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Linda S. Fosnaugh
Midwestem State University

Earl S. Kramer
University of Nebraska- Lincoln

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LINDA S. FOSNAUGH

Division of Mathematics
Midwestern State University
3410 Taft Boulevard
Wichita Falls, TX 76308-2099

and

EARL S. KRAMER

Department of Mathematics
University of Nebraska
Lincoln, Nebraska 68588-0323

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ABSTRACT. The neighborhood $N(T)$ of a tile T is the set of all tiles which meet T in at least one point. If for each tile T there is a different tile T_1 such that $N(T) = N(T_1)$ then we say the tiling has the neighborhood property (NEBP). Grünbaum and Shepard conjecture that it is impossible to have a monohedral tiling of the plane such that every tile T has two different tiles T_1, T_2 with $N(T) = N(T_1) = N(T_2)$. If all tiles are convex we show this conjecture is true by characterizing the convex plane tilings with NEBP. More precisely we prove that a convex plane tiling with NEBP has only triangular tiles and each tile has a 3-valent vertex. Removing 3-valent vertices and the incident edges from such a tiling yields an edge-to-edge planar triangulation. Conversely, given any edge-to-edge planar triangulation followed by insertion of a vertex and three edges that triangulate each triangle yields a convex plane tiling with NEBP. We exhibit an infinite family of nonconvex monohedral plane tilings with NEBP. We briefly discuss tilings of R^3 with NEBP and exhibit a monohedral tetrahedral tiling of R^3 with NEBP.

KEY WORDS AND PHRASES. Tiling and neighborhood.
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1. INTRODUCTION.

A plane tiling denoted by $\mathcal{T}$ is a countable family of closed sets which cover the plane without gaps or overlaps. Tiles can intersect along their boundaries but interiors are disjoint. We assume all of our tiles are (closed) topological disks. A tiling $\mathcal{T}$ is convex if all tiles are convex, and monohedral if every tile in $\mathcal{T}$ is congruent to a fixed tile T, which is called the prototile of $\mathcal{T}$. The neighborhood $N(T)$, in a given tiling, is the set of tiles consisting of T and all tiles which meet T in at least one point. If for each tile T_1 there is a different tile T_2 such that $N(T_1) = N(T_2)$, then we call this the neighborhood property (NEBP). If our tiles are polygons then a tiling is edge-to-edge if the intersection of every pair of nondisjoint tiles is either a vertex or an edge. Grünbaum and Shephard [5] posed the following problem:

Show that there is no monohedral tiling, in which the prototile is a polygonal disk, such that for tile T there exist two other tiles T_1 and T_2 such that $N(T) = N(T_1) = N(T_2)$. We show this is true if the tiles are convex by characterizing the convex plane tilings with the NEBP.

2. CONVEX PLANE TILINGS.

In R^n a cone of support to a closed set A at a point p on its boundary is the intersection of all closed half spaces which contain A and which contain p in their boundary hyperplanes. Let S be a convex set. A point x in S is called an extreme point of S if there exists no nondegenerate line segment in S that contains x in its relative interior. The set of extreme points of S is called the profile of S. It is a theorem that if S is a compact convex subset of R^n, then S is the convex hull of its profile. If $\mathcal{P}$ is a set of points then we let $conv\mathcal{P}$ denote the convex hull of $\mathcal{P}$.

Here we assume that our tiles are convex and (as stated previously) topological disks. Let T_1 and T_2 be different tiles for which $N(T_1) = N(T_2)$. They clearly meet and since they are convex they can be separated by a straight line L. We assume that L is oriented to be the x-axis, and T_1 is above the x-axis. Let μ be a point of T_1 that is at maximum distance from the x-axis. Such a μ exists since T_1 is compact. If a nondegenerate line segment L' in T_1 contained μ in its relative interior or if the supporting cone at μ were a half-space, then there would be a tile T_3 touching T_1 at μ where $T_3 \notin N(T_2)$. Thus the point μ is an extreme point of T_1 and the supporting cone at μ is not a half-space. Let L_1 and L_2 be two rays forming the support cone at μ. It is clear that these two rays L_1 and L_2 must intersect L. Also, the convexity of our tiles implies that at least three tiles meet at μ.

