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ABSTRACT

An optimal method is developed for adaptive multistage image transform coding. The optimality is in the sense of minimizing the mean square reconstruction error with a given total number of bits and a given number of stages. The statistics of the coefficients in different stages and marginal analysis are used to optimize the division of the total number of bits among the stages. Experimental results indicate that, with two stages, more than 14% improvement for one class and more than 10% improvement for multiclasses is achieved in mean square reconstruction error over one stage image transform coding. Higher improvements are achieved with three stages. The reconstructed images with multistage coding are subjectively much more preferable than the reconstructed images with one-stage coding.

1. INTRODUCTION

Transform coding is widely used in coding of images since it gives a very high compression ratio. The effectiveness of transform coding has a lot to do with the properties of decorrelating the pixel values and packing the energy of the signal in a few transform coefficients. Based on these two criteria, the Karhunen-Loeve transform (KLT) is the optimal transform for image coding [1]. However, the KLT is signal dependent and difficult to compute in real time. The discrete cosine transform (DCT) is among the best fast transforms to approximate the KLT in image coding [2]. One technique for improving the efficiency of image coding is to apply adaptivity to the coding procedure by classifying image blocks in a number of classes. An efficient adaptive algorithm for image coding was proposed by Chen and Smith [3].

In this paper, we discuss optimization techniques of transform image coding in the form of a multistage procedure in which the error signal resulting from the quantization of the previous stage is input to the following stage. Multistage transform coding thus involves transform domain quantization in a number of stages such that each stage attempts to correct the errors in the previous stage. The technique to be discussed is different from progressive image coding even though there is some degree of similarity. In progressive image coding, first a low-grade version of the image is sent, and then the image is refined by sending more information in the following stages. A number of different techniques for progressive image coding in both spatial and transform domains have been discussed by Tzou [4], Wang and Goldberg [5,6]. In this technique, the coefficients of each stage are quantized by a predetermined average rate and the number of stages are increased until satisfactory image reconstruction is obtained at the receiver. So far, no adaptive method has been reported in order to adjust the number of bits for each stage based on the statistics of the coefficients of different stages, and for a total given bit rate.

The method to be discussed in this paper involves optimal adaptive multistage transform coding with a fixed total number of bits per pixel and a fixed number of stages. It is optimal in the sense that it minimizes the total final reconstruction error with the given total number of bits and stages. The statistics of the coefficients in different stages are used to optimize the division of the total number of bits among different stages. The adaptivity introduced does not significantly add to the complexity of the coding system since it utilizes the information that is necessary for any kind of multistage transform coding. Simulation results have shown a considerable percentage decrease in reconstruction error despite the simplicity of the coding scheme. In addition, the remaining error image is more noise-like than the error image in one-stage coding, especially with reduced error around the edges. This is believed to be the main reason why multistage transform coding gives subjectively much more pleasing results than one-stage coding at the same bit rate.

There are a number of subjective and objective error measures to quantify the quality of image reconstruction, but the mean square error (MSE) is the most widely used. The MSE is also the measure in this paper to be used to compare experimental results. However, the experimental results will also be discussed in terms of subjective performance.

The paper consists of 6 sections. In Sec. 2, the new proposed method is introduced, and a mathematical expression is derived for the total final reconstruction error which is to be minimized during bit allocation and coding. This expression is based on the mean square error. The optimal bit allocation for different stages to minimize the quantization error is explained in Sec. 3 by using the statistics of the coefficients in different stages. In Sec. 4, the experimental results with the discrete cosine transform (DCT) are discussed with a number of images, and rates, as well as with one class and multiclass adaptive procedures. Sec. 5 is some discussion of problems facing the implementation of multistage transform coding. Sec. 6 is conclusions.

2. OPTIMAL ADAPTIVE MULTISTAGE IMAGE TRANSFORM CODING

The block diagram for adaptive multistage transform coding is shown in Fig. 1. The transform coefficients of the first stage are assumed to have little correlation so that they are quantized and coded independently with an optimal bit map for the first stage as explained below. The two dimensional error signal resulting from the first stage quantization is fed to the second stage and subsequently quantized and coded with another optimal bit map. This procedure is continued for the given total number of stages.

Next, we derive a mathematical expression for the total final reconstruction error based on the mean square error (MSE) measure. We assume that unitary transforms are used for transform coding. Then, the variance of the reconstruction error is equal to that introduced during the quantization of coefficients [7].

Referring to Fig. 1, the following notations are defined:

n : The number of stages.

E_k : The coefficient matrix of size N×N as input to stage k+1, $k = 0, 1, ..., n-1$.

$\hat{E}_k$: The matrix of size N×N for the quantized coefficients as output of stage $k+1$, $k = 0, 1, ..., n-1$.

e_{kij} : The ijth coefficient of the matrix E_k.

$\hat{e}_{kij}$: The ijth coefficient of the matrix $\hat{E}_k$.

b_{kij} : The number of bits used to quantize e_{kij}.

$f_k(b_{kij})$: The mean square distortion of the b_{kij}-bit quantizer for unity variance input (see Sec. 3).

σ^2_{kij}: The variance of e_{kij}.

$\hat{\sigma}^2_{kij}$: The variance of $\hat{e}_{kij}$.

