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Dual properties of the relative belief of singletons

Fabio Cuzzolin
Fabio.Cuzzolin@inrialpes.fr

INRIA Rhone-Alpes

Abstract. In this paper we prove that a recent Bayesian approximation of belief functions, the relative belief of singletons, meets a number of properties with respect to Dempster's rule of combination which mirrors those satisfied by the relative plausibility of singletons. In particular, its operator commutes with Dempster's sum of plausibility functions, while perfectly representing a plausibility function when combined through Dempster's rule. This suggests a classification of all Bayesian approximations into two families according to the operator they relate to.

1 Introduction: A new Bayesian approximation

The theory of evidence (ToE) [1] extends classical probability theory through the notion of *belief function* (b.f.), a mathematical entity which independently assigns probability values to *sets* of possibilities rather than single events. A belief function $b : 2^\Theta \to [0,1]$ on a finite set ("frame") Θ has the form $b(A) = \sum_{B \subseteq A} m_b(B)$ where $m_b : 2^\Theta \to [0,1]$, is called "basic probability assignment" (b.p.a.), and meets normalization $\sum_{A \subseteq \Theta} m_b(A) = 1$ and positivity $m_b(A) \geq 0$ $\forall A \subseteq \Theta$ axioms. Events associated with non-zero basic probabilities are called "focal elements". A b.p.a. can be uniquely recovered from a belief function through Moebius inversion:

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B). \tag{1}$$

As probability measures or *Bayesian* belief functions are just a special class of b.f.s (for which $m(A) = 0$ when $|A| > 1$), the relation between beliefs and probabilities plays a major role in the theory of evidence [2–6]. Tessem [7], for instance, incorporated only the highest-valued focal elements in his m_{klx} approximation. In Smets' "Transferable Belief Model" [8] beliefs are represented at credal level (as convex sets of probabilities), while decisions are made by resorting to a Bayesian belief function called pignistic transformation [9]. More recently, two new Bayesian approximations of b.f.s have been derived from purely geometric considerations [10] in the context of the geometric approach to the ToE.

Another classical approximation is based on the plausibility function (pl.f.) $pl_b : 2^\Theta \to [0,1]$, where

$$pl_b(A) \doteq 1 - b(A^c) = \sum_{B \cap A \neq \emptyset} m_b(B)$$

represent of the evidence not against a proposition A. Voorbraak [11] proposed the so-called *relative plausibility of singletons* (rel.plaus.) $\tilde{pl}_b$ as the unique probability that, given a belief function b with plausibility pl_b, assigns to each singleton $x \in \Theta$ its normalized plausibility:

$$\tilde{pl}_b(x) = \frac{pl_b(x)}{\sum_{y \in \Theta} pl_b(y)}. \tag{2}$$

He proved that $\tilde{pl}_b$ is a perfect representative of b when combined with other probabilities p through Dempster's rule $\oplus$ [12]: $\tilde{pl}_b \oplus p = b \oplus p$. Its properties have been later discussed by Cobb and Shenoy [13].

Another Bayesian approximation based on normalizing the *belief* (instead of plausibility) values of singletons has been recently introduced [14]:

$$\tilde{b}(x) \doteq \frac{b(x)}{\sum_{y \in \Theta} b(y)} = \frac{m_b(x)}{\sum_{y \in \Theta} m_b(y)}. \tag{3}$$

(3) is called *relative belief of singletons* $\tilde{b}$ (rel.bel.). Clearly $\tilde{b}$ exists iff b assigns some mass to singletons:

$$\sum_{x \in \Theta} m_b(x) \neq 0. \tag{4}$$

The different semantics and limitations of the relative belief of singletons have been studied by F. Cuzzolin [14]. In particular, rel.bel. provides a conservative estimate of the evidence supporting each singleton $x \in \Theta$, and can indeed be seen as the relative plausibility of a plausibility function.

