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Kuo-Liang Chang
Utah Valley University, KChang@uvu.edu

Lorraine Males
University of Nebraska-Lincoln, lmales2@unl.edu

Aaron Mosier
Michigan State University, mosieraa@msu.edu

Funda Gonulates
Michigan State University, gonulate@msu.edu

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Kuo-Liang Chang, Lorraine M. Males,
Aaron Mosier, & Funda Gonulates

K.-L. Chang (Corresponding author), Developmental Mathematics Department, Utah Valley University, LA 224B, 800 West University Parkway, Orem, UT 84058, USA; *email* KChang@uvu.edu

L. M. Males & **F. Gonulates,** Division of Science and Mathematics Education, Michigan State University, A722 Wells Hall, East Lansing, MI, USA; F. Gonulates: *email* gonulate@msu.edu

A. Mosier, Division of Science and Mathematics Education, Michigan State University, 143 Erickson Hall, East Lansing, MI, USA; *email* mosieraa@msu.edu

Abstract

Learning to estimate a linear measurement is critical in becoming a successful measurer. Research indicates that the teaching of the estimation of linear measurement is quite open and that instruction does not make explicit to students how to carry out estimation work. Because written curriculum has been identified as one of the main sources affecting teachers' instruction and students' learning, this study examined how estimation of linear measurement tasks were presented to students in three US elementary mathematics curricula to see how much and in what ways these tasks were presented in an open manner. The principal result was that the length estimation tasks were frequently not explicit about which attribute of the object to measure and the requested level of precision of the estimate. Length estimation tasks were also left more open than other measurement tasks like measuring length with rulers.

Keywords: Estimation, Length measurement, Openness, Curriculum analysis

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1 Background

Linear measurement involves the application of a physical standard or non-standard length unit to quantify the length of a given segment, object, or path. The resulting length measure is the number of units required to exhaust (fill up) the space in each case. Sometimes in our measurement activity, we apply tools like rulers to measure a segment, object, or path to some degree of precision. Many other times, we must estimate, because tools are not available, or an estimate is all that is needed or appropriate, or the context makes measurement with physical tools impossible. The estimation of linear measurement involves applying an imagined standard or non-standard length unit to mentally quantify a given segment, object, or path. In contrast to measurement with rulers or non-standard length units, the units of length in estimation must be generated and supplied by the measurer.

Learning to estimate linear measurement is critical because such estimations are used on a daily basis (Joram, Gabriele, and Bertheau 2005; Markovits and Hershkowitz 1997) and because it is critical in becoming a successful measurer. To estimate a linear measure one must perform a physical measurement in the absence of tools (Bright 1976). The absence of a tool means that there are no physical markers for students to rely on, so students must have knowledge of measurement principles and be able to apply them (Joram, Subrahmanyam, and Gelman 1998). The estimation tasks equip students with the means to deal with situations that arise in mathematical and everyday contexts for which precise calculations are not possible or necessary (Levine 1982). In addition, these estimation activities provide students with a means for determining the reasonableness of measurements (Coburn and Shulte 1986). The estimation of linear measurement activities are also useful because they not only help children make connections among measurement concepts, but these activities also contribute to the development of other mathematical concepts such as number and counting (Bright 1976) and fraction and ratio (Joram, Gabriele, and Bertheau 2005). Finally, the estimation tasks help students develop higher order thinking and problem solving skills (Ainley 1991; Dowker 1992).

1.1 Estimation of linear measurement

The importance of the estimation of linear measurement is widely recognized. The National Council of Teachers of Mathematics (NCTM) advocates the teaching of measurement estimation as early as pre-kindergarten and emphasizes it in Grades 3–5 (NCTM 2000). Although the estimation of linear measurement is seen as an important topic of study in its own right in the USA, it also serves as a tool for engaging students with the underlying concepts that allow them to be successful measurers. According to Lehrer (2003), to measure is not only to do, but to ''imagine qualities of the world, such as length and time'' (p. 179). This imagining is a key piece in the estimation of linear measurement; it is through this imagining that students are engaged in the underlying concepts that help in developing a robust understanding of unit and what it means to measure something, such as unit iteration, unit-measure compensation[1] and the meaning of length measure. For example, when students are asked to estimate the length of an item like a pencil, a student must understand that length quantifies the distance along a path, in this case the path from one end of the pencil to the other. Furthermore, students must imagine a unit and mentally iterate this unit along the path of the pencil, being sure not to overlap or leave gaps between the units, and making sure to exhaust the path. It is also common for students to be asked to estimate the same attribute using different units, invoking the concept of unit-measure compensation, the idea that when one uses a smaller unit of measure, one obtains a larger measure. These mental activities may be necessary, not only in the context of a problem that asks students to produce an estimate of a linear measure, but also when asked to produce a measure, as Piaget believed that having an understanding of measure meant that a person must be able to successively mentally restructure space (Piaget and Inhelder 1948/1956; Piaget, Inhelder, and Szeminska 1960).