There are different kinds of quantizers such as optimum mean square (Lloyd-Max) and uniform optimal quantizer [8]. The optimum mean square quantizer is used in this paper. Suppose e_{kij} and $\hat{e}_{kij}$ are the input and the output of the optimum mean square quantizer. They have the following properties [9]:

$$E(e_{kij}) = E(\hat{e}_{kij}) \rightarrow E(e_{kij} - \hat{e}_{kij}) = E(e_{(k+1)ij}) = 0 \tag{1}$$

$$E(e_{kij}\hat{e}_{kij}) = E(\hat{e}^2_{kij}) \rightarrow E\left[\hat{e}_{kij}(e_{kij} - \hat{e}_{kij})\right] = 0 \tag{2}$$

$$\hat{\sigma}^2_{kij} = \left[1 - f_k(b_{kij})\right]\sigma^2_{kij} \tag{3}$$

The following equation can be derived from Eqs. (1),(2) and (3):

$$E\,(e_{kij}\hat{e}_{kij}) = E\,(\hat{e}^2_{kij}) = \hat{\sigma}^2_{kij} = \left[1 - f_k(b_{kij})\right]\sigma^2_{kij} \quad \text{for } k = 1, 2, ..., n-1. \tag{4}$$

Referring to Fig. 1, the final reconstructed image is formed by taking the inverse transform of $(\hat{E}_0 + \hat{E}_1 + \cdots + \hat{E}_{n-1})$. Therefore the mean square error (MSE) is given by

$$\mathrm{MSE} = \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E\left[e_{0ij} - (\hat{e}_{0ij} + \hat{e}_{1ij} + \cdots + \hat{e}_{(n-1)ij})\right]^2. \tag{5}$$

For simplicity, we begin with the cases of n = 2 and n = 3, and then extend the results to any value of n. For n = 2, the expectation in Eq. (5) can be written as

$$E\left[e_{0ij} - (\hat{e}_{0ij} + \hat{e}_{1ij})\right]^2 = E\left[(e_{0ij} - \hat{e}_{0ij}) - \hat{e}_{1ij}\right]^2$$
$$= E\left[(e_{0ij} - \hat{e}_{0ij})^2\right] - E\left[\hat{e}_{1ij}(e_{1ij} - \hat{e}_{1ij})\right] - E(e_{1ij}\hat{e}_{1ij}). \tag{6}$$

The first expectation on the right side of Eq. (6) is the expectation of the squared error in the first stage as shown in Fig. 1. By using Eqs. (1), (2), (3), and the fact that the average value of the coefficients in the first stage, excluding the DC coefficient, is zero, it can be written as

$$E(e_{0ij}^2) + E(\hat{e}_{0ij}^2) - 2\,E(e_{0ij}\,\hat{e}_{0ij}) = E(e_{0ij}^2) - E(\hat{e}_{0ij}^2)$$
$$= \sigma_{0ij}^2 - \hat{\sigma}_{0ij}^2$$
$$= \sigma_{0ij}^2\, f_0(b_{0ij}) \tag{7}$$

The second expectation on the right side of Eq. (6) is zero according to Eq. (2), and the third expectation on the right side of Eq. (6) is equal to $\sigma_{1ij}^2 \left[1 - f_1(b_{1ij})\right]$ according to Eq. (4). Therefore the MSE for the case n = 2 is given by

$$\mathrm{MSE} = \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[\sigma_{0ij}^2\, f_0(b_{0ij}) + \sigma_{1ij}^2 \left[f_1(b_{1ij}) - 1\right]\right]. \tag{8}$$

For n = 3, the expectation in Eq. (5) is written as

$$E\left[e_{0ij}-(\hat{e}_{0ij}+\hat{e}_{1ij}+\hat{e}_{2ij})\right]^2 = E\left[(e_{0ij}-\hat{e}_{0ij}-\hat{e}_{1ij})-\hat{e}_{2ij}\right]^2$$

$$= E(e_{0ij}-\hat{e}_{0ij}-\hat{e}_{1ij})^2 + E\left[\hat{e}_{2ij}^2 - 2(e_{0ij}-\hat{e}_{0ij})\hat{e}_{2ij} + 2(\hat{e}_{1ij}\hat{e}_{2ij}).\right] \tag{9}$$

The first expectation on the right side of Eq. (9) is equal to the left side of Eq. (6), and its value is given by the expression inside the bracket in Eq. (8). Knowing that $e_{0ij} - \hat{e}_{0ij} = e_{1ij}$ and $e_{1ij} = e_{2ij} + \hat{e}_{1ij}$, the second expectation on the right side of Eq. (9) is found to be

$$E(\hat{e}_{2ij}^2 - 2e_{1ij}\hat{e}_{2ij} + 2\hat{e}_{1ij}\hat{e}_{2ij})$$

$$= E\left[\hat{e}_{2ij}^2 - 2(e_{2ij}+\hat{e}_{1ij})\hat{e}_{2ij} + 2\hat{e}_{1ij}\hat{e}_{2ij}\right]$$

$$= E(\hat{e}_{2ij}^2 - 2e_{2ij}\hat{e}_{2ij}) = E\left[\hat{e}_{2ij}(\hat{e}_{2ij}-e_{2ij})\right] - E(e_{2ij}\hat{e}_{2ij}) \tag{10}$$

$$= \sigma_{2ij}^2\left[f_2(b_{2ij})-1\right]$$

where Eqs. (2) and (4) were used in the last step.

Therefore the MSE in the case of n = 3 is given as

$$MSE = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left[\sigma_{0ij}^2\, f_0(b_{0ij}) + \sigma_{1ij}^2\left[f_1(b_{1ij})-1\right] + \sigma_{2ij}^2\left[f_2(b_{2ij})-1\right]\right]. \tag{11}$$

Eq. (11) can be easily extended to arbitrary n, and the result is

$$MSE = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left[\sigma_{0ij}^2\, f_0(b_{0ij}) + \sigma_{1ij}^2\left[f_1(b_{1ij})-1\right] + \cdots + \sigma_{(n-1)ij}^2\left[f_{n-1}(b_{(n-1)ij})-1\right]\right]. \tag{12}$$

Eq. (12) is the objective function that is to be minimized in order to achieve the minimum mean square error. The procedure of bit allocation in order to minimize this function is given in the next section.

