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# Unbounded Derivations of C*-algebras and the Heisenberg Commutation Relation

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# UNBOUNDED DERIVATIONS OF $C^*$-ALGEBRAS AND THE HEISENBERG COMMUTATION RELATION

by

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UNBOUNDED DERIVATIONS OF $C^*$-ALGEBRAS AND

THE HEISENBERG COMMUTATION RELATION

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This dissertation investigates the properties of unbounded derivations on $C^*$-algebras, namely the density of their analytic vectors and a property we refer to as "kernel stabilization." We focus on a weakly-defined derivation $\delta_D$ which formalizes commutators involving unbounded self-adjoint operators on a Hilbert space. These commutators naturally arise in quantum mechanics, as we briefly describe in the introduction.

A first application of kernel stabilization for $\delta_D$ shows that a large class of abstract derivations on unbounded $C^*$-algebras, defined by O. Bratteli and D. Robinson, also have kernel stabilization. A second application of kernel stabilization provides a sufficient condition for when a pair of self-adjoint operators which satisfy the Heisenberg Commutation Relation on a Hilbert space must both be unbounded.

A directly related classification program is of pairs of unitary group representations which satisfy the Weyl Commutation Relation on a Hilbert space. The famous Stone-von Neumann Theorem classifies these pairs when the group is locally compact abelian. In collaboration with L. Huang, we extend the Stone-von Neumann Theorem to a uniqueness statement for representations of $C^*$-dynamical systems on Hilbert $\mathcal{K}(\mathcal{H})$-modules.

# DEDICATION

*For Grandma & Grandpa Ismert and Nanny & Papa McCurdy*:

> "Now they'll walk on my arm through the distant night,
> and I won't let them stray from my heart.
> Through the wind, through the dark, through the winter light,
> I will read all their dreams to the stars."
>
> — Those You've Known, *Spring Awakening*

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# Chapter 1

# Introduction

## 1.1 Quantum Mechanics and Operators on Hilbert Space

A quantum system can be represented by a Hilbert space $\mathcal{H}$ with time evolution of the system modeled by a strongly continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$ on $\mathcal{H}$. By time evolution, we mean that the state of the system at time $t$ is given by $\psi_t = U_{-t}\psi_o$, where $\psi_o \in \mathcal{H}$ is the system's initial state. Stone's Theorem provides a (possibly unbounded) self-adjoint operator $D$ whose functional calculus implements $\{U_t\}_{t\in\mathbb{R}}$; specifically, $e^{itD} = U_t$ for each $t \in \mathbb{R}$. The operator $D$ is called the *Hamiltonian* of the system. If $D$ is unbounded, the domain of $D$ is only a proper dense subspace of $\mathcal{H}$. Consequently, domains of sums and compositions involving $D$ may not be dense. Nonetheless, quantum mechanics necessitates taking such sums and compositions.

An observable of a quantum system modeled by $\mathcal{H}$ is a self-adjoint operator that represents a measurable quantity such as the position or momentum of a particle. Like the Hamiltonian, a general observable $x$ might also be unbounded, but we restrict our attention to bounded observables. Ehrenfest's Theorem (Eqn. 6.2 of [20]) states that the commutator $[iD, x] = i(Dx - xD)$ determines the time-dependence of the observable $x$. Without supplemental conditions on $x$, however, the density of the domain of $[iD, x]$ is not guaranteed, so Ehrenfest's Theorem requires some formalization. To better understand the definedness and boundedness of $[iD, x]$, let us investigate how the commutator arises in Ehrenfest's Theorem as the descriptor of time evolution.

The *expected value* of an observable $x \in \mathcal{B}(\mathcal{H})$ at time $t$ is given by $\langle x\psi_t, \psi_t \rangle$ . Notice how

$$\langle x\psi_t, \psi_t \rangle = \left\langle xe^{-itD}\psi_0, e^{-itD}\psi_0 \right\rangle = \left\langle e^{itD}xe^{-itD}\psi_0, \psi_0 \right\rangle$$

shifts the time dependence from the vector $\psi_t$ to the operator $e^{itD}xe^{-itD}$. These two perspectives are known as the Schrödinger picture and the Heisenberg picture, respectively. For $t \in \mathbb{R}$, define

$$\alpha_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \qquad \text{by} \qquad \alpha_t(x) := e^{itD}xe^{-itD} \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

The family $\{\alpha_t\}_{t\in\mathbb{R}}$ is a norm-continuous group of $*$-automorphisms of $\mathcal{B}(\mathcal{H})$. Informally,

$$\frac{d}{dt}\left(\alpha_t(x)\right) = \frac{d}{dt}\left(e^{itD}xe^{-itD}\right) = iD\left(e^{itD}xe^{-itD}\right) - \left(e^{itD}xe^{-itD}\right)iD = [iD, \alpha_t(x)].$$

We now interpret Ehrenfest's Theorem to mean $\frac{d}{dt}\left(\alpha_t(x)\right)|_{t=0} = [iD, x]$, but *the topology in which the derivative is taken is really the heart of the matter.* The work of E. Christensen in [6] and [5] seeks to connect the topology in which this derivative is taken to the domain of $[iD, x]$ via a *derivation* on $\mathcal{B}(\mathcal{H})$. In section 3.1, we introduce this derivation and its desirable properties.

## 1.2 Derivations on $C^*$-algebras

Given a complex $*$-algebra $\mathcal{A}$, a *derivation* on $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ which satisfies the Leibniz rule: $\delta(bc) = \delta(b)c + b\delta(c)$ for all $b, c \in \mathcal{A}$. We can easily construct a derivation on $\mathcal{A}$ by fixing an element $a \in \mathcal{A}$ such that $a = a^*$ and defining a map

$$\begin{aligned} \delta_a : \quad \mathcal{A} &\rightarrow \quad \mathcal{A} \\ b &\mapsto [ia, b]. \end{aligned}$$

The map $\delta_a$ is a $*$-*derivation*, that is, $\delta_a(b^*) = \delta_a(b)^*$ for all $b \in \mathcal{A}$. Conversely, for an arbitrary $*$-derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$, certain conditions on the algebra and the derivation imply $\delta = \delta_a$ for some $a \in \mathcal{A}$ satisfying $a = a^*$. The correspondence between derivations on algebras and their representation as commutators has a rich history and is deeply connected to the mathematical formulation of quantum mechanics.

We wish to define a derivation $\delta_D : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which implements the derivative informally taken in the previous section: $\delta_D(x) := [iD, x]$ for $x \in \mathcal{B}(\mathcal{H})$. However, as not every $x \in \mathcal{B}(\mathcal{H})$ makes the commutator $[iD, x]$ defined and bounded on a dense subspace of $\mathcal{H}$, the definition of the derivation "$\delta_D$" is ambiguous. A plethora of literature is dedicated to exploring the various definitions of $\delta_D$ and their corresponding domains. In each situation, if $D$ is unbounded then the domain of $\delta_D$ is a proper subspace of $\mathcal{B}(\mathcal{H})$. In turn, further research has been dedicated to the more general study of unbounded derivations on an abstract $C^*$-algebra. The unboundedness of such a derivation creates complexities that are not found with bounded derivations, i.e., derivations defined on the entire $C^*$-algebra. In [10], Kadison summarizes three of the many significant results pertaining to bounded derivations:

1. Every bounded derivation on a commutative $C^*$-algebra is 0. (This follows from the Singer-Wermer Theorem from 1955 in [23].)

2. Sakai (1959) showed in [19] that any everywhere-defined derivation of a $C^*$-algebra is automatically bounded, thus affirmatively settling a 1953 conjecture of Kaplansky.

3. In [12], Kaplansky showed every bounded derivation $\delta$ of a type $I$ von Neumann algebra

$M$ is *inner*, i.e., there exists $a \in M$ such that $\delta = \delta_a$.

We turn our attention to densely-defined derivations on $C^*$-algebras. In section 3.1 we give a formal definition of $\delta_D$, its domain, domains of its higher powers, and state its desirable properties. In particular, Christensen shows in [6] that the domain of $\delta_D$ is strong operator topology (SOT)-dense in $\mathcal{B}(\mathcal{H})$.

In section 3.4 we generalize Christensen's SOT-density result for $\mathrm{Dom}(\delta_D)$ to include SOT-density of $\mathrm{Dom}(\delta_D^n)$ for all $n \in \mathbb{N}$, and we further strengthen this result by proving SOT-density of the analytic vectors for $\delta_D$. Both of these proofs utilize the norm-density of $\mathrm{Dom}(D^n)$ and the analytic vectors for $D$ in $\mathcal{H}$, which displays a nice parallel between the domain of a self-adjoint operator $D$ on a Hilbert space and the domain of the derivation $\delta_D$ that $D$ implements.

**Theorem 1.1.** *The set of analytic vectors for $\delta_D$ is SOT-dense in $\mathcal{B}(\mathcal{H})$.*

On the other hand, our second main result pertaining to $\delta_D$ shows that $\delta_D$ has a property which is not analogous to properties of self-adjoint operators.

**Theorem 1.2.** *If $\mathcal{H}$ is a Hilbert space and $D$ is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$, then $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$.*

The oddity of this result is illustrated by a simple example from calculus: if $f(z) = z$, then $f''(z) = 0$, but $f'(z) = 1 \neq 0$. In other words, the function $f$ belongs to the kernel of the second-derivative, but not to the first. Notice, however, that due to unboundedness of $f$ on $\mathbb{C}$ that an analogue of $f$ inside of $\mathcal{B}(\mathcal{H})$ does not exist. Given $x \in \ker \delta_D^n$, the operator $x$ is both bounded *and* analytic for $\delta_D$. The implication of Theorem 1.2 is that $x$ must belong to $\ker \delta_D$, or that $x$ is a "constant." So, perhaps kernel stabilization is suggestive of a Liouville Theorem for bounded operators on a Hilbert space.

In chapter 4, we prove Theorem 1.2, and in section 4.3, we give two applications. The first application extends the property of kernel stabilization to a class of unbounded $*$-derivations on $C^*$-algebras described in the following theorem.

**Theorem 1.3** (Bratteli-Robinson, [3]). *Let $\delta$ be a derivation of a $C^*$-algebra $\mathcal{A}$, and assume there exists a state $\omega$ on $\mathcal{A}$ which generates a faithful cyclic representation $(\pi, H, f)$ satisfying*

$$\omega(\delta(a)) = 0 \textit{ for all } a \in \mathrm{Dom}(\delta).$$

*Then $\delta$ is closable and there exists a symmetric operator $S$ on $\mathcal{H}$ such that*

$$\mathrm{Dom}(S) = \{h \in \mathcal{H} : h = \pi(a)f \;\; \textit{for some } a \in \mathcal{A}\}$$

*and $\pi(\delta(a))h = [S, \pi(a)]h$ for all $a \in \mathrm{Dom}(\delta)$ and all $h \in \mathrm{Dom}(S)$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic vectors for $\delta$ is dense in $\mathcal{A}$, then $S$ is essentially self-adjoint. For $x \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$, define*

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

*where $\overline{S}$ denotes the self-adjoint closure of $S$. It follows that $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ for all $t \in \mathbb{R}$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ is a strongly continuous group of $*$-automorphisms with closed infinitesimal generator $\widetilde{\delta}$ equaling the closure of $\pi \circ \delta|_{\mathsf{A}(\delta)}$.*

**Theorem 1.4.** *Let $\mathcal{A}$ be a $C^*$-algebra, $\delta$ a derivation on $\mathcal{A}$, and $\omega$ a state on $\mathcal{A}$ which satisfy the hypotheses of Theorem 1.3. For every $n \in \mathbb{N}$, $\ker \delta^n = \ker \delta$.*

As a second application of kernel stabilization, we provide a sufficient condition for when a pair of self-adjoint operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

**Definition 1.5.** Let $A$ and $B$ be two (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$. We say $A$ and $B$ *satisfy the Heisenberg Commutation Relation* (HCR) if there is a dense subspace $K$ of $\mathcal{H}$ such that

(i) $K \subseteq \text{Dom}([A, B])$ and

(ii) $[A, B]k = ik$ for all $k \in K$.

We include the condition that the HCR be satisfied on a dense subspace of $\mathcal{H}$ because of the possible unboundedness of $A$ and $B$. In general,

$$\text{Dom}([A, B]) = \{h \in \text{Dom}(A) \cap \text{Dom}(B) : Ah \in \text{Dom}(B),\ Bh \in \text{Dom}(A)\}.$$

Even if $\text{Dom}(A) \cap \text{Dom}(B)$ were dense in $\mathcal{H}$, $\text{Dom}([A, B])$ may fail to be dense. If, however, $K$ is a dense subspace of $\mathcal{H}$ such that $K \subseteq \text{Dom}([A, B])$, the equality $[A, B]|_K = iI|_K$ implies $[A, B]$ continuously extends to the bounded and everywhere-defined operator $iI$. The condition on $K$ that we give in Theorem 1.6 is that $K$ be a *core* for both $A$ and $B$.

**Theorem 1.6.** *If $A$ and $B$ satisfy the HCR on a common core for $A$ and $B$, then both $A$ and $B$ must be unbounded.*

## 1.3 The Heisenberg and Weyl Commutation Relations

We adopt the following formal definition of a Heisenberg pair.

**Definition 1.7.** A pair of (possibly unbounded) self-adjoint operators $(A, B)$ on a Hilbert space $\mathcal{H}$ form a *Heisenberg pair* if $A$ and $B$ satisfy the HCR.

By Stone's Theorem, $A$ and $B$ yield strongly-continuous one-parameter unitary groups $R$ and $S$, which are families are bounded operators. Thus, one common method in the classification of Heisenberg pairs is to find sufficient conditions on $A$ and $B$ for when $R$ and $S$ form a *Heisenberg representation* of $\mathbb{R}$.

**Definition 1.8.** Let $G$ be a locally compact abelian group and $\widehat{G}$ its Pontryagin dual. A pair of strongly-continuous unitary groups $R = \{R_x\}_{x \in G}$ and $S = \{S_\gamma\}_{\gamma \in \widehat{G}}$ satisfy the *Weyl Commutation Relation* (WCR) if

$$S_\gamma R_x = \gamma(x) R_x S_\gamma \text{ for all } x \in G, \gamma \in \widehat{G}.$$

The pair $(R, S)$ is a *Heisenberg representation of* $G$ (not to be confused with a Heisenberg *pair*).

**Definition 1.9.** Let $\mu$ be a Haar measure for $G$, and denote $L^2(G, \mu)$ by $L^2(G)$. Consider the maps $\lambda : G \to \mathcal{U}(L^2(G))$ and $V : \widehat{G} \to \mathcal{U}(L^2(G))$, where for each $x \in G$, $\gamma \in \widehat{G}$, and $f \in C_c(G)$,

$$[\lambda_x f](y) := f(x^{-1}y) \text{ and } [V_\gamma f](y) := \gamma(y) f(y) \text{ for all } y \in G.$$

The pair $(\lambda, V)$ is a Heisenberg representation of $G$ called the *Schrödinger representation.*

**Theorem 1.10** (Stone-von Neumann Theorem)**.** *Every Heisenberg representation of* $G$ *is unitarily equivalent to a direct sum of copies of the Schrödinger representation.*

Since Heisenberg representations of a locally compact group $G$ are classified by the Stone-von Neumann Theorem, classification of Heisenberg pairs whose generated unitary groups form a Heisenberg representation of $\mathbb{R}$ are immediately classified.

Chapter 5 of this dissertation is joint work with Leonard Huang (University of Nevada, Reno), in which we state and prove a “Covariant Stone-von Neumann Theorem.” Our result generalizes the classical Stone-von Neumann Theorem in two ways. First, we consider representations of $C^*$*-dynamical systems* involving locally compact abelian groups as opposed to just locally compact abelian groups. We also consider representations of these dynamical systems on *Hilbert* $\mathcal{K}(\mathcal{H})$*-modules* as opposed to representations only on Hilbert spaces. Requisite background for $C^*$-dynamical systems and Hilbert $C^*$-modules is in Chapter 2.

**Theorem 1.11.** *Every* $(G, \mathcal{K}(\mathcal{H}), \alpha)$*-Heisenberg representation is unitarily equivalent to a direct sum of copies of the* $(G, \mathcal{K}(\mathcal{H}), \alpha)$*-Schrödinger representation.*

In Chapter 5, we define a $(G, \mathcal{A}, \alpha)$-Heisenberg representation and the $(G, \mathcal{A}, \alpha)$-Schrödinger representation for an arbitrary $C^*$-algebra, and we show that the $(G, \mathcal{A}, \alpha)$-Schrödinger representation is a $(G, \mathcal{A}, \alpha)$-Heisenberg representation. We then provide and prove some results about Hilbert $\mathcal{K}(\mathcal{H})$-modules that are necessary to prove Theorem 1.11.

While interesting in a purely mathematical context, our generalization of the Stone-von Neumann Theorem has a rich interpretation from the perspective of quantum mechanics. Namely, representations of dynamical systems allow for the consideration of an inherit time-dependence of the space of observables in addition to the time-dependence of the state space. This occurs when the Hamiltonian of the system is time-dependent, i.e., the energies influencing the system are not constant. Informally, we obtain a new description of the time-evolution of $x$:

$$\frac{dx}{dt}\Big|_{t=0} = [iD, x] + \frac{\partial x}{\partial t}\Big|_{t=0}, \qquad \text{[Eqn. 3.22 [26]]}$$

where the partial term $\frac{\partial x}{\partial t}|_{t=0}$ is the addition of time-dependence for the observable $x$ in the presence of a time-dependent Hamiltonian. If the Hamiltonian is time-independent, this term vanishes, and we recover the time-independent version of Ehrenfest's Theorem. The time-dependence of $x$ indicated by a nonzero partial derivative term can be modeled by an action of $\mathbb{R}$ on the $C^*$-algebra $\mathcal{A}$ of observables. More generally, we may consider a locally compact abelian group $G$ acting on $\mathcal{A}$ via a continuous group homomorphism $\alpha : G \to \operatorname{Aut}(\mathcal{A})$, which we call a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$.

The goal of representing these dynamical systems on Hilbert $\mathcal{K}(\mathcal{H})$-modules is motivated in large part by the flexibility of modeling quantum field theory (where relativity may be in play) with representations on Hilbert $C^*$-modules. Tangent to this physical motivation is the goal of generalizing major theorems for operators on Hilbert spaces, such as Stone's Theorem and Stinespring's Theorem, to the setting of Hilbert $C^*$-modules. Works in this realm include [1] and [24]. A drawback of our work is that our main result pertains only to $C^*$-dynamical systems $(G, \mathcal{K}(\mathcal{H}), \alpha)$, where $G$ is locally compact abelian, represented on Hilbert $\mathcal{K}(\mathcal{H})$-modules. Ideally our results hold in a more general context, but our current techniques rely heavily on this choice of $C^*$-algebra. Nonetheless, our result is a nontrivial extension of the classical Stone-von Neumann Theorem.

# Chapter 2

# Background

## 2.1 $\mathcal{B}(\mathcal{H})$ and $C^*$-algebras

Throughout we take $\mathcal{H}$ to be a complex Hilbert space, and we denote the continuous linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Recall that $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra with respect to the adjoint operation and the operator norm. In addition to the operator norm, there are two other topologies we consider on $\mathcal{B}(\mathcal{H})$:

**Definition 2.1.** The *strong operator topology* (SOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms $\{x \mapsto \|xh\| : h \in \mathcal{H}\}$. Equivalently, a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{B}(\mathcal{H})$ converges in the strong operator topology to $x \in \mathcal{B}(\mathcal{H})$ if and only if $\lim_{\lambda\to\infty} \|x_\lambda h - xh\| = 0$ for all $h \in \mathcal{H}$.

**Definition 2.2.** The *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ is the topology induced by the seminorms $\{x \mapsto |\langle xh, k\rangle| : h, k \in \mathcal{H}\}$. Equivalently, a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{B}(\mathcal{H})$ converges in the weak operator topology to $x \in \mathcal{B}(\mathcal{H})$ if and only if $\lim_{\lambda\to\infty} |\langle x_\lambda h, k\rangle - \langle xh, k\rangle| = 0$ for all $h, k \in \mathcal{H}$.

**Remark 2.3.** The norm topology on $\mathcal{B}(\mathcal{H})$ is finer than the strong operator topology, and the strong operator topology is finer than the weak operator topology.

**Definition 2.4.** A *von Neumann algebra* is a SOT-closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

## 2.2 Unbounded Symmetric Operators on Hilbert Space

Let $\mathcal{H}$ be a Hilbert space, $K_1$ and $K_2$ subspaces of $\mathcal{H}$, and $T : K_1 \to K_2$ a linear map. We call $K_1$ the *domain* of $T$, denoted $\mathrm{Dom}(T)$.

**Definition 2.5.** A linear operator $T$ is *densely-defined* if $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$.

If $\mathrm{Dom}(T) = \mathcal{H}$ and $T$ is continuous, then $T$ is simply an element of $\mathcal{B}(\mathcal{H})$. If $\mathrm{Dom}(T)$ is only dense in $\mathcal{H}$, but $T$ is bounded on $\mathrm{Dom}(T)$, we may extend $T$ by continuity to a bounded operator on all of $\mathcal{H}$. Thus, the domain of a densely-defined bounded linear operator can always be extended to all of $\mathcal{H}$, but this is not the case for densely-defined linear operators which are unbounded.

**Example 2.6.** For each $f \in C_c(\mathbb{R})$, the continuous compactly supported functions on $\mathbb{R}$, define

$$[Qf](x) := xf(x) \text{ for all } x \in \mathbb{R}.$$

Clearly, $Qf \in C_c(\mathbb{R})$ and $Q$ is linear, so $Q$ defines a linear operator on the $\|\cdot\|_2$-dense subspace $C_c(\mathbb{R})$ of the Hilbert space $L^2(\mathbb{R})$. However, $Q$ is not extendable to an everywhere-defined operator on $L^2(\mathbb{R})$ because $Q$ is not bounded on $C_c(\mathbb{R})$.

For each $k \in \mathbb{N}$, choose $f_k \in C_c(\mathbb{R})$ with $\mathrm{Supp}(f_k) \subseteq [k, k+1]$. Then

$$\|Qf_k\|_2 = \left(\int_{[k,k+1]} |xf_k(x)|^2 \, dm(x)\right)^{1/2} \geq k \left(\int_{[k,k+1]} |f_k(x)|^2 \, dm(x)\right)^{1/2} = k \|f_k\|_2 .$$

Thus, $\|Q\| \geq k$ for all $k \in \mathbb{N}$, which implies $Q$ is unbounded. The largest subspace of $L^2(\mathbb{R})$ on which $Q$ is defined is

$$\mathrm{Dom}(Q) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |xf(x)|^2 \, dm(x) < \infty \right\}.$$

While $Q$ is not extendable to all of $L^2(\mathbb{R})$, $Q$ *is* continuous in a certain sense.

**Definition 2.7.** A linear operator $T$ is *closed* if the *graph of $T$*, $\Gamma(T) := \{(h, Th) : h \in \mathrm{Dom}(T)\}$, is closed in $\mathcal{H} \oplus \mathcal{H}$.

The operator $Q$ in Example 2.6 is closed.

**Definition 2.8.** Given a closed linear operator $T$ on a Hilbert space $\mathcal{H}$, a *core* for $T$ is a subspace $\mathscr{C} \subseteq \mathrm{Dom}(T)$ such that

$$\overline{\Gamma(T|_{\mathscr{C}})}^{\mathcal{H}\oplus\mathcal{H}} = \Gamma(T).$$

**Example 2.9.** For $f \in C_c^\infty(\mathbb{R})$, define $Pf := -if'$. Then $P$ with domain

$$\mathrm{Dom}(P) := \{f \in L^2(\mathbb{R}) : f \text{ is absolutely continuous on every interval } [a, b] \text{ and } f' \in L^2(\mathbb{R})\}$$

is a closed operator.

In addition to being closed, the operators $Q$ and $P$ are *self-adjoint*.

**Definition 2.10** (Conway, X.1.5 [7])**.** Let $T$ be a densely-defined linear operator on $\mathcal{H}$, and let

$$\mathrm{Dom}(T^*) = \{k \in \mathcal{H} : h \mapsto \langle Th, k\rangle \text{ defines a bounded linear functional on } \mathrm{Dom}(T)\}.$$

By density of $\mathrm{Dom}(T)$ in $\mathcal{H}$, for each $k \in \mathrm{Dom}(T^*)$ the Riesz Representation Theorem provides a unique $f \in \mathcal{H}$ such that $\langle Th, k\rangle = \langle h, f\rangle$ for all $h \in \mathrm{Dom}(T)$. Let $T^*k := f$. Then,

$$\langle Th, k\rangle = \langle h, T^*k\rangle \text{ for all } h \in \mathrm{Dom}(T) \text{ and } k \in \mathrm{Dom}(T^*).$$

**Definition 2.11.** A densely-defined linear operator $D$ is *self-adjoint* if

(i) $\langle Dh, k\rangle = \langle h, Dk\rangle$ for all $h, k \in \mathrm{Dom}(D)$ (i.e., $D$ is *symmetric*)

(ii) and $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$.

**Definition 2.12.** A densely-defined linear operator $S$ on $\mathcal{H}$ is *essentially self-adjoint* if the closure of the graph $\Gamma(S)$ in $\mathcal{H} \oplus \mathcal{H}$ defines the graph of a self-adjoint operator.

A symmetric operator automatically satisfies $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D^*)$. In fact, when $D$ is bounded, symmetry implies condition (ii). When $D$ is unbounded, however, condition (ii) requires $D$ to have an adequately large domain—as large as the domain of its adjoint. The domains of higher powers of a self-adjoint operator is one of the properties that make self-adjoint operators so desirable.

**Notation 2.13.** Let $S$ be a linear operator on a Banach space $X$. For each $n \in \mathbb{N}$,

$$\mathrm{Dom}(S^n) := \{x \in \mathrm{Dom}\big(S^{n-1}\big) : S^{n-1}x \in \mathrm{Dom}(S)\}.$$

**Definition 2.14.** Let $S$ be a linear operator on a Banach space $X$. A vector $x \in X$ is an *analytic vector* for $S$ if

(i) $x \in \mathrm{Dom}(S^n)$ for all $n \in \mathbb{N}$ and

(ii) $\sum_{n=0}^{\infty} \frac{\|S^n x\|}{n!} t^n < \infty$ for some $t > 0$.

Denote the set of analytic vectors for $S$ by $\mathsf{A}(S)$.

Given a *densely*-defined operator $T$, domains of higher powers of $T$ may fail to be dense as

$$\mathrm{Dom}(T) \supsetneq \mathrm{Dom}\big(T^2\big) \supsetneq \mathrm{Dom}\big(T^3\big) \supsetneq \ldots$$

When $T$ is self-adjoint, however, $\mathrm{Dom}(T^n)$ is dense in $\mathcal{H}$ for all $n \in \mathbb{N}$. In fact, the set of analytic vectors for $T$ is dense in $\mathcal{H}$.

**Theorem 2.15** (Nelson, [16]). *A densely-defined operator on a Hilbert space $\mathcal{H}$ is essentially self-adjoint if and only if its set of analytic vectors is dense in $\mathcal{H}$.*

This remarkable fact is known as "Nelson's Analytic Vector Theorem." Additionally, self-adjoint operators are the infinitesimal generators of a special type of one-parameter family.