1 Larger units of length produce smaller measures of length; smaller units of length produce larger measures.

1.2 Openness in the teaching of estimation of linear measurement

The teaching of linear estimation has been described as quite open—that instruction does not make explicit to students how to estimate or what decisions to attend to in carrying out estimation work. Joram et al. (2005) indicated that mathematics educators and teachers have often simply asked students to guess when estimating a given dimension of an object. Lang (2001) indicated that teachers often encountered difficulty in teaching measurement estimation to young children, noting that ‘‘most teachers are uncertain about how to build students’ understanding of what constitutes a reasonable estimate’’ (p. 462). Forrester and Pike (1998) examined two teachers’ teaching and their students’ learning of measurement estimation in separate primary schools to examine the discussions and activities in classrooms. Their results showed that measurement estimation discussed in the classes was associated with vagueness and guessing. They also found the orientation to the right answer regarding an estimate ‘‘was never accompanied by any discussion of proximity, purpose, or underlying rationale’’ (p. 352).

In this study, we use the term ‘‘open’’ to apply to tasks whose essential elements or components are not clearly specified for students. For example, in teaching estimation of linear measurement, a teacher may ask students to ‘‘Estimate the length of your shoe.’’ In response to this request, students may wonder which attribute of their shoe they should estimate, what unit of length they should use, and how precise a ‘‘correct’’ estimate needs to be. Because of the nature of estimates (in this case, approximate linear measures), some openness is to be expected. But arguably, too much openness can leave students unclear what the reasoning process is that they should undertake or, worse, think that any ‘‘guess’’ is an acceptable estimate.

1.3 The benefits and risks of openness

An appropriate level of openness in mathematical tasks can help students think about, explore, and investigate mathematical ideas. For example, Sullivan (2009) notes ‘‘opening up tasks can encourage pupils to investigate, make decisions, generalize, seek patterns and connections, communicate, discuss, and identify alternatives’’ (p. 726). Alro and Skovsmose (1998) note "we can see the openness of the situation as an invitation in practice to bring the students into a discussion of the teaching intentions. Such discussions are all too seldom present in mathematics education" (p. 50). In addition, openness facilitates creativity. Openness offers the opportunity for one to investigate and create solutions. Silver (1997) notes "The development of students' creative fluency is also likely to be encouraged through the classroom use of ill-structured, open-ended problems that are stated in a manner that permits the generation of multiple specific goals and possibly multiple correct solutions, depending upon one's interpretation" (p. 77).

However, too much openness can also cause confusion, misunderstanding, or misconceptions in students' learning (Christiansen 1997; Morin and Franks 2010; Suzawa 2003; Voigt 1994). Alro and Skovsmose (1998) note "openness can lead to confusion, which makes it difficult for the students to see the intentions and thus also makes it difficult for them to take part in a joint effort" (p. 49). Sullivan (2003) notes that some pupils may be disadvantaged by an open-ended approach because the style of interaction may require appreciation of features such as the desired ways of thinking and interacting, the kinds of reasoning valued, and semantic structures used. To illustrate, Christiansen (1997) notes that when an open task that asked students to describe a pattern from a provided set of points, confusion surfaced because the students and teacher were not focused on the same aspects of the problem. The teacher expected them to perceive a linear pattern, but instead students focused more narrowly on the patterns of the slopes between individual points and claimed that there was no linear pattern.

1.4 The purpose of study

Given the research cited above about the importance of linear estimation and the presentation of estimation of linear measurement tasks in classrooms, we were interested in exploring the possibility that relatively open enacted estimation of linear measurement tasks might arise from relatively open written estimation of linear measurement tasks. We were interested in examining which aspects of the estimation of linear measurement process were more likely to be explicit in textbooks' presentation of the estimation of linear measurement and which were more frequently left open. This approach was suggested by the fact that, in the USA at least, K-12 mathematics instruction relies heavily on textbooks for mathematical content (McCrory, Francis, & Young 2008).

1.5 Four elements of estimation of linear measurement

To make contact with existing literature on estimation, we draw the four essential components of estimation of linear measurement from the literature on students' estimation strategies.