3. ERROR MODELS AND OPTIMAL BIT ALLOCATION

In general, an analytic expression for the quantizer error is desirable. Usually, such an expression is given in terms of the variance of the input sequence to the quantizer, the number of bits (or the number of levels) used for quantization, and some parameters that depend on the distribution of the input. A closed form expression for the MSE is very difficult to derive, and most reported results have been obtained either by numerical or approximate means. In the case of the optimum mean square (Lloyd-Max) quantizer, the MSE is usually expressed in the form of $\sigma_{kij}^2 f_k(b_{kij})$ where $f_k(b_{kij})$ is a function of b_{kij} and the probability density function (pdf) of the input signal to the quantizer. One such expression for $f_k(b_{kij})$ for Gaussian distribution is given by [8]

$$f_k(b_{kij}) = \begin{cases} 1.32\,(2^{-1.74 b_{kij}}) & \text{for } b_{kij} \text{ around } 2, \\ 2.21\,(2^{-1.96 b_{kij}}) & \text{for } b_{kij} \text{ around } 5.17. \end{cases} \tag{13}$$

Another approximate model in the case of Gaussian distribution is given by [10]

$$f_k(b_{kij}) = \begin{cases} 2^{-1.57 b_{kij}} & \text{for } b_{kij} \le 2.32 \\ \dfrac{2.698\; 2^{2 b_{kij}}}{(2^{2 b^{kij}} + 0.8532)^3} & \text{for } b_{kij} \ge 2.32 \end{cases} \tag{14}$$

Some other error functions have been reported for both Gaussian and Laplacian distributions in Ref. [9] and is given in Table 1.

All of the above models are either for the Gaussian or the Laplacian distribution. In practice, the input to the quantizer usually has neither Gaussian nor Laplacian distribution exactly but some distribution close to one of them.

Recently it was reported that most of AC coefficients for the first stage of the DCT transform have Laplacian pdf [11]. It was also mentioned that the DC coefficients have a pdf close to Gaussian. We performed the Kolmogorov-Smirnov(K-S) [12] test for the coefficients of the second stage. The results indicate that the 8-bit quantization error for the DC coefficients has a pdf close to the uniform distribution. All the AC coefficients which have been allocated 2 or more bits (4 or more quantization levels) have a pdf close to Gaussian, and most coefficients which have been allocated 1 bit (2 quantization levels) have a pdf close to Gaussian as well. Of course, those coefficients which have not been allocated any bit at the first stage have a Laplacian pdf in most cases. Overall, a large number of coefficients which will receive non-zero bits in the second stage have a Gaussian pdf. Therefore in our simulations, we assume Gaussian pdf for the second stage and use the error model given in Table 1.

Having the error models for each stage, the total final error given in Eq. (12) can be minimized through an optimal bit allocation procedure. Coefficients in each stage usually have different variances, and their variances are also different from stage to stage. Therefore, different number of bits should be assigned to each coefficient. The major constraint that should be satisfied is that the total number of bits is fixed. There are a number of methods for bit allocation, and they are not necessarily optimal in minimizing the MSE. Some methods assume the number of bits (or the number of levels) to be a continuous variable in order to get an optimal and closed form expression, but the result has to be rounded to the nearest integer and is no longer optimal. The procedure for obtaining optimal non-integer number of bits was discussed in Ref. [13]. In this paper, we use marginal analysis described in Ref. [14] to develop an optimal method with integer number of bits. The piecewise error models given in Table 1 are strictly convex function and guarantee that the global minimum is achieved. Here, we give the necessary steps for bit allocation with 2 stages where the generalization for more stages being straightforward. The steps involved in bit allocation according to marginal analysis are as follows:

1. Set $b_{kij} = 0$, for $k = 0,1$ and $i,j = 0,1,...,N-1$.
2. Calculate the marginal return, Δ_{kij}, which is the reduction in the total final error given by Eq. (12) if 1 bit is assigned to the coefficient e_{kij}, for $k=0,1$, and $i,j=1,...,N$.
3. Allocate one bit to the coefficient e_{kij} which has the largest marginal return Δ_{kij}.

4. If the total number of assigned bits is equal to or greater than the total number of bits, stop; otherwise go to Step 2 to decide for the next bit.

If ties happen in step 3, the same procedure is repeated among the coefficients which have the same value for Δ_{kij} by assigning another bit to these coefficients and looking for the winner.

The above procedure for bit allocation can be applied to find the bit map that minimize the total final reconstruction error for the multistage transform coding if the multiclass adaptive method is not used for each stage. In deriving the estimated total final error given in Eq. (12), we assume that the coefficient e_{kij} is the resulting error of quantizing the coefficient $e_{(k-1)ij}$. On the other hand, if the multiclass adaptive method is used, the class map of each stage is possibly different, so the above assumption does not hold.

For multiclasses, we introduce another method of optimization to minimize the total error. In this method, we first derive a relation between the total average rate, R, and the average rate for each stage, R_k, $k = 0,...,n-1$. When the average rate of each stage is known, the bit allocation procedure for each stage can be done independently. It is also possible to use different number of classes for the following stages since the spectra in those stages are more flat than the first stage.

First we will find the relation between R and R_k, $k = 0,...,n-1$ for $n = 2$ (two stages). Then, we will show that for $n \geq 3$ the procedure is straightforward. For $n = 2$, the problem is

$$\begin{cases} \text{minimize} \quad \dfrac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[\sigma_{0ij}^2 \, f_0(b_{0ij}) + \sigma_{1ij}^2 \left[f_1(b_{1ij}) - 1 \right] \right] \\[2ex] \text{subject to} \quad \dfrac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[b_{0ij} + b_{1ij} \right] = R. \end{cases} \tag{15}$$

The variances σ_{1ij}^2 depend on the bits b_{0ij}, allocated to the first stage, which is not known in advance. In order to get around this problem, we will first assume that the variances of the coefficients of both stages, σ_{0ij}^2 and σ_{1ij}^2, for $i,j = 1,...,N-1$, are available. Based on this, we will derive the optimal bit rates for the two stages. Once the rates are known, the new values of σ_{1ij}^2 will be computed. The process is iterated with these new values until the optimum point is reached. In practice, we found that two or three iterations are sufficient.