**Definition 2.16.** A family $\{U_t\}_{t\in\mathbb{R}}$ of operators on a Hilbert space $\mathcal{H}$ which satisfies

(i) $U_t$ is unitary for each $t \in \mathbb{R}$, that is, $U_t^* U_t = I = U_t U_t^*$,

(ii) $U_o = I$,

(iii) $U_s U_t = U_{s+t}$ for all $s, t \in \mathbb{R}$, and

(iv) $\lim_{t\to 0} \|U_t h - h\| = 0$ for all $h \in \mathcal{H}$

is a *strongly-continuous one-parameter group of unitaries.*

**Theorem 2.17** (Stone's Theorem). *Given a self-adjoint operator $D$, the family $\{e^{itD}\}_{t\in\mathbb{R}}$ is a strongly-continuous one-parameter group of unitaries. Conversely, given a strongly-continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$, there exists a self-adjoint operator $D$ such that $U_t = e^{itD}$ for all $t \in \mathbb{R}$.*

The self-adjoint operator $D$ is called the *infinitesimal generator* for the group $\{e^{itD}\}_{t\in\mathbb{R}}$:

$$\mathrm{Dom}(D) = \left\{ h \in \mathcal{H} : \lim_{t\to 0} \frac{e^{itD}h - h}{t} \text{ exists} \right\},$$

and for $h \in \mathrm{Dom}(D)$,

$$Dh := -i \left( \lim_{t\to 0} \frac{e^{itD}h - h}{t} \right).$$

## 2.3 Unitary Group Representations

Let $\mathcal{U}(\mathcal{H})$ denote the unitary group of $\mathcal{B}(\mathcal{H})$, and let $G$ be a locally compact group. Up to a scalar, $G$ has a unique nonzero left-invariant Radon measure, called a *Haar measure*, which we denote by $\mu$. We may then consider the Hilbert space $L^2(G, \mu)$, which we denote by $L^2(G)$. In the case when $G$ is abelian, $\mu$ is also right-invariant, and its Pontryagin dual is a locally compact abelian group $\widehat{G}$ whose Haar measure we denote by $\hat{\mu}$.

**Definition 2.18.** A *unitary group representation* of $G$ on a Hilbert space $\mathcal{H}$ is a group homomorphism $U : G \to \mathcal{U}(\mathcal{H})$ such that for each $h \in \mathcal{H}$, the map $s \mapsto U_s h$ is continuous.

**Example 2.19.** Any strongly-continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$ on $\mathcal{H}$ defines a unitary group representation $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ by $t \mapsto U_t$.

**Example 2.20.** Let $G$ be a locally compact abelian group. The *left regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$ and representation $V : \widehat{G} \to \mathcal{U}(L^2(G))$ in the Schrödinger representation $(\lambda, V)$ of $G$ (recall Definition 1.9) are examples of unitary group representations.

## 2.4 $C^*$-Dynamical Systems and Crossed Products

The reader is referred to [25] for a detailed treatment of foundational material on $C^*$-dynamical systems and crossed product $C^*$-algebras. Some definitions and facts are included here for convenience. Throughout, $G$ is a locally compact abelian group with Haar measure $\mu$ and $\mathcal{A}$ is a $C^*$-algebra.

**Definition 2.21.** A *$C^*$-dynamical system* is a triple $(G, \mathcal{A}, \alpha)$ where $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is a continuous homomorphism.

**Example 2.22.** Let $C_o(G)$ be the $C^*$-algebra of continuous functions $f : G \to \mathbb{C}$ such that for each $\epsilon > 0$, there is a compact subset $K \subseteq G$ where $\left\| f|_{G\setminus K} \right\|_\infty < \epsilon$. Consider an action of $G$ on $C_o(G)$ via *left translation:*

$$\begin{array}{rccc} \mathsf{lt}: & G & \to & \mathrm{Aut}(C_o(G)) \\ & x & \mapsto & \mathsf{lt}_x, \end{array}$$

where for each $f \in C_o(G)$,

$$[\mathsf{lt}_x f](y) := f(x^{-1}y) \text{ for all } y \in G.$$

Then $(G, C_o(G), \mathsf{lt})$ is a $C^*$-dynamical system.

**Definition 2.23.** A *covariant representation* of a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$ is a pair $(\pi, U)$ consisting of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a unitary group representation $U : G \to \mathcal{U}(\mathcal{H})$ such that

$$\pi(\alpha_x(a)) = U_x \pi(a) U_x^* \text{ for all } x \in G, a \in \mathcal{A}.$$

**Example 2.24** (Williams, 2.12 [25])**.** Let $M : C_o(G) \to \mathcal{B}(L^2(G))$ denoted $f \mapsto M_f$ be given by pointwise multiplication, that is, for each $f \in C_o(G)$ and $h \in C_c(G)$,

$$[M_f h](x) := f(x)h(x) \text{ for all } x \in G.$$

By density of $C_c(G)$ in $L^2(G)$ and boundedness of $M_f|_{C_c(G)}$, we may extend $M_f$ to a bounded linear operator on all of $L^2(G)$. If $\lambda$ denotes the left regular representation of $G$, then the pair $(M, \lambda)$ is a covariant representation of $(G, C_o(G), \mathsf{lt})$.

Given a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$, one can construct the *crossed product* $C^*$-algebra $\mathcal{A} \rtimes_\alpha G$ which is universal with respect to the covariant representations of $(G, \mathcal{A}, \alpha)$. Let $C_c(G, \mathcal{A})$ denote the set of continuous functions $f : G \to \mathcal{A}$ such that for each $f \in C_c(G, \mathcal{A})$, there exists a compact subset $K \subseteq G$ where $\mathrm{Supp}(f) \subseteq K$. The crossed product corresponding to a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$ is constructed by considering representations of $C_c(G, \mathcal{A})$ which are induced by covariant representations of $(G, \mathcal{A}, \alpha)$.

**Definition 2.25.** Given a covariant representation $(\pi, U)$ for $(G, \mathcal{A}, \alpha)$ on $\mathcal{H}$, define the *integrated form* of $(\pi, U)$ to be the $*$-representation $\pi \rtimes U : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ given by

$$[\pi \rtimes U](f) := \int_G \pi(f(x)) U_x \, d\mu(x) \text{ for all } f \in C_c(G, \mathcal{A}).$$

The above integral is $\mathcal{B}(\mathcal{H})$-valued and converges in the WOT, i.e.,

$$\langle [\pi \rtimes U](f)h, k \rangle = \int_G \langle \pi(f(x)) U_x h, k \rangle \; d\mu(x) \text{ for all } h, k \in \mathcal{H}.$$

**Lemma 2.26** (Williams, 2.27 [25])**.** *For each $f \in C_c(G, \mathcal{A})$, define the* universal norm *on $C_c(G, \mathcal{A})$ by*

$$\|f\| := \sup\{\|[\pi \rtimes U](f)\| : (\pi, U) \textit{ is a covariant representation of } (G, \mathcal{A}, \alpha)\}.$$

*The universal norm is dominated by the $L^1(G, \mathcal{A})$-norm and the completion of $C_c(G, \mathcal{A})$ with respect to $\|\cdot\|$ is a $C^*$-algebra which we denote by $\mathcal{A} \rtimes_\alpha G$.*

## 2.5 Hilbert $C^*$-modules

Let $G$ be a locally compact abelian group with Haar measure $\mu$ and $\mathcal{A}$ a $C^*$-algebra.

**Definition 2.27.** An *inner product $\mathcal{A}$-module* is a linear space $\mathsf{X}$ which is a right $\mathcal{A}$-module via an action $\bullet : \mathsf{X} \times \mathcal{A} \to \mathsf{X}$ denoted $(\xi, a) \mapsto \xi \bullet a$ which satisfies

$$\lambda(\xi \bullet a) = (\lambda \xi) \bullet a = \xi \bullet (\lambda a) \text{ for all } \xi \in \mathsf{X},\ a \in \mathcal{A},\ \lambda \in \mathbb{C},$$

together with a map $\langle \cdot \,|\, \cdot \rangle : \mathsf{X} \times \mathsf{X} \to \mathcal{A}$ such that for all $\xi,\ \eta,\ \nu \in \mathsf{X},\ \alpha,\ \beta \in \mathbb{C}$, and $a \in \mathcal{A}$,

(i) $\langle \xi \,|\, \alpha\eta + \beta\nu \rangle = \alpha \langle \xi \,|\, \eta \rangle + \beta \langle \xi \,|\, \nu \rangle$,

(ii) $\langle \xi \,|\, \eta \bullet a \rangle = \langle \xi \,|\, \eta \rangle \, a$,

(iii) $\langle \eta \,|\, \xi \rangle = \langle \xi \,|\, \eta \rangle^*$, and

(iv) $\langle \xi \,|\, \xi \rangle \geq 0$ as an element of $\mathcal{A}$, and if $\langle \xi \,|\, \xi \rangle = 0$, then $\xi = 0$.

We sometimes subscript $\langle \cdot \,|\, \cdot \rangle$ to avoid ambiguity when multiple algebras or modules are present.

**Definition 2.28.** Let $\mathsf{X}$ be an inner product $\mathcal{A}$-module, and define a norm on $\mathsf{X}$ by

$$\|\xi\| := \|\langle \xi \,|\, \xi \rangle\|_{\mathcal{A}}^{1/2} \text{ for all } \xi \in \mathsf{X}.$$

Then $\mathsf{X}$ is a (right) *Hilbert $\mathcal{A}$-module* if $\mathsf{X}$ is complete with respect to $\|\cdot\|$.

Note that when $\mathcal{A} = \mathbb{C}$, a Hilbert $\mathcal{A}$-module is simply a Hilbert space. Left Hilbert $\mathcal{A}$-modules are defined similarly.

**Example 2.29.** For $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := \phi(x)a \text{ for all } x \in G.$$

Then $C_c(G, \mathcal{A})$ along with the action $\bullet$ by $\mathcal{A}$ is a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \,|\, \phi \rangle := \int_G \psi(x)^* \phi(x) \, d\mu(x),$$

where this $\mathcal{A}$-valued integral is characterized by

$$\zeta\left(\langle \psi \,|\, \phi \rangle\right) = \int_G \zeta(\psi(x)^* \phi(x)) \, d\mu(x) \text{ for all } \zeta \in \mathcal{A}^*.$$

One easily checks that $\langle \cdot \,|\, \cdot \rangle$ satisfies the axioms in Definition 2.27, so $C_c(G, \mathcal{A})$ with $\langle \cdot \,|\, \cdot \rangle$ is an inner product $\mathcal{A}$-module. Denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\| := \|\langle \cdot \,|\, \cdot \rangle\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A})$.

**Example 2.30.** Let $(G, \mathcal{A}, \alpha)$ be a dynamical system. For each $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := \phi(x)\alpha_x(a) \text{ for all } x \in G.$$

Then $\bullet$ makes $C_c(G, \mathcal{A})$ into a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \,|\, \phi \rangle_\alpha := \int_G \alpha_{x^{-1}} \left( \psi(x)^* \phi(x) \right) \, d\mu(x).$$

Then $C_c(G, \mathcal{A})$ along with $\langle \cdot \,|\, \cdot \rangle_\alpha$ defines an inner product $\mathcal{A}$-module. Denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\|_\alpha := \|\langle \cdot \,|\, \cdot \rangle_\alpha\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

**Remark 2.31.** When completing $C_c(G, \mathcal{A})$ with respect to $\|\cdot\|_\alpha$, an isomorphic copy of $C_c(G, \mathcal{A})$ exists in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ via an embedding $q : C_c(G, \mathcal{A}) \to \mathsf{L}^2(G, \mathcal{A}, \alpha)$. When considering the dense subalgebra $q(C_c(G, \mathcal{A}))$ inside $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we will suppress the "copy" and simply identify $C_c(G, \mathcal{A})$ inside $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

**Proposition 2.32.** *Let $(G, \mathcal{A}, \alpha)$ be a $C^*$-dynamical system. A norm $\|\cdot\|_2$ can be defined on $C_c(G, \mathcal{A})$ by*

$$\|\phi\|_2 := \left( \int_G \|\phi(x)\|_{\mathcal{A}}^2 \, d\mu(x) \right)^{1/2} \text{ for each } \phi \in C_c(G, \mathcal{A}).$$

*This norm has the property that $\|\phi\|_\alpha \leq \|\phi\|_2$ for all $\phi \in C_c(G, \mathcal{A})$.*

*Proof.* Checking that $\|\cdot\|_2$ is a norm on $C_c(G, \mathcal{A})$ is a simple exercise. For $\phi \in C_c(G, \mathcal{A})$, observe

$$\begin{aligned}
\|\phi\|_\alpha^2 &= \left\| \int_G \alpha_{x^{-1}}(\phi(x)^*\phi(x)) \; d\mu(x) \right\|_{\mathcal{A}} \\
&\leq \int_G \|\alpha_{x^{-1}}(\phi(x)^*\phi(x))\|_{\mathcal{A}} \; d\mu(x) \\
&= \int_G \|\phi(x)^*\phi(x)\|_{\mathcal{A}} \; d\mu(x) \\
&= \int_G \|\phi(x)\|_{\mathcal{A}}^2 \; d\mu(x) \\
&= \|\phi\|_2^2.
\end{aligned}$$

□

**Corollary 2.33.** *Suppose* $\{\psi_\lambda\}_{\lambda\in\Lambda} \subseteq C_c(G,\mathcal{A})$ *converges uniformly to* $\psi \in C_c(G,\mathcal{A})$*, i.e.,* $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} \to 0$ *as* $\lambda \to \infty$*. Then* $\|\psi_\lambda - \psi\|_\alpha \to 0$ *as* $\lambda \to \infty$*.*

*Proof.* By Proposition 2.32, it suffices to prove that $\|\psi_\lambda - \psi\|_2 \to 0$ as $\lambda \to \infty$. Let $\epsilon > 0$, and choose $\lambda_1 \in \Lambda$ such that $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} < \frac{\epsilon}{\sqrt{2\cdot\mu(\mathrm{Supp}(\psi))+1}}$ for all $\lambda \geq \lambda_1$. Also, since $\|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})} \to 0$ as $\lambda \to \infty$, there exists $\lambda_2 \in \Lambda$ such that $\mu(\mathrm{Supp}(\psi_\lambda) \setminus \mathrm{Supp}(\psi)) < \mu(\mathrm{Supp}(\psi))$ for all $\lambda \geq \lambda_2$. Choose $\lambda_o := \max\{\lambda_1, \lambda_2\}$. Then for all $\lambda \geq \lambda_o$,

$$\begin{aligned}
\|\psi_\lambda - \psi\|_2^2 &= \int_G \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \; d\mu(y) \\
&= \int_{\mathrm{Supp}(\psi)} \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \; d\mu(y) + \int_{\mathrm{Supp}(\psi_\lambda)\setminus\mathrm{Supp}(\psi)} \|\psi_\lambda(y) - \psi(y)\|_{\mathcal{A}}^2 \; d\mu(y) \\
&\leq \int_{\mathrm{Supp}(\psi)} \|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})}^2 \; d\mu(y) + \int_{\mathrm{Supp}(\psi_\lambda)\setminus\mathrm{Supp}(\psi)} \|\psi_\lambda - \psi\|_{C_c(G,\mathcal{A})}^2 \; d\mu(y) \\
&< \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi)) + \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi_\lambda) \setminus \mathrm{Supp}(\psi)) \\
&< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2\cdot\mu(\mathrm{Supp}(\psi))+1} \cdot \mu(\mathrm{Supp}(\psi)) \\
&< \epsilon^2.
\end{aligned}$$

By Proposition 2.32, $\|\psi_\lambda - \psi\|_\alpha \leq \|\psi_\lambda - \psi\|_2 \to 0$ as $\lambda \to \infty$. $\square$

**Notation 2.34.** Let $\mathsf{X}$ be a Hilbert $\mathcal{A}$-module, and let $\mathsf{X}_o$ be a closed $\mathcal{A}$-submodule of $\mathsf{X}$. Denote $\text{Span}\{\xi \bullet a : \xi \in \mathsf{X}_o, a \in \mathcal{A}\}$ by $\mathsf{X}_o \bullet \mathcal{A}$.

**Notation 2.35.** Given a Hilbert $\mathcal{A}$-submodule $\mathsf{X}_o$ of $\mathsf{X}$, define

$$\langle \mathsf{X}_o \,|\, \mathsf{X}_o \rangle := \text{Span}\{\langle \xi \,|\, \eta \rangle : \xi, \eta \in \mathsf{X}_o\}.$$

**Definition 2.36.** A Hilbert $\mathcal{A}$-module $\mathsf{X}$ is *full* if $\langle \mathsf{X} \,|\, \mathsf{X} \rangle$ is dense in $\mathcal{A}$.

**Proposition 2.37.** *The Hilbert $\mathcal{A}$-module $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is full.*

Fullness of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ follows from Green's Imprimitivity Theorem stated in Theorem 4.21 of [25]. We will need Green's Imprimitivity Theorem again later, so we will wait until Chapter 5 to give its statement.

**Definition 2.38.** Given a family $\{\mathsf{X}_j\}_{j \in J}$ of Hilbert $\mathcal{A}$-modules, define

$$\oplus_j \mathsf{X}_j := \left\{ (\xi_j)_{j \in J} : \xi_j \in \mathsf{X}_j \text{ for each } j \in J \text{ and } \sum_{j \in J} \langle \xi_j \,|\, \xi_j \rangle \text{ converges in the norm on } \mathcal{A} \right\}.$$

For $\xi = (\xi_j)_{j \in J}$ and $\eta = (\eta_j)_{j \in J}$ in $\oplus_j \mathsf{X}_j$, define

$$\langle \xi \,|\, \eta \rangle := \sum_{j \in J} \langle \xi_j \,|\, \eta_j \rangle_{\mathsf{X}_j}.$$

It is an exercise in [13] to show that $\oplus_j \mathsf{X}_j$ with this inner product forms a Hilbert $\mathcal{A}$-module.

**Proposition 2.39.** *Given a family of Hilbert $\mathcal{A}$-modules $\{\mathsf{X}_j\}_{j \in J}$, let $\mathsf{Y} := \oplus_j \mathsf{X}_j$. Then*

$$\mathsf{Y}_o := \{(\xi_j)_{j \in J} \in \mathsf{Y} : \xi_j = 0 \textit{ for all but finitely many } j \in J\}$$

*is dense in* $\mathsf{Y}$.

*Proof.* Let $\xi = (\xi_j)_{j \in J} \in \mathsf{Y}$. Then $\sum_{j \in J} \langle \xi_j \,|\, \xi_j \rangle$ converges in $\mathcal{A}$, so in particular, given $\epsilon > 0$, there exists a finite set $F \subseteq J$ such that

$$\left\| \sum_{j \in J \setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} = \left\| \sum_{j \in J} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} - \sum_{j \in F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} < \epsilon^2.$$

Define $(\eta_j)_{j \in J} \in \mathsf{Y}_o$ by $\eta_j = \xi_j$ whenever $j \in F$ and $\eta_j = 0$ otherwise. Then

$$\begin{aligned}
\|\xi - \eta\|_{\mathsf{Y}}^2 &= \|\langle \xi - \eta \,|\, \xi - \eta \rangle_{\mathsf{Y}}\|_{\mathcal{A}} \\
&= \left\| \sum_{j \in J} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j \in F} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} + \sum_{j \in J \setminus F} \langle \xi_j - \eta_j \,|\, \xi_j - \eta_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j \in F} \langle \xi_j - \xi_j \,|\, \xi_j - \xi_j \rangle_{\mathsf{X}_j} + \sum_{j \in J \setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&= \left\| \sum_{j \in J \setminus F} \langle \xi_j \,|\, \xi_j \rangle_{\mathsf{X}_j} \right\|_{\mathcal{A}} \\
&< \epsilon^2.
\end{aligned}$$

Therefore, $\mathsf{Y}_o$ is dense in $\mathsf{Y}$. □

## 2.6 Adjointable Operators on Hilbert $C^*$-modules

Throughout, $\mathsf{X}$ and $\mathsf{Y}$ are (right) Hilbert $\mathcal{A}$-modules. A map $T : \mathsf{X} \to \mathsf{Y}$ which satisfies $T(\xi \bullet a) = (T\xi) \bullet a$ for all $\xi \in \mathsf{X}$ and $a \in \mathcal{A}$ is referred to as $\mathcal{A}$-*linear*.

**Definition 2.40.** A map $T : \mathsf{X} \to \mathsf{Y}$ is *adjointable* if there exists a map $S : \mathsf{Y} \to \mathsf{X}$ such that

$$\langle T\xi \,|\, \eta\rangle_{\mathsf{Y}} = \langle \xi \,|\, S\eta\rangle_{\mathsf{X}} \text{ for all } \xi \in \mathsf{X},\ \eta \in \mathsf{Y}.$$

If $T$ is adjointable, its adjoint is unique and denoted by $T^*$. Denote the set of all adjointable maps from $\mathsf{X}$ to $\mathsf{Y}$ by $\mathcal{L}(\mathsf{X}, \mathsf{Y})$, and denote $\mathcal{L}(\mathsf{X}, \mathsf{X})$ by $\mathcal{L}(\mathsf{X})$.

It is well-known that any adjointable operator is both bounded and $A$-linear. A short proof of this fact is given on page 8 of [13]. Thus, the algebra $\mathcal{L}(\mathsf{X})$ is then closed under the adjoint operation and is complete with respect to the operator norm, so $\mathcal{L}(\mathsf{X})$ is in fact a $C^*$-algebra.

**Definition 2.41.** The *strict topology* on $\mathcal{L}(\mathsf{X})$ is the topology induced by the seminorms

$$\{T \mapsto \|T\xi\| : \xi \in \mathsf{X}\} \text{ and } \{T \mapsto \|T^*\eta\| : \eta \in \mathsf{X}\}.$$

**Notation 2.42.** Given $\xi \in \mathsf{Y}$ and $\eta \in \mathsf{X}$, define $\theta_{\xi,\eta} : \mathsf{X} \to \mathsf{Y}$ by

$$\theta_{\xi,\eta}(\nu) := \xi \bullet \langle \eta \,|\, \nu\rangle_{\mathsf{X}} \text{ for all } \nu \in \mathsf{X}.$$

Then $\theta_{\xi,\eta} \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$. Let $\mathcal{K}(\mathsf{X}, \mathsf{Y})$ denote the closed span of $\{\theta_{\xi,\eta} : \xi \in \mathsf{X}, \eta \in \mathsf{Y}\}$ in $\mathcal{L}(\mathsf{X}, \mathsf{Y})$.

**Definition 2.43.** Let $\{\mathsf{X}_j\}_{j\in J}$ be a collection of Hilbert $\mathcal{A}$-modules, and let $\mathsf{Y} := \oplus_j \mathsf{X}_j$ be the Hilbert $\mathcal{A}$-module formed in Definition 2.38. Given $T_j \in \mathcal{L}(\mathsf{X}_j)$ for each $j \in J$ such that the family $\{T_j\}_{j\in J}$ satisfies $\sup_{j\in J} \|T_j\| < \infty$, define $\oplus_j T_j : \oplus_j \mathsf{X}_j \to \oplus_j \mathsf{X}_j$ by

$$[\oplus_j T_j](\xi_j)_{j\in J} := (T_j\xi_j)_{j\in J} \text{ for all } (\xi_j)_{j\in J} \in \oplus_j \mathsf{X}_j.$$

Then $\oplus_j T_j$ is a well-defined adjointable operator on $\oplus_j \mathsf{X}_j$ with adjoint $\oplus_j T_j^*$.

## 2.7 Representations on Hilbert $C^*$-modules

**Definition 2.44.** An operator $u \in \mathcal{L}(\mathsf{X})$ is *unitary* if $u^*u = I_{\mathsf{X}} = uu^*$.

Let $\mathcal{U}(\mathsf{X})$ denote the unitary group of $\mathcal{L}(\mathsf{X})$.

**Definition 2.45.** A *unitary group representation* of $G$ on a Hilbert $\mathcal{A}$-module $\mathsf{X}$ is a strictly continuous group homomorphism $u : G \to \mathcal{U}(\mathsf{X})$, which we henceforth denote by $x \mapsto u_x$.

Note that the requirement $u : G \to \mathcal{U}(\mathsf{X})$ be strictly continuous is equivalent to requiring that the maps $x \mapsto u_x\xi$ be continuous for each fixed $\xi \in \mathsf{X}$.

**Definition 2.46.** Let $u : G \to \mathcal{U}(\mathsf{X})$ be a unitary group representation, and given an arbitrary index set $J$, let $\oplus_j\mathsf{X} = \oplus_j\mathsf{X}_j$ where $\mathsf{X}_j = \mathsf{X}$ for all $j \in J$. Define

$$\oplus_j u : G \to \mathcal{U}(\oplus_j\mathsf{X}) \quad \text{by} \quad x \mapsto [\oplus_j u]_x := \oplus_j u_x \text{ for each } x \in G,$$

where $\oplus_j u_x$ is as in Definition 2.43. Then $\oplus_j u$ defines a unitary group representation of $G$.

**Definition 2.47.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, and let $\mathsf{X}$ be a Hilbert $\mathcal{B}$-module. A representation $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is *nondegenerate* if $\pi(\mathcal{A})\mathsf{X}$ is dense in $\mathsf{X}$.

**Definition 2.48.** Let $\mathsf{X}$ be a Hilbert $\mathcal{B}$-module and suppose $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation. Let $\mathsf{Y} = \oplus_j\mathsf{X}$, and define

$$\oplus_j\pi : \mathcal{A} \to \mathcal{L}(\mathsf{Y})$$

by $[\oplus_j\pi](a) := \oplus_j\pi(a)$ for each $a \in \mathcal{A}$, as in Definition 2.43. If $\mathsf{Y}_o$ denotes the dense $\mathcal{B}$-submodule of $\mathsf{Y}$ defined in Proposition 2.39, nondegeneracy of $\oplus_j\pi$ is easily established by showing $\mathrm{Span}\{[\oplus_j\pi(a)]\xi : a \in \mathcal{A},\ \xi \in \mathsf{Y}_o\}$ approximates elements of $\mathsf{Y}_o$.

**Definition 2.49.** Let $(G, \mathcal{A}, \alpha)$ be a $C^*$-dynamical system, let $\mathcal{B}$ be a $C^*$-algebra, and let $\mathsf{X}$ be a Hilbert $\mathcal{B}$-module. A *covariant homomorphism* of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$ is a pair $(\pi, u)$ consisting of homomorphisms $\pi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ and a unitary group representation $u : G \to \mathcal{U}(\mathsf{X})$ such that

$$\pi(\alpha_x(a)) = u_x \pi(a) u_x^* \text{ for all } x \in G,\ a \in \mathcal{A}.$$

We say $(\pi, u)$ is *nondegenerate* if $\pi$ is nondegenerate.

