-diagram*. For instance, the following square illustrates an equality $q_1 q_2 = q_1' q_2'$.

$$\begin{tikzcd} \cdot \arrow[d, "q_2"'] & \cdot \arrow[l, "q_1"'] \arrow[d, "q_1'"] \\ \cdot & \cdot \arrow[l, "q_2'"] \end{tikzcd}$$

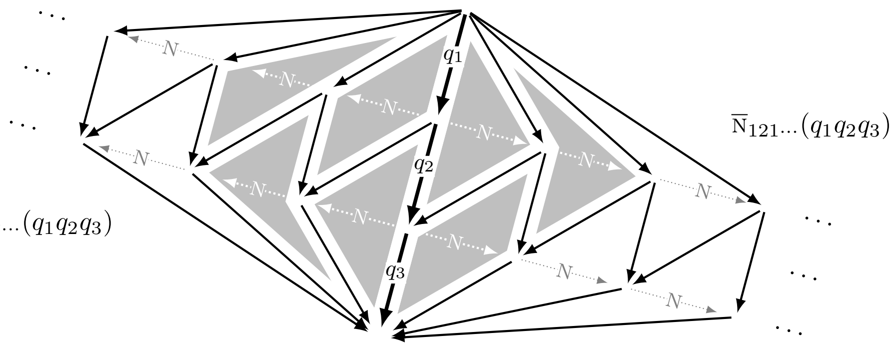


Figure 3 From an initial $\mathcal{Q}$-word $q_1q_2q_3$, one applies normalisations on the first and the second 2-factors alternatively up to stabilisation, beginning either on the first 2-factor q_1q_2 (right-hand side here) or on the second q_2q_3. The gray zone corresponds to Condition ( ) as defined in Definition 7.

For a normalisation $(\mathcal{Q}, \mathrm{N})$, we denote by $\overline{\mathrm{N}}$ the restriction of N to $\mathcal{Q}^2$ and, for $i \geq 1$, by $\overline{\mathrm{N}}_i$ the (partial) map from $\mathcal{Q}^+$ to itself that consists in applying $\overline{\mathrm{N}}$ to the entries in position i and $i+1$. For any finite sequence $\mathbf{i} = i_1 \cdots i_n$ of positive integers, we write $\overline{\mathrm{N}}_{\mathbf{i}}$ for the composite map $\overline{\mathrm{N}}_{i_n} \circ \cdots \circ \overline{\mathrm{N}}_{i_1}$ (so $\overline{\mathrm{N}}_{i_1}$ is applied first).

▶ **Definition 6.** *A normalisation $(\mathcal{Q}, \mathrm{N})$ is* quadratic *if the two following conditions hold:*
- *a $\mathcal{Q}$-word $\mathbf{w}$ is N-normal if, and only if, every length-two factor of $\mathbf{w}$ is;*
- *for every $\mathcal{Q}$-word $\mathbf{w}$, there exists a finite sequence $\mathbf{i}$ of positions, depending on $\mathbf{w}$, such that $\mathrm{N}(\mathbf{w})$ is equal to $\overline{\mathrm{N}}_{\mathbf{i}}(\mathbf{w})$.*

▶ **Definition 7.** *As illustrated in Figure 3, with any quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ is associated its* breadth (d, p) *(called minimal left and right classes in [20, 22]) defined as:*

$$d = \max_{(q_1,q_2,q_3)\in\mathcal{Q}^3} \min\{\, \ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}_{\underbrace{212\cdots}_{\text{length } \ell}}(q_1q_2q_3)\}, \text{ and}$$

$$p = \max_{(q_1,q_2,q_3)\in\mathcal{Q}^3} \min\{\, \ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}_{\underbrace{121\cdots}_{\text{length } \ell}}(q_1q_2q_3)\}.$$

Such a breadth need to be finite provided that $\mathcal{Q}$ is finite, and then satisfies $|d-p| \leq 1$. For $p \leq 3$ (and $d \leq 4$), the quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ is said to satisfy Condition ( ) (its corresponds with the so-called domino rule *in [21] but with a different reading direction).*

The first main result of [22] is an axiomatisation of these quadratic normalisations satisfying Condition ( ) in terms of their restrictions to length-two words: any idempotent map $\overline{\mathrm{N}}$ on $\mathcal{Q}^2$ that satisfies $\overline{\mathrm{N}}_{2121} = \overline{\mathrm{N}}_{121} = \overline{\mathrm{N}}_{1212}$ extends into a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Condition ( ). For larger breadths, a map on length-two words normalising length-three words needs not normalise words of greater length.