In the analysis, we will assume that b_{0ij} and b_{1ij} are continuous. Since we are looking for an analytical expression for the rates R_0 and R_1, the error functions $f_0(.)$ and $f_1(.)$ must be known. The piecewise functions given in Table (1) can be used for marginal analysis bit allocation, but it is not easy to use them in the above minimization problem. Instead, we try to approximate these functions with another function in the form of $f_k(b_{kij}) = 2^{-B_k b_{kij}}$, for $k = 0,1$. We choose the single parameter B_k such that the proposed function is the closest approximation to the corresponding piecewise model in the least mean square sense, or any other kind of measure. Figs. 2 and 3 show the approximation for particular B_k, which is used in this paper. It is observed that the fit is very close. The problem can now be restated in the following form:

$$\begin{cases} \text{minimize} \quad \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[\sigma_{0ij}^2 \, 2^{-B_0 b_{0ij}} + \sigma_{1ij}^2 \left(2^{-B_1 b_{1ij}} - 1 \right) \right] \\ \text{subject to} \quad \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left(b_{0ij} + b_{1ij} \right) = R. \end{cases} \tag{16}$$

We use the Lagrange multiplier method [15] for this optimization problem. Thus,

$$\frac{\partial}{\partial b_{0ij}} \left[\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[\sigma_{0ij}^2 \, 2^{-B_0 b_{0ij}} + \sigma_{1ij}^2 \left[2^{-B_1 b_{1ij}} - 1 \right] \right] - \lambda \left[R - \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left(b_{0ij} + b_{1ij} \right) \right] \right] = 0 \tag{17}$$

and

$$\frac{\partial}{\partial b_{1ij}} \left[\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left[\sigma_{0ij}^2 \, 2^{-B_0 b_{0ij}} + \sigma_{1ij}^2 \left[2^{-B_1 b_{1ij}} - 1 \right] \right] - \lambda \left[R - \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \left(b_{0ij} + b_{1ij} \right) \right] \right] = 0. \tag{18}$$

Eqs. (17) and (18) lead to the following equations, respectively;

$$\lambda = A \, \sigma_{0ij}^2 \, B_0 \, 2^{-B_0 b_{0ij}} \tag{19}$$

and

$$\lambda = A\, \sigma_{1ij}^2\, B_1\, 2^{-B_1 b_{1ij}}\,, \tag{20}$$

where A is some constant.

Using Eqs. (19) and (20), we write

$$b_{0ij} = \frac{B_1}{B_0} b_{1ij} + \frac{1}{B_0}\left[\log_2\left[\frac{\sigma_{0ij}^2}{\sigma_{1ij}^2}\right] + \log_2\left[\frac{B_0}{B_1}\right]\right]. \tag{21}$$

By taking the summation of both sides of Eq. (21), and defining

$$S_0 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\log_2\sigma_{0ij}^2 \tag{22}$$

and

$$S_1 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\log_2\sigma_{1ij}^2 \tag{23}$$

we obtain

$$R_0 = \frac{B_1}{B_0} R_1 + \frac{1}{B_0}\left[S_0 - S_1 + \log_2\left[\frac{B_0}{B_1}\right]\right] \tag{24}$$

where

$$R_0 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} b_{0ij} \tag{25}$$

and

$$R_1 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} b_{1ij}\,. \tag{26}$$

Applying the constraint given in Eq. (16), which is equivalent to $R_0 + R_1 = R$, to Eq. (24), the average rate for the first stage becomes

$$R_0 = \frac{B_1}{B_0 + B_1} R + \frac{1}{B_0 + B_1}\left[S_0 - S_1 + \log_2\left[\frac{B_0}{B_1}\right]\right] \tag{27}$$

Then, R_1 is found as $R - R_0$.

Extending the above procedure to the case $n = 3$ is easy. Suppose we can approximate the error function model of the third stage by $f_2(b_{2ij}) = 2^{-B_2 b_{2ij}}$, for some B_2 (see Sec. 5.) Then, similar to Eq. (24), the following equation is derived:

$$R_1 = \frac{B_2}{B_1} R_2 + \frac{1}{B_1}\left[S_1 - S_2 + \log_2\left[\frac{B_1}{B_2}\right]\right] \tag{28}$$

where

$$R_2 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} b_{2ij} \tag{29}$$

and

$$S_2 = \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \log_2 \sigma^2_{2ij} \tag{30}$$

Solving Eqs. (24) and (28) with the constraint $R_0 + R_1 + R_2 = R$ results in the following relations for R_0 and R_1:

$$\begin{aligned} R_0 = {} & \frac{B_1 B_2}{B_0B_1 + B_0B_2 + B_1B_2} R + \frac{B_1 + B_2}{B_0B_1 + B_0B_2 + B_1B_2}\left[S_0 + \log_2 \frac{B_0}{B_1}\right] \\ & + \frac{B_1}{B_0B_1 + B_0B_2 + B_1B_2}\left[\log_2 \frac{B_1}{B_2} - S_2\right] \\ & - \frac{B_2}{B_0B_1 + B_0B_2 + B_1B_2} S_1 \end{aligned} \tag{31}$$

and

$$\begin{aligned} R_1 = {} & \frac{B_0 B_2}{B_0B_1 + B_0B_2 + B_1B_2} R - \frac{B_2}{B_0B_1 + B_0B_2 + B_1B_2} S_0 \\ & + \frac{B_0 + B_2}{B_0B_1 + B_0B_2 + B_1B_2} S_1 - \frac{B_0}{B_0B_1 + B_0B_2 + B_1B_2} S_2 \\ & - \frac{B_2}{B_0B_1 + B_0B_2 + B_1B_2} \log_2 \frac{B_0}{B_2} \\ & + \frac{B_1}{B_0B_1 + B_0B_2 + B_1B_2} \log_2 \frac{B_1}{B_2} \end{aligned} \tag{32}$$

Again, R_2 is found as $R - R_0 - R_1$.

The above procedure can be generalized to any number of stages. Once the average bit rates for each stage are known, the bit allocation for each stage can be done independently using the marginal analysis for any desired number of classes.

4. EXPERIMENTAL RESULTS

The multistage transform coding technique discussed above was applied to an image of size 256×256 shown in Fig. 4 (a) and another image of size 128×128 which is the head and shoulder part of the previous image. Both images were quantized with 8 bits (256 levels.) We will refer to these two images as "girl256" and "girl128", respectively. The number of stages used were either 2 or 3. The two-dimensional DCT was used as the unitary transform. Coding was carried out with a block size of 16×16. We compared multistage transform coding with one stage coding. The adaptive coding technique of Chen and Smith [Chen] with 4 classes was used for each stage. The total rates used were 1.0, 0.5 and 0.25 bits per pixel (bpp).