**Proposition 2.50** (Williams, 2.39 [25]). *Let $\mathsf{X}$ be a Hilbert $\mathcal{B}$-module and let $(\pi, u)$ be a covariant homomorphism of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$. Consider the integrated form $\pi \rtimes u : C_c(G, \mathcal{A}) \to \mathcal{L}(\mathsf{X})$ defined by*

$$[\pi \rtimes u](f) := \int_G \pi(f(x)) u_x \, d\mu(x) \textit{ for all } f \in C_c(G, \mathcal{A}),$$

*where this integral is the image of the function $x \mapsto \pi(f(x))u_x$ under the linear map described in Lemma 1.91 of [25]. Each $[\pi \rtimes u](f)$ is a well-defined operator in $\mathcal{L}(\mathsf{X})$, and $\pi \rtimes u$ extends to a homomorphism of $\mathcal{A} \rtimes_\alpha G$ which is nondegenerate whenever $\pi$ is nondegenerate. We denote this extension by $\pi \rtimes u$.*

*Conversely, if $L : \mathcal{A} \rtimes_\alpha G \to \mathcal{L}(\mathsf{X})$ is a nondegenerate homomorphism, then there is a unique nondegenerate covariant homomorphism $(\pi, u)$ of $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$ such that $L = \pi \rtimes u$.*

We can further characterize integrals involving continuous compactly supported functions from a locally compact group $G$ into a $C^*$-algebra $\mathcal{A}$ by the following lemma.

**Lemma 2.51** (Raeburn-Williams, C.12 [18]). *Let $\mathsf{X}$ be a Hilbert $\mathcal{A}$-module and $F$ a compactly supported function of $G$ into $\mathcal{L}(\mathsf{X})$ which is continuous for the strict topology. Then for each*

*$\xi, \eta \in \mathsf{X}$, the map $x \mapsto \langle \xi \,|\, F(x)\eta \rangle$ belongs to $C_c(G, \mathcal{A})$ and*

$$\left\langle \xi \,\middle|\, \left( \int_G F(x)\, d\mu(x) \right) \eta \right\rangle = \int_G \langle \xi \,|\, F(x)\eta \rangle \; d\mu(x).$$

**Proposition 2.52.** *Suppose $(\pi, u)$ is a covariant homomorphism for $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\mathsf{X})$ for some Hilbert $\mathcal{B}$-module $\mathsf{X}$. Then $(\oplus_j \pi, \oplus_j u)$ is a covariant homomorphism for $(G, \mathcal{A}, \alpha)$ into $\mathcal{L}(\oplus_j \mathsf{X})$, and*

$$(\oplus_j \pi) \rtimes (\oplus_j u) = \oplus_j (\pi \rtimes u).$$

*Proof.* Covariance of $(\oplus_j \pi, \oplus_j u)$ is straightforward to check. Let $\mathsf{Y} := \oplus_j \mathsf{X}$, and recall from Proposition 2.39 that

$$\mathsf{Y}_o := \{(\xi_j)_{j \in J} \in \mathsf{Y} : \xi_j = 0 \text{ for all but finitely many } j \in J\}$$

is dense in $\mathsf{Y}$. We claim

$$[(\oplus_j \pi) \rtimes (\oplus_j u)](f)|_{\mathsf{Y}_o} = [\oplus_j (\pi \rtimes u)](f)|_{\mathsf{Y}_o} \text{ for all } f \in C_c(G, \mathcal{A}).$$

Fix $f \in C_c(G, \mathcal{A})$, and observe that

$$\begin{aligned} [(\oplus_j \pi) \rtimes (\oplus_j u)]\,(f) &= \int_G [\oplus_j \pi](f(x))[\oplus_j u]_x \, d\mu(x) \\ &= \int_G [\oplus_j \pi(f(x))] \, [\oplus_j u_x] \; d\mu(x) \\ &= \int_G [\oplus_j \pi(f(x)) u_x] \; d\mu(x). \end{aligned}$$

For each $x \in G$, define $F(x) := \oplus_j [\pi(f(x))u_x]$. The maps $x \mapsto [\oplus_j \pi(f(x))]|_{\mathsf{Y}_o}$ and $x \mapsto [\oplus_j u_x]|_{\mathsf{Y}_o}$ from $G$ into $\mathcal{L}(\mathsf{Y})$ are strictly continuous, and density of $\mathsf{Y}_o$ in $\mathsf{Y}$ establishes strict continuity of $x \mapsto [\oplus_j \pi(f(x))] \circ [\oplus_j u_x]$. Therefore, $F : G \to \mathcal{L}(\mathsf{Y})$ is strictly continuous

Let $\eta \in \mathsf{Y}_o$, and let $\mathrm{Supp}(\eta) \subseteq J$ be the finite subset such that $\eta_j = 0$ for all $j \notin \mathrm{Supp}(\eta)$. Then, for any $\xi \in \mathsf{Y}$,

$$\begin{aligned}
\left\langle \xi \,\middle|\, \left( \int_G [\oplus_j \pi(f(x))u_x] \, d\mu(x) \right) \eta \right\rangle_{\mathsf{Y}} &= \left\langle \xi \,\middle|\, \left( \int_G F(x) \, d\mu(x) \right) \eta \right\rangle_{\mathsf{Y}} \\
&= \int_G \langle \xi \,|\, F(x)\eta \rangle_{\mathsf{Y}} \, d\mu(x) \qquad [\text{ Lemma 2.51 }] \\
&= \int_G \left( \sum_{j \in J} \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \right) d\mu(x) \\
&= \int_G \left( \sum_{j \in \mathrm{Supp}(\eta)} \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \right) d\mu(x) \\
&= \sum_{j \in \mathrm{Supp}(\eta)} \int_G \langle \xi_j \,|\, [\pi(f(x))u_x]\eta_j \rangle_{\mathsf{X}} \, d\mu(x) \\
&= \sum_{j \in \mathrm{Supp}(\eta)} \left\langle \xi_j \,\middle|\, \left( \int_G \pi(f(x))u_x \, d\mu(x) \right) \eta_j \right\rangle_{\mathsf{X}} \quad [\text{ Lemma 2.51 }] \\
&= \sum_{j \in J} \langle \xi_j \,|\, [\pi \rtimes u](f) \, \eta_j \rangle_{\mathsf{X}} \\
&= \langle \xi \,|\, [\oplus_j (\pi \rtimes u)(f)]\eta \rangle_{\mathsf{Y}} .
\end{aligned}$$

As $\xi \in \mathsf{Y}$ was arbitrary, we have that $[(\oplus_j \pi) \rtimes (\oplus_j u)](f)\eta = [\oplus_j(\pi \rtimes u)(f)]\eta$ for all $\eta \in \mathsf{Y}_o$. By density of $\mathsf{Y}_o$ in $\mathsf{Y}$ and continuity of both $[(\oplus_j \pi) \rtimes (\oplus_j u)](f)$ and $\oplus_j[\pi \rtimes u](f)$, we have $[(\oplus_j \pi) \rtimes (\oplus_j u)](f) = \oplus_j[\pi \rtimes u](f)$ as adjointable operators on $\mathcal{L}(\mathsf{Y})$. As $f \in C_c(G, \mathcal{A})$ was arbitrary and $C_c(G, \mathcal{A})$ is dense in $\mathcal{A} \rtimes_\alpha G$, we conclude $\oplus_j[\pi \rtimes u] = (\oplus_j \pi) \rtimes (\oplus_j u)$. □

## 2.8 Hilbert $\mathcal{K}(\mathcal{H})$-modules

A substantial portion of the collaboration with L. Huang is in the setting of $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ obtained by closing the finite-rank operators on $\mathcal{H}$ in the norm topology. The following results are used later in the paper and provide some evidence of why

$\mathcal{K}(\mathcal{H})$ was desirable to work with. As a first attractive property, recall that $\mathcal{K}(\mathcal{H})$ is simple, so every nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module $\mathsf{X}$ is full since $\langle \mathsf{X} \,|\, \mathsf{X} \rangle$ forms a nontrivial two-sided ideal in $\mathcal{K}(\mathcal{H})$.

**Lemma 2.53** (Arveson, 1.4.1 [2])**.** *Let $p$ be a nonzero projection in $\mathcal{K}(\mathcal{H})$. Then $p$ is rank-one if and only if $p\mathcal{K}(\mathcal{H})p = \mathbb{C}p$.*

**Corollary 2.54.** *If $p \in \mathcal{K}(\mathcal{H})$ is a rank-one projection, there is a linear functional $f_p : \mathcal{K}(\mathcal{H}) \to \mathbb{C}$ such that $pap = f_p(a)p$ for all $a \in \mathcal{K}(\mathcal{H})$.*

**Corollary 2.55.** *Let $\mathsf{X}$ be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module, and let $p$ be a rank-one projection in $\mathcal{K}(\mathcal{H})$. Then there exists $\psi \in \mathsf{X}$ such that $\langle \psi \,|\, \psi \rangle = p$.*

*Proof.* Let $p \in \mathcal{K}(\mathcal{H})$ be a rank-one projection. Then there exists $\psi_o \in \mathsf{X}$ such that $\psi_o \bullet p \neq 0$ (since $\mathsf{X} \bullet p$ is a full Hilbert $\mathcal{K}(\mathcal{H})$-module). Thus,

$$0 \neq \langle \psi_o \bullet p \,|\, \psi_o \bullet p \rangle = p \, \langle \psi_o \,|\, \psi_o \rangle \, p = f_p \left( \langle \psi_o \,|\, \psi_o \rangle \right) p,$$

where $f_p$ is the linear functional corresponding to $p$ obtained in Corollary 2.54. Let $\lambda := f_p \left( \langle \psi_o \,|\, \psi_o \rangle \right)$, and define $\psi := \lambda^{-1/2}(\psi_o \bullet p)$. Then $\langle \psi \,|\, \psi \rangle = p$. □

**Lemma 2.56.** *Let $\mathsf{X}$ be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module and $p$ a rank-one projection on $\mathcal{H}$. Then $\mathsf{X} \bullet p$ is a nontrivial closed subspace of $\mathsf{X}$ that is also a Hilbert space with inner product*

$$\langle \xi \bullet p \,|\, \eta \bullet p \rangle_{\mathsf{X} \bullet p} = f_p(\langle \xi \,|\, \eta \rangle_{\mathsf{X}}) \text{ for every } \xi, \eta \in \mathsf{X},$$

*where $f_p$ is the linear functional related to $p$ in Corollary 2.54. Furthermore, the norm on $\mathsf{X} \bullet p$ induced by $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X} \bullet p}$ coincides with the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$.*

*Proof.* (Huang) It is obvious that $\mathsf{X} \bullet p$ is a subspace of $\mathsf{X}$. To see that it is closed in $\mathsf{X}$, let $(\zeta_n)_{n \in \mathbb{N}}$ be a sequence in $\mathsf{X} \bullet p$ such that $(\zeta_n)_{n \in \mathbb{N}}$ converges to some $\eta \in \mathsf{X}$. Because $\zeta_n \bullet p = \zeta_n$

for all $n \in \mathbb{N}$, we have

$$\eta = \lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \zeta_n \bullet p = \left[\lim_{n\to\infty} \zeta_n\right] \bullet p = \eta \bullet p.$$

Hence, $\eta \in \mathsf{X} \bullet p$, which proves that $\mathsf{X} \bullet p$ is a closed subspace of $\mathsf{X}$.

Clearly, $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X}\bullet p}$ is a sesquilinear form on $\mathsf{X} \bullet p$, so it remains to prove that it is positive definite and complete. Let $\zeta \in \mathsf{X} \bullet p$. Then $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}}$ is positive in $\mathcal{K}(\mathcal{H})$, which means that

$$f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}}\right) p = p \langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} p = p \langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} p^*$$

is positive in $\mathcal{K}(\mathcal{H})$ as well. As $p(I - p) = 0$, we deduce that $I - p$ is not invertible in $\mathcal{K}(\mathcal{H})$, so $1 \in \sigma_{\mathcal{K}(\mathcal{H})}(p)$. Hence, $f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}}\right) \in \sigma_{\mathcal{K}(\mathcal{H})} \left(f_p \left(\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}}\right) p\right) \subseteq \mathbb{R}_{\geq 0}$, which shows that $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X}\bullet p}$ is at least positive semidefinite. Next, observe that

$$\begin{aligned}
\left|\langle \zeta \,|\, \eta \rangle_{\mathsf{X}\bullet p}\right| &= \left|f_p \left(\langle \zeta \,|\, \eta \rangle_{\mathsf{X}}\right)\right| \\
&= \left\|f_p \left(\langle \zeta \,|\, \eta \rangle_{\mathsf{X}}\right) p\right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\|p \langle \zeta \,|\, \eta \rangle_{\mathsf{X}} p\right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\|\langle \zeta \bullet p \,|\, \eta \bullet p \rangle_{\mathsf{X}}\right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\|\langle \zeta \,|\, \eta \rangle_{\mathsf{X}}\right\|_{\mathcal{K}(\mathcal{H})}. \qquad [\text{As } \zeta \bullet p = \zeta \text{ and } \eta \bullet p = \eta.]
\end{aligned}$$

Consequently, if $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}\bullet p} = 0$ for some $\zeta \in \mathsf{X} \bullet p$, then $\langle \zeta \,|\, \zeta \rangle_{\mathsf{X}} = 0$, which yields $\zeta = 0$. This proves that $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X}\bullet p}$ is positive definite. Incidentally, this also proves that $\|\zeta\|_{\mathsf{X}\bullet p} = \|\zeta\|_{\mathsf{X}}$ for all $\zeta \in \mathsf{X} \bullet p$. As $\mathsf{X} \bullet p$ is a closed subspace of $\mathsf{X}$, it is a Banach space with respect to the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$, and is thus a Banach space with respect to $\|\cdot\|_{\mathsf{X}\bullet p}$. Therefore, $\mathsf{X} \bullet p$ is a Hilbert space whose inner product is given by $\langle \cdot \,|\, \cdot \rangle_{\mathsf{X}\bullet p}$, and the induced norm on $\mathsf{X} \bullet p$ is the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet p$. □

**Theorem 2.57** (Bakić-Guljaš, 5 & 6 [8]). *Given a rank-one projection $p \in \mathcal{K}(\mathcal{H})$, the maps*

$$\Psi : \mathcal{L}(\mathsf{X}) \to \mathcal{B}(\mathsf{X} \bullet p) \quad \textit{and} \quad \Psi|_{\mathcal{K}(\mathsf{X})} : \mathcal{K}(\mathsf{X}) \to \mathcal{K}(\mathsf{X} \bullet p)$$

*given by $T \mapsto T|_{\mathsf{X} \bullet p}$ are $C^*$-isomorphisms.*

**Theorem 2.58** (Magajna, 1 [14]). *Every Hilbert $\mathcal{K}(\mathcal{H})$-module $\mathsf{X}$ is complementable, that is, every closed $\mathcal{K}(\mathcal{H})$-submodule $\mathsf{Y} \subseteq \mathsf{X}$ has an orthogonal complement $\mathsf{Y}^\perp$ such that $\mathsf{X} = \mathsf{Y} \oplus \mathsf{Y}^\perp$.*

**Proposition 2.59.** *Let $\mathsf{X}$ be a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module, let $\mathsf{Y}$ be a nonzero $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$ that is not necessarily closed, and let $p$ be a rank-one projection on $\mathcal{H}$. Then*

$$\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \overline{\mathsf{Y}}.$$

*Proof.* As $\mathsf{Y}$ is a $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$, we have that $\mathsf{Y} \bullet p \subseteq \mathsf{Y}$, and thus, $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ is contained in $\mathsf{Y}$. Hence, $\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})}$ is contained in $\overline{\mathsf{Y}}$. It thus remains to establish the reverse containment.

Note that $\{pa : a \in \mathcal{K}(\mathcal{H}) \setminus \{0\}\}$ is the set of all rank-one projections on $\mathcal{H}$. Let $\zeta \in \mathsf{Y}$ and let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $\mathcal{K}(\mathcal{H})$. Then $\|\zeta \bullet e_\lambda - \zeta\| \to 0$ as $\lambda \to \infty$. Moreover, $\operatorname{Span}\{pa : a \in \mathcal{K}(\mathcal{H})\}$ contains all finite-rank operators on $\mathcal{H}$, so $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ can approximate $\zeta \bullet e_\lambda$ for any choice of $\lambda \in \Lambda$. An $\frac{\epsilon}{2}$-argument shows that the closure of $(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})$ in $\mathsf{X}$ equals the closure of $\mathsf{Y}$. $\square$

# Chapter 3

# Analytic Vectors for $\delta_D$

## 3.1 Definition of Weak $D$-Differentiability

Throughout, $\mathcal{H}$ is a Hilbert space and $D$ is a (possibly unbounded) self-adjoint operator on $\mathcal{H}$. For each $t \in \mathbb{R}$, both Stone's Theorem and the Spectral Theorem for Self-Adjoint Operators yields a strongly-continuous one-parameter group of unitaries $\{e^{itD}\}_{t\in\mathbb{R}}$. For each $t \in \mathbb{R}$, define a map $\alpha_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\alpha_t(x) := e^{itD}xe^{-itD} \text{ for all } x \in \mathcal{B}(\mathcal{H}).$$

Then $\{\alpha_t\}_{t\in\mathbb{R}}$ defines a flow on $\mathcal{B}(\mathcal{H})$ and forms group of $*$-automorphisms on $\mathcal{B}(\mathcal{H})$.

**Definition 3.1.** An operator $x \in \mathcal{B}(\mathcal{H})$ is *weakly $D$-differentiable* if there exists $y \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to 0}\left|\left\langle \left(\frac{\alpha_t(x)-x}{t} - y\right) h, k\right\rangle\right| = 0 \text{ for all } h,k \in \mathcal{H}. \qquad (*)$$

Denote the set of all weakly $D$-differentiable operators by $\mathrm{Dom}(\delta_D)$, and for $x \in \mathrm{Dom}(\delta_D)$, let $\delta_D(x) := y$, where $y$ satisfies condition $(*)$.

**Theorem 3.2** (Christensen, 3.8 [6])**.** *Let $x \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

*(i) $x$ is weakly $D$-differentiable.*

*(ii) There exists* $y \in \mathcal{B}(\mathcal{H})$ *such that for every* $h \in \mathcal{H}$*,*

$$\lim_{t \to 0} \left\| \left( \frac{\alpha_t(x) - x}{t} - y \right) h \right\| = 0.$$

*(iii) There exists* $c > 0$ *such that* $\|\alpha_t(x) - x\| \leq c\,|t|$ *for all* $t \in \mathbb{R}$*.*

*(iv) The commutator* $[iD, x]$ *is defined and bounded on the domain of* $D$*.*

*(v) The commutator* $[iD, x]$ *is defined and bounded on a core for* $D$*.*

*If any of the above conditions hold, then* $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ *and* $\delta_D(x)|_{\mathrm{Dom}(D)} = [iD, x]$*.*

**Theorem 3.3** (Christensen, 3.9 [6])**.** *The domain of definition* $\mathrm{Dom}(\delta_D)$ *is a SOT-dense* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *and* $\delta_D$ *is a* $*$*-derivation into* $\mathcal{B}(\mathcal{H})$*. The graph of* $\delta_D$ *is WOT-closed.*

Theorem 3.3 supports Christensen's argument in [6] for considering differentiability of $x \in \mathcal{B}(\mathcal{H})$ in the weak operator topology as opposed to the norm topology on $\mathcal{B}(\mathcal{H})$. In a subsequent paper, [5], Christensen defines higher weak $D$-differentiability via higher powers of $\delta_D$.

**Definition 3.4.** An operator $x \in \mathcal{B}(\mathcal{H})$ is *n-times weakly D-differentiable* if $x \in \mathrm{Dom}(\delta_D^n)$.

**Proposition 3.5** (Christensen, 2.6 [5])**.** *An operator* $x \in \mathcal{B}(\mathcal{H})$ *is* $n$*-times weakly* $D$*-differentiable if and only if for each pair* $h, k \in \mathcal{H}$*, the function* $t \mapsto \langle \alpha_t(x)h, k \rangle$ *is* $n$*-times continuously differentiable. Moreover, if* $x$ *is* $n$*-times weakly* $D$*-differentiable, then*

$$\frac{d^n}{dt^n} \langle \alpha_t(x)h, k \rangle = \langle \alpha_t[\delta_D^n(x)]h, k \rangle .$$

Analogous to Theorem 3.2, the following proposition and theorem connect higher-order weak $D$-differentiability of $x \in \mathcal{B}(\mathcal{H})$ to definedness and boundedness of iterated commutators $[iD, ..., [iD, x]]$.

**Proposition 3.6** (Christensen, 3.3 [5])**.** *Let $x \in \mathrm{Dom}(\delta_D^n)$. Then for $k = 1, ..., n$,*

*(i)* $\delta_D^{k-1}(x)(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$

*(ii)* $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$

*(iii)* $\mathrm{Dom}\left(\underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}\right) = \mathrm{Dom}(D^k)$

*(iv)* $\delta_D^k(x)|_{\mathrm{Dom}(D^k)} = \underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}$

*(v)* $\delta_D^k(x)$ *is the bounded extension of* $\underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}$ *from* $\mathrm{Dom}(D^k)$ *to all of* $\mathcal{H}$.

**Theorem 3.7** (Christensen, 4.1 [5])**.** *Let $x \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

*(i)* *$x$ is $n$ times weakly $D$-differentiable.*

*(ii)* *For all $k = 1, ..., n$, $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$ and $\underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}$ is defined and bounded on $\mathrm{Dom}(D^k)$ with bounded extension $\delta_D^k(x)$.*

*(iii)* *There exists a core $\mathscr{C}$ for $D$ such that for any $k = 1, ..., n$, the operator $\underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}$ is defined and bounded on $\mathscr{C}$.*

**Notation 3.8.** For notational convenience, for each $k \in \mathbb{N}$ we define

$$d^k(x) := \underbrace{[iD, ..., [iD, x]]}_{k \text{ times}}.$$

## 3.2 Weakly $-i\frac{d}{d\theta}$-Differentiable Multiplication Operators on $L^2(\mathbb{T})$

Consider the operator $D = -i\frac{d}{d\theta}$ on $L^2(\mathbb{T})$ with domain

$$\mathrm{Dom}(D) = \left\{ f \in L^2(\mathbb{T}) : f \text{ is absolutely continuous}, f' \in L^2(\mathbb{T}) \right\}.$$

**Notation 3.9.** Given a $\sigma$-finite measure space $(X, \mu)$, define

$$\begin{array}{rccc} \mathrm{diag}: & L^\infty(X,\mu) & \to & \mathcal{B}(L^2(X,\mu)) \\ & f & \mapsto & M_f \end{array}$$

where $M_f g = fg$ for each $g \in L^2(X, \mu)$.

Proposition 3.11 characterizes the $n$-times weakly $D$-differentiable multiplication operators $M_f \in \mathrm{diag}(L^\infty(\mathbb{T}))$, and Proposition 3.10 provides as the case when $n = 1$.

**Proposition 3.10.** *Let $f \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

*(i) $M_f$ is weakly $D$-differentiable.*

*(ii) $f \in \mathrm{Dom}(D)$ and $Df \in L^\infty(\mathbb{T})$.*

*When either condition is satisfied, $\delta_w^D(M_f) = M_{f'}$.*

*Proof.* ($\Rightarrow$) If $M_f \in \mathrm{Dom}(\delta_D)$, then $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ by Theorem 3.2. Let $\mathbf{1}$ denote the function which takes the value 1 for all $z \in \mathbb{T}$. Then $\mathbf{1}$ is in $\mathrm{Dom}(D)$, and so $f = M_f\mathbf{1} \in \mathrm{Dom}(D)$. In [6], Christensen remarks that in this particular setting, condition (iii) of Theorem 3.2 holds if and only if there exists $c > 0$ such that for all $z \in \mathbb{T}$ and $t \in \mathbb{R}$,

$$\left| f(ze^{it}) - f(z) \right| \leq c\,|t|\,.$$

As $f \in \mathrm{Dom}(D)$, $f$ is absolutely continuous and thus differentiable a.e. Hence, for a.e. $z \in \mathbb{T}$,

$$|f'(z)| = \lim_{t \to 0} \left| \frac{f(ze^{it}) - f(z)}{t} \right| \leq c.$$

Therefore, $\|f'\|_\infty \leq c$, so $f' \in L^\infty(\mathbb{T})$. Hence, $Df = -if' \in L^\infty(\mathbb{T})$.

($\Leftarrow$): Suppose $f \in \mathrm{Dom}(D)$ and $Df \in L^\infty(\mathbb{T})$. We show $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $[iD, M_f]$ agrees with the bounded operator $M_{f'}$ on $\mathrm{Dom}(D)$. Fix $g \in \mathrm{Dom}(D)$. Then $g' \in L^2(\mathbb{T})$, so

$$\|(fg)'\|_2 = \|fg' + f'g\|_2 \leq \|fg'\|_2 + \|f'g\|_2 \leq \|f\|_\infty \|g'\|_2 + \|f'\|_\infty \|g\|_2 < \infty.$$

Also, the product of two absolutely continuous functions is absolutely continuous. Therefore, $fg \in \mathrm{Dom}(D)$. As $g \in \mathrm{Dom}(D)$ was arbitrary, we have $M_f(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. Observe

$$[iD, M_f]g = (fg)' - fg' = f'g + fg' - fg' = f'g = M_{f'}g \text{ for all } g \in \mathrm{Dom}(D).$$

As $f' \in L^\infty(\mathbb{T})$ and $[iD, M_f]|_{\mathrm{Dom}(D)} = M_{f'} \in \mathcal{B}(L^2(\mathbb{T}))$, we have that $[iD, M_f]$ is defined and bounded on $\mathrm{Dom}(D)$. By (i) $\Longleftrightarrow$ (iv) of Theorem 3.2, we conclude $M_f \in \mathrm{Dom}(\delta_D)$ and $\delta_D(M_f) = M_{f'}$. □

**Proposition 3.11.** *Let $f \in L^\infty(\mathbb{T})$. The following statements are equivalent:*

*(i) $M_f$ is $n$-times weakly $D$-differentiable.*

*(ii) $f \in \mathrm{Dom}(D^n)$ and $D^n f \in L^\infty(\mathbb{T})$.*

*When either condition is satisfied, $\delta_D^n(M_f) = M_{f^{(n)}}$.*

*Proof.* Fix $n \in \mathbb{N}$. We proceed by induction. The base case was established in Proposition 3.10.

$(\Rightarrow)$ : Suppose for all $k \leq n-1$, if $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$ then $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$. Let $M_f \in \mathrm{Dom}(\delta_D^n)$, so $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$ for each $k \leq n$. The inductive hypothesis implies $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$ for each $k \leq n-1$.

As in the proof of Proposition 3.10, let $\mathbf{1}$ the function which takes the value 1 for all $z \in \mathbb{T}$. By Proposition 3.6 (ii), $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$, and so $f = M_f\mathbf{1} \in \mathrm{Dom}(D^n)$. To see that $D^n f \in L^\infty(\mathbb{T})$, note $M_f \in \mathrm{Dom}(\delta_D^n)$ implies $\delta_D^{n-1}(M_f) \in \mathrm{Dom}(\delta_D)$. By the inductive hypothesis,

$$\delta_D^{n-1}(M_f) = M_{f^{(n-1)}}.$$

By (i) $\iff$ (iii) of Theorem 3.2, $M_{f^{(n-1)}} \in \mathrm{Dom}(\delta_D)$ if and only if there exists $c > 0$ such that for all $z \in \mathbb{T}$ and $t \in \mathbb{R}$,

$$\left|f^{(n-1)}(ze^{it}) - f^{(n-1)}(z)\right| \leq c\,|t|\,.$$

Now, $f \in \mathrm{Dom}(D^n)$ by definition means $D^{n-1}f \in \mathrm{Dom}(D)$, which is equivalent to $f^{(n-1)} \in \mathrm{Dom}(D)$. In particular, $f^{(n-1)}$ is differentiable a.e., and thus, for almost every $z \in \mathbb{T}$, we have

$$\left|f^{(n)}(z)\right| = \lim_{t\to 0}\left|\frac{f^{(n-1)}(ze^{it}) - f^{(n-1)}(z)}{t}\right| \leq c.$$

Therefore, $\left\|f^{(n)}\right\|_\infty \leq c$, and hence, $f^{(n)} \in L^\infty(\mathbb{T})$. Given $D^n f = (-i)^n f^{(n)}$, we have shown $D^n f \in L^\infty(\mathbb{T})$.