The second main result of [22] involves termination. Every quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ gives rise to a quadratic rewriting system, namely the one with rules $\mathbf{w} \longrightarrow \overline{\mathrm{N}}(\mathbf{w})$ for $\mathbf{w} \in \mathcal{Q}^2$. By Definition 6, such a rewriting system is confluent and normalising, meaning that, for every initial word, there exists a finite sequence of rewriting steps leading to a unique N-normal word, but its convergence, meaning that *any* sequence of rewriting steps is finite, is a quite different problem.

▶ **Theorem 8** ([22])**.** *If $(\mathcal{Q}, \mathrm{N})$ is a quadratic normalisation satisfying Condition ( ), then the associated rewriting system is convergent.*

More precisely, every rewriting sequence starting from a word of $\mathcal{Q}^p$ has length at most $\frac{p(p-1)}{2}$ (*resp.* $2^p - p - 1$) in the case of a breadth $(3,3)$ (*resp.* either $(3,4)$ or $(4,3)$). Theorem 8 is essentially optimal since there exist nonconvergent rewriting systems with breadth $(4,4)$.

The results of Section 5 rely on the special Condition ($\blacklozenge$). As mentioned, this condition was already outlined by Dehornoy and Guiraud (see [22]). However, none of their results (in particular Theorem 8 given above for the sake of completeness) is either applied or needed to establish ours. The current work and its exposition are thus self-contained and our constructions never require any of their stronger hypotheses (neither cancellativity nor absence of nontrivial invertible elements). We want here to emphasise that Condition ($\blacklozenge$) happens to appear as a common denominator from different approaches (see also [29]).

5 From an automatic structure to a self-similar structure

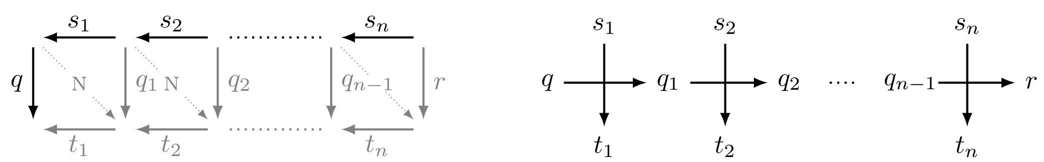
All the ingredients are now in place to effectively and naturally interpret as an automaton monoid any automatic monoid admitting a special language of normal forms – namely, a quadratic normalisation satisfying Condition ($\blacklozenge$). The point is to construct a Mealy automaton encoding the behaviour of its language of normal forms under one-sided multiplication.

▶ **Definition 9.** *Assume that S is a semigroup admitting a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$. We define the Mealy automaton $\mathcal{M}_{\mathcal{Q},\mathrm{N}} = (\mathcal{Q}, \mathcal{Q}, \tau, \sigma)$ such that, for every $(a,b) \in \mathcal{Q}^2$, $\sigma_b(a)$ is the rightmost element of $\mathcal{Q}$ in the normal form $\mathrm{N}(ab)$ of ab and $\tau_a(b)$ is the left one:*

$$\mathrm{N}(ab) = \tau_a(b)\sigma_b(a).$$

The latter correspondence can be simply interpreted via square-diagram *vs* cross-diagram:

For $\mathbf{s} = s_n \cdots s_1$, $\mathbf{t} = t_n \cdots t_1$, $\sigma_q(\mathbf{s}) = \mathbf{t}$, and $\tau_{\mathbf{s}}(q) = r$, we obtain diagrammatically:



We choose on purpose to always draw a normalisation square-diagram backward, such that it coincides with the associated cross-diagram. The function σ_q induced by the state q should map any word $\mathbf{s}$ (read backward) to some word $\mathbf{t}$ (read backward) with $\mathrm{N}(\mathbf{s}q) = \mathrm{N}(r\mathbf{t})$.

