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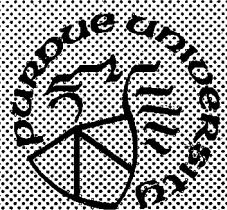*San Diego State University*

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CONTROL OF DYNAMIC SYSTEMS

VIA NEURAL NETWORKS

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ABSTRACT

This report is devoted to the problem of controlling a class of linear time-invariant dynamic systems via controllers based on additive neural network models. In particular, the tracking and stabilization problems are considered. First, we show how to transform the problem of tracking a reference signal by a control system into the stabilization problem. Then, some concepts from the variable structure control theory are utilized to construct stabilizing controllers. In order to facilitate the stability analysis of the closed-loop systems we employ a special state space transformation. This transformation allows us also to reveal connections between the proposed controllers and the additive neural network models.

1. INTRODUCTION

A neural network is a large-scale nonlinear circuit of interconnected simple circuits called cells or neurons. These networks resemble patterns of biological neural networks hence the term neural networks. In fact motivation for studying such circuits came from attempts "to understand how known biophysical properties and (the) architectural organization of neural systems can provide the immense computational power characteristic of the brains of higher animals" (Tank and Hopfield, [23], p. 533). Another reason of a recent resurgence of interest in neural networks is the low execution speed of conventional computers which perform a program of instructions serially or sequentially. In contrast, neural networks operate in parallel. The ability to be interconnected in a regular fashion results in higher computation rates. Furthermore, regular interconnections of the same basic cells leads to easier design and testing of a chip.

Potential applications of neural networks are in such areas as speech and image recognition ([21], [27]), linear and nonlinear optimization ([3], [16], [23]), automatic control ([1], [2], [15]), and in highly parallel computers ([17]) to mention but a few.

The subject of this report is an application of additive neural network models to the control of dynamic processes. We begin with a brief description of a neural network model used in the report. Then we formulate the tracking problem and show how the additive neural network model can be used as a controller. A variable structure systems approach ([7], [25]) is utilized to construct the proposed controllers. This approach allows us to circumvent analysis problems caused by the discontinuous nonlinearity which is used to described neurons. Finally, in the concluding Section we indicate directions for future research in the applications of neural networks to the control problems.

2. A BRIEF DESCRIPTION OF THE HOPFIELD NET

There are many neural network models ([11], [17]). In this report we will be concerned with the simpler additive model, also known as the Hopfield model, in the context of a control system tracking a reference signal. "The additive model has continued to be a cornerstone of neural network research to the present day... Some physicists unfamiliar with the classical status of the additive model in neural network theory erroneously called it the Hopfield model after they became acquainted with Hopfield's first application of the additive model in Hopfield (1984)" [14] - see Grossberg ([11], p. 23). The Hopfield neural network model is represented in Fig. 1.
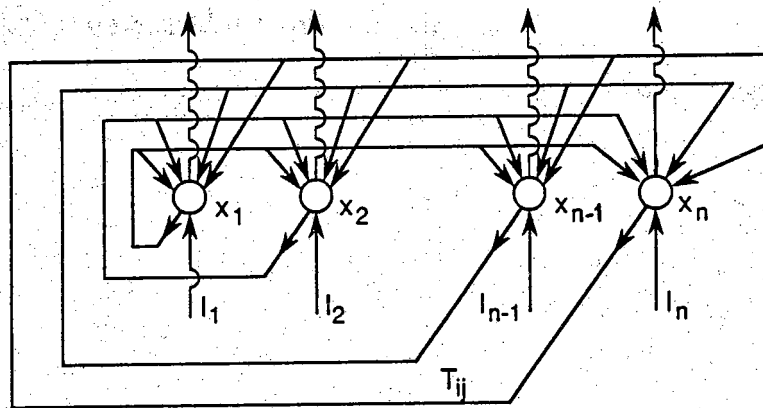

Fig. 1. Hopfield type neural network model.

Nodes or neurons, represented by circles in Fig. 1, can be modeled as shown in Fig. 2.
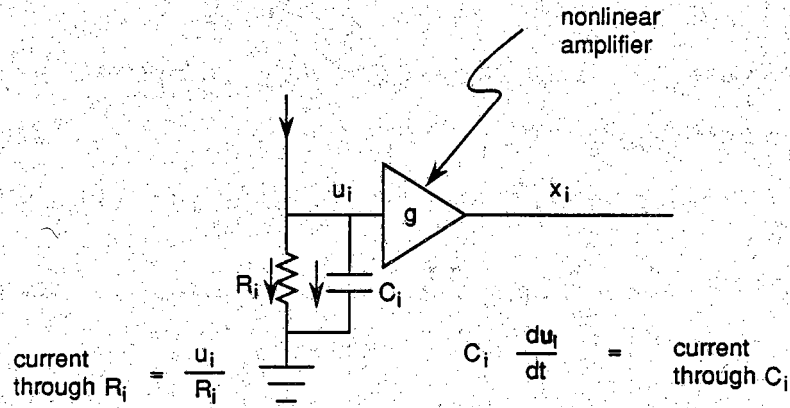
Fig. 2. Model of a neuron in the Hopfield net.