1.1 Aim of the paper

On our side, in this paper we focus on the behavior of the relative belief of singletons with respect to evidence combination in the form of Dempster's combination rule. We prove that rel.bel. meets a number of properties with respect to Dempster's rule of combination which mirrors those satisfied by the relative plausibility of singletons (2). In particular: 1. its operator commutes with Dempster's sum of plausibility functions, and 2. rel.bel. perfectly represents a plausibility function when combined through Dempster's rule.
These results together with those holding for the relative plausibility suggest a clear subdivision of all Bayesian approximations in two families, related to Dempster's sum and affine combination respectively.

After briefly recalling the different semantics of the relative belief of singletons we summarize the properties of rel.plaus. with respect to Dempster's rule, whose dual we are going to prove here for $\tilde{b}$. To this purpose we introduce the notion of "pseudo belief functions", i.e. b.f.s which admit negative b.p.a.s, as the basic tool we need in the course of this work.
We prove that the relative belief operator commutes with respect to Dempster's combination of plausibility functions, and enjoys idempotence properties similar to those met by the relative plausibility. Analogously, convergence results for

rel.bel. can also be proven. In the last Section we prove that the relative belief of singletons perfectly represents the corresponding plausibility function pl_b when combined with any probability through (extended) Dempster's rule.

2 Semantics of the relative belief of singletons

2.1 A conservative estimate

A first insight on the meaning of $\tilde{b}$ comes from the original semantics of belief functions as constraints on the actual allocation of mass of an underlying unknown probability distribution. A focal element A with mass $m_b(A)$ indicates that this mass can "float" around in A and be distributed arbitrarily between the elements of A. In this framework $\tilde{pl}_b$ (2) can be interpreted as follows:

- for each singleton $x \in \Theta$ the most optimistic hypothesis in which the mass of *all* $A \supseteq \{x\}$ focuses on x is considered, yielding $\{pl_b(x), x \in \Theta\}$;
- this assumption, however, is contradictory as it is supposed to hold for all singletons (many of which belong to the same higher-size events);
- nevertheless, the obtained values are normalized to yield a Bayesian belief function.

$\tilde{pl}_b$ is associated with the less conservative (but incoherent) scenario in which all the mass that can be assigned to a singleton is actually assigned to it.
The relative belief of singletons (3) has in turn the following interpretation in terms of mass assignments:

- for each singleton $x \in \Theta$ the most *pessimistic* hypothesis in which only the mass of $\{x\}$ itself actually focuses on x is considered, yielding $\{b(x) = m_b(x), x \in \Theta\}$;
- this assumption is also contradictory, as the mass of all higher-size events is not assigned to any singletons;
- the obtained values are again normalized.

Dually, $\tilde{b}$ reflects the most conservative (but still not coherent) choice of assigning to x only the mass that the b.f. b (seen as a constraint) assures it belong to x, in perfect analogy to the case of rel.plaus.

One can argue that the existence of rel.bel. is subject to quite a strong condition (4). However it can be proven that the case in which $\tilde{b}$ does not exist is indeed pathological, as it excludes a great deal of belief and probability measures [14].

2.2 Convergence under quasi-Bayesianity

A different angle on the utility of $\tilde{b}$ comes from a discussion of what classes of b.f.s are "suitable" to be approximated by means of (3). As it only makes use of the masses of singletons, working with $\tilde{b}$ requires storing n values to represent a belief function. As a consequence, the computational cost of combining new

evidence through Dempster's rule or disjunctive combination [15] is reduced to $O(n)$ as only the mass of singletons has to be calculated.
When the actual values of $\tilde{b}(x)$ are close to those provided by, for instance, pignistic function or rel.plaus. is then more convenient to resort to the relative belief transformation.

Let us call *quasi-Bayesian* b.f.s the belief functions b for which the mass assigned to singletons is very close to one:

$$k_{m_b} \doteq \sum_{x \in \Theta} m_b(x) \to 1.$$

Proposition 1. *For quasi-Bayesian b.f.s all Bayesian approximations converge:*

$$\lim_{k_{m_b} \to 1} BetP[b] = \lim_{k_{m_b} \to 1} \tilde{pl}_b = \lim_{k_{m_b} \to 1} \tilde{b}.$$

For quasi-Bayesian b.f.s the relative belief works as a low-cost proxy for the other Bayesian approximations.