We now aim to strike reasonable (most often optimal) hypotheses for a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ associated with an original semigroup S to generate a semigroup $\langle \mathcal{M}_{\mathcal{Q},\mathrm{N}} \rangle_+$ that approximates S as sharply as possible. Since the generating sets coincide by Definition 9, we shall address the case when S is a quotient of $\langle \mathcal{M}_{\mathcal{Q},\mathrm{N}} \rangle_+$ (top-approximation, Lemma 10), and next, the case when $\langle \mathcal{M}_{\mathcal{Q},\mathrm{N}} \rangle_+$ is a quotient of S (bottom-approximation, Proposition 11).

Before establishing our top-approximation statement (Lemma 10), we could stress how semigroups might appear much more difficult to handle than monoids, especially when it comes to automaticity (see [32]) or self-similarity (see [11, 12]). For a semigroup S with a quadratic normalisation on $\mathcal{Q}$, two situations occur. First, if S is a monoid with unit $\mathtt{1}$, it admits a quadratic normalisation satisfying $\mathrm{N}(\mathtt{1}) = \mathtt{1}$ and

$$\mathrm{N}(\mathtt{1}q) = \mathrm{N}(q\mathtt{1}) = \mathtt{1}q \qquad (\boxed{1})$$

for each $q \in \mathcal{Q}$. Second, if S does not admit a unit, one can adjoin a unit $\mathtt{1}$ to obtain a monoid (if needed) with a quadratic normalisation satisfying Condition ($\boxed{1}$). The choice made for such a condition becomes natural whenever we think of the (adjoined or not) unit $\mathtt{1}$ as some *dummy* element that escapes from the normalisation and simply ensures its length-preserving property.

▶ **Lemma 10.** *If S is a monoid with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Condition ($\boxed{1}$), then the Mealy automaton $\mathcal{M}_{\mathcal{Q},\mathrm{N}}$ generates a monoid of which S is a quotient.*

Although specific to a monoidal framework and then requiring the innocuous Condition ($\boxed{1}$), the previous straightforward result relies only on the definition of a quadratic normalisation and on the well-fitted associated Mealy automaton (Definition 9). For the bottom-approximation statement, we consider an extra assumption, which happens to be necessary and sufficient.

▶ **Proposition 11.** *Assume that S is a semigroup with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$. If Condition (⬢) is satisfied, then the Mealy automaton $\mathcal{M}_{\mathcal{Q},\mathrm{N}}$ generates a semigroup quotient of S. The converse holds provided that Condition ($\boxed{1}$) is satisfied.*

Sketch proof. Let $S = \mathcal{Q}^+/\equiv_{\mathrm{N}}$ and $\mathcal{M}_{\mathcal{Q},\mathrm{N}} = (\mathcal{Q}, \mathcal{Q}, \tau, \sigma)$ as in Definition 9.

($\Leftarrow$) Assume that Condition (⬢) is satisfied and that there exists $(a, b, c, d) \in \mathcal{Q}^4$ with $ab \equiv_{\mathrm{N}} cd$. We have to prove $\sigma_{ab} = \sigma_{cd}$. Without loss of generality, the word ab can be supposed to be N-normal, that is, $\mathrm{N}(ab) = \mathrm{N}(cd) = ab$ holds.

Let $\mathbf{u} = q\mathbf{v} \in \mathcal{Q}^n$ for some $n > 0$ and $q \in \mathcal{Q}$. We shall prove both $\sigma_{ab}(\mathbf{u}) = \sigma_{cd}(\mathbf{u})$ (letterwise) and $\tau_{\mathbf{u}}(ab) \equiv_{\mathrm{N}} \tau_{\mathbf{u}}(cd)$ by induction on $n > 0$. For $n = 1$, we obtain the two square-diagrams on Figure 4 (left). With these notations, we have to prove $q''_0 = q''_1$ and $a'b' \equiv_{\mathrm{N}} c'd'$, the latter meaning $\mathrm{N}(a'b') = \mathrm{N}(c'd')$, that is, with the notations from Figure 4, the conjunction of $a'' = c''$ and $b'' = d''$. Now these three equalities hold whenever $(\mathcal{Q}, \mathrm{N})$ satisfies Condition (⬢), as shown on Figure 4 (right).

This allows to proceed the induction and to prove the implication ($\Leftarrow$).