For the first stage, the optimum mean square error quantizer for Laplacian distribution was used. The DC coefficient in the first stage was quantized by the optimum mean square error quantizer for Gaussian distribution. The optimum mean square quantizer for Gaussian distribution was also used for the second stage. This choice was based on the statistical tests explained in Sec. 3.

For two stages with one class (without using Chen and Smith adaptive method), the total number of bits were allocated according to the marginal analysis method discussed in Sec. 3 to minimize the total final error function given by Eq. (12). For this case, two scheme are possible. Either the variances of the second stage, σ_{1ij}^2, can be estimated by the known variances of the first stage by $\sigma_{1ij}^2 = \sigma_{0ij}^2 \, f_0(b_{0ij})$, or we can start from initial rates for the first and second stages and then iterate once the variances of the second stage are known. Our experiments showed that the second scheme is not as efficient as the first scheme. In addition, the first scheme is much better in terms of computational cost. Therefore, we chose the first scheme.

For the two-stage multiclass adaptive method, we used Eq. (27) to allocate the total bits between two stages. In this case, we started with the initial rates $R_0 = R$ and $R_1 = 0$. This choice was based on our observation that, for optimum rate division, R_0 is always greater than R_1. In most cases two iterations were sufficient to get the optimum rates R_0 and R_1.

difference images for the one stage coding. In addition, the differences on and around the edges are less for the multistage coding case.

In the two-stage experiments, we tested all possible combinations of R_0 and R_1 for rates equal to 0.5 and 0.25 bits per pixel, and some of the results are shown in Figs. 6. It is clear that the optimum points are very close to what we found by either minimizing Eq. (12) directly for the one-class case or dividing the total rate by Eq. (27) for the multi-class case.

We also tested three stages, at the total rate 0.5 bpp and 4 classes, with the image shown in Fig. 4 (a). The results showed 13.88% improvement over one stage. This is 5.33% more than the improvement with two stages, and the same type of improvement is expected for other cases.

5. DISCUSSION

As mentioned in Sec. 4, a large number, but not all, of coefficients in the second stage have a pdf close to Gaussian. Since one kind of pdf is usually assumed during quantization, we chose the Gaussian pdf in Secs. 3 and 4. However, more than one choice of pdf is possible, but it increases the overhead information that should be known during decoding. Thus, for one kind of pdf assumption, we have the error of mismatch between the assumed pdf and the real pdf for some coefficients. This error was experimentally studied by Mauersberger [17]. The reported results show that the error resulting from using Gaussian quantizer for a random variable with Laplacian pdf is more than the error resulting from using Laplacian quantizer for a random variable with Gaussian pdf (assuming the same variance and number of levels.) In practice, the total error depends on the number of mismatch cases. For the third stage in multistage image transform coding, our statistical tests showed that the coefficients have a mixture of uniform, Gaussian and Laplacian pdf. Again, since more than half of them had Gaussian pdf, we used the Gaussian quantizer. We are investigating further how the mismatch error can be minimized for multistage image transform coding.

It must be mentioned that the multistage procedure discussed in this paper will slightly increase the overhead information. The main part of overhead information is the bit map. For the optimal multistage image transform coding, we assume that the total number of bits (or the corresponding total average rate) is fixed. When the total rate is divided between stages, more number of pixels per stage assume zero bits. Thus, the overhead information will not be doubled. Usually, about 0.03 bpp is needed in one-stage coding for overhead information, including error protection bits, for the 0.5 bpp case with an image of size 256×256 [3]. For estimating the net improvement, we assumed 0.015 bpp for extra overhead in two-stage coding. When we increased the bit rate by this amount in one-stage coding, we found that the net improvement was about 1.5% less than what is given in Table (2).

With sequential video images, it may be possible to use the same variances in corresponding blocks of successive images to reduce the computation in the iterative procedure of finding the bit rates $R_0, R_1, \cdots$ in the multiclass problem.

6. CONCLUSIONS

Both theoretical and experimental results indicate that optimal adaptive multistage image transform coding is quite effective in reducing mean square reconstruction error over what is possible with one stage transform coding. Optimality is achieved by minimization of the total final error expression using marginal analysis. This minimization determines how to allocate bits to the coefficients in each stage. After the first stage, the pdf of the coefficients appear to be either Gaussian or uniform. Reconstruction of the quantized image is obtained by adding together the quantized transform domain coefficients from all the stages and inverse transforming. Further improvements in the techniques described are expected to reduce reconstruction error further.

In this paper, we considered MSE as the performance criterion. However, the difference images shown in Fig. 5 indicate that the reconstruction errors are more noise-like in multistage coding than in one-stage coding, with especially reduced errors at the edges. This is believed to be the reason why the reconstructed images with the multistage method are subjectively much more preferable than the reconstructed images with the one-stage method at the same bit rate.

Although the proposed method was tested for DCT and monochrome images, it can be easily applied to other transforms and color images.

APPENDIX

The error variances in the (k + 1)th stage can be computed in terms of the variances of the input coefficients from the kth stage as follows:

$$\begin{aligned}\sigma^2_{(k+1)ij} &= E\left[e^2_{(k+1)ij}\right] = E(e_{kij} - \hat{e}_{kij})^2 \\ &= E(e^2_{kij}) + E(\hat{e}^2_{kij}) - 2E(e_{kij}\hat{e}_{kij}) \\ &= E(e^2_{kij}) - E(e_{kij}\hat{e}_{kij}) \\ &= \sigma^2_{kij} - \left[1 - f_k(b_{kij})\right]\sigma^2_{kij} \\ &= \sigma^2_{kij} f_k(b_{kij}) \end{aligned} \quad (A.1)$$