The most commonly used estimation strategy, as noted by Joram et al. (1998), is the *unit iteration* strategy. According to Joram et al. (1998) ''the estimator segments the to be estimated into units, verbally counts the number of units, and then finds the place on the mental number line that corresponds to the total number of units counted''(p. 425). In this strategy, one needs the following fundamental measurement elements to perform the strategy: length *Attribute* (a one-dimensional feature of an object, such as width or height), *Start/End* points (well-defined locations where the length attribute starts and ends), *Unit* of measure (a unit for quantifying the length attribute, such as inches or meters), and *Precision* (the level of approximation, such as half or quarter inch).

Another commonly used estimation strategy is the *reference point* or *benchmark* strategy. This strategy involves ''imagining an object whose measurement is known (e.g., a paper clip known to be 1-in. long), and comparing it with the to-be-estimated object (e.g., the length of a pen)'' (Joram et al., 2005, p. 5). One may not need to perform the unit iteration strategy while employing this strategy. For example, when the length of a referent is close to the length of the to-be-estimated object (e.g., use a paper clip as a referent to estimate the length of an eraser when knowing the length of the longer side of a paper clip is about 1 in.), one can claim the length of the to-be-estimated object is about the length of the referent (e.g., the eraser is about 1 in.). One needs to employ the unit iteration strategy when the length of the to-be-estimated object is greater than the length of a referent (e.g., a paper clip as a referent for estimating the width of a book). Whether one needs to iterate a referent or not, one still needs the aforementioned four measurement elements for performing this strategy.

Decomposition/recomposition is another estimation strategy. Joram et al. (1998) described decomposition/ recomposition as a strategy in which an estimator decomposes the to-be-estimated object into several smaller parts before estimating, and then the estimator either uses the *unit iteration* strategy or *reference point* strategy to generate estimates of the small parts, and then the estimates of the smaller parts are either added up, or multiplied in a process. The difference between the decomposition/ recomposition strategy and the unit iteration strategy or the reference point strategy is that the to-be-estimated attribute needs to be partitioned into smaller parts before estimating and the estimate of the to-be-estimated attribute is attained by adding up the estimates of the smaller parts. An estimator who employs this strategy still needs to use other estimation strategies (e.g., unit iteration strategy or reference point strategy) to get the estimates of the smaller parts. Therefore, the fundamental measurement elements needed for performing each of the three strategies remain the same as the four elements described above.

According to the strategies discussed above, the four fundamental concepts play a role in the estimation of linear measurement tasks. These four fundamental estimation concepts (Attribute, Start/End, Unit, and Precision) are the Fundamental Estimation Elements. Tasks with varying degrees of openness can be created by leaving these four Fundamental Estimation Elements open. Doing this provides a variety of alternatives for students to choose from. See **Table 1** for the illustrations of alternatives for each of the four elements.

Table 1. The four elements and some of their alternatives

Element	*Focal question*	*Alternatives*
Attribute	Is there more than one length attribute in the object that can be estimated?	Length, width, or height of a table
Start/End	Is there more than one starting or ending point when estimating an object?	Starting and ending at the front door or back door when estimating the distance between the classroom and the library
Unit	Is there more than one standard or non-standard length unit that can be used to estimate an object?	Eraser (shorter or longer side), foot, inch, centimeter
Precision	Is there more than one precision level for a measure?	Whole-number, fractional estimates

If each element of estimation is clearly specified, the estimation task is not open. However, varying the specification of the elements can create tasks with varying degrees of openness. For example, if a task asks students to estimate an object by saying ''How long is the rectangular table?'', then the task could be considered open on three of the four elements (except the Start/End points) for the following reasons:

- Attribute: The question ''how long'' could refer to the measure of several one-dimensional attributes of the table, such as width, length or height.
- Start/End: If the attribute is clearly selected, the start and end points might not be open. For example, if the width of the table is selected, the start and end points may be determined by its unique set of end points.
- Unit: The unit for estimation is not described, so an estimator could use any unit, such as feet, centimeters or arm spans.
- Precision: The precision of the attribute is not specified, so an estimator could respond with the nearest whole unit or half unit.

When given the estimation task ''How long is the rectangular table?'' one estimator could estimate the length of the table to the nearest arm span, while another could estimate the width of the table to the nearest hand span. These different selections are possible through openness in the elements.

1.6 Research questions

The purpose of this study was to examine instances of estimation of linear measurement in US written curriculum materials for their clarity and explicitness. Specifically, we were interested in the frequency with which each instance was clear and explicit on each of the four elements of the estimation task, and whether there was more commonality or difference among instances of estimation of linear measurement in each of the three textbook series we examined. Second, we were interested in how the estimation tasks compared to more traditional measurement tasks (like measuring with a ruler) in their openness.