($\Rightarrow$) Consider an arbitrary length 3 word over $\mathcal{Q}$, say qcd. Let a, b denote the elements in $\mathcal{Q}$ satisfying $\mathrm{N}(cd) = ab$. By definition, we deduce $ab \equiv_{\mathrm{N}} cd$. This implies $\sigma_{ab} = \sigma_{cd}$ by hypothesis. In particular, the images of any word $q\mathbf{v}$ under σ_{ab} and σ_{cd} coincide. Now $\sigma_{ab}(q\mathbf{v}) = \sigma_{cd}(q\mathbf{v})$ decomposes into

$$\sigma_{ab}(q) = q''_0 = q''_1 = \sigma_{cd}(q) \quad \text{and} \quad \sigma_{\tau_q(ab)}(\mathbf{v}) = \sigma_{a'b'}(\mathbf{v}) = \sigma_{c'd'}(\mathbf{v}) = \sigma_{\tau_q(cd)}(\mathbf{v})$$

(with notations of Figure 4). The last equality holds for any original word $\mathbf{v} \in \mathcal{Q}^*$ and implies $\sigma_{a'b'} = \sigma_{c'd'}$. Whenever, Condition ($\boxed{1}$) is satisfied, we deduce $\mathrm{N}(a'b') = \mathrm{N}(c'd')$ according to Lemma 10. For any such arbitrary word $qcd \in \mathcal{Q}^3$, we obtain $\overline{\mathrm{N}}_{121}(qcd) = \overline{\mathrm{N}}_{2121}(qcd)$. Therefore $(\mathcal{Q}, \mathrm{N})$ satisfies Condition (⬢). ◀

Gathering Lemma 10 and Proposition 11, we obtain the following main result.

▶ **Theorem 12.** *Assume that S is a monoid with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Conditions ($\boxed{1}$) and (⬢). The Mealy automaton $\mathcal{M}_{\mathcal{Q},\mathrm{N}}$ generates a monoid isomorphic to S.*

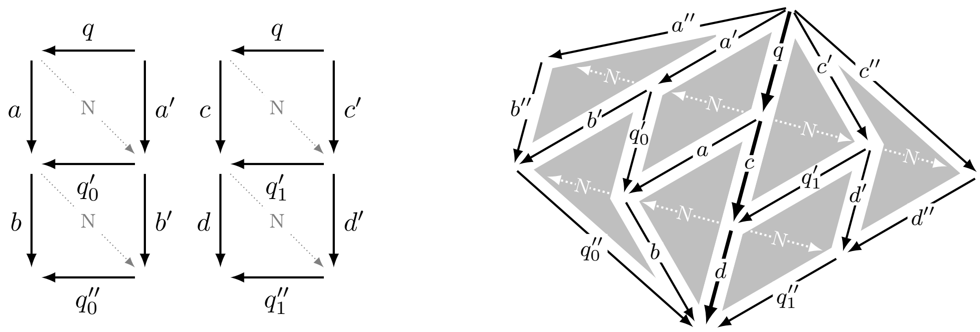


Figure 4 Proof of Proposition 11: initial data (left) can be pasted into Condition ( ) (right).

Proof. By construction, S and $\langle\, \mathcal{M}_{\mathcal{Q},\mathrm{N}}\, \rangle_+^1$ share a same generating subset $\mathcal{Q}$. Now, any defining relation for S maps to a defining relation for $\langle\, \mathcal{M}_{\mathcal{Q},\mathrm{N}}\, \rangle_+^1$ by Proposition 11, and conversely by Lemma 10. ◀

▶ **Corollary 13.** *Any monoid with a quadratic normalisation satisfying Conditions* ($\boxed{1}$) *and* ( ) *is residually finite.*

To conclude this main section, we come back to that remark (following Definition 3) about the transducer approach by Thurston.

▶ **Definition 14.** *With any quadratic normalisation* $(\mathcal{Q}, \mathrm{N})$ *is associated its* Thurston transducer *defined as the Mealy automaton* $\mathcal{T}_{\mathcal{Q},\mathrm{N}}$ *with stateset* $\mathcal{Q}$*, alphabet* $\mathcal{Q}$*, and transitions as follows:*

▶ **Corollary 15.** *Assume that* S *is a monoid with a quadratic normalisation* $(\mathcal{Q}, \mathrm{N})$ *satisfying Conditions* ($\boxed{1}$) *and* ( ). *The Thurston transducer* $\mathcal{T}_{\mathcal{Q},\mathrm{N}}$ *and the Mealy automaton* $\mathcal{M}_{\mathcal{Q},\mathrm{N}}$ *being dual automaton,* S *possesses both the explicitly dual properties of automaticity and self-similarity.*

These rather unexpected results provide the very first bridge between two fundamental areas that have always been widely seen as irreconcilable: automatic semigroups *vs* automaton semigroups. We choose to conclude by gathering several carefully selected examples, counterexamples, and open problems.