$(\Leftarrow)$ : Let $f \in \mathrm{Dom}(D^n)$ and suppose $D^n f \in L^\infty(\mathbb{T})$. Further, suppose for all $k \leq n-1$, if $f \in \mathrm{Dom}\big(D^k\big)$ and $D^k f \in L^\infty(\mathbb{T})$, then $M_f \in \mathrm{Dom}\big(\delta_D^k\big)$. To prove $M_f \in \mathrm{Dom}(\delta_D^n)$, by Theorem 3.7, it suffices to prove $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$ and the commutator

$$d^n(M_f) = \underbrace{[iD, ..., [iD, M_f]]}_{n \text{ times}}$$

is bounded on $\mathrm{Dom}(D^n)$. Given $g \in \mathrm{Dom}(D^n)$, showing $M_f g \in \mathrm{Dom}(D^n)$ amounts to proving

(i) $fg \in \mathrm{Dom}(D^{n-1})$,

(ii) $D^{n-1}(fg)$ is absolutely continuous, and

(iii) $(D^{n-1}(fg))' \in L^2(\mathbb{T})$.

Since $M_f \in \mathrm{Dom}\big(\delta_D^{n-1}\big)$, Proposition 3.6 implies $M_f(\mathrm{Dom}(D^{n-1})) \subseteq \mathrm{Dom}(D^{n-1})$. Hence, $g \in \mathrm{Dom}(D^n) \subseteq \mathrm{Dom}(D^{n-1})$ implies $M_f g = fg \in \mathrm{Dom}(D^{n-1})$. Now,

$$D^{n-1}(fg) = (-i)^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(n-1-j)} g^{(j)}.$$

Each term of the above sum is the product of absolutely continuous functions because $D^{n-1-j} f \in \mathrm{Dom}(D)$ and $D^j g \in \mathrm{Dom}(D)$ for all $j = 0, ..., n-1$. The product of any two absolutely continuous functions on a bounded interval is again absolutely continuous, and thus the entire sum is as well. Therefore, (ii) is satisfied. Also,

$$\left\|(D^{n-1}(fg))'\right\|_2 = \|D^n(fg)\|_2 \leq \sum_{j=0}^{n} \binom{n}{j} \left\|f^{(n-j)} g^{(j)}\right\|_2 \leq \sum_{j=0}^{n} \binom{n}{j} \left\|f^{(n-j)}\right\|_\infty \left\|g^{(j)}\right\|_2.$$

As $\|f^{n-j}\|_\infty = \left\|D^{(n-j)} f\right\|_\infty < \infty$ and $g \in \mathrm{Dom}(D^n)$ ensures $g^{(j)} \in L^2(\mathbb{T})$ for all $j = 0, ..., n$, we conclude that $\|(D^{n-1}(fg))'\|_2 < \infty$. Therefore, $M_f(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(D^n)$.

Having established that $d^n(M_f)$ is defined on $\mathrm{Dom}(D^n)$, we now show $d^n(M_f)$ is bounded on $\mathrm{Dom}(D^n)$. In Proposition 3.10 we observed $[iD, M_f]|_{\mathrm{Dom}(D)} = M_{f'}$. Since $f' \in L^\infty(\mathbb{T})$, we concluded $\delta_D(M_f) = M_{f'}$. Following this same argument, we have $d^k(M_f) = M_{f^{(k)}}|_{\mathrm{Dom}\left(D^k\right)}$, so $\delta_D^k(M_f) = M_{f^{(k)}}$ for all $k \leq n-1$. As $\mathrm{Dom}(D^n) \subseteq \mathrm{Dom}(D^{n-1})$,

$$d^n(M_f)|_{\mathrm{Dom}(D^n)} = d(d^{n-1}(M_f))|_{\mathrm{Dom}(D^n)} = d(M_{f^{(n-1)}})|_{\mathrm{Dom}(D^n)} = [iD, M_{f^{(n-1)}}]|_{\mathrm{Dom}(D^n)}.$$

Furthermore, $[iD, M_{f^{(n-1)}}]|_{\mathrm{Dom}(D^n)} = M_{f^{(n)}}$. By assumption, $D^n f \in L^\infty(\mathbb{T})$, which is equivalent to $f^{(n)} \in L^\infty(\mathbb{T})$. Therefore, the commutator $d^n(M_f)$ agrees with the bounded operrator $M_{f^{(n)}}$ on $\mathrm{Dom}(D^n)$, which establishes by Theorem 3.7 that $M_f \in \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(M_f) = M_{f^{(n)}}$. □

## 3.3 Domains of Higher Powers

Throughout this section, $D$ denotes an arbitrary self-adjoint operator on a Hilbert space $\mathcal{H}$. While Theorem 3.7 extends Theorem 3.2 by connecting $n$-times weak $D$-differentiability of a bounded operator $x$ to definedness and boundedness of an iterated commutator of $x$ with $iD$, there is no analogous theorem to Theorem 3.3 stating that $\mathrm{Dom}(\delta_D^n)$ remains SOT-dense in $\mathcal{B}(\mathcal{H})$. The purpose of this section is to give a constructive proof of SOT-density of $\mathrm{Dom}(\delta_D^n)$ for all $n \in \mathbb{N}$.

Given $f, g \in \mathcal{H}$, recall the rank-one operator $f \otimes g^* : \mathcal{H} \to \mathcal{H}$ is defined as

$$(f \otimes g^*)(v) := \langle v, g \rangle \, f \text{ for all } v \in \mathcal{H}.$$

Fix $n \in \mathbb{N}$. We use the facts that $\mathrm{Span}\{f \otimes g^* : f, g \in \mathcal{H}\}$ is norm-dense in $\mathcal{K}(\mathcal{H})$ and that $\mathcal{K}(\mathcal{H})$ is SOT-dense in $\mathcal{B}(\mathcal{H})$ to prove $\mathrm{Dom}(\delta_D^n)$ is SOT-dense in $\mathcal{B}(\mathcal{H})$.

**Lemma 3.12.** *Let $n \in \mathbb{N}$. If $h, k \in \mathrm{Dom}(D^n)$, then $h \otimes k^* \in \mathrm{Dom}(\delta_D^n)$ and*

$$\delta_D^n(h \otimes k) = \sum_{j=0}^{n} (iD)^{n-j} h \otimes [(iD)^j k]^*.$$

*Proof.* Let $h, k \in \mathrm{Dom}(D^n)$. First, observe that for all $f, g \in \mathcal{H}$,

$$\begin{aligned}
\langle \alpha_t(h \otimes k^*)f, g \rangle &= \left\langle e^{itD}(h \otimes k^*)e^{-itD}f, g \right\rangle \\
&= \left\langle (h \otimes k)e^{-itD}f, e^{-itD}g \right\rangle \\
&= \left\langle \left\langle e^{-itD}f, k \right\rangle h, e^{-itD}g \right\rangle \\
&= \left\langle e^{itD}h, g \right\rangle \left\langle f, e^{itD}k \right\rangle .
\end{aligned}$$

Let us consider the case when $n = 1$. By Proposition 3.5, $h \otimes k^* \in \mathrm{Dom}(\delta_D)$ if and only if for every $f, g \in \mathcal{H}$ the map $t \mapsto \langle \alpha_t(h \otimes k^*)f, g \rangle$ is continuously differentiable. Thus, it suffices to prove that

$$t \mapsto \langle \alpha_t(h \otimes k^*)f, g \rangle = \left\langle f, e^{itD}k \right\rangle \left\langle e^{itD}h, g \right\rangle$$

is $n$-times continuously differentiable for all $f, g \in \mathcal{H}$.

Fix $f, g \in \mathcal{H}$. By Stone's Theorem,

$$\lim_{t \to 0} \left\| \frac{e^{itD}h - h}{t} - iDh \right\| = 0 \quad \text{and} \quad \lim_{t \to 0} \left\| \frac{e^{itD}k - k}{t} - iDk \right\| = 0.$$

By the Schwarz inequality, the maps $t \mapsto \left\langle f, e^{itD}k \right\rangle$ and $t \mapsto \left\langle e^{itD}h, g \right\rangle$ are continuously differentiable with derivatives $t \mapsto \left\langle f, e^{itD}(iDk) \right\rangle$ and $t \mapsto \left\langle e^{itD}(iDh), g \right\rangle$, respectively. Since the product of two continuously differentiable functions is continuously differentiable, $t \mapsto \langle \alpha_t(h \otimes k^*)f, g \rangle = \left\langle e^{itD}h, g \right\rangle \left\langle f, e^{itD}k \right\rangle$ is continuously differentiable. As $f, g \in \mathcal{H}$ were arbitrary, we conclude $h \otimes k^* \in \mathrm{Dom}(\delta_D)$.

Furthermore, Proposition 3.5 states that for all $f, g \in \mathcal{H}$,

$$\frac{d}{dt} \langle \alpha_t(h \otimes k^*)f, g \rangle \big|_{t=0} = \langle \delta_D(h \otimes k^*)f, g \rangle .$$

Hence,

$$\begin{aligned}
\langle \delta_D(h \otimes k^*)f, g\rangle &= \frac{d}{dt}\left(\left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}h, g\right\rangle\right)\big|_{t=0} \\
&= \left\langle f, e^{itD}iDk\right\rangle \left\langle e^{itD}h, g\right\rangle \big|_{t=0} + \left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}iDh, g\right\rangle \big|_{t=0} \\
&= \langle f, iDk\rangle \langle h, g\rangle + \langle f, k\rangle \langle iDh, g\rangle \\
&= \langle \langle f, iDk\rangle h, g\rangle + \langle \langle f, k\rangle iDh, g\rangle \\
&= \langle [h \otimes (iDk)^*]f, g\rangle + \langle [(iDh) \otimes k^*]f, g\rangle \\
&= \langle [(iDh) \otimes k^* + h \otimes (iDk)^*]f, g\rangle
\end{aligned}$$

As $f, g \in \mathcal{H}$ were arbitrary, $\delta_D(h \otimes k^*) = (iDh) \otimes k^* + h \otimes (iDk)^*$.

For general $n \in \mathbb{N}$, the rank-one operator $h \otimes k^*$ is $n$-times weakly $D$ differentiable if and only if for every $f, g \in \mathcal{H}$ the map $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ is $n$-times continuously differentiable. As above, $\langle \alpha_t(h \otimes k^*)f, g\rangle = \left\langle f, e^{itD}k\right\rangle \left\langle e^{itD}h, g\right\rangle$ and, given $h, k \in \mathrm{Dom}(D^n)$, the functions $t \mapsto \left\langle f, e^{itD}k\right\rangle$ and $t \mapsto \left\langle e^{itD}h, g\right\rangle$ are $n$-times continuously differentiable, where

$$\frac{d^j}{dt^j}\left\langle f, e^{itD}k\right\rangle = \left\langle f, e^{itD}[(iD)^j k]\right\rangle \quad \text{and} \quad \frac{d^j}{dt^j}\left\langle e^{itD}h, g\right\rangle = \left\langle e^{itD}[(iD)^j h], g\right\rangle$$

for each $j = 1, ..., n$. Since the product of two $n$-times continuously differentiable functions is $n$-times continuously differentiable, $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ is $n$-times continuously differentiable. As $f, g \in \mathcal{H}$ were arbitrary, $h \otimes k^* \in \mathrm{Dom}(\delta_D^n)$, and a computation similar to the $n = 1$ case yields

$$\delta_D^n(h \otimes k^*) = \sum_{j=0}^{n} (iD)^{n-j} h \otimes [(iD)^j k]^*.$$

□

**Notation 3.13.** Given a subset $S \subseteq \mathcal{H}$, let $\mathcal{F}(S) := \mathrm{Span}\{f \otimes g^* : f, g \in S\}$.

**Lemma 3.14.** *If $S \subseteq \mathcal{H}$ is a dense subspace, then $\mathcal{F}(S)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.*

The proof is an easy exercise which we leave to the reader.

**Corollary 3.15.** *For each* $n \in \mathbb{N}$, $\mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$ *is norm-dense in* $\mathcal{K}(\mathcal{H})$*.*

*Proof.* By Lemma 3.12, $\mathcal{F}(\mathrm{Dom}(D^n)) \subseteq \mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$. As $\mathrm{Dom}(D^n)$ is dense in $\mathcal{H}$ for each $n \in \mathbb{N}$ by Nelson's Analytic Vector Theorem, Lemma 3.14 implies $\mathcal{F}(\mathrm{Dom}(D^n))$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Therefore, $\mathrm{Dom}(\delta_D^n) \cap \mathcal{F}(\mathcal{H})$ is norm-dense in $\mathcal{K}(\mathcal{H})$. □

**Theorem 3.16.** *For each* $n \in \mathbb{N}$, $\mathrm{Dom}(\delta_D^n)$ *is SOT-dense in* $\mathcal{B}(\mathcal{H})$*.*

*Proof.* As the norm topology is finer than the SOT on $\mathcal{B}(\mathcal{H})$,

$$\overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\mathrm{SOT}} \supseteq \overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\|\cdot\|} = \mathcal{K}(\mathcal{H})$$

by Corollary 3.15. Therefore, $\overline{\mathcal{F}(\mathcal{H}) \cap \mathrm{Dom}(\delta_D^n)}^{\mathrm{SOT}} = \overline{\mathcal{K}(\mathcal{H})}^{\mathrm{SOT}} = \mathcal{B}(\mathcal{H})$. □

## 3.4 $C_o$–Groups of Isometries and their Infinitesimal Generators

Theorem 3.16 strengthens Christensen's Theorem 3.3 and provides a way to construct elements in $\mathrm{Dom}(\delta_D^n)$ using elements from $\mathrm{Dom}(D^n)$. Given that the analytic vectors for $D$ are dense in $\mathcal{H}$, we were led to wonder if the analytic vectors for $\delta_D$ (which are operators in $\mathcal{B}(\mathcal{H})$) were SOT-dense in $\mathcal{B}(\mathcal{H})$.

To relate the analytic vectors for $D$ and $\delta_D$ as we related $\mathrm{Dom}(D^n)$ and $\mathrm{Dom}(\delta_D^n)$ in Lemma 3.12, we exploit an equivalent notion of analyticity for the one-parameter families for which $D$ and $\delta_D$ are infinitesimal generators: $\{e^{itD}\}_{t\in\mathbb{R}}$ and $\{\alpha_t\}_{t\in\mathbb{R}}$, respectively. We first introduce the notion of analytic vectors for a general one-parameter family on a Banach space, and then we specialize to our setting.

**Definition 3.17.** Let $X$ be a Banach space and let $Y$ be a closed subspace of $X^*$. A one-parameter family $\{\tau_t\}_{t\in\mathbb{R}}$ of isometries on $X$ into itself is called a $\sigma(X,Y)$-*continuous group of isometries of* $X$ if

1. $\tau_0 = I$,
2. $\tau_{s+t} = \tau_s\tau_t$ for all $s,t \in \mathbb{R}$,
3. $t \mapsto \tau_t(x)$ is $\sigma(X,Y)$-continuous for all $x \in X$, i.e., $t \mapsto \psi(\tau_t(x))$ is continuous for all $x \in X$ and $\psi \in Y$, and
4. $x \mapsto \tau_t(x)$ is $\sigma(X,Y)$-continuous for all $t \in \mathbb{R}$.

Note that condition (4) in Definition 3.17 is needed as $Y$ may not be invariant under the Banach space adjoint of $\tau_t$ acting on $X^*$. Given $\lambda > 0$, set $\Omega_\lambda := \{z \in \mathbb{C} : \mathrm{Im}(z) < \lambda\}$.

**Definition 3.18.** Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, an element $x \in X$ is *analytic for* $\{\tau_t\}_{t\in\mathbb{R}}$ if there exists $\lambda > 0$ and a function $\varphi : \Omega_\lambda \to X$ such that

1. $\varphi(t) = \tau_t(x)$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \psi(\varphi(z))$ is analytic on $\Omega_\lambda$ for all $\psi \in Y$.

**Definition 3.19.** Given a $\sigma(X,Y)$-continuous group of isometries $\{\tau_t\}_{t\in\mathbb{R}}$, the *infinitesimal generator* $S$ for $\{\tau_t\}_{t\in\mathbb{R}}$ is the operator whose domain consists of all elements $x \in X$ such that there exists $x' \in X$ which satisfies

$$\lim_{t\to 0} \psi\left(\frac{\tau_t(x) - x}{t} - x'\right) = 0 \text{ for all } \psi \in Y. \qquad (*)$$

If $x \in \mathrm{Dom}(S)$, set $Sx := x'$, where $x'$ satisfies condition $(*)$.

Proposition 3.20 below states that the two notions of analyticity in Definitions 2.14 and 3.18 are equivalent when $S$ is the infinitesimal generator of $\{\tau_t\}_{t\in\mathbb{R}}$.

**Proposition 3.20** (Bratteli-Robinson, [4]). *If* $\{\tau_t\}_{t\in\mathbb{R}}$ *is a* $\sigma(X,Y)$*-continuous group of isometries with infinitesimal generator* $S$*, then* $x$ *is analytic for* $\{\tau_t\}_{t\in\mathbb{R}}$ *if and only if* $x$ *is an analytic vector for* $S$.

Consider the Banach space $\mathcal{B}(\mathcal{H})$ along with the one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ given by $\alpha_t(x) = e^{itD}xe^{-itD}$ for all $x \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$. The closed subspace of elementary vector functionals $Y$ in $\mathcal{B}(\mathcal{H})^*$ recovers the WOT on $\mathcal{B}(\mathcal{H})$ as the $\sigma(X,Y)$-topology.

**Proposition 3.21.** *The family* $\{\alpha_t\}_{t\in\mathbb{R}}$ *is a WOT-continuous group of* $*$*-automorphisms with infinitesimal generator* $\delta_D$.

It is straightforward to check WOT-continuity of the automorphism group $\{\alpha_t\}_{t\in\mathbb{R}}$ using the SOT-continuity of the unitary group $\{e^{itD}\}_{t\in\mathbb{R}}$. Furthermore, $\delta_D$ is the corresponding infinitesimal generator for $\{\alpha_t\}_{t\in\mathbb{R}}$ simply by definition of weak $D$-differentiability. As a corollary of Propositions 3.20 and 3.21, we have the following:

**Corollary 3.22.** *An element* $x \in \mathcal{B}(\mathcal{H})$ *is analytic for* $\{\alpha_t\}_{t\in\mathbb{R}}$ *if and only if* $x \in \mathsf{A}(\delta_D)$*, where* $\mathsf{A}(\delta_D)$ *denotes the set of analytic operators for* $\delta_D$.

## 3.5 The Riesz Map and Density of Analytic Vectors

Initially, our strategy for proving SOT-density of the set of analytic vectors for $\delta_D$ in $\mathcal{B}(\mathcal{H})$ was to mimic the steps of Lemma 3.12—given $h, k \in \mathsf{A}(D)$, we wanted $h \otimes k^*$ to be analytic for $\delta_D$. If $h, k \in \mathsf{A}(D)$, the equivalent notion of analyticity from Proposition 3.20 implies that for each $f, g \in \mathcal{H}$, the maps $t \mapsto \langle e^{itD}h, g\rangle$ and $t \mapsto \langle e^{itD}k, f\rangle$ extend to analytic functions on some strip in the complex plane. But then, the map $t \mapsto \langle f, e^{itD}k\rangle$ is co-analytic, and since $\langle \alpha_t(h \otimes k^*)f, g\rangle = \langle e^{itD}h, g\rangle \langle f, e^{itD}k\rangle$ is the product of an analytic function and a co-analytic function, we could not necessarily extend the map $t \mapsto \langle \alpha_t(h \otimes k^*)f, g\rangle$ to an analytic function on a strip in the complex plane.

To remedy the issue of co-analyticity for the function involving $k$, we utilize the Riesz map $\mathcal{R} : H \to H^*$ given by $h \mapsto \psi_h$, where

$$\psi_h(f) := \langle f, h \rangle \quad \text{for all } f \in \mathcal{H}.$$

Note that $\mathcal{R}$ is *anti*-unitary: $\langle \mathcal{R}f, \mathcal{R}g \rangle_{\mathcal{H}^*} = \langle g, f \rangle_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$.

It is clear that conjugating a self-adjoint operator $D$ by a unitary $U$ results in another self-adjoint operator. Below we verify that conjugating $D$ by $\mathcal{R}$ results in a self-adjoint operator.

**Lemma 3.23.** *Define $D^\# : \mathcal{R}(\mathrm{Dom}(D)) \to \mathcal{H}^*$ by $D^\#(\mathcal{R}h) := \mathcal{R}(Dh)$ for all $h \in \mathrm{Dom}(D)$. The map $D^\# := \mathcal{R}D\mathcal{R}^{-1}$ with $\mathrm{Dom}\big(D^\#\big) = \mathcal{R}(\mathrm{Dom}(D))$ is self-adjoint.*

*Proof.* We first show $D^\#$ is a linear symmetric operator. Given $h \in \mathrm{Dom}(D)$ and $\lambda \in \mathbb{C}$, observe

$$D^\#(\lambda \mathcal{R}h) = \left[\mathcal{R}D\mathcal{R}^{-1}\right](\lambda \mathcal{R}h) = [\mathcal{R}D](\overline{\lambda}h) = \mathcal{R}(\overline{\lambda}Dh) = \lambda[\mathcal{R}D\mathcal{R}^{-1}](\mathcal{R}h) = \lambda D^\#(\mathcal{R}h).$$

As $h \in \mathrm{Dom}(D)$ was arbitrary and $\mathrm{Dom}\big(D^\#\big) = \mathcal{R}(\mathrm{Dom}(D))$, we have $D^\#(\lambda\psi) = \lambda D^\#\psi$ for all $\psi \in \mathrm{Dom}\big(D^\#\big)$ and $\lambda \in \mathbb{C}$. It's easy to check additivity of $D^\#$, so $D^\#$ is linear. For $f, h \in \mathrm{Dom}(D)$,

$$\left\langle D^\#\mathcal{R}h, \mathcal{R}f \right\rangle = \langle \mathcal{R}Dh, \mathcal{R}f \rangle = \langle f, Dh \rangle = \langle Df, h \rangle = \langle \mathcal{R}h, \mathcal{R}Df \rangle = \left\langle \mathcal{R}h, D^\#\mathcal{R}f \right\rangle.$$

As $f, h \in \mathrm{Dom}(D)$ were arbitrary and $\mathrm{Dom}\big(D^\#\big) = \mathcal{R}(\mathrm{Dom}(D))$,

$$\left\langle D^\#\psi, \phi \right\rangle = \left\langle \psi, D^\#\phi \right\rangle \text{ for all } \psi, \phi \in \mathrm{Dom}\big(D^\#\big).$$

Therefore, $D^{\#}$ is symmetric. Note that $\mathcal{R}(\mathrm{Dom}(D))$ is dense in $\mathcal{H}^*$ since $\mathrm{Dom}(D)$ is dense in $\mathcal{H}$ and $\mathcal{R}$ is a continuous bijection. Thus, it suffices to show $\mathrm{Dom}\big((D^{\#})^*\big) \subseteq \mathrm{Dom}\big(D^{\#}\big)$. Recall that the domain of the adjoint of $D^{\#}$ is the set

$$\begin{aligned}
\mathrm{Dom}\big((D^{\#})^*\big) &= \{\phi \in \mathcal{H}^* : \text{ the map } \mathrm{Dom}\big(D^{\#}\big) \to \mathbb{C};\ \psi \mapsto \big\langle D^{\#}\psi, \phi\big\rangle \text{ is bounded}\} \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle D^{\#}(\mathcal{R}h), \phi\big\rangle \text{ is bounded}\}. \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, \mathcal{R}^{-1}D^{\#}(\mathcal{R}h)\big\rangle \text{ is bounded}\}. \\
&= \{\phi \in \mathcal{H}^* : \text{ the map } \mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C};\ \ \mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, Dh\big\rangle \text{ is bounded}\}.
\end{aligned}$$

Hence, given $\phi \in \mathrm{Dom}\big((D^{\#})^*\big)$, the map $\mathcal{R}(\mathrm{Dom}(D)) \to \mathbb{C}$ defined by

$$\mathcal{R}h \mapsto \big\langle \mathcal{R}^{-1}\phi, Dh\big\rangle \text{ for all } h \in \mathrm{Dom}(D)$$

is a bounded linear functional. Then, since $\mathcal{R}$ is isometric, the composition

$$\begin{array}{ccccc}
\mathrm{Dom}(D) & \longrightarrow & \mathcal{R}(\mathrm{Dom}(D)) & \longrightarrow & \mathbb{C} \\
h & \mapsto & \mathcal{R}h & \mapsto & \langle \mathcal{R}^{-1}\phi, Dh\rangle
\end{array}$$

defines a bounded linear functional on the domain of $D$. By the definition of the domain of $D^*$, this implies $\mathcal{R}^{-1}\phi$ belongs to $\mathrm{Dom}(D^*)$. Further, self-adjointness of $D$ implies $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)$, so $\mathcal{R}^{-1}\phi \in \mathrm{Dom}(D)$. Since $\mathcal{R}$ is bijective, we conclude $\phi \in \mathcal{R}(\mathrm{Dom}(D)) = \mathrm{Dom}\big(D^{\#}\big)$. Therefore, $D^{\#}$ is self-adjoint. $\square$

By Nelson's Analytic Vector Theorem, the set of analytic vectors $\mathsf{A}(D^{\#})$ is dense in $\mathcal{H}^*$. As $\mathcal{R}^{-1} : \mathcal{H}^* \to \mathcal{H}$ is a continuous bijection, it follows that $\mathcal{R}^{-1}[\mathsf{A}(D^{\#})]$ is dense in $\mathcal{H}$.

**Notation 3.24.** Given subsets $S_1, S_2 \subseteq \mathcal{H}$, let

$$\mathcal{F}(S_1, S_2) := \mathrm{Span}\{f \otimes g^* : f \in S_1, g \in S_2\}.$$

Denote $\mathcal{F}(S_1, S_1)$ by $\mathcal{F}(S_1)$.

**Lemma 3.25.** *If $S_1, S_2 \subseteq \mathcal{H}$ are dense, then $\mathcal{F}(S_1, S_2)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.*

The proof of Lemma 3.25 is a simple modification of the case when $S_1 = S_2$ in Lemma 3.14. By Lemma 3.25, $\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^\#)]\right)$ is norm-dense in $\mathcal{K}(\mathcal{H})$.