The nonlinear amplifier's input-output characteristic is described by the sigmoid function. The sigmoid function $x = g(u)$ ($g:\mathbb{R}\rightarrow\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers) is defined by the properties:

(a) $|g(u)| \le M$, where $0 < M < \infty$ is a constant, and

(b) $\dfrac{dg(u)}{du} \ge 0.$

A possible circuit implementation of the Hopfield network proposed by Smith and Portmann [22] is presented in Fig. 3.
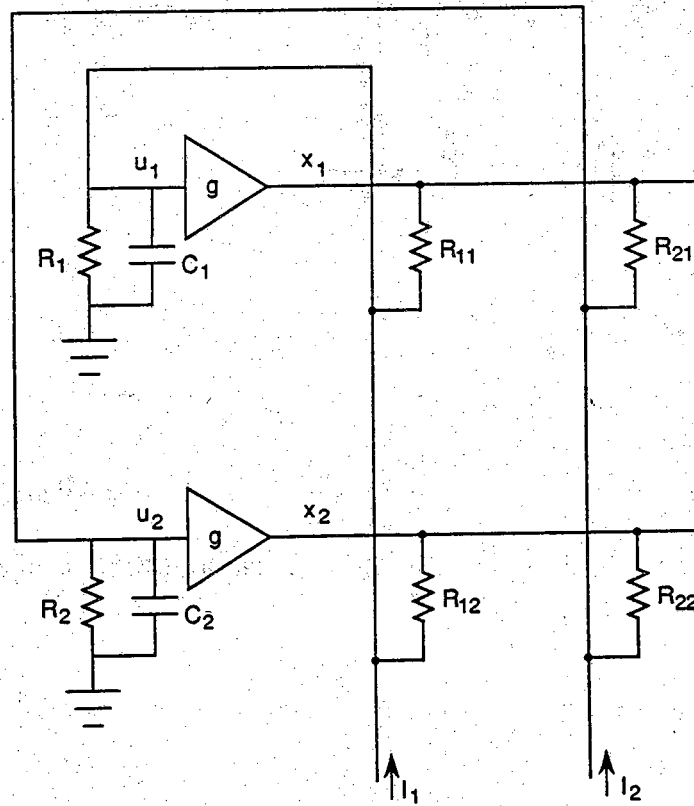
Fig. 3.   Circuit realization of the Hopfield net.

An equivalent representation of the circuit from Fig. 3. is given in Fig. 4.
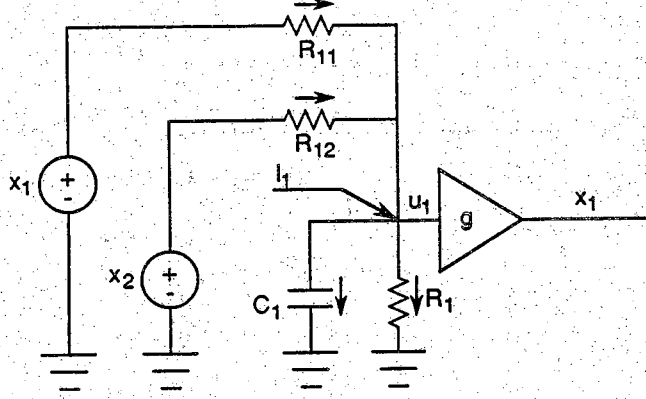
Fig. 4.   Smith and Portmann's [22] equivalent representation of the Hopfield network circuit realization.

This equivalent representation allows us to write down the equations governing the dynamical behavior of the net in a straightforward manner. Indeed, applying the Kirchoff current law at the input node of the amplifier and utilizing the fact that the input current into the amplifier is negligible (high input impedance) we obtain

$$C_i \frac{du_i}{dt} + \frac{u_i}{R_i} = \frac{x_1 - u_i}{R_{i1}} + \frac{x_2 - u_i}{R_{i2}} + \dots + \frac{x_n - u_i}{R_{in}} + I_i, \quad i = 1, 2, \dots, n . \quad (2.1)$$

Let

$$T_{ij} = \frac{1}{R_{ij}},$$

$$\frac{1}{r_i} = \sum_{j=1}^{n} T_{ij} + \frac{1}{R_i} .$$

Utilizing the above notation we can represent equations (2.1) in the following form

$$C_i \frac{du_i}{dt} + \frac{u_i}{r_i} = \sum_{j=1}^{n} T_{ij}\, x_j + I_i , \quad i = 1,2,...n , \qquad (2.2)$$

where $x_i = g(u_i)$. We can rewrite the equations (2.2) as

$$\frac{du_i}{dt} = -b_i u_i + \sum_{j=1}^{n} A_{ij}\, g(u_j) + U_i \quad i = 1,2,...,n \qquad (2.3)$$

where

$$A_{ij} = \frac{T_{ij}}{C_i} , \quad U_i = U_i(t) = \frac{I_i(t)}{C_i} , \quad \text{and} \quad b_i = \frac{1}{r_i C_i} .$$

The qualitative analysis of the neural network model represented by (2.3) was performed among others, by Hopfield [14] and Michel et al. [20], see also [10] and [11].