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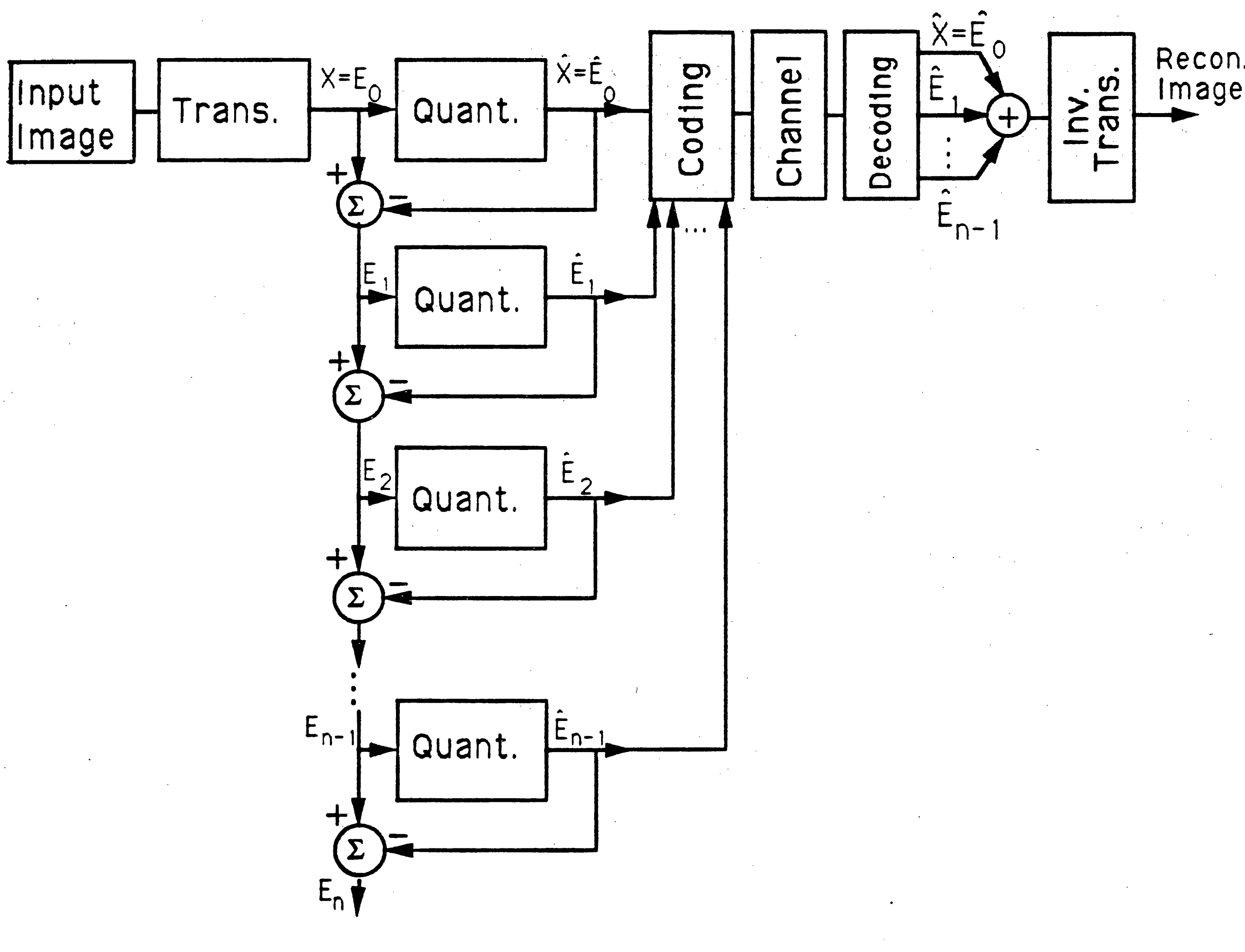


Fig. 1. Block diagram of multistage image transform coding.

Table 1. The error model for Gaussian and Laplacian distribution of the form $f_k(b_{kij}) = A2^{-Bb_{kij}}$ given in Ref. [9].

Distribution	$0 \le b_{kij} \le 2.32$		$2.32 < b_{kij} \le 5.17$		$5.17 < b_{kij} \le 9$	
	A	B	A	B	A	B
Gaussian	1	1.5047	1.5253	1.8274	2.2573	1.9626
Laplacian	1	1.1711	2.0851	1.7645	3.6308	1.9572

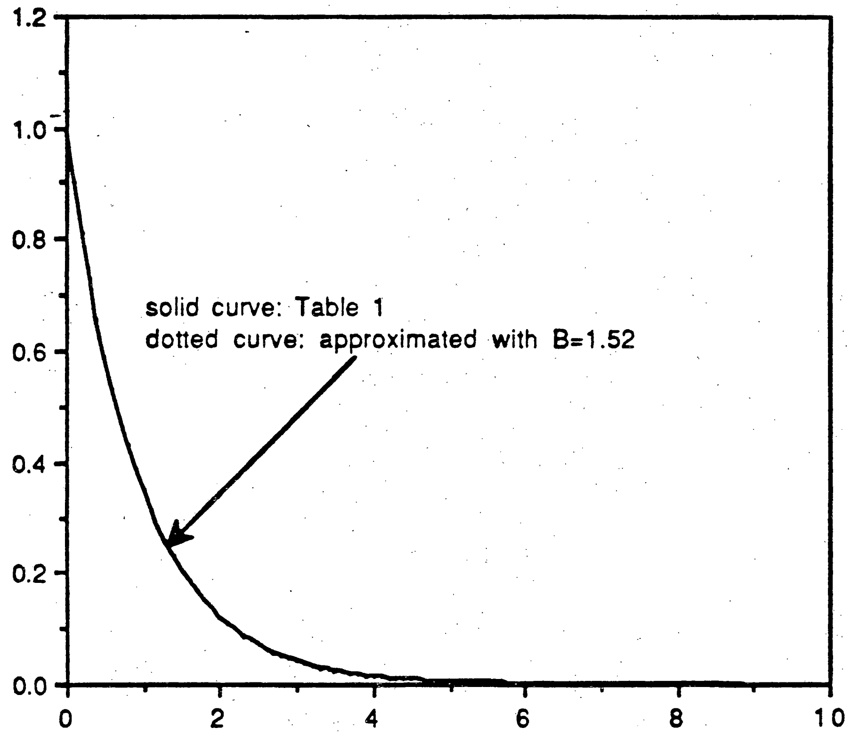


Fig. 2. Approximation of the error model for the Gaussian pdf given in Table 1 with $B_k = 1.52$ in the form of $f_k(b_{kij}) = 2^{-B_k b_{kij}}$.