**Proposition 3.26.** *If $h \in \mathsf{A}(D)$ and $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$, then $h \otimes k^* \in \mathsf{A}(\delta_D)$.*

*Proof.* Let $h \in \mathsf{A}(D)$ and $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$. By Corollary 3.22, $h \otimes k^* \in \mathsf{A}(\delta_D)$ if and only if $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$. To prove $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$, we must find $\lambda > 0$ and a function $\varphi : \Omega_\lambda \to \mathcal{B}(\mathcal{H})$ such that

1. $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \langle \varphi(z) f, g\rangle$ is analytic on $\Omega_\lambda$ for all $f, g \in \mathcal{H}$.

We construct $\varphi$ using the two functions obtained from analytic properties of $h$ and $k$. As $h \in \mathsf{A}(D)$, Proposition 3.20 implies $h$ is analytic for $\{e^{itD}\}_{t\in\mathbb{R}}$. Thus, there exists $\lambda_h > 0$ and a function $\varphi_h : \Omega_{\lambda_h} \to \mathcal{H}$ such that

1. $\varphi_h(t) = e^{itD}h$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \langle \varphi_h(z), g\rangle$ is analytic on $\Omega_{\lambda_h}$ for all $g \in \mathcal{H}$.

As $k \in \mathcal{R}^{-1}[\mathsf{A}(D^\#)]$, there exists a unique $\zeta_k \in \mathsf{A}(D^\#)$ such that $k = \mathcal{R}^{-1}\zeta_k$. Since $\zeta_k$ is analytic for $D^\#$, it is analytic for $\{e^{itD^\#}\}_{t\in\mathbb{R}}$ by Proposition 3.20. Hence, there exists $\lambda_k > 0$ and a function $\varphi_k : \Omega_{\lambda_k} \to \mathcal{H}^*$ such that

1. $\varphi_k(t) = e^{itD^\#}\zeta_k$ for all $t \in \mathbb{R}$ and

2. $z \mapsto \langle \varphi_k(z), \mathcal{R}f \rangle$ is analytic on $\Omega_{\lambda_k}$ for all $f \in \mathcal{H}$.

Note that in (2), we simply identified $\mathcal{H}^*$ with $\mathcal{R}(\mathcal{H})$. Set $\lambda := \min\{\lambda_h, \lambda_k\}$, and fix $z \in \Omega_\lambda$. Define a map $[\cdot,\cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$[f, g] := \langle \varphi_h(z), g \rangle \langle \varphi_k(z), \mathcal{R}f \rangle \text{ for all } f, g \in \mathcal{H}.$$

Sesquilinearity of the inner products on $\mathcal{H}$ and $\mathcal{H}^*$ and antilinearity of $\mathcal{R}$ establishes that $[\cdot,\cdot]$ is a sesquilinear form. Moreover, for any $f, g \in \mathcal{H}$,

$$|[f, g]| = |\langle \varphi_h(z), g \rangle| \, |\langle \varphi_k(z), \mathcal{R}f \rangle| \leq \|\varphi_h(z)\| \, \|g\| \, \|\varphi_k(z)\| \, \|f\| \, .$$

As $h, k$, and $z$ are fixed, $[\cdot,\cdot]$ defines a bounded sesquilinear form on $\mathcal{H}$. Thus, for each $z \in \Omega_\lambda$, the Riesz Representation Theorem for Bounded Sesquilinear Forms yields an operator $\varphi(z) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \varphi(z)f, g \rangle = [f, g] = \langle \varphi_h(z), g \rangle \langle \varphi_k(z), \mathcal{R}f \rangle \text{ for all } f, g \in \mathcal{H}.$$

As the two maps $z \mapsto \langle \varphi_h(z), g \rangle$ and $z \mapsto \langle \varphi_k(z), \mathcal{R}f \rangle$ are analytic on $\Omega_\lambda$ for all $f, g \in \mathcal{H}$, their product $z \mapsto \langle \varphi(z)f, g \rangle$ is analytic on $\Omega_\lambda$ for all $f, g \in \mathcal{H}$. Furthermore, for each $t \in \mathbb{R}$,

$$\langle \varphi(t)f, g \rangle = \left\langle e^{itD}h, g \right\rangle \left\langle e^{itD^\#}\zeta_k, \mathcal{R}f \right\rangle = \left\langle e^{itD}h, g \right\rangle \left\langle f, e^{itD}k \right\rangle = \langle \alpha_t(h \otimes k^*)f, g \rangle \, .$$

As $f, g \in \mathcal{H}$ were arbitrary, we have $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$. Therefore, $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t\in\mathbb{R}}$ in the WOT. By equivalence of analyticity for $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\delta_D$, we conclude $h \otimes k^* \in \mathsf{A}(\delta_D)$. □

**Theorem 1.1.** *The set of analytic vectors for $\delta_D$ is SOT-dense in $\mathcal{B}(\mathcal{H})$.*

*Proof.* Proposition 3.26 implies $\mathcal{F}(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})])$ is contained in $\mathsf{A}(\delta_D)$, so

$$\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})]\right) \subseteq \mathsf{A}(\delta_D) \cap F(\mathcal{H}).$$

By Lemma 3.25 and Nelson's Analytic Vector Theorem, $\mathcal{F}\left(\mathsf{A}(D), \mathcal{R}^{-1}[\mathsf{A}(D^{\#})]\right)$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Thus, $\mathsf{A}(\delta_D) \cap \mathcal{F}(\mathcal{H})$ is norm-dense in $\mathcal{K}(\mathcal{H})$. Therefore, $\mathsf{A}(\delta_D)$ is SOT-dense in $\mathcal{B}(\mathcal{H})$. $\square$

# Chapter 4

# Kernel Stabilization

The main theorem of this chapter, Theorem 1.2, states that for any self-adjoint operator $D$ on a Hilbert space, $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$. We call this property *kernel stabilization.*

## 4.1 Motivating Example

Throughout section 4.1, we denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$ by $\{\epsilon_j : j \in \mathbb{Z}\}$, and we denote the matrix representation of an operator $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ with respect to the standard orthonormal basis by $[x_{rc}]$ where

$$x_{rc} := \langle x\epsilon_c, \epsilon_r\rangle \text{ for all } r, c \in \mathbb{Z}.$$

**Example 4.1.** Define $(Df)(j) := jf(j)$ for $f \in \mathrm{Dom}(D)$, where

$$\mathrm{Dom}(D) := \{f \in \ell^2(\mathbb{Z}) : \sum_{j\in\mathbb{Z}} j^2 \left|f(j)\right|^2 < \infty\}.$$

Then

(i) the operator $D$ is self-adjoint.

(ii) an operator $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is $n$-times weakly $D$-differentiable if and only if for every

$k \leq n$, $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$ and the matrix $[i^k(r-c)^k x_{rc}]$ with dense domain $\mathrm{Dom}(D^k)$ extends to a bounded operator on $\ell^2(\mathbb{Z})$. When either condition is satisfied,

$$[\delta_D^n(x)_{rc}]|_{\mathrm{Dom}(D^n)} = [i^n(r-c)^n x_{rc}].$$

(iii) for any $g \in \ell^\infty(\mathbb{Z})$, $\delta_D(M_g) = 0$.

(iv) for all $n \in \mathbb{N}$, $\ker \delta_D^n = \mathrm{diag}(\ell^\infty(\mathbb{Z}))$.

*Proof.* (i) See Example 7.1.5 of [22].

(ii) Matrix multiplication shows for any $r, c \in \mathbb{Z}$,

$$d^k(x)_{rc} = i^k(r-c)^k x_{rc}.$$

Given $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ such that $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$ for each $k \leq n$, the domain of $d^k(x)$ is $\mathrm{Dom}(D^k)$. Theorem 3.7 states $x$ is $n$-times weakly $D$-differentiable if and only if for every $k \leq n$, $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$ and $d^k(x)$ is bounded on $\mathrm{Dom}(D^k)$. It follows that $x$ is $n$-times weakly $D$-differentiable if and only if $x(\mathrm{Dom}(D^k)) \subseteq \mathrm{Dom}(D^k)$ and $[d^k(x)_{rc}] = [i^k(r-c)^k x_{rc}]$ is bounded on $\mathrm{Dom}(D^k)$. As $D$ is self-adjoint, $\mathrm{Dom}(D^k)$ is dense in $\ell^2(\mathbb{Z})$ for all $k \in \mathbb{N}$. Therefore, $[d^k(x)_{rc}]$ extends to a bounded matrix on all of $\ell^2(\mathbb{Z})$. By Theorem 3.7, the closure $\delta_D^n(x)$ is the extension of $[i^n(r-c)^n x_{rc}]$ to all of $\ell^2(\mathbb{Z})$.

(iii) Fix $g \in \ell^\infty(\mathbb{Z})$, and let $f \in \mathrm{Dom}(D)$. We show $M_g f \in \mathrm{Dom}(D)$. Observe

$$\sum_{j \in \mathbb{Z}} |j(M_g f)(j)|^2 = \sum_{j \in \mathbb{Z}} |j g(j) f(j)|^2 \leq \|g\|_\infty^2 \left( \sum_{j \in \mathbb{Z}} |j f(j)|^2 \right) < \infty.$$

As $f \in \mathrm{Dom}(D)$ was arbitrary, $M_g(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and hence, the commutator $[iD, M_g]$ is a well-defined linear operator on $\mathrm{Dom}(D)$. Furthermore, $iD$ and $M_g$ are diagonal matrices with complex entries (which commute), so the commutator $[iD, M_g]$ is simply a restriction of the 0 operator to $\mathrm{Dom}(D)$. Theorem 3.2 implies $M_g \in \mathrm{Dom}(\delta_D)$ and $\delta_D(M_g)$ is the extension of $[iD, M_g]$ to all of $\mathcal{H}$. In particular, $\delta_D(M_g) = 0$. Hence, $M_g \in \ker \delta_D$, and since $g \in \ell^\infty(\mathbb{Z})$ was arbitrary, $\mathrm{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_D$.

(iv) Part (c) quickly implies $\mathrm{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_D^n$ for all $n \in \mathbb{N}$. We now show if $\delta_D^n(x) = 0$, then $x \in \mathrm{diag}(\ell^\infty(\mathbb{Z}))$. If $x \in \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(x) = 0$, then $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and $\delta_D^n(x)_{rc} = 0$ for every $r, c \in \mathbb{Z}$. By part (b),

$$[\delta_D^n(x)_{rc}]|_{\mathrm{Dom}(D^n)} = [i^n(r-c)^n x_{rc}],$$

thus, $i^n(r-c)^n x_{rc} = 0$ for every $r, c \in \mathbb{Z}$. If $r \neq c$, it must be that $x_{rc} = 0$, i.e., $x$ must be zero off the diagonal. As $x \in \mathcal{B}(\ell^2(\mathbb{Z}))$, we conclude $x \in \mathrm{diag}(\ell^\infty(\mathbb{Z}))$. Therefore, $\ker \delta_D^n = \mathrm{diag}(\ell^\infty(\mathbb{Z}))$ for all $n \in \mathbb{N}$.

□

This kernel stabilization phenomenon initially appears unique to the setting of Example 4.1; the self-adjoint operator has a complete set of eigenvectors which forms our choice of orthonormal basis. However, Theorem 1.2 shows that this example is not unique; kernel stabilization holds for every self-adjoint operator on any Hilbert space.

## 4.2 General Kernel Stabilization of $\delta_D$

**Proposition 4.2.** *Let $\mathcal{H}$ be a Hilbert space and $D$ a self-adjoint operator. The algebra* $\ker \delta_D$ *is a von Neumann algebra.*

*Proof.* The identity $I$ of $\mathcal{B}(\mathcal{H})$ is easily shown to be in $\ker \delta_D$. Let $x \in \ker \delta_D$. As $\mathrm{Dom}(\delta_D)$ is a $*$-algebra by Theorem 3.3, $x^* \in \mathrm{Dom}(\delta_D)$. Since $\delta_D$ is a $*$-derivation, $\delta_D(x^*) = \delta_D(x)^* = 0$. Therefore, $x^* \in \ker \delta_D$. Finally, if $x, y \in \ker \delta_D$, then $xy \in \mathrm{Dom}(\delta_D)$ and $\delta_D(xy) = \delta_D(x)y + x\delta_D(y) = 0$, so $xy \in \ker \delta_D$.

Let $(x_\lambda)_{\lambda \in \Lambda} \subset \ker \delta_D$ be a net converging in the WOT to some $x \in \mathcal{B}(\mathcal{H})$. We show $x \in \mathrm{Dom}(\delta_D)$ and $\delta_D(x) = 0$. Because $\delta_D(x_\lambda) = 0$ for all $\lambda \in \Lambda$, we trivially have $\delta_D(x_\lambda) \overset{\mathrm{WOT}}{\to} 0$ as $\lambda \to \infty$. By Theorem 3.3, the graph of $\delta_D$ is WOT-closed. Therefore, $x \in \mathrm{Dom}(\delta_D)$ and $\delta_D(x) = 0$. We conclude $\ker \delta_D$ is a von Neumann algebra. □

**Notation 4.3.** Let $\mathscr{P}_D$ denote the collection of all spectral projections for $D$ obtained through the Spectral Theorem for Unbounded Self-Adjoint Operators. Also, let

$$\mathcal{M}_D := \mathscr{P}_D''.$$

**Lemma 4.4.** *Suppose $x \in \mathcal{B}(\mathcal{H})$ satisfies $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. If $P \in \mathscr{P}_D$, then*

$$[P, [D, x]]h = [D, [P, x]]h \text{ for all } h \in \mathrm{Dom}(D).$$

*Proof.* Let $\mathcal{B}(\mathbb{R})$ be the bounded Borel functions on $\mathbb{R}$, and for $R \in \mathbb{R}$, define $\mathrm{id}_R : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{id}_R(t) := \begin{cases} t; & \text{-}R \leq t \leq R \\ 0; & else \end{cases}.$$

The Spectral Theorem, stated as in Theorem 7.2.8 of [22], provides a bounded Borel functional calculus for $D$, that is, a $*$-homomorphism $\Phi_D : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ satisfying $\Phi_D(1) = I$,

$$\mathrm{Dom}(D) = \{h \in \mathcal{H} : \lim_{R \to \infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty\},$$

and

$$Dh = \lim_{R\to\infty} \Phi_D(\mathrm{id}_R)h \quad \text{for all} \quad h \in \mathrm{Dom}(D).$$

We claim for each $P \in \mathscr{P}_D$, $P(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $PDh = DPh$ for all $h \in \mathrm{Dom}(D)$. Given $P \in \mathscr{P}_D$, there exists some Borel set $E \subseteq \mathbb{R}$ such that $P = \Phi_D(\chi_E)$. Note that

$$(\mathrm{id}_R \cdot \chi_E)(t) = \begin{cases} t; & \mathrm{t} \in E \cap [-R, R] \\ 0; & else \end{cases}.$$

Thus, for any $h \in \mathrm{Dom}(D)$,

$$\lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)Ph\| = \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)\Phi_D(\chi_E)h\| = \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \leq \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)h\| < \infty.$$

Therefore, $Ph \in \mathrm{Dom}(D)$, and as $h \in \mathrm{Dom}(D)$ was arbitrary, $P(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$. Furthermore,

$$\begin{aligned} \|DPh - PDh\| &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R)\Phi_D(\chi_E)h - \Phi_D(\chi_E)\Phi_D(\mathrm{id}_R)h\| \\ &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\chi_E \cdot \mathrm{id}_R)h\| \\ &= \lim_{R\to\infty} \|\Phi_D(\mathrm{id}_R \cdot \chi_E)h - \Phi_D(\mathrm{id}_R \cdot \chi_E)h\| \\ &= 0. \end{aligned}$$

Given $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, for any $h \in \mathrm{Dom}(D)$ we observe

$$\begin{aligned}
[P,[D,x]]h &= P(Dx - xD)h - (Dx - xD)Ph \\
&= PDxh - PxDh - DxPh + xDPh \\
&= DPxh - PxDh - DxPh + xPDh \\
&= DPxh - DxPh + xPDh - PxDh \\
&= D(Px - xP)h + (xP - Px)Dh \\
&= D(Px - xP)h - (Px - xP)Dh \\
&= [D,[P,x]]h
\end{aligned}$$

Hence, $[P,[D,x]]h = [D,[P,x]]h$ for all $h \in \mathrm{Dom}(D)$, and as $P \in \mathscr{P}_D$ was arbitrary, this equality holds for any spectral projection of $D$. ☐

**Proposition 4.5.** $\mathcal{M}_D \subseteq \ker \delta_D = \mathcal{M}_D'$.

*Proof.* Let $P \in \mathscr{P}_D$. By the previous lemma, $[D,P] = 0$ on $\mathrm{Dom}(D)$, so $P \in \mathrm{Dom}(\delta_D)$ by Theorem 3.2. Moreover, $\delta_D(P)$ is the bounded extension of $i(DP - PD)$ to all of $\mathcal{H}$, which is 0. Therefore, $P \in \ker \delta_D$. Because $\mathcal{M}_D$ is generated as a von Neumann algebra by the projections in $\mathscr{P}_D$, Proposition 4.2 implies $\mathcal{M}_D \subseteq \ker \delta_D$.

Let $x \in \ker \delta_D$. By Theorem 3.7, $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$ and $\delta_D(x)|_{\mathrm{Dom}(D)} = [D,x]|_{\mathrm{Dom}(D)} = 0$. Then, by Theorem X.4.11 of [7], $xf(D) \subseteq f(D)x$ for any $f \in \mathcal{B}(\mathbb{R})$. In particular, when $f = \chi_E$ for some Borel subset $E \subseteq \mathbb{R}$ and $P$ denotes the corresponding spectral projection for $D$, $xP = Px$. Hence, $x$ commutes with all projections in $\mathscr{P}_D$, and as $\mathcal{M}_D$ is generated as a von Neumann algebra by these projections, it follows that $x \in \mathcal{M}_D'$.

Let $x \in \mathcal{M}_D'$. For each $t \in \mathbb{R}$, $e^{itD} \in \mathcal{M}_D$. Thus, $\alpha_t(x) = e^{itD}xe^{-itD} = x$ for all $t \in \mathbb{R}$. In particular, for any $h,k \in \mathcal{H}$, the function $t \mapsto \langle \alpha_t(x)h, k\rangle = \langle xh, k\rangle$ is constant, and thus is continuously differentiable with derivative 0. Therefore, $x \in \ker \delta_D$ by Proposition 3.5. ☐

We now present our kernel stabilization result.

**Theorem 1.2.** *If $D$ is any self-adjoint operator on a Hilbert space $\mathcal{H}$, then for every $n \in \mathbb{N}$,*

$$\ker \delta_D^n = \ker \delta_D.$$

*Proof.* We first show $\ker \delta_D^2 = \ker \delta_D$. The inclusion $\ker \delta_D \subseteq \ker \delta_D^2$ is clear. Let $x \in \ker \delta_D^2$. Proposition 4.5 states $\ker \delta_D = \mathcal{M}'_D$. Thus, it suffices to prove $x \in \mathcal{M}'_D$, which holds if and only if $[P, x] = 0$ for every $P \in \mathscr{P}_D$. By Proposition 3.6, if $x \in \mathrm{Dom}(\delta_D^2)$, then

(i) $x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$,

(ii) $\delta_D(x)(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and

(iii) $\delta_D^2(x)|_{\mathrm{Dom}(D)} = [iD, \delta_D(x)]$.

Since $\delta_D^2(x) = 0$, it must be that $[iD, \delta_D(x)] = 0$. Thus, Theorem X.4.11 of [7] implies $\delta_D(x)$ commutes with the bounded Borel functional calculus for $D$, so, in particular, $[P, \delta_D(x)] = 0$ for every $P \in \mathscr{P}_D$. Because $\delta_D(x)$ and $P$ both preserve the domain of $D$, so does the commutator $[P, \delta_D(x)]$. Thus, Lemma 4.4 implies

$$0 = [P, \delta_D(x)]|_{\mathrm{Dom}(D)} = [P, [iD, x]]|_{\mathrm{Dom}(D)} = [iD, [P, x]]|_{\mathrm{Dom}(D)}.$$

As $[P, x] \in \mathcal{B}(\mathcal{H})$, $[P, x](\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$, and $[iD, [P, x]]$ is bounded on the domain of

$D$, Theorem 3.2 implies $[P,x] \in \ker \delta_D$. Hence, by Proposition 4.5, $[P,x] \in \mathcal{M}'_D$. Therefore,

$$\begin{aligned}
[P,x] &= (P+P^{\perp})[P,x](P+P^{\perp}) \\
&= P[P,x]P + P[P,x]P^{\perp} + P^{\perp}[P,x]P + P^{\perp}[P,x]P^{\perp} \\
&= P[P,x]P + PP^{\perp}[P,x] + P^{\perp}P[P,x] + P^{\perp}[P,x]P^{\perp} \\
&= P(Px-xP)P + 0 + 0 + P^{\perp}(Px-xP)P^{\perp} \\
&= PxP - PxP + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

As $P \in \mathscr{P}_D$ was arbitrary, $x \in \mathcal{M}'_D$. By Proposition 4.5, $x \in \ker \delta_D$.

We proceed by induction on $n$. The case when $n = 1$ is vacuous. Suppose $\ker \delta_D^k = \ker \delta_D$ for some $k \in \mathbb{N}$. Let $x \in \ker \delta_D^{k+1}$. Then $\delta_D(x) \in \ker \delta_D^k$, which equals $\ker \delta_D$ by the inductive hypothesis. Hence, $x \in \ker \delta_D^2$. Since we have already shown $\ker \delta_D^2 = \ker \delta_D$, we have $x \in \ker \delta_D$. Therefore, $\ker \delta_D^n = \ker \delta_D$ for all $n \in \mathbb{N}$. □

**Remark 4.6.** Let $n \in \mathbb{N}$ be arbitrary, and let $x \in \mathcal{B}(\mathcal{H})$. By Christensen's Theorem 3.7, kernel stabilization of $\delta_D$ is equivalent to the following statement: If

(i) the domains of the iterated commutators $d^k(x)$ for $k = 1, ..., n$ contain a common core $\mathscr{C}$ for $D$,

(ii) $d^k(x)$ is bounded on $\mathscr{C}$ for all $k = 1, ..., n$, and

(iii) the continuous bounded extension of $d^n(x)$ to all of $\mathcal{H}$ belongs to $\mathcal{M}'_D$,

then $[iD,x]|_{\mathscr{C}} = 0$.

Less formally, if $\underbrace{[iD, ..., [iD,x]]}_{n \text{ times}}$ and all lower commutators are well-defined and bounded on a common core for $D$, then

$$\underbrace{[iD, ..., [iD, x]]}_{n \text{ times}} = 0 \text{ implies } [iD, x] = 0.$$

This rephrasing of Theorem 1.2 in the case when $n = 2$ is equivalent to Theorem 1.6.3 of [17] in the self-adjoint setting. Putnam's proof relies on techniques in the proof of Fuglede's Theorem, whereas our proof is direct. Establishing the equivalence of these statements requires use of Christensen's work in [5].

**Equivalence of Kernel Stabilization to a Result of C.R. Putnam**

**Theorem 4.7** (Putnam, 1.6.3 [17])**.** Suppose $D$ is normal and $x, y \in \mathcal{B}(\mathcal{H})$. If

1. $xD + y \subset Dx$ and

2. $yD \subset Dy$,

then $y = 0$.

We claim that when $D$ is self-adjoint, Theorem 4.7 is equivalent to Theorem 1.2 in the case when $n = 2$. To show this, we show hypotheses (1) and (2) of Putnam's Theorem 4.7 are equivalent to the hypothesis in Theorem 1.2.

(1) Note that the domain of $xD + y$ is $\text{Dom}(D)$ because $y$ is bounded, and

$$\text{Dom}(D)x = \{f \in \mathcal{H} : xf \in \text{Dom}(D)\}.$$

To say $xD + y \subset Dx$ is to say that there is an inclusion of these operators' graphs.

Hence,

$$\begin{aligned}\Gamma(xD+y)\subset\Gamma(Dx) &\iff \{(h,xDh+yh):h\in\mathrm{Dom}(D)\}\subset\{(k,Dxk):k\in\mathrm{Dom}(Dx)\}\\ &\iff \mathrm{Dom}(D)\subset\mathrm{Dom}(Dx)\ \textbf{and}\ xDh+yh=Dxh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff \mathrm{Dom}(D)\subset\{f\in H: xf\in\mathrm{Dom}(D)\}\ \textbf{and}\ [D,x]h=yh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff x(\mathrm{Dom}(D))\subset\mathrm{Dom}(D)\ \textbf{and}\ [D,x]h=yh\ \ \forall h\in\mathrm{Dom}(D).\end{aligned}$$

(2) Similarly, $yD\subset Dy$ is an inclusion of these operators' graphs. Note that the domain of $yD$ is the domain of $D$, while

$$\mathrm{Dom}(Dy)=\{f\in\mathcal{H}: yf\in\mathrm{Dom}(D)\}.$$

Thus,

$$\begin{aligned}\Gamma(yD)\subset\Gamma(Dy) &\iff \{(h,yDh):h\in\mathrm{Dom}(D)\}\subset\{(k,Dyk):k\in\mathrm{Dom}(Dy)\}\\ &\iff \mathrm{Dom}(D)\subset\mathrm{Dom}(Dy)\ \textbf{and}\ yDh=Dyh\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff \mathrm{Dom}(D)\subset\{f\in H: yf\in\mathrm{Dom}(D)\}\ \textbf{and}\ [D,y]h=0\ \ \forall h\in\mathrm{Dom}(D)\\ &\iff y(\mathrm{Dom}(D))\subset\mathrm{Dom}(D)\ \textbf{and}\ [D,y]h=0\ \ \forall h\in\mathrm{Dom}(D).\end{aligned}$$

The content of Theorem 1.2 in the case when $n=2$ is $\ker\delta_D^2\subseteq\ker\delta_D$. We break the hypothesis that $x\in\ker_D^2$ into two simpler hypotheses:

(I) $x\in\mathrm{Dom}(\delta_D)$

(II) $y:=\delta_D(x)\in\mathrm{Dom}(\delta_D)$ **and** $\delta_D(y)=0$.

Below we rewrite (I) and (II) using Christensen's Theorem 3.2.

(I) By Theorem 3.2,

$$\begin{aligned}
x \in \mathrm{Dom}(\delta_D) &\iff \exists y \in \mathcal{B}(\mathcal{H}) \text{ st. } [iD,x]|_{\mathrm{Dom}(D)} = y|_{\mathrm{Dom}(D)} \\
&\iff Dx - xD \text{ is well-defined on } \mathrm{Dom}(D) \\
&\qquad \textbf{and } \exists y \in \mathcal{B}(\mathcal{H}) \text{ s.t. } [iD,x]|_{\mathrm{Dom}(D)} = y|_{\mathrm{Dom}(D)} \\
&\iff x(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \textbf{ and } \exists y \in \mathcal{B}(\mathcal{H}) \text{ s.t. } [iD,x]h = yh \;\; \forall h \in \mathrm{Dom}(D) \\
&\iff (1).
\end{aligned}$$

(II) Again by Theorem 3.2,

$$\begin{aligned}
y \in \mathrm{Dom}(\delta_D) \textbf{ and } \delta_D(y) = 0 &\iff [D,y] \text{ is well-defined on } \mathrm{Dom}(D) \textbf{ and } [D,y]|_{\mathrm{Dom}(D)} = 0 \\
&\iff y(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \textbf{ and } [D,y]h = 0 \;\; \forall h \in \mathrm{Dom}(D) \\
&\iff (2).
\end{aligned}$$

We have established that the statement of Theorem 1.2 in the $n = 2$ case is equivalent to Theorem 4.7 in the self-adjoint setting.