Suppose T_3 and T_4 meet T_1 at μ and are separated from T_1 by lines L_3 and L_4, respectively. Arguing symmetrically for T_2, there would be a T_3' and a T_4' which would meet T_2 at an extreme point μ', and which are separated from T_2 byL_3' and L_4'. It is then elementary to establish that the lines L, L_3, L_3' intersect in a common point p_3, and $p_3 \in T_1 \cap T_2 \cap T_3 \cap T_3'$. Analogously, L, L_4, L_4' intersect in a common point p_4, and $p_4 \in T_1 \cap T_2 \cap T_4 \cap T_4'$.

Now $\{p_3, p_4 \mu\} \subset T_1, T_1$ is convex, and T_1 is bounded by L_3, L_4, L, hence T_1 is the triangle with vertices μ, p_3 and p_4. Analogously, T_2 is a triangle with vertices μ', p_3 and p_4. Clearly T and T' share an edge. Suppose another tile T_5 met T_1 at μ. Then T_5 could meet T_2 only at p_3 or p_4. But this is clearly impossible because of the location of T_3, T_4 and the fact that T_5 is convex. We have proven:

PROPOSITION 2.1. Let $\mathcal{T}$ be a convex plane tiling with NEBP. Then the tiles of $\mathcal{T}$ are triangles. If $T \neq T'$ and $N(T) = N(T')$, then T and T' share an edge. In addition, exactly three tiles meet at vertex $\mu(\mu')$ of $T(T')$ which is not on the shared edge.

PROPOSITION 2.2. The tiling is edge-to-edge.

PROOF. Let e_3 be the edge of T_3 and e_1 be the edge of T_1 which are both contained in the line L_3 that separates T_3 and T_1. The proof of Proposition 2.1 shows that $e_1 \subset e_3$. If $N(T_3) = N(T_1)$, then $e_3 = e_1$ by Proposition 2.1. If $N(T_3) \neq N(T_1)$, let T_3 play the role of T_1 in Proposition 2.1 to get that $e_3 \subset e_1$. Thus $e_3 = e_1$. A symmetric argument shows that the edge of T_1 and that of T_4 that lie along L_4 must coincide. The result follows since T_1 is an arbitrary tile.

PROPOSITION 2.3. Let $\mathcal{T}$ be a convex plane tiling with NEBP. Then each tile (which is a triangle) has exactly one 3-valent vertex and the other two vertices are at least 6-valent. Further, the edge opposite the 3-valent vertex in a tile is shared by two tiles whose neighborhoods are equal.

PROOF. By Proposition 2.1 the valency of vertex μ in T_1 is 3 and since tiles are convex, the third vertex of T_3, besides p_3 and μ, must be at a strictly greater distance from L than μ. Let this vertex be p_5 and let the analogous additional vertex of T_3', besides p_3 and μ', be p_5'. Note that $T_3 = conv\{\mu, p_3, p_5\}$ and $T_3' = conv\{\mu', p_3, p_5'\}$, $T_4 = conv\{\mu, p_4, p_5\}$ and $T_4' = conv\{\mu', p_4, p_5'\}$. But there is no tile with vertices p_3, p_5, p_5' since such a tile would have no vertex of valency three. Hence the valency of p_3, and analogously p_4, must be at least 6. The final statement of our result is then immediate.

Define a *plane triangulation refinements* as follows Begin with any edge-to-edge tiling of the plane by triangles. Insert a point p_T in the interior of each triangle T and draw an edge from p_T to each of the vertices of T. Do this for each of the tiles in the original plane triangulation.

THEOREM 2.1. A convex plane tiling with NEBP is equivalent to a plane triangulation refinement.

PROOF. A plane triangulation refinement is easily checked to be a convex plane tiling with NEBP. Conversely, let T be a convex plane tiling with NEBP. Delete all vertices that are initially 3-valent and the incident edges. By Proposition 2.3 every tile in $\mathcal{T}$ is removed by this process. What remains is an edge-to-edge triangulation of the plane Using the discarded 3-valent vertices as the p_T's gives a refinement that recreates $\mathcal{T}$. Our result is clear.