3 Relative plausibility and Dempster's rule

Rel.bel. and rel.plaus. are then strictly related. In this paper we prove indeed that $\tilde{b}$ and $\tilde{pl}_b$ share also an intimate relationship with Dempster's evidence combination rule $\oplus$, as they meet a set of dual properties with respect to $\oplus$.

Definition 1. *The* orthogonal sum *or* Dempster's sum *of two belief functions b_1, b_2 on the same frame Θ is a new belief function $b_1 \oplus b_2$ on Θ with b.p.a.*

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B)\, m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B)\, m_{b_2}(C)} \qquad (5)$$

where m_{b_i} denotes the b.p.a. associated with b_i.

We denote with $k(b_1, b_2)$ the denominator of Equation (5).

Cobb and Shenoy [13] proved that the relative plausibility function $\tilde{pl}_b$ commutes with respect to Dempster's rule. More precisely, they proved that the relative plausibility of singletons meets the following properties[1] which relates to Dempster's combination rule.

Proposition 2. *1. if $b = b_1 \oplus \cdots \oplus b_m$ then $\tilde{pl}_b = \tilde{pl}_{b_1} \oplus \cdots \oplus \tilde{pl}_{b_m}$. In other words, Dempster's sum and the relative plausibility operator*

$$\begin{array}{rl} \tilde{pl} : \mathcal{B} \to & \mathcal{P} \\ b \mapsto & \tilde{pl}[b] = \tilde{pl}_b \end{array} \qquad (6)$$

commute.

[1] Original statements from [13] have been reformulated according to the notation of this paper.

2. *if* m_b *is idempotent with respect to Dempster's rule, i.e.* $m_b \oplus m_b = m_b$*, then* $\tilde{pl}_b$ *is idempotent with respect to* $\oplus$.
3. *let us define the limit of a belief function* b *as*

$$b^\infty \doteq \lim_{n\to\infty} b^n \doteq \lim_{n\to\infty} b \oplus \cdots \oplus b \quad (n \text{ times}); \tag{7}$$

   if $\exists x \in \Theta$ *such that* $pl_b(x) > pl_b(y)\ \forall y \neq x, y \in \Theta$*, then* $\tilde{pl}_{b^\infty}(x) = 1$, $\tilde{pl}_{b^\infty}(y) = 0\ \forall y \neq x$.
4. *if* $\exists A \subseteq \Theta$ $(|A| = k)$ *s.t.* $pl_b(x) = pl_b(y)\ \forall x, y \in A$, $pl_b(x) > pl_b(z)\ \forall x \in A, z \in A^c$*, then* $\tilde{pl}_{b^\infty}(x) = \tilde{pl}_{b^\infty}(y) = 1/k\ \forall x, y \in A$, $\tilde{pl}_{b^\infty}(z) = 0\ \forall z \in A^c$.

On his side, Voorbraak has shown [11] that the relative plausibility function perfectly represents a belief function when combined with a probability.

Proposition 3. *The relative plausibility of singletons* $\tilde{pl}_b$ *is a perfect representative of* b *in the probability space when combined through Dempster's rule, i.e.*

$$b \oplus p = \tilde{pl}_b \oplus p, \ \forall p \in \mathcal{P}.$$

4 Pseudo belief functions

The study of the properties of $\tilde{b}$ requires first to extend the set of objects we work on from that of b.f.s to the more general class of "pseudo belief functions". Namely, the b.p.a. m_b associated with a b.f. b meets the positivity axiom: $m_b(A) \geq 0\ \forall A \subseteq \Theta$. If we relax this condition we get functions ς of the form

$$\varsigma(A) = \sum_{B\subseteq A} m_\varsigma(B).$$

or *sum function* [16] whose Moebius inverse (1) $m_\varsigma : 2^\Theta \setminus \emptyset \to \mathbb{R}$ may assume negative values: $m_\varsigma(B) \not\geq 0\ \forall B \subseteq \Theta$. Sum functions meeting the normalization axiom $\sum_{\emptyset \subsetneq A \subseteq \Theta} m_\varsigma(A) = 1$, or *pseudo belief functions* (p.b.f.s) [17], are then natural extensions of belief functions.