2 Method

2.1 Data sources

This study is a part of the work of the STEM research project.[2] Data sources include three elementary mathematics curricula: University of Chicago School Mathematics Project's (2007) *Everyday Mathematics*, Scott Foresman-Addison Wesley's (2008) *Michigan Mathematics*, and Saxon Publishers' (2004) *Saxon Math*. The three curricula were chosen based on the following two criteria: (1) evidence of wide use as indicated by market share (Dossey et al. 2008), and (2) substantial differences in basic design principles. Particularly, we chose Scott Foresman-Addison Wesley's *Michigan Mathematics* (SFAW) because it was a ''publisher-developed traditional'' curriculum with the largest market share. We chose *Everyday Mathematics* (EM) because it was a ''reform'' curriculum written to achieve the vision of the 1989 NCTM Standards, and it also had the largest market share of its type. Finally, we chose *Saxon Math* (Saxon) because its approach was quite different (e.g., highly scripted with more teacher-directed lessons) from the reform and traditional publisher-created materials. We analyzed the curricula from Grade K-3 because linear measurement is introduced in Grade K, developed in Grades 1 and 2, and transformed as part of area measurement in Grade 3 (e.g., finding the length and width in service of computing the area of a rectangle).

The estimation instances were identified by the STEM project. Particularly, three criteria were created by the STEM project for the identification of estimation of linear measurement instances from the three curricula (See **Table 2**). This study added a fourth criterion to limit the estimation instances identified by the STEM project to the cases that required generating an estimated measure.

The traditional measurement tasks were identified by the STEM project. Particularly, this study selected measuring with standard units (e.g., ruler) or non-standard units (e.g., paper clips) tasks identified by the STEM project for the comparison of openness between

2 The analysis was supported by the broader work of the STEM (Strengthening Tomorrow's Education in Measurement) research project. This project analyzed elementary and middle school mathematics curricula to estimate the extent to which deficits in written curricula may contribute to the overall national problem in learning spatial measurement.

Table 2. Steps in identifying instances of estimation of linear measurements

STEM criterion	*Description*
1. Is it spatial measurement?	Some instances may look like spatial measurement, but they may be closer to numerical or geometrical reasoning rather than spatial measurement reasoning
2. Is it linear measurement?	Some instances combine length, area or volume estimation together. Differentiating linear measurement from the other two is necessary
3. Is it estimation of linear measurement?	Some instances look like estimation of linear measurement, but they could be done without mental estimation
This study	
4. Is generating an estimated measure likely?	Some instances look like estimation of linear measurement, but they could be done by visually comparing two lengths instead of generating an estimated measure

the estimation of linear measurement tasks and traditional measurement tasks.

2.2 Procedure

We generated a coding scheme based on the definition of estimation of linear measurement and the elements of the estimation process described above. Four elements (Attribute, Start/End, Unit, and Precision) with two coding items in each (clarity of expression and means of designation), yield a total of eight coding items. The coding items were used to code each of the estimation of linear measurement instances in the three curricula. The clarity of expression item recorded whether an intended focus among alternatives was specified. Particularly, the code O (Open) or E (Explicit) was assigned when the intended focus was not specified or specified, respectively. The means of designation recorded *how* an intended focus among alternatives was specified. For example, Attribute may be specified by words (e.g., width, height, perimeter) or pictures (e.g., a rectangle with one side marked).

Four coders (doctoral students majoring in mathematics education at a large mid-western university in the USA) were cross-paired for a 20% inter-rater reliability coding, where the estimation instances were randomly assigned to each coding pair. The overall inter-rater reliability rate was above 80% except one coding item (75%). This lower rate (75%) was mainly caused by overlooking one criterion defined in the coding scheme, and this mistake was corrected before the remaining 80% of the estimation instances were coded. After the 20% inter-rater reliability coding, the 80% estimation instances were divided into four sets and each individual coder coded one set of the estimation instances.

2.2.1 Examples of clarity of expressions and means of designation

For each estimation element, we describe instances from the teacher's guide that represent items that were coded as Open and Explicit, and describe the particular Means of Designation of each instance.

With respect to Attribute, the Means of Designation varied from the inclusion of a picture with the attribute clearly indicated by boundaries such as lines or end points to others that included words that specified the attribute. Open instances were void of anything that clearly defined the attribute to be estimated. For example, an instance coded as Explicit from SFAW, Grade 3 includes lines that clearly indicate to students that they are to estimate the longer attribute of the nail. An instance coded as Open from EM asks students to estimate two shadows. It is unclear what attribute is being estimated. For example, students may estimate the difference between the two shadows in different ways (e.g., head-to-head, toe-to-toe, head-to-toe).