6 Examples and counterexamples

Our very first example is straightforward, but enlightening.

▶ **Example 16.** Every finite monoid $\mathcal{J}$ (in particular every finite group) is an *automaticon* monoid, that is, both an automatic and an automaton monoid. Consider its quadratic normalisation $(\mathcal{J}, \mathrm{N})$ with $\mathrm{N}(ab) = 1(ab)$ for every $(a, b) \in \mathcal{J}^2$. Figure 5 shows how to compute its breadth $(3, 2)$, witness of Condition ( ) for applying Theorem 12.

As mentioned in Section 1 and appearing on Figure 1, there exist automatic semigroups that cannot be automaton semigroups.

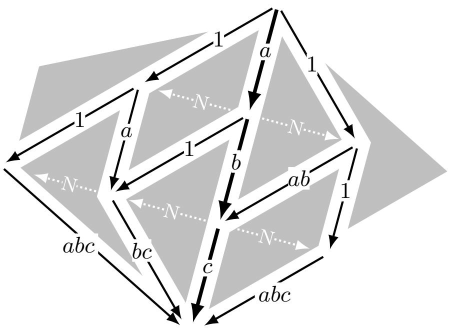


Figure 5 Computing the breadth $(3, 2)$ for any finite monoid $\mathcal{J}$ as in Example 16.

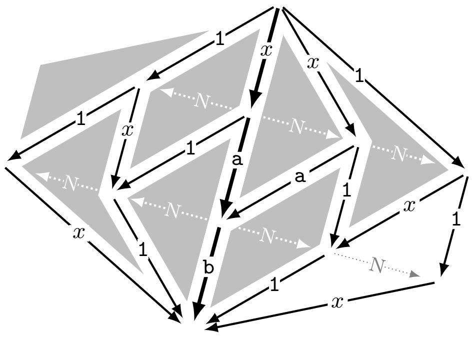


Figure 6 Computing the breadth $(3, 4)$ for the bicylic monoid **B** from Example 17.

▶ **Example 17.** The bicyclic monoid $\mathbf{B} = \langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{1} \,\rangle_+^1$ is known to be automatic and not residually finite, hence cannot be an automaton monoid. Choose for **B** the quadratic normalisation $(\{\mathtt{a}, \mathtt{b}, \mathtt{1}\}, \mathtt{N})$ with $\mathtt{N}(\mathtt{ab}) = \mathtt{11}$, $\mathtt{N}(x\mathtt{1}) = \mathtt{1}x$ for $x \in \{\mathtt{a}, \mathtt{b}\}$, and $\mathtt{N}(xy) = xy$ otherwise. Figure 6 illustrates the computation (on the witness word $x\mathtt{ab}$ with $x \in \{\mathtt{a}, \mathtt{b}\}$) of its breadth $(3, 4)$. The Condition ( ) is hence not satisfied and Theorem 12 cannot apply. Precisely, according to the proof of Proposition 11 and Figure 4, we have $\sigma_{\mathtt{ab}}(x) = \mathtt{1} \neq x = \sigma_{\mathtt{11}}(x)$ for $x \in \{\mathtt{a}, \mathtt{b}\}$, hence $\sigma_{\mathtt{ab}} \neq \sigma_{\mathtt{1}} = \sigma_{\mathtt{11}}$.

By contrast, one of the simplest nontrivial examples could be the following.

▶ **Example 18.** The automatic monoid $\langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{a} \,\rangle_+^1$ admits the quadratic normalisation $(\{\mathtt{a}, \mathtt{b}, \mathtt{1}\}, \mathtt{N})$ with $\mathtt{N}(\mathtt{ab}) = \mathtt{1a}$, $\mathtt{N}(x\mathtt{1}) = \mathtt{1}x$ for $x \in \{\mathtt{a}, \mathtt{b}\}$, and $\mathtt{N}(xy) = xy$ otherwise. Condition ($\boxed{1}$) is satisfied, and the breadth is (3,3) according to the graph on Figure 7. By Theorem 12, it is therefore an automaton monoid, generated by the Mealy automaton displayed on Figure 7.