The proofs of both Theorems 1.2 and 4.7 rely heavily on the Spectral Theorem for normal operators. However, the kernel stabilization result depends only on independently-proven facts about commutators of $x \in \mathcal{B}(\mathcal{H})$ with spectral projections for $D$, while Putnam's theorem is stated as a corollary to Fuglede's Theorem. Of course, Fuglede's Theorem makes great use of spectral projections for normal operators, but our proof is more direct. We then applied a simple inductive argument to get kernel stabilization for all higher powers of $\delta_D$.

## 4.3 Applications

### Abstract Derivations on $C^*$-algebras

Given a self-adjoint operator $D$, our proof of kernel stabilization of $\delta_D$ relied on the relationship between $\delta_D$ and commutation with $D$. Intuitively, then, kernel stabilization is likely to occur for a derivation $\delta$ on an abstract $C^*$-algebra that can be implemented, under an appropriate representation, as commutation with a self-adjoint operator. Theorem 1.3 provides sufficient conditions for when a derivation on a $C^*$-algebra has such a representation.

Under this representation, Bratteli and Robinson construct an essentially self-adjoint operator $S$ which implements the derivation's action as commutation with $S$. Once this essentially self-adjoint operator is defined, we use its self-adjoint closure $D = \overline{S}$ to generate a corresponding weak-$D$ derivation $\delta_D$. We shall show $\delta_D$ extends $\delta \circ \pi$ and then apply Theorem 1.2 (kernel stabilization of $\delta_D$) to obtain kernel stabilization of $\delta$.

**Theorem 1.3** (Bratteli-Robinson, 4 [3]). *Let $\delta$ be a derivation of a $C^*$-algebra $\mathcal{A}$, and assume there exists a state $\omega$ on $\mathcal{A}$ which generates a faithful cyclic representation $(\pi, H, f)$ satisfying*

$$\omega(\delta(a)) = 0 \textit{ for all } a \in \mathrm{Dom}(\delta).$$

*Then $\delta$ is closable and there exists a symmetric operator $S$ on $\mathcal{H}$ such that*

$$\mathrm{Dom}(S) = \{h \in \mathcal{H} : h = \pi(a)f \ \ \textit{for some } a \in \mathcal{A}\}$$

*and $\pi(\delta(a))h = [S, \pi(a)]h$, for all $a \in \mathrm{Dom}(\delta)$ and all $h \in \mathrm{Dom}(S)$. Moreover, if the set $\mathsf{A}(\delta)$ of analytic vectors for $\delta$ is dense in $\mathcal{A}$, then $S$ is essentially self-adjoint on $\mathrm{Dom}(S)$. For $x \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$, define*

$$\alpha_t(x) := e^{i\overline{S}t} x e^{-i\overline{S}t}$$

*where* $\overline{S}$ *denotes the self-adjoint closure of* $S$*. It follows that* $\alpha_t(\pi(\mathcal{A})) = \pi(\mathcal{A})$ *for all* $t \in \mathbb{R}$*, and* $\{\alpha_t\}_{t\in\mathbb{R}}$ *is a strongly continuous group of* $*$*-automorphisms with closed infinitesimal generator* $\widetilde{\delta}$ *equaling the closure of* $\pi \circ \delta|_{\mathsf{A}(\delta)}$.

The condition that there exist a state $\omega$ on $\mathcal{A}$ which satisfies $\omega(\delta(a)) = 0$ for all $a \in \mathrm{Dom}(\delta)$ physically represents the presence of an *equilibrium state* for the $C^*$-algebra $\mathcal{A}$ of observables for a physical system with time evolution described by $\delta$. If $\delta$ were the infinitesimal generator for a one-parameter group of $*$-automorphisms $\{\beta_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$, then $\omega(\beta_t(a)) = \omega(a)$ for all $t \in \mathbb{R}$ would be an equivalent condition to require, and this condition more commonly describes an equilibrium state. However, $\delta$ is an abstract derivation on $\mathcal{A}$ with norm-dense domain, so there may not be a one-parameter group of $*$-automorphisms for which $\delta$ is the infinitesimal generator.

Under the representation $\pi$, however, $\delta$ is implemented by commutation with $S$, whose closure provides unitaries from which we can build a one-parameter group of $*$-automorphisms $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathcal{B}(\mathcal{H})$. We relate the infinitesimal generator $\widetilde{\delta}$ for $\{\alpha_t\}_{t\in\mathbb{R}}$ in Theorem 1.3 to a derivation $\delta_u$ studied by Christensen.

**Definition 4.8.** Let $D$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. An operator $x \in \mathcal{B}(\mathcal{H})$ is *uniformly* $D$*-differentiable* if there exists $y \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to 0} \left\| \frac{e^{itD}xe^{-itD} - x}{t} - y \right\| = 0. \qquad (*)$$

We denote this by $x \in \mathrm{Dom}(\delta_u)$ and set $\delta_u(x) = y$, where $y$ satisfies condition $(*)$.

**Remark 4.9.** Let $S$ and $\widetilde{\delta}$ be as in Theorem 1.3, and let $D = \overline{S}$. Then $\widetilde{\delta}$ from Theorem 1.3 and $\delta_u$ from Definition 4.8 are the same derivations with the same domains.

**Proposition 4.10.** *If* $D$ *is a self-adjoint operator, then* $\ker \delta_u = \ker \delta_D$.

*Proof.* Theorem 4.1 of [6] states $x \in \mathrm{Dom}(\delta_u)$ if and only if $x \in \mathrm{Dom}(\delta_D)$ and $t \mapsto \alpha_t(\delta_D(x))$ is norm-continuous. Moreover, $\delta_D$ extends $\delta_u$. Thus, $\ker \delta_u \subseteq \ker \delta_D$.

Let $x \in \ker \delta_D$. Then $t \mapsto \alpha_t(\delta_D(x)) = 0$ is norm-continuous, and hence, $x \in \mathrm{Dom}(\delta_u)$. Moreover, $\delta_u(x) = [\delta_D|_{\mathrm{Dom}(\delta_u)}](x) = 0$. Therefore, $x \in \ker \delta_u$. We conclude $\ker \delta_D = \ker \delta_u$. $\square$

**Corollary 4.11.** *For all $n \in \mathbb{N}$, $\ker \delta_u^n = \ker \delta_u$.*

*Proof.* Fix $n \in \mathbb{N}$ and let $x \in \ker \delta_u^n$. Then $x \in \mathrm{Dom}(\delta_u^n) \subseteq \mathrm{Dom}(\delta_D^n)$ and $\delta_D^n(x) = \delta_u^n(x) = 0$. Therefore, $x \in \ker \delta_D^n$, so by Theorem 1.2, $x \in \ker \delta_D$. By Proposition 4.10, $\ker \delta_D = \ker \delta_u$, so we conclude $x \in \ker \delta_u$. Thus, $\ker \delta_u^n = \ker \delta_u$ for all $n \in \mathbb{N}$, as claimed. $\square$

**Lemma 4.12.** *If $\delta$, $\mathcal{A}$, $\pi$, and $\widetilde{\delta}$ are as in Theorem 1.3, then*

$$\ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta)) = \pi(\ker \delta^n) \textit{ for all } n \in \mathbb{N}.$$

*Proof.* Fix $n \in \mathbb{N}$. If $a \in \mathsf{A}(\delta)$, then $a \in \mathrm{Dom}(\delta^n)$ and $\delta^n(a) \in \mathsf{A}(\delta)$. Theorem 1.3 states $\widetilde{\delta}(\pi(b)) = \pi(\delta(b))$ for all $b \in \mathsf{A}(\delta)$. Thus, as $\delta^n(a) \in \mathsf{A}(\delta)$, we have $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a))$. Suppose $\widetilde{\delta}^n(\pi(a)) = 0$. Then $\pi(\delta^n(a)) = \widetilde{\delta}^n(\pi(a)) = 0$, and since $\pi$ is faithful, $\delta^n(a) = 0$. Therefore, $\pi(a) \in \pi(\ker \delta^n)$.

Conversely, suppose $a \in \ker \delta^n$. Then $a \in \mathsf{A}(\delta)$ because $\delta^j(a) = 0$ for all $j \geq n$ and $\sum_{k=0}^{\infty} \frac{t^k}{k!} \left\|\delta^k(a)\right\| = \sum_{k=0}^{n-1} \frac{t^k}{k!} \left\|\delta^k(a)\right\| < \infty$ for any choice of $t > 0$. Similar to above, $\widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a)) = \pi(0) = 0$. Therefore, $\pi(a) \in \ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta))$. As $a \in \mathcal{A}$ was arbitrary, $\ker \widetilde{\delta}^n \cap \pi(\mathsf{A}(\delta)) = \pi(\ker \delta^n)$. Finally, because $n \in \mathbb{N}$ was arbitrary, this equality holds for all $n \in \mathbb{N}$. $\square$

**Theorem 4.13.** *If $\delta$, $\mathcal{A}$, $\pi$, $\widetilde{\delta}$, and $S$ are as in Theorem 1.3, then* $\ker \delta^n = \ker \delta$.

*Proof.* Fix $n \in \mathbb{N}$, and let $a \in \ker \delta^n$. Then $a \in \mathsf{A}(\delta)$ and $\pi(a) \in \ker \widetilde{\delta}^n$ by Lemma 4.12. Note $\widetilde{\delta} = \delta_u$ where $D = \overline{S}$, so Proposition 4.11 implies $\ker \widetilde{\delta}^n = \ker \widetilde{\delta}$ for all $n \in \mathbb{N}$. Hence,

$\pi(a) \in \ker \widetilde{\delta} \cap \pi(\mathsf{A}(\delta))$. By another application of Lemma 4.12, we get $a \in \ker \delta$. Therefore, $\ker \delta^n = \ker \delta$ for all $n \in \mathbb{N}$. □

**The Heisenberg Commutation Relation**

Our second application of Theorem 1.2 gives a sufficient condition for when two self-adjoint operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

**Definition 1.5.** Let $A$ and $B$ be two (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$, with domains $\mathrm{Dom}(A)$ and $\mathrm{Dom}(B)$, respectively. We say $A$ and $B$ *satisfy the Heisenberg Commutation Relation* (HCR) if there is a dense subspace $K$ of $\mathcal{H}$ such that

(i) $K \subseteq \mathrm{Dom}([A, B])$ and

(ii) $[A, B]k = ik$ for all $k \in K$.

**Definition 4.14.** The classical example of a pair satisfying the HCR is the *Schrödinger pair*, the quantum mechanical position operator $Q$ and momentum operator $P$ on $L^2(\mathbb{R})$ from Examples 2.6 and 2.9.

Let $S(\mathbb{R})$ denote the *Schwartz space* on $\mathbb{R}$:

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall m, n \in \mathbb{N},\ \|Q^m P^n f\|_\infty < \infty\}.$$

Proposition X.6.5 of [7] shows $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, and it is clear from its definition that $S(\mathbb{R})$ is contained in $\mathrm{Dom}(Q) \cap \mathrm{Dom}(P)$ and is invariant under both $Q$ and $P$. Hence, $S(\mathbb{R}) \subseteq \mathrm{Dom}([Q, P])$. Furthermore, $[Q, P]g = ig$ for all $g \in S(\mathbb{R})$. Therefore, $Q$ and $P$ satisfy the HCR.

If two operators are unitarily equivalent to a direct sum of copies of the Schrödinger pair, they are certainly both unbounded, and it is well-known that no two bounded operators may satisfy the HCR. Below is a well-known example of a pair of operators satisfying the HCR where one operator is bounded.

**Example 4.15.** For $f \in L^2([0,1])$, define $(Af)(x) = xf(x)$ for a.e. $x \in [0,1]$. In contrast to its unbounded analogue $Q$, the operator $A$ is contractive. Let $AC([0,1])$ denote the set of functions which are absolutely continuous on $[0,1]$, and let

$$\mathrm{Dom}(B) = \{f \in AC[0,1] : f' \in L^2([0,1]),\ f(0) = f(1)\}.$$

For $g \in \mathrm{Dom}(B)$, define $Bg = -ig'$. Example X.1.12 of [7] shows the operator $B$ with this particular domain is self-adjoint. Due to boundedness of $A$,

$$\mathrm{Dom}([A,B]) = \{f \in \mathrm{Dom}(B) : Af \in \mathrm{Dom}(B)\}.$$

Choose

$$K := \{f \in AC([0,1]) : f' \in L^2([0,1]),\ f(0) = f(1) = 0\}.$$

Example X.1.11 of [7] shows $K$ is dense in $L^2([0,1])$ as it contains all polynomials $p$ on $[0,1]$ satisfying $p(0) = p(1) = 0$. Furthermore, we claim $K$ is invariant for $A$. Indeed, products of absolutely continuous functions are again absolutely continuous, so $(Ag)(x) = xg(x)$ for a.e. $x \in [0,1]$ defines an absolutely continuous function. The a.e.-defined derivative of $Ag$ is equivalent to $Ag' + g$ by the product rule. Moreover, $Ag' + g$ belongs to $L^2([0,1])$ as $g' \in L^2([0,1])$ and $A \in \mathcal{B}(L^2([0,1]))$. Lastly,

$$(Ag)(0) = 0 \cdot g(0) = 0 = 1 \cdot 0 = 1 \cdot g(1) = (Ag)(1).$$

Thus, $AK \subseteq K$. As a result, $K \subseteq \mathrm{Dom}([A,B])$. For $k \in K$, observe

$$[A,B]k = A(-ik') - B(Ak) = -iAk' - (-i)[Ak' + k] = ik.$$

Therefore, $A$ and $B$ satisfy the HCR.

We claim the boundedness of the operators in Examples 4.14 and 4.15 is due to the relative size of $\mathrm{Dom}([Q,P])$ in $L^2(\mathbb{R})$ versus $\mathrm{Dom}([A,B])$ in $L^2([0,1])$. In particular, $\mathrm{Dom}([A,B])$ does not contain a core for $A$ or $B$, while $\mathrm{Dom}([Q,P])$ contains a core for both $Q$ and $P$.

**Theorem 1.6.** *Let $A$ and $B$ be self-adjoint operators which satisfy the HCR on a dense subspace $K \subseteq \mathcal{H}$. If $K$ is a core for $A$ and $B$, then $A$ and $B$ are both unbounded.*

*Proof.* Suppose that $K$ is a core for both $A$ and $B$. It is well-known that $A$ and $B$ cannot both be bounded and satisfy the Heisenberg Relation. Thus, without loss of generality, the only possibilities are that $A$ is bounded and $B$ is unbounded, or both $A$ and $B$ are unbounded. Suppose that $A \in \mathcal{B}(\mathcal{H})$. Note that $[A,B]k = ik$ for all $k \in K$ if and only if $[iB,A]k = k$ for all $k \in K$.

As $K$ is a core for $B$ and $\|[iB,A]|_K\| = 1$, we have that $A \in \mathrm{Dom}(\delta_B)$. Furthermore, $\delta_B(A)$ is the continuous extension of the bounded and densely-defined operator $[iB,A]|_K$ to all of $\mathcal{H}$, and thus, $\delta_B(A) = I$. Trivially, $I \in \mathrm{Dom}(\delta_B)$ and $\delta_B(I) = 0$, so $A \in \mathrm{Dom}(\delta_B^2)$ and $\delta_B^2(A) = 0$. Since $A \in \ker \delta_B^2$, Theorem 1.2 implies $A \in \ker \delta_B$. But then

$$0 = \delta_B(A)|_K = [iB,A]|_K = I|_K,$$

which is absurd. Therefore, $A$ cannot be bounded. We conclude that if $A$ and $B$ satisfy the HCR on a common core for $A$ and $B$, then $A$ and $B$ must both be unbounded. □

# Chapter 5

# A Covariant Stone-von Neumann Theorem

## 5.1 $(G, \mathcal{A}, \alpha)$-Heisenberg and Schrödinger Representations

Throughout, $G$ is a locally compact abelian group with Haar measure $\mu$ and dual group $\widehat{G}$ with Haar measure $\hat{\mu}$. As defined in Definition 1.8, the Schrödinger representation $(\lambda, V)$ for a locally compact abelian group $G$ is an example of a Heisenberg representation for $G$. We seek to generalize the definition of this pair to a representation of a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$ on a Hilbert $\mathcal{A}$-module.

**Definition 5.1.** A $(G, \mathcal{A}, \alpha)$-*Heisenberg representation* is a quadruple $(\mathsf{X}, \rho, r, s)$ with the following properties:

(i) $\mathsf{X}$ is a full Hilbert $\mathcal{A}$-module.

(ii) $\rho : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation.

(iii) $r : G \to \mathcal{U}(\mathsf{X})$ is a (strictly continuous) unitary group representation.

(iv) $s : \widehat{G} \to \mathcal{U}(\mathsf{X})$ is a (strictly continuous) unitary group representation.

(v) $s_\gamma r_x = \gamma(x) r_x s_\gamma$ for all $x \in G$ and $\gamma \in \widehat{G}$.

(vi) $(\rho, r)$ is a nondegenerate covariant homomorphism of $(G, \mathcal{A}, \alpha)$ into $\mathsf{X}$.

(vii) $\rho(a)s_\gamma = s_\gamma\rho(a)$ for all $a \in \mathcal{A}$ and $\gamma \in \widehat{G}$.

When $\mathcal{A} = \mathbb{C}$, we recover the definition of a classical Heisenberg representation. To define the $(G, \mathcal{A}, \alpha)$-Schrödinger representation, consider the right Hilbert $\mathcal{A}$-module $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, defined in Example 2.30, which we recall here for convenience. For each $\phi \in C_c(G, \mathcal{A})$ and $a \in \mathcal{A}$, define

$$[\phi \bullet a](x) := f(x)\alpha_x(a) \text{ for all } x \in G.$$

Then $\bullet$ makes $C_c(G, \mathcal{A})$ into a right $\mathcal{A}$-module. For $\phi, \psi \in C_c(G, \mathcal{A})$, define

$$\langle \psi \,|\, \phi \rangle := \int_G \alpha_{x^{-1}} \left(\psi(x)^* \phi(x)\right) \, d\mu(x).$$

We denote the completion of $C_c(G, \mathcal{A})$ with respect to the induced norm $\|\cdot\|_\alpha := \|\langle \cdot \,|\, \cdot \rangle\|_{\mathcal{A}}^{1/2}$ by $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Next, consider the map $\mathsf{M} : \mathcal{A} \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ defined on $\phi \in C_c(G, \mathcal{A})$ by

$$[\mathsf{M}(a)\phi](x) := a\phi(x) \text{ for all } x \in G.$$

**Proposition 5.2.** $\mathsf{M} : \mathcal{A} \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ *is a well-defined nondegenerate* $*$*-representation.*

*Proof.* Fix $a \in \mathcal{A}$. First we show $\mathsf{M}(a)|_{C_c(G,\mathcal{A})}$ is bounded with respect to $\|\cdot\|_\alpha$, and by $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, A, \alpha)$, we may continuously extend $\mathsf{M}(a)$ to all of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

Recall that for any element $d$ of a unital $C^*$-algebra $\mathcal{B}$ with unit $e$, $d^*d \leq_{\mathcal{B}} \|d\|^2 e$, where $\leq_{\mathcal{B}}$ is the ordering on the positive elements in $\mathcal{B}$. Let $\phi \in C_c(G, \mathcal{A})$. Using an approximate identity argument and Theorem 2.2.5(b) of [15], we have that

$$\phi(x)^*(a^*a)\phi(x) \leq_{\mathcal{A}} \phi(x)^* \|a^*a\| \phi(x) = \|a\|^2 \phi(x)^*\phi(x).$$

Observe

$$\begin{aligned}\langle \mathsf{M}(a)\phi \,|\, \mathsf{M}(a)\phi\rangle &= \int_G \alpha_{x^{-1}}((a\phi(x))^* a\phi(x))\, d\mu(x)\\ &= \int_G \alpha_{x^{-1}}(\phi(x)^* a^* a\phi(x))\, d\mu(x)\\ &\leq_{\mathcal{A}} \int_G \alpha_{x^{-1}}(\|a\|_{\mathcal{A}}^2\, \phi(x)^*\phi(x))\, d\mu(x)\\ &= \|a\|_{\mathcal{A}}^2 \langle \phi \,|\, \phi\rangle\end{aligned}$$

Theorem 2.2.5(c) of [15] implies $\|\langle \mathsf{M}(a)\phi \,|\, \mathsf{M}(a)\phi\rangle\|_{\mathcal{A}} \leq \|a\|_{\mathcal{A}}^2 \|\langle \phi \,|\, \phi\rangle\|_{\mathcal{A}}$. Therefore,

$$\|\mathsf{M}(a)\phi\|_{\alpha}^2 \leq \|a\|_{\mathcal{A}}^2 \|\phi\|_{\alpha}^2 ,$$

so $\mathsf{M}(a)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_{\alpha}$-continuous. Similarly, so is $\mathsf{M}(a^*)$. For $\phi, \psi \in C_c(G, \mathcal{A})$,

$$\langle \psi \,|\, \mathsf{M}(a)\phi\rangle = \int_G \alpha_{x^{-1}}(\psi(x)^* a\phi(x))\, d\mu(x) = \int_G \alpha_{x^{-1}}([a^*\psi(x)]^*\phi(x))\, d\mu(x) = \langle \mathsf{M}(a^*)\psi \,|\, \phi\rangle .$$

As $\mathsf{M}(a)$ and $\mathsf{M}(a^*)$ are both $\|\cdot\|_{\alpha}$-continuous, this equality of inner products holds on arbitrary elements of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Therefore $\mathsf{M}(a^*) = \mathsf{M}(a)^*$, so $\mathsf{M}(a) \in \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$. Moreover, $\mathsf{M}$ is clearly linear, multiplicative, and $*$-preserving, so $\mathsf{M}$ is a well-defined $*$-representation of $\mathcal{A}$. We now show $\mathsf{M}$ is nondegenerate.

Fix $\phi \in C_c(G, \mathcal{A})$. As $\text{Range}(\phi) \subseteq \phi[\text{Supp}(\phi)] \cup \{0_{\mathcal{A}}\}$, and as $\overline{\text{Supp}(\phi)}$ is a compact subset of $G$, we see that $\text{Range}(\phi)$ is contained in a compact subset of $\mathcal{A}$. Compact subsets of metric spaces are separable, and subsets of separable subsets of metric spaces are separable, so in particular, $\text{Range}(\phi)$ is a separable subset of $\mathcal{A}$. Let $D$ be a countable dense subset of $\text{Range}(\phi)$. If $\mathcal{B}$ denotes the $C^*$-subalgebra of $\mathcal{A}$ generated by $\text{Range}(\phi)$, then $\mathcal{B}$ is also the $C^*$-subalgebra of $\mathcal{A}$ generated by $D$. Hence, $\mathcal{B}$ is a separable $C^*$-algebra, which means that it possesses a sequential approximate identity $(e_n)_{n\in\mathbb{N}}$. Now, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|\phi - \mathsf{M}(e_n)\phi\|_\alpha &= \|\langle \phi - \mathsf{M}(e_n)\phi \mid \phi - \mathsf{M}(e_n)\phi\rangle\|_{\mathcal{A}}^{1/2} \\
&= \left\| \int_G \alpha_{x^{-1}}([\phi(x) - e_n\phi(x)]^*[\phi(x) - e_n\phi(x)]) \; d\mu(x) \right\|_{\mathcal{A}}^{\frac{1}{2}} \\
&\leq \left[ \int_G \|\alpha_{x^{-1}}([\phi(x) - e_n\phi(x)]^*[\phi(x) - e_n\phi(x)])\|_{\mathcal{A}} \; d\mu(x) \right]^{\frac{1}{2}} \\
&= \left[ \int_G \|[\phi(x) - e_n\phi(x)]^*[\phi(x) - e_n\phi(x)]\|_{\mathcal{A}} \; d\mu(x) \right]^{\frac{1}{2}} \\
&= \left[ \int_G \|\phi(x) - e_n\phi(x)\|_{\mathcal{A}}^2 \; d\mu(x) \right]^{\frac{1}{2}}.
\end{aligned}$$

Next, notice for all $n \in \mathbb{N}$ and $x \in G$ that

$$\begin{aligned}
\|\phi(x) - e_n\phi(x)\|_{\mathcal{A}}^2 &\leq [\|\phi(x)\|_{\mathcal{A}} + \|e_n\phi(x)\|_{\mathcal{A}}]^2 \\
&\leq [\|\phi(x)\|_{\mathcal{A}} + \|e_n\|_A\|\phi(x)\|_{\mathcal{A}}]^2 \\
&\leq [\|\phi(x)\|_{\mathcal{A}} + \|\phi(x)\|_A]^2 \qquad (\text{As } \|e_n\|_{\mathcal{A}} \leq 1.) \\
&= 4\|\phi(x)\|_{\mathcal{A}}^2.
\end{aligned}$$

Hence, the $\mathbb{R}$-valued sequence of functions $\{\|\phi(\cdot) - e_n\phi(\cdot)\|_{\mathcal{A}}^2\}_{n\in\mathbb{N}}$ is dominated by the integrable function $x \mapsto 4\|\phi(x)\|_{\mathcal{A}}^2$. As this sequence converges pointwise to 0, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \|\phi - \mathsf{M}(e_n)\phi\|_\alpha = 0.$$

Finally, an $\frac{\epsilon}{3}$-argument shows that for any $\Phi \in \mathsf{L}^2(G, \mathcal{A}, \alpha)$ and any $\epsilon > 0$, there exists an $a \in \mathcal{A}$ such that $\|\Phi - \mathsf{M}(a)\Phi\|_\alpha < \epsilon$. Therefore, $\mathsf{M}$ is nondegenerate. □

Next we define $u : G \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$, where for each $\phi \in C_c(G, \mathcal{A})$,

$$[u_x\phi](y) := \alpha_x(\phi(x^{-1}y)) \text{ for all } y \in G.$$

A similar argument as in Proposition 5.2 shows that $u_x \in \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ with adjoint $u_x^* = u_{x^{-1}}$ for each $x \in G$. Note $u_x|_{C_c(G,\mathcal{A})} = \alpha_x \circ \mathsf{lt}_x$. Thus, as $\alpha_x \in \mathrm{Aut}(\mathcal{A})$ and $\mathsf{lt}_x \in \mathrm{Aut}(C_o(G, \mathcal{A}))$ are norm-continuous, strict continuity of the map $x \mapsto u_x|_{C_c(G,\mathcal{A})}$ follows immediately. Finally, $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ implies strict continuity holds for the mapping $x \mapsto u_x$. Therefore, $u : G \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a strictly continuous unitary group representation.

Last, consider $v : \widehat{G} \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ given by $\gamma \mapsto v_\gamma$, which acts on $\phi \in C_c(G, \mathcal{A})$ by

$$[v_\gamma\phi](y) := \gamma(y)\phi(y) \text{ for all } y \in G.$$

Note $\|v_\gamma\phi - \phi\|_{C_c(G,\mathcal{A})} = \|\gamma \cdot \phi - \phi\|_{C_c(G,\mathcal{A})} = \|\gamma - \mathbf{1}\|_\infty \cdot \|\phi\|_{C_c(G,\mathcal{A})} \to 0$ as $\gamma \to 0$. By Corollary 2.33, we have $\|v_\gamma\phi - \phi\|_\alpha \to 0$ as $\gamma \to 0$. Therefore, $\gamma \mapsto v_\gamma|_{C_c(G,\mathcal{A})}$ is strongly, and thus strictly, continuous. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, strict continuity holds for the mapping $\gamma \mapsto v_\gamma$. We conclude $v : \widehat{G} \to \mathcal{U}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ is a strictly continuous unitary group representation.