PROPOSITION 2.4. For any fixed positive integer k there is a convex plane tiling with NEBP using exactly k different tile shapes

PROOF. Begin with an edge-to-edge tiling of the plane with equilateral triangles. For $k = 1$ we produce a refinement by choosing p_T to be the centroid of each triangle. For $k = 2$ we produce a refinement by choosing p_T on the line through the centroid and perpendicular to the base of T, but p_T is not set at the centroid. For $k = 3$ choose p_T so three distinct triangles tile T. Clearly, by choosing the placement of the p_T's we can produce a refinement with any given positive number of k distinct tile shapes.

3. CONVEX MONOHEDRAL PLANE TILINGS WITH NEBP.

We are now in position to completely characterize the monohedral convex plane tilings with NEBP.

THEOREM 3.1. There is a unique monohedral convex plane tiling $\mathcal{T}$ with NEBP.

PROOF. We need a simple fact that is easy to verify. Let T be a triangle and let T_1, T_2 and T_3 be three triangles that tile T edge-to-edge. If T_1 is congruent to T_2 then T_3 is an isosceles triangle. If all three tiles are equal, then each is the triangle with angles 30°, 30° and 120°. It is now clear that removal of all 3-valent vertices from $\mathcal{T}$ along with incident edges produces an edge-to-edge tiling of the plane with equilateral triangles. Such a tiling is unique, so our tiling $\mathcal{T}$ is obviously unique. This tiling is exhibited in Figure 3.1.

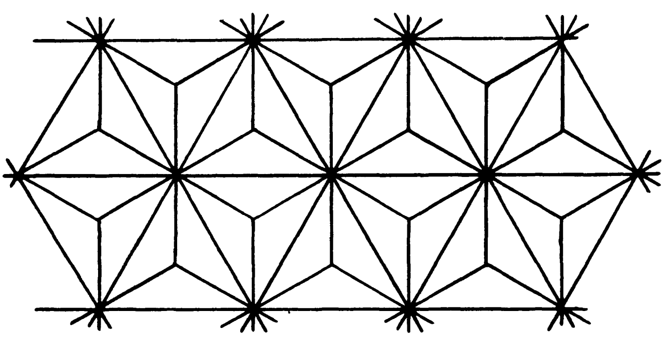

Figure 3 1.

The automorphism group of a tiling $\mathcal{T}$ is the set of isometrics that preserve the tiling Such a group partitions the tiles into transitivity classes where any two tiles within a class can be mapped to each other via some automorphism. A tiling $\mathcal{T}$ is isohedral if there is only one transitivity class. The plane isohedral tilings are completely known (see [3] or [4]) and among the isohedral plane tilings only the tiling of Figure 3.1 has NEBP. Thus we have:

PROPOSITION 3.1. There is a unique isohedral plane tiling with NEBP

Note that Theorem 3.1 and Proposition 3.1 prove the Grünbaum and Shephard conjecture (stated in the introduction) for plane tilings when the tilings are either convex or isohedral.

4. NONCONVEX PLANE TILINGS WITH NEBP.

One simple example of a nonconvex plane tiling with NEBP is given as follows: With center C on a straight line L draw circles with varying radii that become arbitrarily large The tiles are the simply-connected regions. The pair of tiles sharing the same two edges lying along L will have the same neighborhood.

We now produce monohedral nonconvex plane tiling with NEBP Figure 4 1 shows how to "dent in" the edges of each triangle in the tiling of Figure 3.1 so that the tiling is still monohedral.

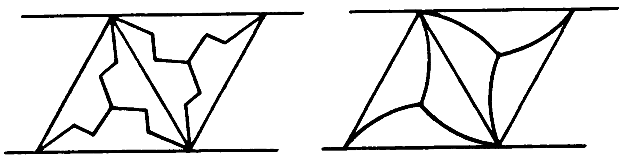

Figure 4.1.