With respect to Start/End, the Means of Designation included different ways of identifying the starting and ending points of a dimension to be estimated. Explicit was used to code a Saxon, Grade 3 instance because the problem displayed a rectangle whose sides had clear Start/End points and the sides were labeled with a blank in which students were to write their estimates. In contrast, Open was used to code an SFAW, Grade K instance because students were asked to estimate the distance from the school to other locations and the Start/End points of the school and the various locations were left open.

With respect to Unit, the Means of Designation varied from instances in which the unit was indicated by words (e.g., centimeters, 1-in. square tiles) or a picture, similar to the Means of Designation for Attribute. Instances that included these means were coded as Explicit. For example, an SFAW, Grade 2 instance that indicated students should estimate in inches and in feet was coded as Explicit. In addition, sometimes requests were made for students to use a two-dimensional unit to estimate. This was coded as Explicit only if a picture or text was used to indicate which attribute of this unit to use. For example, an EM, Grade 3 instance was coded as Explicit because even though it asked students to use sheets of notebook paper, which each have a shorter and longer side, a picture indicated that students were to use the longer side of the paper to produce their estimate. Instances coded as Open included no unit or included a unit that had more than one dimension that could be used to obtain the estimate. In contrast to the earlier EM, Grade 3 instance with notebook paper where a picture was provided, in Grade 3 EM also asks students to use paper clips to obtain estimates. Although the unit was provided, no picture or text indicated whether students should use the longer or shorter side of the paperclip to obtain their estimates (See **Fig. 1**).

With respect to Precision, there was only one Means of Designation, the word ‘‘nearest’’ followed by the unit. Therefore, if Unit was coded as Open, so was Precision.

Using our coding scheme, we provide an example (See Table 3) of coding the elements for the following instance: ‘‘Estimate how many cubes long an eraser is’’ (SFAW, Grade 1, p. 365A).

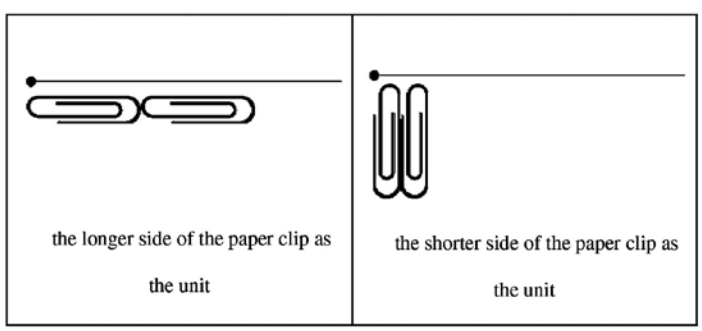

Fig. 1 Example of using a unit with more than one dimension

2.3 Analysis

Each instance of estimation of linear measurement was coded as Open or Explicit with respect to the four elements. As a result, each estimation instance is associated with a four-tuple and a degree of openness based on the number of ''O'' codes in the four-tuple. For example, an instance of estimation associated with the four-tuple OEEO, such as the SFAW example in **Table 3**, is coded as Open with respect to Attribute and Precision, but not with Unit and Start/End. This instance would be considered to have a degree of openness of 2, as it was open in two elements. We present the data using descriptive statistics without combining categories (e.g., curricula, grade levels, degrees of ambiguity). When testing if aggregation of categories was appropriate, the independence model and the conditional models (Wickens, 1989) were rejected, leading us to conclude that the categories were too different to merit aggregation.

Table 3. An example of coding for SFAW, Grade 1, p. 365A

Fundamental elements	*Coding item*	*Code*	*Reasoning of coding*
Attribute	Clarity of expression	Open	The indicator of dimension to be estimated is the word ''long'', which can indicate the length or width of an eraser. There is no indication of which dimension is to be measured
	Means of designation	Word – ''long''	
Start/End	Clarity of expression	Explicit	Each of the possible to-be-estimated dimensions (length or width) has a start point to an end point
	Means of designation	Point to point	
Unit	Clarity of expression	Explicit	The unit for estimation (linking cube) is given
	Means of designation	Linking cubes	
Precision	Clarity of expression	Open	There is no indication for how precise this estimate should be, for example, whole unit or nearest half unit
	Means of designation	Nothing specified	

3 Results

When the instances of estimation of linear measurements were all coded, we were left with a data set that was broken down by curriculum and grade, as well as the clarity of expression for each element. In each of the curricula, the frequency of instances of estimation of linear measurements generally increased as grade level increased (see Table 4), which could relate to increases in the number of pages where linear measurement was present in each curriculum. The SFAW curriculum had more instances of estimation in every grade than either of the other curricula, which may be a result of having more pages where linear measurement was present than the other curricula.