**Definition 5.3.** The $(G, \mathcal{A}, \alpha)$-*Schrödinger representation* is the quadruple $(\mathsf{L}^2(G, \mathcal{A}, \alpha), \mathsf{M}, u, v)$.

When $\mathcal{A} = \mathbb{C}$, we recover the classical Schrödinger representation $(\lambda, V)$ of $G$.

**Proposition 5.4.** *The $(G, \mathcal{A}, \alpha)$-Schrödinger representation is a $(G, \mathcal{A}, \alpha)$-Heisenberg representation.*

*Proof.* Fullness of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is established in Theorem 5.7, and nondegeneracy of $\mathsf{M}$ is given in Proposition 5.2. By above, $u$ and $v$ are (strictly continuous) unitary group representations of $G$ and $\widehat{G}$, respectively. Fix $x \in G$ and $\gamma \in \widehat{G}$. Then for all $y \in G$ and $\phi \in C_c(G, \mathcal{A})$,

$$\begin{aligned}([v_\gamma u_x]\phi)](y) &= \gamma(y) \cdot [u_x \phi](y) \\ &= \gamma(xx^{-1}y) \cdot \alpha_x(\phi(x^{-1}y)) \\ &= \gamma(x) \cdot \gamma(x^{-1}y) \cdot \alpha_x(\phi(x^{-1}y)) \\ &= \gamma(x) \cdot \alpha_x([\gamma \cdot \phi](x^{-1}y)) \\ &= \gamma(x)[u_x v_\gamma \phi](y).\end{aligned}$$

As $y \in G$ was arbitrary, $[v_\gamma u_x]\phi = \gamma(x) \cdot [u_x v_\gamma]\phi$ for all $\phi \in C_c(G, \mathcal{A})$, and as $\phi \in C_c(G, \mathcal{A})$ was arbitrary, this holds for any $\phi \in C_c(G, \mathcal{A})$. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ and $\|\cdot\|_\alpha$-continuity of both $u_x$ and $v_\gamma$, we have $v_\gamma u_x = \gamma(x) \cdot u_x v_\gamma$. As $x \in G$ and $\gamma \in \widehat{G}$ were arbitrary, this equality holds for all $x \in G$ and $\gamma \in \widehat{G}$, so the pair $(u, v)$ satisfies the Weyl Commutation Relation.

Next we show $(\mathsf{M}, u)$ is a covariant homomorphism for $(G, \mathcal{A}, \alpha)$. Fix $x \in G$ and $a \in \mathcal{A}$. For any $\phi \in C_c(G, \mathcal{A})$ and $y \in G$, observe

$$([u_x \mathsf{M}(a)]\phi)(y) = \alpha_x(a\phi(x^{-1}y)) = \alpha_x(a)\alpha_x(\phi(x^{-1}y)) = ([\mathsf{M}(\alpha_x(a))u_x]\phi)(y).$$

As $y \in G$ was arbitrary, $[u_x \mathsf{M}(a)]\phi = [\mathsf{M}(\alpha_x(a))u_x]\phi$. As $\phi \in C_c(G, \mathcal{A})$ was arbitrary, this holds for all $\phi \in C_c(G, \mathcal{A})$. By $\|\cdot\|_\alpha$-density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ and $\|\cdot\|_\alpha$-continuity of the adjointable operators $u_x$, $\mathsf{M}(a)$, and $\mathsf{M}(\alpha_x(a))$, we have $u_x \mathsf{M}(a) = \mathsf{M}(\alpha_x(a))u_x$. Since $x \in G$ and $a \in \mathcal{A}$ were arbitrary, this equality holds for all $x \in G$ and $a \in \mathcal{A}$. Therefore, $(\mathsf{M}, u)$ is a covariant homomorphism.

Last, for fixed $\gamma \in \widehat{G}$ and $a \in \mathcal{A}$, note that for each $\phi \in C_c(G, \mathcal{A})$,

$$([v_\gamma \mathsf{M}(a)]\phi)(y) = \gamma(y) \cdot a\phi(y) = a(\gamma(y) \cdot \phi(y)) = ([\mathsf{M}(a)v_\gamma]\phi)(y) \text{ for all } y \in G.$$

By similar reasoning as above, we have that $v_\gamma \mathsf{M}(a) = \mathsf{M}(a) v_\gamma$ for any $\gamma \in \widehat{G}$ and $a \in \mathcal{A}$. It follows that $v$ and $\mathsf{M}$ are commuting representations. Therefore, $(\mathsf{L}^2(G, \mathcal{A}, \alpha), \mathsf{M}, u, v)$ is a $(G, \mathcal{A}, \alpha)$-Heisenberg representation. □

The ultimate goal of this chapter is to prove Theorem 1.11, which states that every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation. We call this the "Covariant Stone-von Neumann Theorem."

## 5.2 Green's Imprimitivity Theorem

The Stone-von Neumann Theorem relies on the $C^*$-isomorphism $C_o(G) \rtimes_{\mathsf{lt}} G \cong \mathcal{K}(L^2(G))$. In [25] this isomorphism is given by the integrated form of the covariant pair $(M, \lambda)$, where $M : C_o(G) \to \mathcal{B}(L^2(G))$ takes $f \in C_o(G)$ to the bounded multiplication operator $M_f$ and $\lambda : G \to \mathcal{U}(L^2(G))$ is the left regular representation. Our required generalization of this isomorphism is achieved via Green's Imprimitivity Theorem and Proposition 3.8 of [18].

**Definition 5.5** (Rieffel)**.** Suppose $\mathcal{C}$ and $\mathcal{D}$ are $C^*$-algebras and $\mathsf{X}$ is a left Hilbert $\mathcal{C}$-module, a right Hilbert $\mathcal{D}$-module, and a $\mathcal{C}$-$\mathcal{D}$ bimodule. Then $\mathsf{X}$ is a $\mathcal{C}$-$\mathcal{D}$ *imprimitivity bimodule* if

(i) $\mathsf{X}$ is full as both a Hilbert $\mathcal{C}$-module and Hilbert $\mathcal{D}$-module and

(ii) ${}_{\mathcal{C}}\langle x \mid y \rangle \bullet z = x \bullet \langle y \mid z \rangle_{\mathcal{D}}$ for all $x, y, z \in \mathsf{X}$

where ${}_{\mathcal{C}}\langle \cdot \mid \cdot \rangle$ denotes the inner product on $\mathsf{X}$ as a left Hilbert $\mathcal{C}$-module and $\langle \cdot \mid \cdot \rangle_{\mathcal{D}}$ denotes the inner product on $\mathsf{X}$ as a right Hilbert $\mathcal{D}$-module.

**Remark 5.6** (Brown-Mingo-Shen, 1.9 [21])**.** As a consequence of (ii), a $\mathcal{C}$-$\mathcal{D}$ imprimitivity bimodule $\mathsf{X}$ also satisfies

$$ {}_{\mathcal{C}}\langle x \bullet d \mid y \rangle = {}_{\mathcal{C}}\langle x \mid y \bullet d^* \rangle \text{ for all } x, y \in \mathsf{X},\ d \in \mathcal{D} $$

and

$$\langle c \bullet x \,|\, y \rangle_{\mathcal{D}} = \langle x \,|\, c^* \bullet y \rangle_{\mathcal{D}} \text{ for all } x, y \in \mathsf{X},\ c \in \mathcal{C}.$$

Moreover, the norms induced on $\mathsf{X}$ by $\mathcal{C}$ and $\mathcal{D}$ coincide: $\|x\|_{\mathcal{C}} = \|x\|_{\mathcal{D}}$ for all $x \in \mathsf{X}$.

Given a $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$, let $\sigma$ denote the "diagonal action" on $C_o(G, \mathcal{A})$ by $G$, i.e., for each $x \in G$, $\sigma_x = \alpha_x \circ \mathsf{lt}_x$. Below we state Green's Imprimitivity Theorem in our specific context.

**Theorem 5.7** (Green's Imprimitivity Theorem)**.** *Let $\mathcal{B}_o := C_c(G \times G, \mathcal{A})$. If $(G, \mathcal{A}, \alpha)$ is a $C^*$-dynamical system, then $C_c(G, \mathcal{A})$ is a $\mathcal{B}_o$-$\mathcal{A}$ pre-imprimitivity bimodule with actions*

$$(b \bullet f)(y) = \int_G b(x, y)[\sigma_x(f)](y)\, d\mu(x) \textit{ for all } b \in \mathcal{B}_o,\ y \in G,$$

$$(f \bullet a)(x) = f(x)\alpha_x(a) \textit{ for all } a \in \mathcal{A},\ x \in G,$$

*and inner products*

$$[{}_{\mathcal{B}_o}\langle f \,|\, g \rangle](x, y) = [f \cdot \sigma_x(\overline{g})](y) = f(y)\alpha_x[g(x^{-1}y)^*] \textit{ for all } x, y \in G$$

$$\langle f \,|\, g \rangle_{\mathcal{A}} = \int_G \alpha_{x^{-1}}(f(x)^* g(x))\, d\mu(x).$$

*Moreover, the completion $\mathsf{Z}$ of $C_c(G, \mathcal{A})$ with respect to the norms induced by $\mathcal{B}_o$ and $\mathcal{A}$ (which coincide) is a $\mathcal{B}$-$\mathcal{A}$ imprimitivity bimodule, where $\mathcal{B} := C_o(G, \mathcal{A}) \rtimes_\sigma G$ contains a dense copy of $\mathcal{B}_o$ and acts on $\mathsf{Z}$ by the extension of the action of $\mathcal{B}_o$ on $C_c(G, \mathcal{A})$.*

Note that $\mathsf{Z}$ as a right Hilbert $\mathcal{A}$-module is precisely $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, so Green's Imprimitivity Theorem actually says $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a $C_o(G, \mathcal{A}) \rtimes_\sigma G$-$\mathcal{A}$ imprimitivity bimodule.

**Proposition 5.8** (Raeburn-Williams, 3.8 [18])**.** *If* $\mathsf{X}$ *is a* $\mathcal{C}$*-*$\mathcal{D}$ *imprimitivity bimodule, the map* $\Phi : \mathcal{C} \to \mathcal{L}(\mathsf{X}_{\mathcal{D}})$ *defined by* $\Phi(c)x := c \bullet x$ *for all* $x \in \mathsf{X}$ *is an isomorphism of* $\mathcal{C}$ *onto* $\mathcal{K}(\mathsf{X}_{\mathcal{D}})$.

Since $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is a $C_o(G, \mathcal{A}) \rtimes_\sigma \mathcal{A}$ - $\mathcal{A}$ imprimitivity bimodule, Proposition 5.8 implies $C_o(G, \mathcal{A}) \rtimes_\sigma G \cong \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$, where $\mathsf{L}^2(G, \mathcal{A}, \alpha)$ is viewed as a right Hilbert $\mathcal{A}$-module. We now give an explicit definition of $\Phi$ in this setting. Consider the map $\Xi : C_o(G, \mathcal{A}) \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ defined on $\phi \in C_c(G, \mathcal{A})$ by

$$([\Xi(f)]\phi)(x) := f(x)\phi(x) \text{ for all } x \in G.$$

Note $\|[\Xi(f)]\phi\|_{C_c(G,\mathcal{A})} = \|f\phi\|_{C_c(G,\mathcal{A})} \leq \|f\|_{C_o(G,\mathcal{A})} \cdot \|\phi\|_{C_c(G,\mathcal{A})}$, so the operator $\Xi(f)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_{C_c(G,\mathcal{A})}$-continuous. Following an argument similar to the proof of Proposition 5.2, $\Xi(f)|_{C_c(G,\mathcal{A})}$ is $\|\cdot\|_\alpha$-continuous, so we may continuously extend $\Xi(f)$ to act on all of $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Checking $\Xi(f)^* = \Xi(f^*)$, where $f^*(x) = f(x^{-1})^*$ for each $x \in G$, confirms that $\Xi(f)$ is an adjointable operator on $\mathsf{L}^2(G, \mathcal{A}, \alpha)$. Therefore, $\Xi$ is a well-defined $*$-representation of $C_o(G, \mathcal{A})$ on $\mathsf{L}^2(G, \mathcal{A}, \alpha)$.

To explicitly describe $\Phi : C_o(G, \mathcal{A}) \rtimes_\sigma G \xrightarrow{\cong} \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we also require the $\mathcal{A}$-valued *Fourier transform* $\mathcal{F}$ for $\widehat{G}$, where $\mathcal{F} : C_c(\widehat{G}, \mathcal{A}) \to C_o(G, \mathcal{A})$ is defined on $f \in C_c(\widehat{G}, \mathcal{A})$ by

$$[\mathcal{F}f](x) := \int_{\widehat{G}} f(\gamma)\gamma(x)\, d\hat{\mu}(\gamma) \text{ for all } x \in G.$$

Denote $\mathcal{F}f$ by $\hat{f}$. Consider the $C^*$-dynamical system $(\widehat{G}, \mathcal{A}, \iota)$ with trivial action $\iota$. Note that $\mathcal{F}$ is just the restriction of the $C^*$-isomorphism $\varphi_2 : \mathcal{A} \rtimes_\iota \widehat{G} \xrightarrow{\cong} C_o(G, \mathcal{A})$ in Lemma 7.3 of [25] to the dense $*$-subalgebra $C_c(\widehat{G}, \mathcal{A})$ of $\mathcal{A} \rtimes_\iota \widehat{G}$.

**Lemma 5.9.** *The* $*$*-representation* $\Xi : C_o(G, \mathcal{A}) \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ *is equal to* $(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, *where* $\mathsf{M} \times v$ *is the integrated form of the covariant homomorphism* $(\mathsf{M}, v)$ *for* $(\widehat{G}, \mathcal{A}, \iota)$.

*Proof.* Note that $\mathcal{F}(C_c(\widehat{G},\mathcal{A}))$ is dense in $C_o(G,\mathcal{A})$. Fix $f \in C_c(\widehat{G},\mathcal{A})$. For $\phi \in C_c(G,\mathcal{A})$,

$$[(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}](\hat{f})\phi = [(\mathsf{M} \rtimes v)(f)]\phi = \underbrace{\left(\int_{\widehat{G}} \mathsf{M}(f(\gamma))v_\gamma \, d\hat{\mu}(\gamma)\right)}_{\in \mathcal{L}(\mathsf{L}^2(G,\mathcal{A},\alpha))} \phi = \underbrace{\int_{\widehat{G}} \mathsf{M}(f(\gamma))v_\gamma \phi \, d\hat{\mu}(\gamma)}_{\in C_c(G,\mathcal{A})},$$

where the last equality is a standard property of this vector-valued integral. The reader is referred to Section 1.5 of [25] for details. Since point evaluation is a linear functional on $C_o(G,\mathcal{A})$,

$$\begin{aligned}
\int_{\widehat{G}} \mathsf{M}(f(\gamma))[v_\gamma\phi](x) \, d\hat{\mu}(\gamma) &= \int_{\widehat{G}} f(\gamma)\gamma(x)\phi(x) \, d\hat{\mu}(\gamma) \\
&= \left(\int_{\widehat{G}} f(\gamma)\gamma(x) \, d\hat{\mu}(\gamma)\right)\phi(x) \\
&= \hat{f}(x)\phi(x) \\
&= [\Xi(\hat{f})\phi](x)
\end{aligned}$$

for every $x \in G$. As $x \in G$ was arbitrary, as was $\phi \in C_c(G,\mathcal{A})$, we have that

$$\Xi(\hat{f})|_{C_c(G,\mathcal{A})} = [(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}](\hat{f})|_{C_c(G,\mathcal{A})}.$$

By density of $C_c(G,\mathcal{A})$ in $\mathsf{L}^2(G,\mathcal{A},\alpha)$, this equality holds on $\mathsf{L}^2(G,\mathcal{A},\alpha)$. Then, by density of $\mathcal{F}(C_c(\widehat{G},\mathcal{A}))$ in $C_o(G,\mathcal{A})$ and continuity of $\Xi$ and $(\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, we have $\Xi(g) = [(M \rtimes v) \circ \mathcal{F}^{-1}](g)$ for all $g \in C_o(G,\mathcal{A})$. $\square$

Having established $\Xi = (\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$, we know that $\Xi$ is nondegenerate. We now show $(\Xi, u)$ is a covariant homomorphism of $(G, C_o(G,\mathcal{A}), \sigma)$ into $\mathcal{L}(\mathsf{L}^2(G,\mathcal{A},\alpha))$. Fix $x \in G$ and

$f \in C_c(\widehat{G}, \mathcal{A})$. Let $\phi \in C_c(G, \mathcal{A})$ be arbitrary. Then for all $y \in G$,

$$\begin{aligned}([u_x \Xi(\hat{f})]\phi)(y) &= \alpha_x(\hat{f}(x^{-1}y)\phi(x^{-1}y)) \\ &= \alpha_x(\hat{f}(x^{-1}y))\alpha_x(\phi(x^{-1}y)) \\ &= [\sigma_x(\hat{f})](y)\alpha_x(\phi(x^{-1}y)) \\ &= ([\Xi(\sigma_x(\hat{f}))u_x]\phi)(y).\end{aligned}$$

As $y \in G$ was arbitrary, $[u_x \Xi(\hat{f})]\phi = [\Xi(\sigma_x(\hat{f}))u_x]\phi$. Also, $\phi \in C_c(G, \mathcal{A})$ was arbitrary, and $C_c(G, \mathcal{A})$ is $\|\cdot\|_\alpha$-dense in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, so $u_x \Xi(\hat{f}) = \Xi[\sigma_x(\hat{f})]u_x$ as adjointable operators. By density of $\mathcal{F}(C_c(\widehat{G}, \mathcal{A}))$ in $C_o(G, \mathcal{A})$, $\|\cdot\|_{C_o(G,\mathcal{A})}$-continuity of $\Xi$ and $\sigma_x$ suffice to conclude $u_x \Xi(g) = [\Xi(\sigma_x(g))]u_x$ for all $g \in C_o(G, \mathcal{A})$. Thus, $(\Xi, u)$ is a nondegenerate covariant homomorphism for $(G, C_o(G, \mathcal{A}), \sigma)$ whose integrated form yields a nondegenerate $*$-representation $\Xi \rtimes u : C_o(G, \mathcal{A}) \rtimes_\sigma G \to \mathcal{L}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$.

**Proposition 5.10.** *The isomorphism* $\Phi : C_o(G, \mathcal{A}) \rtimes_\sigma G \to \mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ *in Proposition 5.8 is the integrated form* $\Xi \rtimes u$.

*Proof.* It suffices to check that $\Phi(F) = (\Xi \rtimes u)F$ for all $F \in C_c(G, C_o(G, \mathcal{A}))$ by density of $C_c(G, C_o(G, \mathcal{A}))$ in $C_o(G, \mathcal{A}) \rtimes_\sigma G$. Let $\phi, \psi \in C_c(G, \mathcal{A})$, and observe

$$
\begin{aligned}
\langle \phi \,|\, [\Xi \rtimes u](F)\psi \rangle &= \left\langle \phi \,\middle|\, \left( \int_G \Xi(F_y) u_y \, d\mu(y) \right) \psi \right\rangle \\
&= \int_G \langle \phi \,|\, [\Xi(F_y) u_y](\psi) \rangle \; d\mu(y) \qquad [\text{ by Lemma 2.51 }] \\
&= \int_G \left( \int_G \alpha_{x^{-1}} \left[ \phi(x)^* \; F_y(x) \; \alpha_y(\psi(y^{-1}x)) \right] \; d\mu(x) \right) \; d\mu(y) \\
&= \int_G \left( \int_G \alpha_{x^{-1}} \left[ \phi(x)^* \; F_y(x) \; \alpha_y(\psi(y^{-1}x)) \right] \; d\mu(y) \right) \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[ \phi(x)^* \left( \int_G F_y(x) \; \alpha_y(\psi(y^{-1}x)) \, d\mu(y) \right) \right] \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[ \phi(x)^* \left( \int_G F_y(x) \; [\sigma_y(\psi)](x) \, d\mu(y) \right) \right] \; d\mu(x) \\
&= \int_G \alpha_{x^{-1}} \left[ \phi(x)^* (F \bullet \psi)(x) \right] \; d\mu(x) \qquad [\text{ by Green's Imprimitivity Theorem }] \\
&= \langle \phi \,|\, \Phi(F)\psi \rangle \, .
\end{aligned}
$$

By density of $C_c(G, \mathcal{A})$ in $\mathsf{L}^2(G, \mathcal{A}, \alpha)$, we conclude $\Phi(F) = [\Xi \rtimes u](F)$. Moreover, $C_c(G, C_o(G, \mathcal{A}))$ is dense in $C_o(G, \mathcal{A}) \rtimes_\sigma G$, so we finally establish that $\Phi = \Xi \rtimes u$. □

The isomorphism $C_o(G) \rtimes_{\mathsf{lt}} G \cong \mathcal{K}(L^2(G))$ relates nondegenerate $*$-representations of $C_o(G) \rtimes_{\mathsf{lt}} G$ with the nicely classified nondegenerate $*$-representations of $\mathcal{K}(L^2(G))$. For our purposes, then, the utility of Proposition 5.8 follows only from having an analogous classification of representations of $\mathcal{K}(\mathsf{X})$ where $\mathsf{X}$ is a Hilbert $\mathcal{A}$-module for some $C^*$-algebra $\mathcal{A}$. Without more assumptions on $\mathcal{A}$, however, such a classification for representations of $\mathcal{K}(\mathsf{X})$ does not exist. Hence, we restrict our attention to Hilbert $\mathcal{K}(\mathcal{H})$-modules.

## 5.3 Representations of Hilbert $\mathcal{K}(\mathcal{H})$-modules

Henceforth, $\mathsf{X}$ denotes a Hilbert $\mathcal{K}(\mathcal{H})$-module. The main result of this section, Theorem 5.14, generalizes the following theorem to representations of $\mathcal{K}(\mathsf{X})$ as adjointable operators on

Hilbert $\mathcal{K}(\mathcal{H})$-modules. It will be useful to keep Lemma 2.56 in mind.

**Theorem 5.11** (Arveson, 1.4.4 [2])**.** *Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{K}(\mathcal{H})$, and let $\pi$ be any nondegenerate representation of $\mathcal{A}$. Then there is an orthogonal family $\{\pi_i\}$ of irreducible subrepresentations of $\pi$ such that $\pi = \sum_i \pi_i$, and each $\pi_i$ is equivalent to a subrepresentation of the identity representation $id : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

**Definition 5.12.** Let $\mathcal{A}$ be a $C^*$-algebra. A projection $p \in \mathcal{A}$ is called *minimal* if and only if $p \neq 0_{\mathcal{A}}$ and the only sub-projections of $p$ in $\mathcal{A}$ are $0_{\mathcal{A}}$ and $p$ itself.

Note that the minimal projections in $\mathcal{K}(\mathcal{H})$ are simply the rank-one operators, and recall that **every** nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module is full by simplicity of $\mathcal{K}(\mathcal{H})$.

**Lemma 5.13.** *The $C^*$-algebra $\mathcal{K}(\mathsf{X})$ acts irreducibly on $\mathsf{X}$, that is, $\mathsf{X}$ has no nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodules.*

*Proof.* Suppose $\mathsf{Y}$ were a nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$. Let $p$ be a rank-one projection in $\mathcal{K}(\mathcal{H})$. By Lemma 2.56, $\mathsf{Y} \bullet p$ and $\mathsf{X} \bullet p$ are Hilbert spaces, and furthermore, $\mathsf{Y} \bullet p$ is a closed subspace of $\mathsf{X} \bullet p$. We claim that $\mathsf{Y} \bullet p$ is $\mathcal{K}(\mathsf{X} \bullet p)$-invariant. Let $b \in \mathcal{K}(\mathsf{X} \bullet p)$. By Theorem 2.57, $b$ has the form $a|_{\mathsf{X}\bullet p}$ for some $a \in \mathcal{K}(\mathsf{X})$. Thus,

$$b[\mathsf{Y} \bullet p] = a|_{\mathsf{X}\bullet p}[\mathsf{Y} \bullet p] = a[\mathsf{Y} \bullet p] = (a\mathsf{Y}) \bullet p \subseteq \mathsf{Y} \bullet p$$

by $\mathcal{K}(\mathcal{H})$-linearity of $a$. As $b \in \mathcal{K}(\mathsf{X}\bullet p)$ was arbitrary, $\mathsf{Y}\bullet p$ is $\mathcal{K}(\mathsf{X}\bullet p)$-invariant. Furthermore,

$$\overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \mathsf{Y}$$

by Proposition 2.59, so since $\mathsf{Y}$ is nontrivial, $\mathsf{Y} \bullet p$ must be nontrivial. Last, $\mathsf{Y} \bullet p$ is a proper subspace of $\mathsf{X} \bullet p$. Indeed, if $\mathsf{Y} \bullet p = \mathsf{X} \bullet p$, then applying Proposition 2.59 twice implies

$$\mathsf{Y} = \overline{(\mathsf{Y} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \overline{(\mathsf{X} \bullet p) \bullet \mathcal{K}(\mathcal{H})} = \mathsf{X},$$

which contradicts the assumption that $\mathsf{Y}$ is a proper $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$. Therefore, $\mathsf{Y} \bullet p$ is a $\mathcal{K}(\mathsf{X} \bullet p)$-invariant proper nontrivial closed subspace of $\mathsf{X} \bullet p$. This is a contradiction to the fact that given any Hilbert space $\mathcal{H}$, there are no $\mathcal{K}(\mathcal{H})$-invariant proper nontrivial closed subspaces of $\mathcal{H}$. Since $\mathsf{X} \bullet p$ is a Hilbert space, we have reached a contradiction. Thus, there can exist no nontrivial $\mathcal{K}(\mathsf{X})$-invariant closed $\mathcal{K}(\mathcal{H})$-submodules of $\mathsf{X}$, so $\mathcal{K}(\mathsf{X})$ acts irreducibly on $\mathsf{X}$. □

**Theorem 5.14.** *Let $\mathsf{X}$ and $\mathsf{Y}$ be Hilbert $\mathcal{K}(\mathcal{H})$-modules. If $\widetilde{\pi} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y})$ is a nondegenerate $*$-representation, then $\widetilde{\pi}$ is unitarily equivalent to a direct sum of copies of the identity representation $id : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{X})$.*

*Proof.* Our proof is an adaptation of Arveson's proof of Theorem 5.11. Fix a rank-one projection $p \in \mathcal{K}(\mathcal{H})$, and consider the composition $\pi$ given by

$$\pi : \mathcal{K}(\mathsf{X} \bullet p) \xrightarrow{\cong} \mathcal{K}(\mathsf{X}) \xrightarrow{\widetilde{\pi}} \mathcal{L}(\mathsf{Y}) \xrightarrow{\cong} \mathcal{B}(\mathsf{Y} \bullet p),$$

where $[(\Psi_{\mathsf{X}})|_{\mathcal{K}(\mathsf{X}\bullet p)}]^{-1} : \mathcal{K}(\mathsf{X} \bullet p) \xrightarrow{\cong} \mathcal{K}(\mathsf{X})$ and $\Psi_{\mathsf{Y}} : \mathcal{L}(\mathsf{Y}) \xrightarrow{\cong} \mathcal{B}(\mathsf{Y} \bullet p)$ are provided by Theorem 2.57. As $\widetilde{\pi}$ is nondegenerate and $\pi$ is the composition of $\widetilde{\pi}$ with $C^*$-isomorphisms, $\pi$ is also nondegenerate. Note that $\mathsf{X} \bullet p$ and $\mathsf{Y} \bullet p$ are both Hilbert spaces by Lemma 2.56, so in fact, $\pi$ is a nondegenerate $*$-representation of the compact operators on the Hilbert space $\mathsf{X} \bullet p$ as bounded operators on the Hilbert space $\mathsf{Y} \bullet p$. Thus, by Theorem 5.11, there exists an index set $J$ and a unitary $W : \oplus_{j \in J} \mathsf{X} \bullet p \to \mathsf{Y} \bullet p$ such that $\pi(a) = \mathrm{ad}_W \circ \oplus_j a$ for all $a \in \mathcal{K}(\mathsf{X} \bullet p)$. However, Theorem 2.57 does not necessarily lift $W$ to a unitary $w : \oplus_j \mathsf{X} \to \mathsf{Y}$, so we proceed to construct the desired unitary $w : \oplus_j \mathsf{X} \to \mathsf{Y}$.