4.1 Plausibilities as pseudo belief functions

Plausibility functions are p.b.f.s, as they meet the normalization constraint $pl_b(\Theta) = 1$ for all b. Their Moebius inverse [18]

$$\mu_b(A) \doteq \sum_{B\subseteq A} (-1)^{|A\setminus B|} pl_b(B) = (-1)^{|A|+1} \sum_{B\supseteq A} m_b(B) \tag{8}$$

when $A \neq \emptyset$, $\mu_b(\emptyset) = 0$ is called *basic plausibility assignment* (b.pl.a.).
Each pl.f. is an affine combination of *basis* belief functions

$$b_A \doteq b \in \mathcal{B} \ s.t.\ m_b(A) = 1, m_b(B) = 0\ \forall B \neq A \tag{9}$$

with coefficients given by its b.pl.a. [18]:

$$pl_b = \sum_{A\subseteq\Theta} \mu_b(A) b_A. \tag{10}$$

4.2 Dempster's sum of pseudo belief functions

The orthogonal sum can be naturally extended to pseudo b.f.s by applying (5) to the Moebius inverses m_{ς_1}, m_{ς_2} of a pair of p.b.f.s. As Cuzzolin has proven [19]

Proposition 4. *Dempster's rule defined as in Equation (5) when applied to a pair of pseudo belief functions ς_1, ς_2 yields again a pseudo belief function.*

We denote the orthogonal sum of two p.b.f.s ς_1, ς_2 by $\varsigma_1 \oplus \varsigma_2$.

5 Dual results for relative belief operator

5.1 The relative belief operator

As pl.f.s are pseudo b.f.s, Dempster's rule can then be formally applied to pl.f.s too. We can then prove a dual commutativity result for relative beliefs, once introduced (in full analogy to what done for the other Bayesian approximations) the *relative belief operator*

$$\begin{array}{rl} \tilde{b} : \mathcal{PL} \rightarrow & \mathcal{P} \\ pl_b \mapsto & \tilde{b}[pl_b] \end{array}$$

where

$$\tilde{b}[pl_b](x) \doteq \frac{m_b(x)}{\sum_{y \in \Theta} m_b(y)} \qquad \forall x \in \Theta \tag{11}$$

is defined as usual for b.f.s b such that $\sum_x m_b(x) \neq 0$.
As a matter of fact, since b and pl_b are in 1-1 correspondence, we could indifferently define two operators mapping respectively a belief function b onto its relative belief, *or* the unique plausibility function pl_b associated with b onto $\tilde{b}$. We chose to consider the operator in this second form as this is instrumental to prove the following theorem, the dual of point 1. in Proposition 2.

5.2 Commutativity

A useful property of μ_b is that [14]

Lemma 1. $m_b(x) = \sum_{A \supseteq \{x\}} \mu_b(A)$.

Theorem 1. *The relative belief operator commutes with respect to Dempster's combination of plausibility functions, namely*

$$\tilde{b}[pl_1 \oplus pl_2] = \tilde{b}[pl_1] \oplus \tilde{b}[pl_2].$$

Theorem 1 implies that

$$\tilde{b}[(pl_b)^n] = (\tilde{b}[pl_b])^n. \tag{12}$$

5.3 Idempotence

Another consequence of Theorem 1 is an idempotence property which is the dual of point 2. of Proposition 2.