The primary goal of this report is to show how the Hopfield type of neural networks can be applied to the control of dynamic processes, in particular to the tracking a reference signal by a control system.

The formulation of the tracking problem is the subject of the next Section.

3. FORMULATION OF THE TRACKING PROBLEM

Suppose we have a model of a dynamic process, plant, given by the following equations

$$\left. \begin{array}{l} \dot{x} = Fx + Gu \\ y = Hx \end{array} \right\} \qquad (3.1)$$

where $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$, and $H \in \mathbb{R}^{m \times n}$. We wish to design a controller so that the closed-loop system can
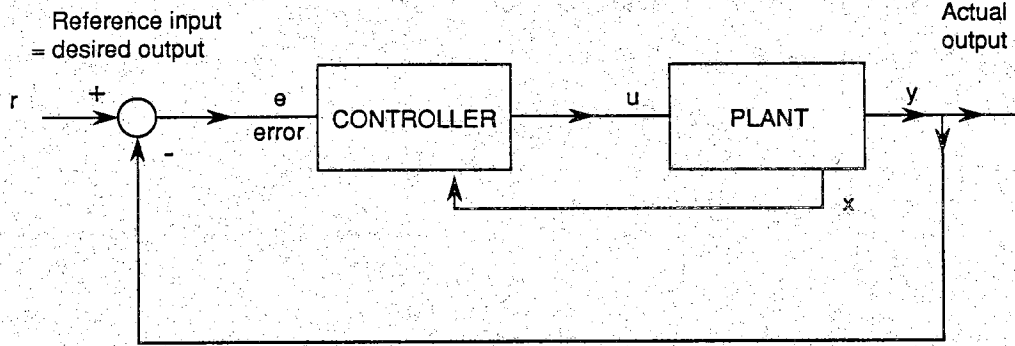
Fig. 5. Tracking system structure.

Let the reference signals be described, as in Davison [5], by the following differential equations

$$r_i^{(p)}(t) + \alpha_p r_i^{(p-1)}(t) + ... + \alpha_2 \dot{r}_i(t) + \alpha_1 r_i(t) = 0 , \quad i = 1,2,...,m \qquad (3.2)$$

where the initial conditions

$$r_i(0) , \quad \dot{r}_i(0) , \quad ... , \quad r_i^{(p-1)}(0)$$

are specified.

Equations (3.2) can be rewritten in the form of the matrix differential equation

$$r^{(p)}(t)I_m + \alpha_p r^{(p-1)}(t)I_m + ... + \alpha_2 \dot{r}(t)I_m + \alpha_1 r(t)I_m = 0 \qquad (3.3)$$

where I_m is the m×m identify matrix, and r(·) is a scalar function.

The tracking error is defined as

$$e(t) = y(t) - r(t) \qquad (3.4)$$

The problem of tracking $r(t) = [r_1(t),...,r_m(t)]^T$ can be viewed as a designing exercise of a control strategy which provides regulation of the error (Franklin et al. [9], p. 390),

that is, the error e(t) should tend to zero as time gets large.

One way to tackle this problem is to include the equations which are satisfied by the reference signal as a part of the control, (stabilization), problem in an error state space.

In particular, we differentiate the error equation p times and then introduce the error as a state. In what follows we extend results of Franklin et al. [9].

We have

$$
\begin{aligned}
e^{(p)} &= y^{(p)} - r^{(p)} \\
&= \mathbf{H}x^{(p)} + \alpha_p r^{(p-1)} I_m + \alpha_{p-1} r^{(p-2)} I_m + ... + \alpha_1 r I_m .
\end{aligned} \tag{3.5}
$$

We then replace the plant state vector by the following vector

$$
\xi = x^{(p)} + \alpha_p x^{(p-1)} + ... + \alpha_1 x \tag{3.6}
$$

and define the new control vector as

$$
\mu = u^{(p)} + \alpha_p u^{(p-1)} + ... + \alpha_1 u \tag{3.7}
$$

We now rewrite (3.5) as

$$
\begin{aligned}
e^{(p)} &= \alpha_p r^{(p-1)} I_m - \alpha_p y^{(p-1)} + \alpha_p y^{(p-1)} + ... \\
&\quad + ... + \alpha_1 r I_m - \alpha_1 y + \alpha_1 y + \mathbf{H}x^{(p)} \\
&= -\alpha_p e^{(p-1)} - ... - \alpha_1 e + \mathbf{H}\left[x^{(p)} + \alpha_p x^{(p-1)} + ... + \alpha_1 x\right].
\end{aligned} \tag{3.8}
$$