The latter happens to be the common smallest nontrivial member of the family of Baumslag–Solitar monoids (see [32] for instance), namely $\mathbf{BS}_+^1(1, 0)$, and of a wide family of right-cancellative semigroups, that we readily call *Artin–Krammer* monoids and that have been introduced and studied in [39] (see also [30, 31, 48]), namely $\mathbf{AK}_+^1(\Gamma)$ associated with the Coxeter-like matrix $\Gamma = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

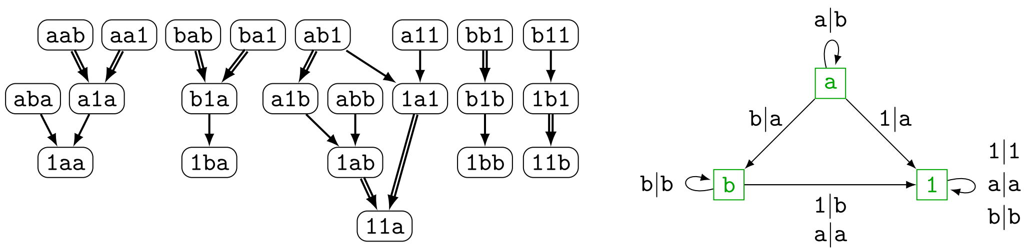


Figure 7 The $\overline{\mathtt{N}}$-graph for the quadratic normalisation associated with $\langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{a} \,\rangle_+^1$ from Example 18: simple arrows correspond to $\overline{\mathtt{N}}_1$ and double arrows to $\overline{\mathtt{N}}_2$, while loops are simply omitted for better readability. The breadth is $(3, 3)$ as well. On the right is the associated Mealy automaton.

Examples 19 and 20 describe important members from both these families.

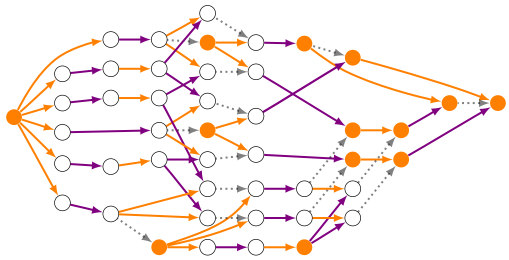


Figure 8 The minimal Garside family of the monoid $\mathbf{AK}_{+}^{1}\left(\begin{bmatrix}1&3&2\\4&1&3\\2&4&1\end{bmatrix}\right)$ from Example 19.

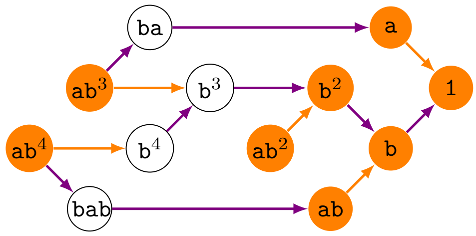


Figure 9 The minimal Garside family of the monoid $\mathbf{BS}_{+}^{1}(3,2)$ from Example 20.

▶ **Example 19.** The following Artin–Krammer monoid is emblematic:

$$\mathbf{AK}_{+}^{1}\left(\begin{bmatrix}1&3&2\\4&1&3\\2&4&1\end{bmatrix}\right)=\left\langle\ \mathtt{a},\mathtt{b},\mathtt{c}:\begin{array}{c}\mathtt{abab}=\mathtt{aba}\\ \mathtt{ac}=\mathtt{ca}\\ \mathtt{bcbc}=\mathtt{bcb}\end{array}\right\rangle_{+}^{1}.$$

As displayed on Figure 8, its minimal so-called Garside family forms like a flint which encodes its whole combinatorics and, according to Theorem 12, makes it an automaticon monoid.

▶ **Example 20.** Consider the Baumslag–Solitar monoid $\mathbf{BS}_{+}^{1}(3,2)=\langle\ \mathtt{a},\mathtt{b}:\mathtt{ab}^3=\mathtt{b}^2\mathtt{a}\ \rangle_{+}^{1}$. Displayed on Figure 9, its minimal Garside family contains eight elements (orange vertices) and makes it an automaticon monoid. This is an example of a group-embeddable automaton monoid whose enveloping group is not an automaton group. Indeed, the Baumslag–Solitar group $\mathbf{BS}(3,2)$ is precisely known as an example of non-residually finite group, hence cannot be an automaton group. The question remains open for those automaton semigroups whose enveloping group is a group of fractions, see Problem **C** below.