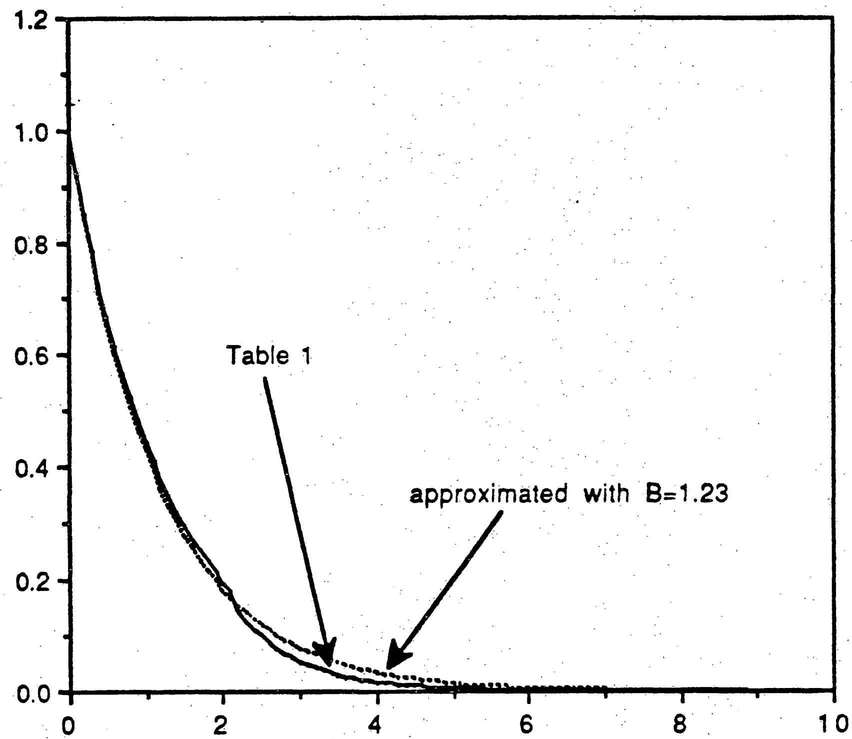


Fig. 3. Approximation of the error model for the Laplacian pdf given in Table 1 with $B_k = 1.23$ in the form of $f_k(b_{kij}) = 2^{-B_k b_{kij}}$.

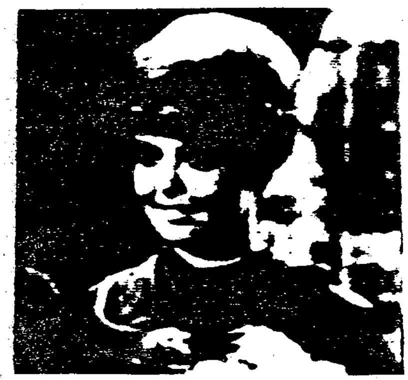
Table 2. Simulation results for multistage transform coding.

Images	Rates	1.00		0.50		0.25	
	# of classes	1	4	1	4	1	4
"girl128"	MSE for 1 stage	31.83	25.88	67.76	53.35	126.3	98.82
	NMSE(%) for 1 stage	0.450	0.366	0.959	0.755	1.787	1.398
	MSE for 2 stages	30.10	24.23	62.89	47.78	116.9	88.31
	NMSE(%) for 2 stages	0.426	0.343	0.890	0.676	1.654	1.250
	Improvement in MSE (%)	5.44	5.99	7.19	10.44	7.44	10.64
	Improvement in dB	0.243	0.286	0.324	0.479	0.336	0.488
	R_0	0.707	0.827	0.344	0.338	0.176	0.052
	R_1	0.293	0.173	0.156	0.162	0.074	0.198
"girl256"	MSE for 1 stage	30.78	18.16	62.78	36.04	116.9	70.29
	NMSE(%) for 1 stage	0.426	0.251	0.868	0.498	1.616	0.972
	MSE for 2 stages	27.66	18.12	55.35	32.96	99.77	63.60
	NMSE(%) for 2 stages	0.382	0.251	0.765	0.456	1.379	0.879
	Improvement in MSE (%)	10.14	0.22	11.83	8.55	14.65	9.52
	Improvement in dB	0.464	0.010	0.547	0.388	0.688	0.434
	R_0	0.695	0.86	0.352	0.34	0.176	0.148
	R_1	0.305	0.14	0.148	0.16	0.074	0.102

(a)

(b)

(c)



(d)

(e)

(f)

Fig. 4. (a) Original image of "girl256". (b) Reconstructed image for one stage coding with rate 0.5 bpp and 1 class. (c) Reconstructed image for two stage coding with rate 0.5 bpp and 1 class. (d) Reconstructed image for one stage coding with rate 0.5 bpp and 4 classes. (e) Reconstructed image for two stage coding with rate 0.5 bpp and 4 classes. (f) Reconstructed image for one stage coding with rate 0.25 bpp and 4 classes.

(g)

Fig. 4. (con't) (g) Reconstructed image for two stage coding with rate 0.25 bpp and 4 classes.