Theorem 2. *If pl_b is idempotent with respect to Dempster's rule, i.e. $pl_b \oplus pl_b = pl_b$, then $\tilde{b}[pl_b]$ is itself idempotent: $\tilde{b}[pl_b] \oplus \tilde{b}[pl_b] = \tilde{b}[pl_b]$.*

Proof. By Theorem 1 $\tilde{b}[pl_b] \oplus \tilde{b}[pl_b] = \tilde{b}[pl_b \oplus pl_b]$, and if $pl_b \oplus pl_b = pl_b$ the thesis immediately follows. □

5.4 Convergence

The dual statements of the convergence results of Proposition 2 can be proven in a similar fashion.

Theorem 3. *If $\exists x \in \Theta$ such that $b(x) > b(y)$ $\forall y \neq x, y \in \Theta$ then*

$$\tilde{b}[pl_b^\infty](x) = 1, \; \tilde{b}[pl_b^\infty](y) = 0 \quad \forall y \neq x.$$

A similar proof can be provided for the following generalization of Theorem 3.

Theorem 4. *if $\exists A \subseteq \Theta$ ($|A| = k$) s.t. $b(x) = b(y)$ $\forall x, y \in A$, $b(x) > b(z)$ $\forall x \in A, z \in A^c$ then*

$$\tilde{b}[pl_b^\infty](x) = \tilde{b}[pl_b^\infty](y) = 1/k \quad \forall x, y \in A, \qquad \tilde{b}[pl_b^\infty](z) = 0 \quad \forall z \in A^c.$$

5.5 Example

Let us consider the belief function b on the frame of size four $\Theta = \{x, y, z, w\}$ defined by the following basic probability assignment:

$$m_b(\{x, y\}) = 0.4, \quad m_b(\{y, z\}) = 0.4, \quad m_b(w) = 0.2. \tag{13}$$

The corresponding b.pl.a. is by (8)

$$\begin{array}{lll} \mu_b(x) = 0.4, & \mu_b(y) = 0.8, & \mu_b(z) = 0.4, \\ \mu_b(w) = 0.2, & \mu_b(\{x, y\}) = -0.4, & \mu_b(\{y, z\}) = -0.4. \end{array} \tag{14}$$

To check the validity of Theorems 1 and 3 let us then compute the series $(\tilde{b}[pl_b])^n$ and $\tilde{b}[(pl_b)^n]$. By applying Dempster's rule to the b.pl.a. (14) ($pl_b^2 = pl_b \oplus pl_b$) we get a new b.pl.a. μ_b^2 with (see Figure 1)

$$\begin{array}{lll} \mu_b^2(x) = 4/7, & \mu_b^2(y) = 8/7, & \mu_b^2(z) = 4/7, \\ \mu_b^2(w) = -1/7, & \mu_b^2(\{x, y\}) = -4/7, & \mu_b^2(\{y, z\}) = -4/7. \end{array}$$

To compute the corresponding relative belief $\tilde{b}[pl_b^2]$ we first need to get the plausibility values

$$\begin{array}{l} pl_b^2(\{x, y, z\}) = \mu_b^2(x) + \mu_b^2(y) + \mu_b^2(z) + \mu_b^2(\{x, y\}) + \mu_b^2(\{y, z\}) = 8/7, \\ pl_b^2(\{x, y, w\}) = pl_b^2(\{x, z, w\}) = pl_b^2(\{y, z, w\}) = 1 \end{array}$$

	{x}	{y}	{z}	{w}	{x,y}	{y,z}
{y,z}		{y}	{z}		{y}	{y,z}
{x,y}	{x}	{y}			{x,y}	{y}
{w}				{w}		
{z}			{z}			{z}
{y}		{y}			{y}	{y}
{x}	{x}				{x}	

Fig. 1. Intersection of focal elements in Dempster's combination of the b.pl.a. (14) with itself. Non-zero mass events for each addendum $\mu_1 = \mu_2 = \mu_b$ correspond to rows/columns of the table, each entry of the table hosting the related intersection.