Concerning again group-embeddability, the following gives now an example of a cancellative automaton semigroup which is not group-embeddable.

▶ **Example 21.** The monoid $\mathbf{T}=\langle\ \mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{a}',\mathtt{b}',\mathtt{c}',\mathtt{d}':\mathtt{ab}=\mathtt{cd},\mathtt{a}'\mathtt{b}'=\mathtt{c}'\mathtt{d}',\mathtt{a}'\mathtt{d}=\mathtt{c}'\mathtt{b}\ \rangle_{+}^{1}$ is known (by Malcev work [41, 42, 43]) to be cancellative but not group-embeddable: from these three relations, we cannot deduce the relation $\mathtt{ad}'=\mathtt{cb}'$ that holds in the enveloping group. The quadratic normalisation $(\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{a}',\mathtt{b}',\mathtt{c}',\mathtt{d}'\},\mathrm{N})$ defined by $\mathrm{N}(\mathtt{ab})=\mathtt{cd}$, $\mathrm{N}(\mathtt{a}'\mathtt{b}')=\mathtt{c}'\mathtt{d}'$, and $\mathrm{N}(\mathtt{a}'\mathtt{d})=\mathtt{c}'\mathtt{b}$ for instance has breadth $(3,3)$, hence satisfies Condition ( ) and Theorem 12 applies. This answers in particular a question by Cain [15].

Some classes of neither left- nor right-cancellative monoids have been studied and shown to admit nice normal forms yielding biautomatic structures (see [16]):

▶ **Example 22.** According to Schützenberger [54], plactic monoids are among the most fundamental monoids. The rank 2 plactic monoid is $\mathbf{P}_2=\langle\ \mathtt{a},\mathtt{b}:\mathtt{aba}=\mathtt{baa},\mathtt{bab}=\mathtt{bba}\ \rangle_{+}^{1}$. As noted in [20, 22], $\mathbf{P}_2$ admits the quadratic normalisation $(\mathcal{Q},\mathrm{N})$ with $\mathcal{Q}=\{\mathtt{1},\mathtt{a},\mathtt{b},\mathtt{ba}\}$, $\mathrm{N}(\mathtt{ba})=\mathtt{1}(\mathtt{ba})$, $\mathrm{N}((\mathtt{ba})\mathtt{a})=\mathtt{a}(\mathtt{ba})$, $\mathrm{N}((\mathtt{ba})\mathtt{b})=\mathtt{b}(\mathtt{ba})$, $\mathrm{N}(\mathtt{1}x)=x\mathtt{1}$ for $x\in\mathcal{Q}$, and $\mathrm{N}(xy)=xy$ otherwise. The latter has a breadth $(3,3)$, hence satisfies Condition ( ) and Theorem 12 ensures that $\mathbf{P}_2$ is an automaton monoid. Note that, for a higher rank plactic monoid $\mathbf{P}_X$, it suffices to take again for $\mathcal{Q}$ the set of *columns*, that is, the strictly decreasing products of elements of X. Chinese monoids admit quadratic normalisations with breadth $(4,3)$, hence satisfy Condition ( ), and Theorem 12 ensures that they are automaticon monoids [16, 51].

To conclude, we would like to illustrate the duality between "being an automatic semigroup" and "being an automaton semigroup" by highlighting a paradigmatic example.

▶ **Example 23.** The braid monoids were used by Thurston [24, Chapter 9] to describe his idea to build a single transducer that computes the so-called Adjan–Garside–Thurston normal form via iterated runs. The n-strand braid monoid is

$$\mathbf{B}^1_{n+} = \left\langle \sigma_1, \ldots, \sigma_{n-1} : \begin{array}{cl} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| \leq 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \end{array} \right\rangle^1_+ .$$

Garside theory allows to build a suitable generating set $\mathcal{Q}$ of size $n!$ and a corresponding quadratic normalisation with breadth $(3,3)$. According to Corollary 15, its Thurston transducer and its Mealy automaton make therefore $\mathbf{B}^1_{n+}$ both an automatic and an automaton monoid. Such an approach may hopefully shed some light on the question of whether or not the braid groups are self-similar (Problem **B**). In particular, a positive answer to our following Problem **C** would imply a positive answer to Problem **B**.

Problem C. Is the group of fractions of an automaton monoid an automaton group?

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