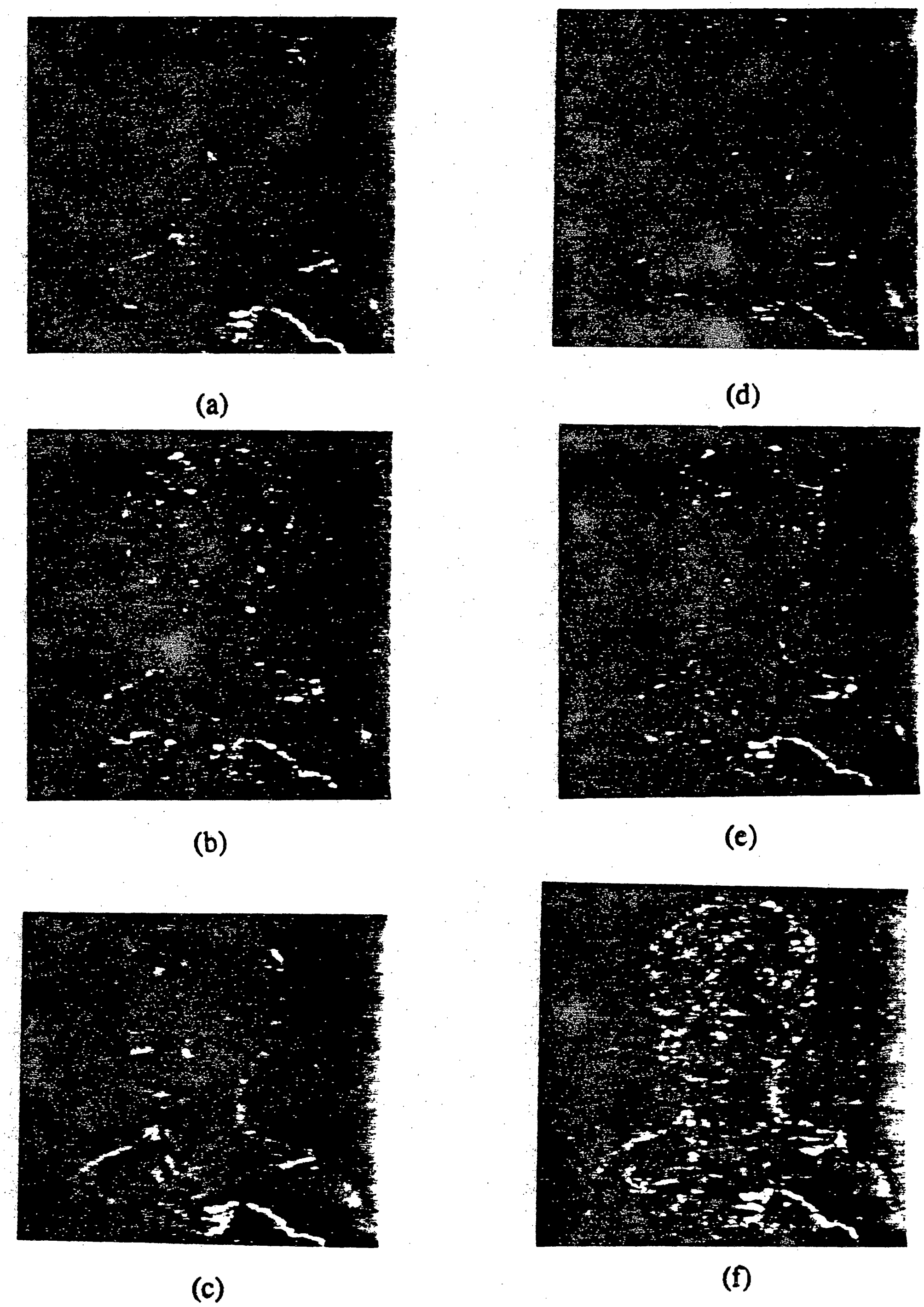


Fig. 5. The difference images between: (a) Fig. 4 (a) and Fig. 4 (b); (b) Fig. 4 (a) and Fig. 4 (c); (c) Fig. 4 (a) and Fig. 4 (d); (d) Fig. 4 (a) and Fig. 4 (e); (e) Fig. 4 (a) and Fig. 4 (f); (f) Fig. 4 (a) and Fig. 4 (g).

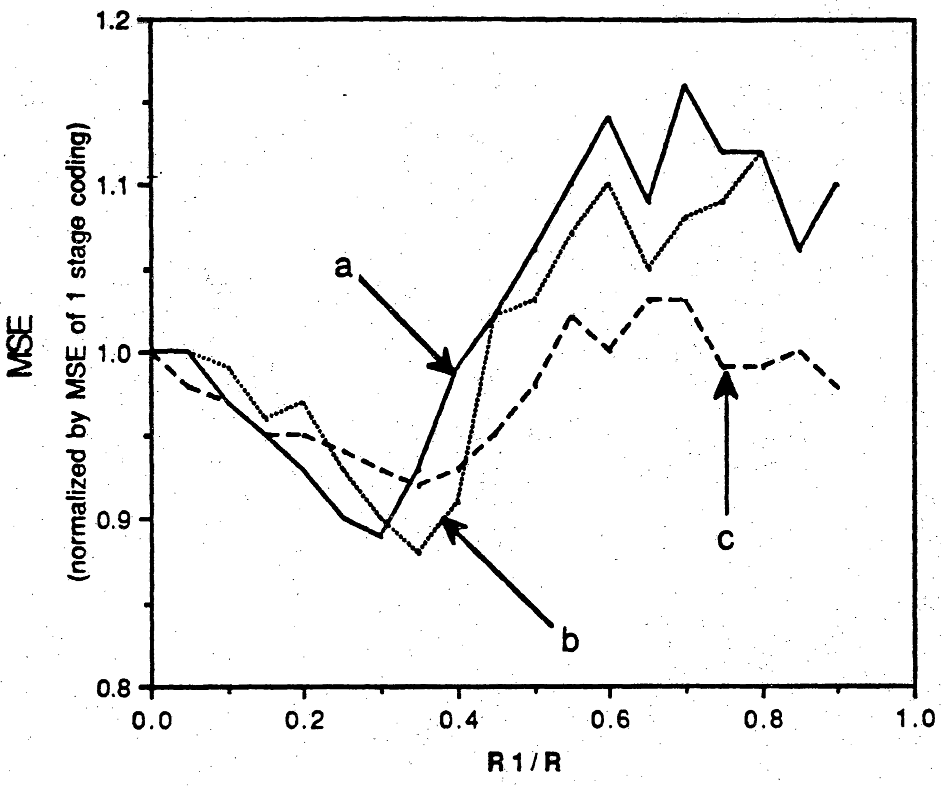


Fig. 6. Experimental results with two stage coding for all possible values for R_0 and R_1 ($R = R_0 + R_1$): (a) with "girl256", rate 0.5 bpp and 1 class; (b) with "girl256", rate 0.25 bpp and 1 class; (c) with "girl128", rate 1.0 bpp and 4 classes.