which imply by Definition $pl_b(A) \doteq 1 - b(A^c)$

$$b^2(w) = -1/7, \quad b^2(z) = 0, \quad b^2(y) = 0, \quad b^2(x) = 0$$

i.e. $\tilde{b}[pl_b^2] = [0,0,0,1]'$.
Theorem 1 is confirmed as by (13) (being $\{w\}$ the only singleton with non-zero mass) $\tilde{b} = [0,0,0,1]'$ so that $\tilde{b} \oplus \tilde{b} = [0,0,0,1]'$ and $\tilde{b}[.]$ commutes with $pl_b\oplus$.
By combining pl_b^2 with pl_b one more time we get the b.pl.a.

$$\begin{array}{ll} \mu_b^3(x) = \mu_b^3(z) = 16/31, & \mu_b^3(y) = 32/31, \\ \mu_b^3(w) = -1/31, & \mu_b^3(\{x,y\}) = \mu_b^3(\{y,z\}) = -16/31 \end{array}$$

which corresponds to

$$\begin{array}{ll} pl_b^3(\{x,y,z\}) = 32/31, & pl_b^3(\{x,y,w\}) = 1, \\ pl_b^3(\{x,z,w\}) = 1, & pl_b^3(\{y,z,w\}) = 1 \end{array}$$

i.e.

$$b^3(w) = -1/31, \;\; b^3(z) = 0, \;\; b^3(y) = 0, \;\; b^3(x) = 0$$

and $\tilde{b}[pl_b^3] = [0,0,0,1]'$ which again is equal to $\tilde{b} \oplus \tilde{b} \oplus \tilde{b}$ as Theorem 1 guarantees.
Clearly the series of the basic plausibilities $(\mu_b)^n$ converges to

$$\begin{array}{lll} \mu_b^n(x) \to 1/2^+, & \mu_b^3(y) \to 1^+, & \mu_b^3(z) \to 1/2^+, \\ \mu_b^3(w) \to 0^-, & \mu_b^3(\{x,y\}) \to -1/2^-, & \mu_b^3(\{y,z\}) \to -1/2^- \end{array}$$

associated with the following plausibility values

$$\begin{array}{lll} \lim_{n \to \infty} pl_b^n(\{x,y,z\}) = 1^+, & pl_b^n(\{x,y,w\}) = 1, & \\ pl_b^n(\{x,z,w\}) = 1, & pl_b^n(\{y,z,w\}) = 1 & \forall n \geq 1 \end{array}$$

which correspond to $\lim_{n\rightarrow\infty} b^n(w) = 0^-, b^n(z) = b^n(y) = b^n(x) = 0 \ \forall n \geq 1$, so that

$$\begin{array}{ll}\lim_{n\rightarrow\infty} \tilde{b}[pl_b^\infty](w) = \lim_{n\rightarrow\infty} \frac{b^n(w)}{b^n(w)} = 1 \\ \lim_{n\rightarrow\infty} \tilde{b}[pl_b^\infty](x) & = \lim_{n\rightarrow\infty} \tilde{b}[pl_b^\infty](y) = \\ \lim_{n\rightarrow\infty} \tilde{b}[pl_b^\infty](z) & = \lim_{n\rightarrow\infty} \frac{0}{b^n(w)} = \lim_{n\rightarrow\infty} 0 = 0\end{array}$$

in agreement with Theorem 3.

5.6 Combination of plausibilities versus combination of beliefs

It is crucial to notice that Theorem 1 (and by consequence Theorem 3) are about combination of *plausibility functions* (as pseudo b.f.s) and *not* combinations of belief functions. Hence, it is *not* true in general that $\widetilde{b^\infty} = (\tilde{b})^\infty$ or for that matters that commutativity holds. If we go back to the above example, it is straightforward to see that the combination $b \oplus b$ of b with itself has b.p.a.