The following lemma is an easy exercise and the subsequent proposition is immediate.

LEMMA 4.1. If the denting process is applied to Figure 3.1 to produce a monohedral polygonal nonconvex plane tiling with NEBP, then each tile has an odd number of sides.

PROPOSITION 4.1. For each odd integer $k \geq 3$, there is a monohedral plane tiling with NEBP using tiles with k sides.

One has to wonder here whether all monohedral plane tilings with NEBP arise from Figure 3.1 by a suitable denting process. From the proof of Proposition 2.4 and the "denting" procedure we have:

THEOREM 4.1. For any fixed positive integer k there is a nonconvex plane tiling with NEBP using exactly k different tile shapes.

5. SOME TILINGS OF R^3 WITH NEBP.

We can extend a tiling of the plane to a tiling of R^3 by a natural lifting process (mentioned to us by B. Grünbaum). Let $\mathcal{T}$ be a plane tiling (taken to be in the xy-coordinate plane) and d a fixed positive distance. Define $L(\mathcal{T}, d)$ as follows: for each T in $\mathcal{T}$, construct a prism with base T and height d. This produces a slab and now we stack such slabs (face-to-face) to fill R^3. Our next result is easy to verify.

PROPOSITION 5.1. Let $\mathcal{T}$ be a plane tiling with NEBP. Then $L(\mathcal{T}, d)$ is a tiling of R^3 with NEBP.

From Proposition 4.1 and Proposition 5.1, we immediately have:

COROLLARY 5.1. For each odd positive integer $k \geq 5$ there is a monohedral tiling of R^3 with NEBP where the tiles have exactly k faces.

We generalize the refinement process described prior to Theorem 2.1. Begin with a face-to-face tiling $\mathcal{T}$ or R^3 using tetrahedral and/or hexahedra with triangular faces. Insert a point p_T in the interior of each tile T. Clearly T can be subdivided into tetrahedra that are determined by p_T and each face of T. Do this for each T in $\mathcal{T}$ to get a refinement of the original face-to-face tiling. The following theorems is easy to verify:

THEOREM 5.1. The refinement of a face-to-face tiling of R^3 using tetrahedra and/or hexadehra with triangular faces, is a tetrahedral tiling of R^3 with NEBP.

A tetrahedron is isosceles if opposite edges are equal. An isosceles tetrahedron clearly has congruent faces. Also, a tetrahedron is isosceles iff the medians are equal, i e., iff the centroid is equidistant from the vertices of the tetrahedron.

THEOREM 5.2. If there is a monohedral face-to-face tiling of R^3 with isosceles tetrahedra, then there is a monohedral face-to-face tiling with tetrahedra of R^3 with NEBP.

PROOF. The medians of the isosceles tetrahedra are equal and the faces are congruent. If we choose p_T to be the centroid of a given tetrahedron, then the resulting tetrahedra in the refinement will be congruent. Our result follows by Theorem 5.1.

The isosceles tetrahedron T_0 with side lengths 2, 2, $\sqrt{3}$, $\sqrt{3}$, $\sqrt{3}$, and $\sqrt{3}$, tiles E^3 face-to-face (see [2]). In Figure 5.1 three copies of T_0 are shown to tile

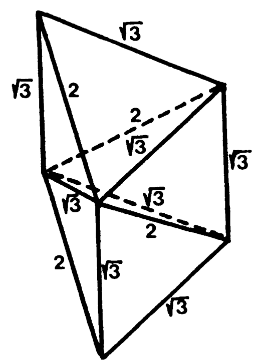

Figure 5.1.

a prism. This prism can easily be stacked and translated to produce a face-to-face tiling of R^3 using T_0. Using Theorem 5.2 the following is then clear:

COROLLARY 5.2. There is a monohedral tetrahedral tiling of R^3 with NEBP.

6. FINAL REMARKS.

One question posed earlier is whether all monohedral plane tilings with NEBP arise from Figure 3.1 by a suitable denting process. Another natural question is whether there is a succinct characterization of convex tilings of R^3 with NEBP as there was for convex plane tilings with NEBP.

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