3.1 The extent of openness within each element

3.1.1 Attribute

As can be seen in **Table 4**, in six of the twelve curriculum documents, more than half of the instances of estimation were open with respect to Attribute. Each of the curricula had different patterns to their proportions of openness, but it should be noted that such values can be quite volatile when the bases for the computation of the percentage are small (e.g., EM K has only one code). Based on the quantities in Table 4, over half of the EM and Saxon instances of estimation were open with respect to Attribute, with SFAW close behind with approximately 40%.

3.1.2 Start/End

All three curricula at all grades had relatively small proportions of instances that were open with respect to Start/ End when compared to Attribute. Saxon had the greatest proportion of openness of each of the curricula (appearing in Grade 2). Based on the values in Table 4, it can be noted that, overall, EM had a negligible number of instances of estimation that were open with respect to Start/End (less than 2%), where Saxon and SFAW still had less than 10% of their instances of estimation open in this category.

Table 4. The frequencies and percentages of openness for each element by curriculum and grade

	Focus elements					*Frequency*
Curriculum	*Grade*	*Attribute*	*Start/End*	*Unit*	*Precision*	
EM	K	1	0	1	1	1
	100%	0%	100%	100%		
	1	10	1	2	15	15
	67%	7%	13%	100%		
	2	1	0	3	13	13
	8%	0%	23%	100%		
	3	28	0	33	41	45
	62%	0%	73%	91%		
Saxon	K	1	0	0	4	4
	25%	0%	0%	100%		
	1	5	0	3	5	5
	100%	0%	60%	100%		
	2	16	4	6	16	16
	100%	25%	38%	100%		
	3	11	0	8	30	32
	34%	0%	25%	94%		
SFAW	K	15	3	23	29	29
	52%	10%	79%	100%		
	1	24	2	8	44	45
	31%	4%	18%	98%		
	2	20	1	8	56	58
	34%	2%	14%	97%		
	3	30	1	53	60	65
	46%	2%	82%	92%		

3.1.3 Unit

Each curriculum had a moderately substantial proportion of openness with respect to Unit. Based on the values in Table 4, we can say that, overall, more than half of EM and nearly half of SFAW instances of estimation were open with respect to Unit, but Saxon had a much smaller proportion of open codes in this regard, at 30%. There was substantial variation in the extent of openness in the Unit category for each of the curricula.

3.1.4 Precision

Nearly all of the tasks in every grade of each of the curricula were open with respect to Precision, though the extent of openness with respect to Precision appears to decrease slightly as the grade levels progress. SFAW included language like "to the nearest" in some estimation tasks starting in Grade 2, while the other curricula did not include such language until Grade 4.

3.1.5 Overall

On the whole, Precision was the element with the greatest extent of openness, whereas Start/End was the element with the least extent of openness. Between 40 and 50% of the instances of estimation were open with respect to Attribute in each grade, whereas between 20 and 70% of the instances were open with respect to Unit.

3.2 The extent of openness of instances of estimation

The extent of openness of an instance of estimation is a number between 0 and 4, where degree of 0 means that none of the focus elements were open, a degree of 1 means that one of the focus elements was open, and a degree of 4 means all of the focus elements were open.

In Sect. 3.1.4, we noted that almost every instance (98%) of estimation was open with respect to Precision. As a deductive result, the number of instances of estimation with degree of openness 0 is small. Similarly, we noted above that almost every instance (96%) of estimation was explicit with respect to the Start/End focus category, meaning that the number of instances of estimation with degree of openness 4 is also small.

Table 5 contains percentages of instances of estimation as they sort into degrees of openness within each grade and curriculum. The information about cumulative frequencies noted in the previous section is illustrated in the right-most column of Table 5. While almost all of the instances of estimation had a non-zero degree of openness, about a quarter of the instances were open for only one element in each curriculum. More than a third were open in two of the focus elements and about another third of the instances of estimation were open in all but one of the elements.

Table 5. Percentages of instances in degrees of openness within each grade and curriculum

	Degree					*Total frequencies*
	0 (%)	*1 (%)*	*2 (%)*	*3 (%)*	*4 (%)*	
EM						
K	0	100	0	0	0	1
1	0	13	60	27	0	15
2	0	0	31	69	0	13
3	0	47	33	20	0	45
Saxon						
K	0	0	33	67	0	3
1	0	60	40	0	0	5
2	6	50	44	0	0	16
3	0	13	34	47	6	32
SFAW						
K	7	38	45	10	0	29
1	0	13	24	62	0	45
2	0	9	29	62	0	58
3	2	34	54	6	5	65

3.3 The extent of openness of ruler measurement

Since our results indicated that the estimation of linear measurement tasks was often left open, this led us to wonder whether the openness described here was specific to the estimation tasks or if all measurement tasks were left open with respect to the fundamental measurement elements. Our small analysis of tasks involving measuring with standard and non-standard units indicated that these non-estimation linear measurement tasks were not as open. This was evidenced by lower percentages of openness for each of the fundamental measurement elements (Attribute, 25.6%; Start/End: 0.0%; Unit, 10.2%; Precision, 79.5%).