$$\begin{array}{ll}m_{b\oplus b}(\{x,y\}) = \dfrac{m_b(\{x,y\})m_b(\{x,y\})}{k(b,b)} = \dfrac{0.16}{0.68} = 0.235, \\ m_{b\oplus b}(\{y,z\}) = \dfrac{m_b(\{y,z\})m_b(\{y,z\})}{k(b,b)} = \dfrac{0.16}{0.68} = 0.235, \\ m_{b\oplus b}(w) \quad\quad = \dfrac{m_b(w)m_b(w)}{k(b,b)} = \dfrac{0.04}{0.68} = 0.058, \ m_{b\oplus b}(y) \\ \quad\quad\quad\quad\quad\quad = \dfrac{m_b(\{x,y\})m_b(\{y,z\}) + m_b(\{y,z\})m_b(\{x,y\})}{k(b,b)} = \dfrac{0.32}{0.68} = 0.47\end{array}$$

which obviously yields

$$\widetilde{b \oplus b} = \left[0, \frac{0.47}{0.528}, 0, \frac{0.058}{0.528}\right]' \neq \tilde{b} \oplus \tilde{b} = [0,0,0,1]'.$$

The basic reason for this is that the plausibility function of a sum of two belief functions is *not* the sum of the associated plausibilities:

$$[pl_{b_1} \oplus pl_{b_2}] \neq pl_{b_1 \oplus b_2}.$$

6 Representation theorem for relative beliefs

A dual of the representation theorem (Proposition 3) for relative beliefs can also be proven, once we recall a result on Dempster's sum of affine combinations [19].

Proposition 5. *The orthogonal sum* $b \oplus \sum_i \alpha_i b_i$, $\sum_i \alpha_i = 1$ *of a b.f.* b *with any*[2] *affine combination of b.f.s can be written as* $b \oplus \sum_i \alpha_i b_i = \sum_i \gamma_i (b \oplus b_i)$, *where*

$$\gamma_i = \frac{\alpha_i k(b,b_i)}{\sum_j \alpha_j k(b,b_j)} \tag{15}$$

and $k(b,b_i)$ *is the normalization factor of the combination between* b *and* b_i.

[2] In fact the collection $\{b_i\}$ is required to include *at least* a b.f. which is combinable with b, [19].

Theorem 5. *The relative belief of singletons $\tilde{b}$ represents perfectly the corresponding plausibility function pl_b when combined with any probability through (extended) Dempster's rule:*

$$\tilde{b} \oplus p = pl_b \oplus p$$

for each Bayesian belief function $p \in \mathcal{P}$.

Theorem 5 can be obtained by replacing b with pl_b, and $\tilde{pl}_b$ by $\tilde{b}$ in Proposition 3. It is natural to suppose other properties of upper probabilities could in the future be found by analogous transformations of known propositions on lower probabilities, as a useful mathematical characterization of the relation between them.

7 Conclusions: Two families of Bayesian approximations

In this paper we studied the properties of the relative belief of singletons as a novel Bayesian approximation of a belief function, and discussed its interpretations and applicability. We proved that relative belief and plausibility of singletons form a distinct family of Bayesian approximations related to Dempster's rule, as they both commute with $\oplus$, and meet dual representation and idempotence properties. On one side, this suggests a new mathematical form of the duality which exists between upper and lower probabilities that can be used to prove new results. On the other side, once we recall that [10]

Proposition 6. *Both pignistic function $BetP[b]$ and orthogonal projection $\pi[b]$ commute with respect to affine combination:*

$$\pi\Big[\sum_i \alpha_i b_i\Big] = \sum_i \alpha_i \pi[b_i], \quad BetP\Big[\sum_i \alpha_i b_i\Big] = \sum_i \alpha_i BetP[b_i], \quad \sum_i \alpha_i = 1.$$

the present results bring about a subdivision of all Bayesian approximations in two families, related to Dempster's sum and affine combination respectively.