By Arveson's proof, there is a rank-one projection $q \in \mathcal{K}(\mathsf{X} \bullet p)$ such that $\pi(q) \neq 0$. Furthermore, Theorem 2.57 yields a minimal projection $E \in \mathcal{K}(\mathsf{X})$ such that $q = E|_{\mathsf{X}\bullet p}$. Since $\pi(q) \neq 0$, it must be that $\widetilde{\pi}(E) \neq 0$. By Corollary 2.54, there is a linear functional

$$f_q : \mathcal{K}(\mathsf{X} \bullet p) \to \mathbb{C} \text{ which satisfies } f_q(S)q = qSq \text{ for all } S \in \mathcal{K}(\mathsf{X} \bullet p).$$

Define a linear functional $g : \mathcal{K}(\mathsf{X}) \to \mathbb{C}$ by $g(T) := f_q(T|_{\mathsf{X}\bullet p})$. For each $T \in \mathcal{K}(\mathsf{X})$, notice

$$(ETE)|_{\mathsf{X}\bullet p} = E|_{\mathsf{X}\bullet p}\, T|_{\mathsf{X}\bullet p}\, E|_{\mathsf{X}\bullet p} = q(T|_{\mathsf{X}\bullet p})q = f_q(T|_{\mathsf{X}\bullet p})q = f_q(T|_{\mathsf{X}\bullet p})E|_{\mathsf{X}\bullet p} = [g(T)E]|_{\mathsf{X}\bullet p}.$$

By Theorem 2.57, we conclude $ETE = g(T)E$ for all $T \in \mathcal{K}(\mathsf{X})$.

Consider the $\mathcal{K}(\mathcal{H})$-submodule $E[\mathsf{X}]$ of $\mathsf{X}$. Note that $E[\mathsf{X}]$ is nonzero since $E \neq 0$, and $E[\mathsf{X}]$ is closed because $E$ is a projection. Thus, $E[\mathsf{X}]$ is a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module. Similarly, $\widetilde{\pi}(E)[\mathsf{Y}]$ is a nonzero Hilbert $\mathcal{K}(\mathcal{H})$-module. Hence, by Corollary 2.55, there exist $\xi \in E[\mathsf{X}]$ and $\eta \in \widetilde{\pi}(E)[\mathsf{Y}]$ such that $\langle \xi \,|\, \xi \rangle_{\mathsf{X}} = p$ and $\langle \eta \,|\, \eta \rangle_{\mathsf{Y}} = p$.

Define a map $w' : [\mathcal{K}(\mathsf{X})\xi]\bullet\mathcal{K}(\mathcal{H}) \to [\widetilde{\pi}(\mathcal{K}(\mathsf{X}))\eta]\bullet\mathcal{K}(\mathcal{H})$ by $\sum_{i=1}^{n} T_i(\xi\bullet a_i) \mapsto \sum_{i=1}^{n} \widetilde{\pi}(T_i)(\eta\bullet$

$a_i$). By virtue of being an isometry, $w'$ is well-defined: for $T_1, ..., T_n \in \mathcal{K}(\mathsf{X})$, $a_1, ..., a_n \in \mathcal{K}(\mathcal{H})$,

$$\begin{aligned}
\left\| \left\langle \sum_{i=1}^{n} \widetilde{\pi}(T_i)(\eta \bullet a_i) \,\middle|\, \sum_{j=1}^{n} \widetilde{\pi}(T_j)(\eta \bullet a_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} &= \left\| \sum_{i,j=1}^{n} \langle \widetilde{\pi}(T_i)(\eta \bullet a_i) \,|\, \widetilde{\pi}(T_j)(\eta \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \langle \widetilde{\pi}(T_i)([\widetilde{\pi}(E)\eta] \bullet a_i) \,|\, \widetilde{\pi}(T_j)([\widetilde{\pi}(E)\eta] \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \langle \widetilde{\pi}(T_i E)(\eta \bullet a_i) \,|\, \widetilde{\pi}(T_j E)(\eta \bullet a_j) \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \left\langle \widetilde{\pi}(E T_j^* T_i E)(\eta \bullet a_i) \,\middle|\, \eta \bullet a_j \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \left\langle \widetilde{\pi}(g(T_j^* T_i) E)(\eta \bullet a_i) \,\middle|\, \eta \bullet a_j \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g(T_j^* T_i)} \, \langle \widetilde{\pi}(E)(\eta \bullet a_i) \,|\, \eta \bullet a_j \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g(T_j^* T_i)} \, \langle \eta \bullet a_i \,|\, \eta \bullet a_j \rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g(T_j^* T_i)} \, a_i^* \, \langle \eta \,|\, \eta \rangle_{\mathsf{Y}} \, a_j \right\|_{\mathcal{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g(T_j^* T_i)} \, a_i^* \, p \, a_j \right\|_{\mathcal{K}(\mathcal{H})}
\end{aligned}$$

Following a nearly identical computation yields

$$\left\| \left\langle \sum_{i=1}^{n} T_i(\xi \bullet a_i) \,\middle|\, \sum_{j=1}^{n} T_j(\xi \bullet a_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathcal{K}(\mathcal{H})} = \left\| \sum_{i,j=1}^{n} \overline{g(T_j^* T_i)} \; a_i^* \; p \; a_j \right\|_{\mathcal{K}(\mathcal{H})}.$$

Therefore, $w'$ is a surjective isometry which extends by continuity to $w' : \mathsf{X}' \to \mathsf{Y}'$, where

$$\mathsf{X}' := \overline{[\mathcal{K}(\mathsf{X})\xi] \bullet \mathcal{K}(\mathcal{H})} \;\text{ and }\; \mathsf{Y}' := \overline{[\widetilde{\pi}(\mathcal{K}(\mathsf{X}))\eta] \bullet \mathcal{K}(\mathcal{H})}.$$

Note that $\mathsf{X}'$ is a nonzero closed $\mathcal{K}(\mathsf{X})$-invariant $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$. Thus, by Lemma 5.13, $\mathsf{X} = \mathsf{X}'$. Hence, $w' : \mathsf{X} \to \mathsf{Y}'$ is a surjective isometry, which, moreover, is $\mathcal{K}(\mathcal{H})$-linear. Thus, $w' : \mathsf{X} \to \mathsf{Y}'$ is unitary.

We claim $w'T = [\widetilde{\pi}(T)|_{\mathsf{Y}'}]w'$ for all $T \in \mathcal{K}(\mathsf{X})$. Fix $T \in \mathcal{K}(\mathsf{X})$ and let $T_1, ..., T_n \in \mathcal{K}(\mathsf{X})$ and $a_1, ..., a_n \in \mathcal{K}(\mathcal{H})$ be arbitrary. Then

$$\begin{aligned} w'T\left(\sum_{i=1}^{n} T_i(\xi \bullet a_i)\right) &= w'\left(\sum_{i=1}^{n} TT_i(\xi \bullet a_i)\right) \\ &= \sum_{i=1}^{n} \widetilde{\pi}(TT_i)(\eta \bullet a_i) \\ &= \sum_{i=1}^{n} \widetilde{\pi}(T)\widetilde{\pi}(T_i)(\eta \bullet a_i) \\ &= \widetilde{\pi}(T)\left(\sum_{i=1}^{n} \widetilde{\pi}(T_i)(\eta \bullet a_i)\right) \\ &= \widetilde{\pi}(T)w'\left(\sum_{i=1}^{n} T_i(\xi \bullet a_i)\right) \end{aligned}$$

By density of $[\mathcal{K}(\mathsf{X})\xi] \bullet \mathcal{K}(\mathcal{H})$ in $\mathsf{X}$ and continuity of both $w'T$ and $(\widetilde{\pi}(T)|_{\mathsf{Y}'})w'$, we have $w'T = (\widetilde{\pi}(T)|_{\mathsf{Y}'})w'$. Thus, the map $\mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y}')$ given by $T \mapsto \widetilde{\pi}(T)|_{\mathsf{Y}'}$ is a nondegenerate $*$-representation of $\mathcal{K}(\mathsf{X})$ on $\mathsf{Y}'$ which is unitarily equivalent via $w'$ to the identity representation $\mathrm{id} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{X})$.

Complementability of Hilbert $\mathcal{K}(\mathcal{H})$-modules allows us to apply this argument to the subrepresentation $T \mapsto \widetilde{\pi}(T)|_{(\mathsf{Y}')^{\perp}}$ of $\widetilde{\pi} : \mathcal{K}(\mathsf{X}) \to \mathcal{L}(\mathsf{Y})$. An exhaustive argument and application of Zorn's Lemma yields a family $\{\mathsf{Y}_j\}_{j\in J}$ of closed $\mathcal{K}(\mathcal{H})$-submodules of $\mathsf{Y}$ and unitaries $\{w_j : \mathsf{X} \to \mathsf{Y}_j\}_{j\in J}$ such that $\mathsf{Y} = \oplus_j \mathsf{Y}_j$. Then $w := \oplus_j w_j$ is a unitary from $\oplus_j \mathsf{X}$ onto $\mathsf{Y}$ such that $w[\oplus_j T] = \widetilde{\pi}(T)w$ for all $T \in \mathcal{K}(\mathsf{X})$. This completes the proof. $\square$

## 5.4 Correspondence of $(G, C_o(G,\mathcal{A}), \mathsf{lt}\otimes\alpha)$-Covariant Homomorphisms and $(G,\mathcal{A},\alpha)$-Heisenberg Representations

Let $(G,\mathcal{A},\alpha)$ be a dynamical system. Suppose $s : \widehat{G} \to \mathcal{U}(\mathsf{X})$ is a unitary group representation on a Hilbert $\mathcal{A}$-module $\mathsf{X}$ and $\rho : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation such that $\rho(a)s_\gamma = s_\gamma\rho(a)$ for all $a \in \mathcal{A}$, $\gamma \in \widehat{G}$. Then the integrated form $\rho \rtimes s : \mathcal{A} \rtimes_\iota \widehat{G} \to \mathcal{L}(\mathsf{X})$ is a nondegenerate $*$-representation by Proposition 2.50. Define $\Pi_{\rho,s}$ to be the composition $\Pi_{\rho,s} : C_o(G,\mathcal{A}) \xrightarrow{\mathcal{F}^{-1}} \mathcal{A} \rtimes_\iota \widehat{G} \xrightarrow{\rho\rtimes s} \mathcal{L}(\mathsf{X})$. As $\mathcal{F}^{-1}$ is a $C^*$-isomorphism and $\rho \rtimes s$ is a nondegenerate $*$-representation of $\mathcal{A} \rtimes_\iota \widehat{G}$, the map $\Pi_{\rho,s}$ is a nondegenerate $*$-representation of $C_o(G,\mathcal{A})$.

**Theorem 5.15.** *If $(\mathsf{X},\rho,r,s)$ is a $(G,\mathcal{A},\alpha)$-Heisenberg representation, then $(\Pi_{\rho,s},r)$ is a nondegenerate covariant homomorphism for $(G, C_o(G,\mathcal{A}),\sigma)$ into $\mathcal{L}(\mathsf{X})$.*

*Proof.* Fix $x \in G$ and $f \in C_c(\widehat{G},\mathcal{A})$, so $\hat{f} \in C_o(G,\mathcal{A})$. Let $\hat{x} : \widehat{G} \to \mathbb{C}$ denote the copy of $x \in G$ acting as an element of the dual of $\widehat{G}$ by $\hat{x}(\gamma) = \overline{\gamma(x)}$ for each $\gamma \in \widehat{G}$. For all $y \in G$, note

$$[\mathcal{F}(f\cdot\hat{x})](y) = \int_{\widehat{G}} f(\gamma)\overline{\gamma(x)}\gamma(y)\,d\hat{\mu}(\gamma) = \int_{\widehat{G}} f(\gamma)\gamma(x^{-1}y)\,d\hat{\mu}(\gamma) = \hat{f}(x^{-1}y) = [\mathsf{lt}_x(\hat{f})](y).$$

It follows that $\alpha_x \circ \mathcal{F}(f \cdot \hat{x}) \overset{(\star)}{=} \sigma_x(\hat{f})$ since $\sigma_x = \alpha_x \circ \mathsf{lt}_x$. Thus,

$$
\begin{aligned}
r_x \Pi_{\rho,s}(\hat{f}) &= r_x \left( \int_{\widehat{G}} \rho(f(\gamma)) s_\gamma \, d\hat{\mu}(\gamma) \right) \\
&= \int_{\widehat{G}} r_x \rho(f(\gamma)) s_\gamma \, d\hat{\mu}(\gamma) \\
&= \int_{\widehat{G}} \rho[\alpha_x(f(\gamma))] \, r_x s_\gamma \, d\hat{\mu}(\gamma) \qquad [\text{ by covariance of } (\rho, r) \text{ }] \\
&= \int_{\widehat{G}} \rho[\alpha_x(f(\gamma))] \, \overline{\gamma(x)} s_\gamma r_x \, d\hat{\mu}(\gamma) \qquad [\text{ as } r \text{ and } s \text{ satisfy the WCR }] \\
&= \left( \int_{\widehat{G}} \rho[\alpha_x(f(\gamma) \, \overline{\gamma(x)})] s_\gamma \, d\hat{\mu}(\gamma) \right) r_x \\
&= \left( \int_{\widehat{G}} \rho[\alpha_x([f \cdot \hat{x}](\gamma))] s_\gamma \, d\hat{\mu}(\gamma) \right) \circ r_x \\
&= [(\rho \rtimes s)(\alpha_x \circ (f \cdot \hat{x}))] r_x \\
&= [(\rho \rtimes s) \circ \mathcal{F}^{-1}][\mathcal{F}(\alpha_x \circ (f \cdot \hat{x}))] r_x \\
&= \Pi_{\rho,s}[\alpha_x \circ \mathcal{F}(f \cdot \hat{x})] r_x \\
&= \Pi_{\rho,s}(\sigma_x(\hat{f})) r_x \qquad [\text{ by } (\star) \text{ }].
\end{aligned}
$$

As $f \in C_c(\widehat{G}, \mathcal{A})$ was arbitrary and $\mathcal{F}(C_c(\widehat{G}, \mathcal{A}))$ is dense in $C_o(G, \mathcal{A})$, $\|\cdot\|_{C_o(G,\mathcal{A})}$-continuity of both $\Pi_{\rho,s}$ and $\sigma_x$ imply $r_x \Pi_{\rho,s}(g) = \Pi_{\rho,s}(\sigma_x(g)) r_x$ for all $g \in C_o(G, \mathcal{A})$. Therefore, since $x \in G$ was arbitrary, $(\Pi_{\rho,s}, r)$ is a nondegenerate covariant homomorphism for $(G, C_o(G, \mathcal{A}), \sigma)$. □

## 5.5 Proof of the Covariant Stone-von Neumann Theorem

**Definition 5.16.** Two $(G, \mathcal{A}, \alpha)$-Heisenberg representations $(\mathsf{X}, \rho, r, s)$ and $(\mathsf{Y}, \tau, u, v)$ are *unitarily equivalent* if there exists a unitary $w : \mathsf{X} \to \mathsf{Y}$ such that

(i) $\tau = \mathrm{ad}_w \circ \rho$, that is, $\tau(a) = w \rho(a) w^{-1}$ for all $a \in \mathcal{A}$,

(ii) $u_x = w r_x w^{-1}$ for all $x \in G$, and

(iii) $v_\gamma = w s_\gamma w^{-1}$ for all $\gamma \in \widehat{G}$.

**Theorem 1.11.** *Every $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation is unitarily equivalent to a direct sum of copies of the $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Schrödinger representation.*

*Proof.* Given a $(G, \mathcal{K}(\mathcal{H}), \alpha)$-Heisenberg representation $(\mathsf{X}, \rho, r, s)$, Theorem 5.15 states $(\Pi_{\rho,s}, r)$ is a covariant homomorphism for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$. Since $\Pi_{\rho,s}$ is nondegenerate, the integrated form $\Pi_{\rho,s} \rtimes r$ is a nondegenerate $*$-representation of $C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G$ into $\mathcal{L}(\mathsf{X})$. Let $\mathsf{Z} := \mathsf{L}^2(G, \mathcal{K}(\mathcal{H}), \alpha)$, and recall Proposititons 5.8 and 5.10 yield the isomorphism $\Xi \rtimes u : C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \xrightarrow{\cong} \mathcal{K}(\mathsf{Z})$. Thus, the composition

$$\Theta : \mathcal{K}(\mathsf{Z}) \stackrel{(\Xi \rtimes u)^{-1}}{\longrightarrow} C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \stackrel{\Pi_{\rho,s} \rtimes r}{\longrightarrow} \mathcal{L}(\mathsf{X})$$

is a nondegenerate $*$-representation of $\mathcal{K}(\mathsf{Z})$ as adjointable operators on the Hilbert $\mathcal{K}(\mathcal{H})$-module $\mathsf{X}$. As $\mathsf{Z}$ and $\mathsf{X}$ are Hilbert $\mathcal{K}(\mathcal{H})$-modules, Theorem 5.14 implies $\Theta$ is unitarily equivalent to a direct sum of copies of the identity representation $\mathrm{id} : \mathcal{K}(\mathsf{Z}) \to \mathcal{L}(\mathsf{Z})$. Specifically, there exists a unitary $w : \mathsf{X} \to \oplus_j \mathsf{Z}$ such that $\mathrm{ad}_w \circ \Theta = \oplus_j \mathrm{id}$.

We claim $\mathrm{ad}_w \circ \rho = \oplus_j \mathsf{M}$, $\mathrm{ad}_w \circ r = \oplus_j u$, and $\mathrm{ad}_w \circ s = \oplus_j v$. Note that for any covariant homomorphism $(\pi, q)$ for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$ into $\mathcal{L}(\mathsf{X})$, we have

$$(\mathrm{ad}_w \circ \pi) \rtimes (\mathrm{ad}_w \circ q) = \mathrm{ad}_w \circ (\pi \rtimes q) : C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G \to \mathcal{L}(\oplus_j \mathsf{Z}).$$

Thus, Proposition 2.52 implies

$$\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} \rtimes (\mathrm{ad}_w \circ r) = \mathrm{ad}_w \circ (\Pi_{\rho,s} \rtimes r) = \oplus_j (\Xi \rtimes u) = [\oplus_j \Xi] \rtimes [\oplus_j u].$$

By Proposition 2.50, the covariant homomorphisms $(\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)}, \mathrm{ad}_w \circ r)$ and $(\oplus_j \Xi, \oplus_j u)$ for $(G, C_o(G, \mathcal{K}(\mathcal{H})), \sigma)$ must coincide since their integrated forms are the same nondegenerate $*$-representation of $C_o(G, \mathcal{K}(\mathcal{H})) \rtimes_\sigma G$ into $\mathcal{L}(\oplus_j \mathsf{Z})$. Therefore,

$$\Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} = \oplus_j \Xi \quad \text{and} \quad \mathrm{ad}_w \circ r = \oplus_j u.$$

Recall $\Xi = (\mathsf{M} \rtimes v) \circ \mathcal{F}^{-1}$ by Lemma 5.9. Hence,

$$[(\mathrm{ad}_w \circ \rho) \rtimes (\mathrm{ad}_w \circ s)] \circ \mathcal{F}^{-1} = \Pi_{(\mathrm{ad}_w \circ \rho),\, (\mathrm{ad}_w \circ s)} = \oplus_j \Xi = [\oplus_j (\mathsf{M} \rtimes v)] \circ \mathcal{F}^{-1} = ([\oplus_j \mathsf{M}] \rtimes [\oplus_j v]) \circ \mathcal{F}^{-1}.$$

By another application of Proposition 2.50, we have that $\mathrm{ad}_w \circ \rho = \oplus_j \mathsf{M}$ and $\mathrm{ad}_w \circ s = \oplus_j v$, as desired. We conclude $(\mathsf{X}, \rho, r, s)$ is unitarily equivalent to a direct sum of copies of $(\mathsf{Z}, \mathsf{M}, u, v)$. $\square$

# Chapter 6

# Conclusions and Future Directions

## 6.1 Weak $D$-Antidifferentiability and Extended Derivations

Given an unbounded self-adjoint operator $D$ on a Hilbert space $\mathcal{H}$, Christensen's work in [6] and [5] gives multiple equivalent conditions for when an operator $x \in \mathcal{B}(\mathcal{H})$ makes the commutator $[iD, x]$ defined and bounded on $\mathrm{Dom}(D)$. Recall that this family of operators is precisely $\mathrm{Dom}(\delta_D)$. A lingering question is when an operator $y \in \mathcal{B}(\mathcal{H})$ arises as the continuous extension of $[iD, x]|_{\mathrm{Dom}(D)}$ for some $x \in \mathcal{B}(\mathcal{H})$, which, by Christensen's work, is simply when $y \in \mathrm{Range}(\delta_D)$.

If $y \in \ker \delta_D$ is nonzero, then $y \notin \mathrm{Range}(\delta_D)$. Indeed, if $y = \delta_D(x)$ for some $x \in \mathrm{Dom}(\delta_D)$, then $\delta_D^2(x) = \delta_D(y) = 0$. Thus, $x \in \ker \delta_D^2$, which, by Theorem 1.2, implies $x \in \ker \delta_D$. This contradicts the assumption that $\delta_D(x) = y \neq 0$, so $\ker \delta_D \cap \mathrm{Range}(\delta_D) = \{0\}$. We are led to ask:

(1) If we extended $\delta_D$ to act on unbounded operators that are *affiliated* with $\mathcal{B}(\mathcal{H})$, would kernel stabilization for the extension $\Delta_D$ of $\delta_D$ still hold?

(2) Would operators in $\ker \Delta_D$ be weakly $D$-antidifferentiable if we allow for antiderivatives to be unbounded operators which are affiliated to $\mathcal{B}(\mathcal{H})$?

Our resounding answer to (1) is "no," and consequently our answer to (2) is "yes." Let

$P$ be the momentum operator on $L^2(\mathbb{R})$ defined in Example 2.9, and let $Q$ be the position operator on $L^2(\mathbb{R})$ defined in Example 2.6. Recall that the domains of $P$ and $Q$ contain the class of Schwartz functions $S(\mathbb{R})$, which is a core for both $P$ and $Q$. Let $\mathscr{C}$ be any common core for $P$ and $Q$. Ideally, we would define $\Delta_P$ so that $Q \in \text{Dom}(\Delta_P)$, and

$$\Delta_P(Q)|_{\mathscr{C}} = [iP, Q]|_{\mathscr{C}} = I|_{\mathscr{C}}.$$

As $\mathscr{C}$ is dense in $L^2(\mathbb{R})$, we have $\Delta_P(Q) = I$, but $\Delta_P^2(Q) = \Delta_P(I) = 0$, so $\ker \Delta_P^2 \neq \ker \Delta_P$. Furthermore, we could say that a weak $P$-antiderivative of $I$ is $Q$, or more generally, $Q + y$ where $y$ is any element of $\ker \Delta_P$.

The notion of defining or extending a derivation on an algebra $\mathcal{A}$ of bounded operators to unbounded operators which are affiliated with $\mathcal{A}$ is studied in [11] of R. Kadison and Z. Liu. Specifically, Kadison and Liu consider the extensions of an arbitrary derivation $\delta$ on a von Neumann algebras $\mathcal{M}$ to a derivation $\Delta$ on the affiliated *Murray-von Neumann algebra* $\mathscr{A}_{\text{f}}(\mathcal{M})$. The definition of their extended derivation in the case when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\delta = \delta_D$ may be a fruitful place to begin in the quest for $\Delta_D$.

## 6.2 Further Generalizations of the Stone-von Neumann Theorem

Thanks to D. Pitts, the Covariant Stone-von Neumann Theorem has an interesting interpretation we've not yet explored. Given a $C^*$-dynamical system $(G, \mathcal{K}(\mathcal{H}), \alpha)$, note that for each $x \in G$, $\alpha_x \in \text{Aut}(\mathcal{K}(\mathcal{H}))$ must be implemented by unitary conjugation, i.e., there exists a unitary $U_x \in \mathcal{B}(\mathcal{H})$ such that $\alpha_x(a) = U_x a U_x^*$ for all $a \in \mathcal{K}(\mathcal{H})$. While $\{\alpha_x\}_{x \in G}$ is a norm-continuous group, the family $\{U_x\}_{x \in G}$ need not form a group. It does, however, satisfy a *2-cocycle* condition: $U_x U_y = \sigma(x, y) U_{xy}$ for all $x, y \in G$, where $\sigma : G \times G \to \mathbb{T}$ is a 2-cocycle. Then, the representation $G \to \mathcal{U}(\mathcal{H})$ given by $x \mapsto U_x$ defines a *projective unitary group representation*. So, we could consider our classification of representations of dynamical systems of the form $(G, \mathcal{K}(\mathcal{H}), \alpha)$ as a classification of projective unitary group representations.

Delving more deeply into this interpretation may offer some insight on how we can extend our Covariant Stone-von Neumann Theorem without attempting to replace $\mathcal{K}(\mathcal{H})$ with a more general $C^*$-algebra. On the other hand, if $\mathcal{A}$ were a $C^*$-algebra such that any nondegenerate $*$-representation of $\mathcal{K}(\mathsf{L}^2(G, \mathcal{A}, \alpha))$ decomposed as in Theorem 5.14, our statement of Theorem 1.11 would hold if we replaced $\mathcal{K}(\mathcal{H})$ with $\mathcal{A}$. Identifying $C^*$-algebras with this desirable representation property may require tools such as Morita equivalence and KK-theory.

As an application of Theorem 1.11 in its current form, we are able to classify all pairs of self-adjoint operators $(A, B)$ on a Hilbert $\mathcal{K}(\mathcal{H})$-module $\mathsf{X}$ which satisfy the HCR on some dense $\{A, B\}$-analytic $\mathcal{K}(\mathcal{H})$-submodule of $\mathsf{X}$. This extends Huang's main result in [9], and will appear in an article on the `arXiv` this summer.

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