4 Discussion and implications

4.1 Openness in estimation tasks

The results from coding all of the instances of estimation of linear measurement in three elementary curricula indicated that the estimation instances were often left open with respect to multiple fundamental measurement elements (i.e., Attribute, Start/End, Unit, Precision). Start/End was often explicit (under 10% of the instances were unspecified except for Saxon, Grade 2 with 25%) and Precision was often left open (over 91% of the instances unspecified). We acknowledge that these results might make sense, particularly with Precision since issues of Precision might be related to grade level (i.e., fractional units introduced in higher grades) or the inherent features of an estimation task (i.e., estimates may be assumed to be rounded to the nearest whole unit). The results related to Attribute and Unit might present us with more information as to the level of openness of these estimation tasks as written in the curriculum materials. Attribute varied considerably from under 10% of instances being left open in EM, Grade 2–100% in EM, Grade K and Saxon, Grades 1 and 2. Unit also varied considerably ranging from Saxon, Grade K in which none of the instances left the unit up for interpretation, and EM, Grade K having all instances unspecified with respect to Unit.

While there were very few estimation instances of degree 4 or degree 0, we found that approximately one quarter of all estimation instances were left open for one of the fundamental measurement elements, more than a third were left open in two, and about one-third of all estimation instances were left open for three of the fundamental measurement elements. This means that close to 60% of all estimation instances were not explicit for more than one of the fundamental measurement elements. Narrowing in on Attribute and Unit, results indicated that in many instances both the choice of Attribute and Unit were left open. In four grades (EM, Grade K; EM, Grade 3; Saxon, Grade 1; SFAW, Grade K) over 40% of the estimation instances did not specify the Attribute to be estimated or the Unit to be used to obtain the estimate.

We also found that the estimation of linear measurement tasks was left more open than other measurement tasks. Our small analysis of key measurement tasks related to finding lengths using standard and non-standard units indicated that for all of the fundamental estimation elements that the measurement tasks were less open. For each element, a smaller percentage of the tasks were left unspecified as to the Attribute, Start/End, Unit, and Precision to be used.

4.2 Openness in estimation tasks: from written to enacted

As reported by both Forrester and Pike (1998) and Joram, Subrahmanyam, and Gelman (1998), the teaching and learning of estimation of linear measurement has been plagued by vagueness, incompleteness, and confusion. This study points to one of the potential sources of this confusion as textbooks serve as a dominant source of curricular knowledge for mathematics teachers (Remillard, 2005; Stein, Remillard, & Smith, 2007; Schmidt, et al., 1996) and in the textbooks we analyzed the estimation tasks were often not explicit about the fundamental measurement elements needed to create an estimate. If teachers rely heavily on textbooks, it follows that if the instances of estimation in textbooks are left so open, it may be hard for students to know where to begin (focusing on an attribute or choosing a length unit). However, if teachers are aware of the ways in which the estimation tasks have been left open, they may engage their students in rich and productive discussions related to these elements, including using the openness in more intentional ways, such as discussing what units might be appropriate for the estimation. For example, if a teacher encountered an estimation task in which the unit was not specified, this might be an avenue for discussing the inverse relationship between unit size and measure. When two students use units of different sizes they will get different measures and a teacher could ask ''Why do you think you got different measures?'' If the unit was specified in this task, a discussion around this essential concept of unit-measure compensation would not be possible. If teachers were made aware of how varying the specification of these fundamental measurement elements could highlight the important conceptual underpinnings of measurement, teachers could begin to vary the specification to fit their own instructional goals. Recognizing, using, and creating open tasks such as the one described above requires that teachers do more than just use the task as is, but that they value the openness or lack of specification embedded in the tasks enough to think deeply about how the unspecified elements may contribute to students exploration and understanding of the underlying concepts related to measurement. A teacher who reads these open tasks with this in mind cannot be solely concerned with students getting the right answer.