Appendix

Proof of Theorem 1

The basic plausibility assignment of $pl_1 \oplus pl_2$ is, according to (5),

$$\mu_{pl_1 \oplus pl_2}(A) = \frac{1}{k(pl_1, pl_2)} \sum_{X \cap Y = A} \mu_1(X)\mu_2(Y)$$

so that the corresponding relative belief of singletons $\tilde{b}[pl_1 \oplus pl_2](x)$ (11) is proportional to

$$\begin{aligned} m_{pl_1 \oplus pl_2}(x) = \sum_{A \supseteq \{x\}} \mu_{pl_1 \oplus pl_2}(A) &= \frac{\sum_{A \supseteq \{x\}} \sum_{X \cap Y = A} \mu_1(X)\mu_2(Y)}{k(pl_1, pl_2)} \\ &= \frac{\sum_{X \cap Y \supseteq \{x\}} \mu_1(X)\mu_2(Y)}{k(pl_1, pl_2)} \end{aligned} \qquad (16)$$

where $m_{pl_1 \oplus pl_2}(x)$ denotes the b.p.a. of the (pseudo) b.f. corresponding to the pl.f. $pl_1 \oplus pl_2$. As $\sum_{X \supseteq \{x\}} \mu_b(X) = m_b(x)$ by Lemma 1,

$$\tilde{b}[pl_1](x) \propto m_1(x) = \sum_{X \supseteq \{x\}} \mu_1(X), \quad \tilde{b}[pl_2](x) \propto m_2(x) = \sum_{X \supseteq \{x\}} \mu_2(X)$$

so that their Dempster's combination is

$$(\tilde{b}[pl_1] \oplus \tilde{b}[pl_2])(x) \propto \Big(\sum_{X \supseteq \{x\}} \mu_1(X) \Big) \Big(\sum_{Y \supseteq \{x\}} \mu_2(Y) \Big) = \sum_{X \cap Y \supseteq \{x\}} \mu_1(X) \mu_2(Y)$$

and by normalizing we get (16).

Proof of Theorem 3

Taking the limit on both sides of Equation (12) we get

$$\tilde{b}[pl_b^\infty] = (\tilde{b}[pl_b])^\infty. \tag{17}$$

Let us now focus on the quantity on the right hand side: $(\tilde{b}[pl_b])^\infty = \lim_{n \to \infty} (\tilde{b}[pl_b])^n$. Since $(\tilde{b}[pl_b])^n(x) = K(b(x))^n$ (where K is a constant independent on x) and x is the unique most believed state, it follows that

$$(\tilde{b}[pl_b])^\infty(x) = 1, \quad (\tilde{b}[pl_b])^\infty(y) = 0 \;\; \forall y \neq x. \tag{18}$$

Hence by (17) $\tilde{b}[pl_b^\infty](x) = 1$, and $\tilde{b}[pl_b^\infty](y) = 0 \; \forall y \neq x$.

Proof of Theorem 5

Once expressed a plausibility function in terms of its basic plausibility assignment (10) we can apply the commutativity property (Proposition 5), obtaining

$$pl_b \oplus p = \sum_{A \subseteq \Theta} \nu(A) p \oplus b_A \tag{19}$$

where

$$\nu(A) = \frac{\mu_b(A) k(p, b_A)}{\sum_{B \subseteq \Theta} \mu_b(B) k(p, b_B)}, \quad p \oplus b_A = \frac{\sum_{x \in A} p(x) b_x}{k(p, b_A)}$$

with $k(p, b_A) = \sum_{x \in A} p(x)$. Once replaced these expressions in (19) we get

$$pl_b \oplus p = \frac{\sum\limits_{A \subseteq \Theta} \mu_b(A) \big(\sum\limits_{x \in A} p(x) b_x \big)}{\sum\limits_{B \subseteq \Theta} \mu_b(B) \big(\sum\limits_{y \in B} p(y) \big)} = \frac{\sum\limits_{x \in \Theta} p(x) \big(\sum\limits_{A \supseteq \{x\}} \mu_b(A) \big) b_x}{\sum\limits_{y \in \Theta} p(y) \big(\sum\limits_{B \supseteq \{y\}} \mu_b(B) \big)} = \frac{\sum\limits_{x \in \Theta} p(x) m_b(x) b_x}{\sum\limits_{y \in \Theta} p(y) m_b(y)}$$

again by Lemma 1. But this is exactly $\tilde{b} \oplus p$, as a direct application of Dempster's rule (5) shows.

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