4.3 Implications for curriculum development

Curricula may be enhanced by attending more to the role of the teacher in engaging students in the estimation of linear measurement. Educative curriculum materials (Ball & Cohen, 1996; Davis & Krajcik, 2005; Stein & Kim, 2009; Stein et al. 2007), or materials that provide more explicit opportunities for teachers to learn, may provide teachers with the necessary resources they need to teach estimation more effectively. Teachers' notes, such as those that highlight the ways in which the estimation tasks attend to the fundamental measurement elements, as well as those that include ways in which teachers can set up activities involving estimation to allow for more exploration and openness, examples of student work related to estimation, or information about possible misconceptions regarding the estimation of linear measurement, may help teachers understand the intention of these items and improve their teaching. Additional supports are also particularly important when open tasks are provided. Curriculum authors could draw attention to the elements of tasks that have been left open and provide more rationale for their approach so that teachers may better understand why certain aspects of tasks are not explicit. This may help teachers use these estimation tasks more effectively and in addition, may also help teachers learn to develop and use more open tasks in their teaching.

4.4 Implications for teacher education

Research indicates that prospective teachers, when not allowed the opportunity to understand their role as interpreters of curriculum, may remain text-bound (Ben-Peretz, 1990; Ball & Feiman-Nemser, 1988). According to Ben- Peretz (1990), prospective teachers are ''using textbook and teacher guides because they 'are there' without attempting adaptation or enrichment of existing materials'' (p. 109). This is particularly important in light of our current study involving elementary school mathematics textbooks. For teachers to use the estimation tasks in ways that are appropriate, they will need to learn to become critical consumers and flexible users of their textbooks. This means that teachers may need to engage their students in discussions regarding the fundamental measurement elements that are not explicit or left open for the interpretation, whether the textbook indicates this or not. In addition, teachers may need to adapt the estimation tasks. As written, many of the estimation tasks from the curricula in this study were quite open and teachers may need to learn how to make decisions about the appropriateness of these tasks for students of varying ages and how they might design lessons that align the open estimation tasks in their textbooks with what they know of students' prior knowledge, developmental level, and their goals for instruction. It is within teacher education courses and professional development activities that this learning can occur.

4.5 For further study

This study attended to the instances of estimation of linear measurement in three US elementary school curricula. While this is an important first step, there is still much to be done. First, an important question that came to us in our investigation is whether the level of openness present in the tasks described here was specific to estimation tasks or if all measurement tasks were this open. Further investigation is needed with a larger sample of non-estimation measurement items to confirm that the estimation tasks are more generally more open. Second, this framework can be used to explore other curricula, including curricula from different countries, and the estimation of other spatial measures, such as area and volume. We expect that we might see similar results for Unit or Precision in the estimation of area or volume tasks. However, it is not clear how other elements may or may not be specified. Third, other factors that may contribute to the confusion involved in the learning and teaching of estimation of linear measurement need to be examined. The enacted curriculum is worthy of our exploration as it is within this that the written curriculum is transformed (Stein et al. 2007). Finally, further experimental study is needed to examine whether the open framing of estimation of linear measurement tasks in textbooks is related to the vagueness identified in the teaching of estimation of linear measurement.

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Appendix: Data analysis

We would have preferred to report our results by making statements about these curricula as a whole. Therefore, we investigated whether this sort of aggregation should be done. We intentionally selected curricula for perceived differences, so the argument to combine curriculum categories had to be statistically justified. For both of the types of data (openness by focus Elements and degree of openness), we tested the models where the three variables (curriculum, grade, and openness or degree of openness) were independent. We rejected the independent model for each of the focus Elements (Attribute, Start/End, Unit and Precision), as the null hypothesis was rejected in each case (df = 17, Chi-squared = 72.08, $p < 0.001$; df = 17, Chi-squared = 63.01, $p < 0.001$; df = 17, Chi-squared = 154.80, $p < 0.001$, respectively). To use the same model for degrees of openness, we first had to reduce the number of degrees of openness categories from 5 (degrees 0 through 4) to 3 (Low, Medium, High) in order to fit the requirements to use the chi-squared distribution. With that change, we still rejected the null hypothesis (df = 28,Chi-squared = 138.26, $p < 0.001$). Thus, we determined that there were relationships between our variables that would not warrant aggregating across curricula without further investigation.

Because those independent models were all rejected, we tested a more constrained model to see if curriculum was independent of grade and openness (i.e., a conditional model). The data for the precision category did not fit the necessary conditions for the test, but each of the other models was rejected (df = 14, Chi-squared = 70.22, $p < 0.001$; df = 14, Chi-squared = 45.54, $p < 0.001$; df = 14, Chi-squared = 83.49, $p < 0.001$). Similarly, we tested a conditional model to see if curriculum was independent of grade and degree of openness. That model was also rejected (df = 28, Chi-squared = 138.26, $p < 0.001$). With the conditional independence models rejected, we could not conclude that we could aggregate the curricula. Therefore, these three curricula are sufficiently different, so we are unable to make statements about them as a whole. We present these data using descriptive statistics without combining curricula.

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