Taking into account (3.6) yields

$$
e^{(p)} = -\alpha_p e^{(p-1)} - ... - \alpha_1 e + \mathbf{H}\xi . \tag{3.9}
$$

Differentiating ξ gives

$$\dot{\xi} = x^{(p+1)} + \alpha_p x^{(p)} + ... + \alpha_1 \dot{x}$$

$$= Fx^{(p)} + Gu^{(p)} + \alpha_p Fx^{(p-1)} + \alpha_p Gu^{(p-1)}$$

$$+ \alpha_1 Fx + \alpha_1 Gu$$

$$= F(x^{(p)} + \alpha_p x^{(p-1)} + ... + \alpha_1 x)$$

$$+ G(u^{(p)} + \alpha_p u^{(p-1)} + ... + \alpha_1 u) \qquad (3.10)$$

Hence, in view of (3.7),

$$\dot{\xi} = F\xi + G\mu \qquad (3.11)$$

Combining (3.9) and (3.11) yields

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \\ \vdots \\ e^{(p-1)} \\ e^{(p)} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & I_m & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & I_m & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & I_m & 0 \\ -\alpha_1 I_m & \cdot & \cdot & \cdot & & -\alpha_p I_m & H \\ 0 & 0 & 0 & 0 & & 0 & F \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \vdots \\ e^{(p-1)} \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ G \end{bmatrix} \mu . \qquad (3.12)$$

Let $z = [e^T \dot{e}^T , ... , \xi^T]$,

$$A = \begin{bmatrix} 0 & I_m & ... & 0 & 0 \\ \vdots & & & & \vdots \\ -\alpha_1 I_m & & ... & -\alpha_p I_m & H \\ 0 & & ... & 0 & F \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G \end{bmatrix} .$$

Then (3.12) can be represented as

$$\dot{z} = Az + B\mu . \qquad (3.13)$$

We refer to the system modeled by (3.13) as the error system. Thus the problem of the regulation of the error is reduced to the problem of the stabilization of the error system. Different approaches to the stabilization problem of dynamic systems modeled by (3.13)

exist. For example see DeCarlo [6] or Franklin et al. [9]. We propose two new approaches to designing stabilizing control strategies for (3.13) involving the Hopfield type neural networks. Prior to presenting these control laws we will introduce the necessary apparatus for further analysis. This is the subject of the following Section.

4. AUXILIARY RESULTS

In the following analysis we utilize certain concepts from the theory of variable structure control.

Variable structure control (VSC), the control of dynamical systems with discontinuous state feedback controllers, has been developed over the last 25 years. See [7], and [25] for surveys and [12], [24], [28] for applications. This theory rests on the concept of changing the structure of the controller in response to the changing states of the system to obtain a desired response. This is accomplished by the use of a high speed switching control law which forces the trajectories of the system onto a chosen manifold, where they are maintained thereafter. The system is insensitive to certain parameter variations and disturbances while the trajectories are on the manifold. If the state vector is not accessible, then a suitable estimate must be used.

We use the following notation. If $x \in \mathbb{R}^n$, then $\|x\|$ denotes the Euclidean norm, that is, $\|x\| = (x_1^2 + \dots + x_n^2)^{1/2}$. If A is a matrix, then $\|A\|$ is the spectral norm defined by

$$\|A\| = \max_x \{\|Ax\| \mid \|x\| \leq 1\} .$$

For any square matrix A, we let $\lambda_{\min}(A)$ be the minimum eigenvalue of A and $\lambda_{\max}(A)$ be the maximum eigenvalue of A.

With this notation,

$$\|A\|^2 = \lambda_{\max}(A^T A) .$$

Rayleigh principle states that if P is a real symmetric positive definite matrix, then

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2 .$$

An important concept in variable structure control is that of an attractive manifold on which certain desired dynamical behavior is guaranteed. Trajectories of the system should be steered towards the manifold and subsequently constrained to remain on it.

Definition 4.1. ([25])

A domain Δ in the manifold $\{x \mid \sigma(x) = 0\}$ is a sliding mode domain if for each $\epsilon > 0$ there exists a $\delta > 0$ such that any trajectory starting in the n-dimensional δ-neighborhood of Δ may leave the n-dimensional ϵ-neighborhood of Δ only through the n-dimensional ϵ-neighborhood of the boundary of Δ. (see Fig. 6.)
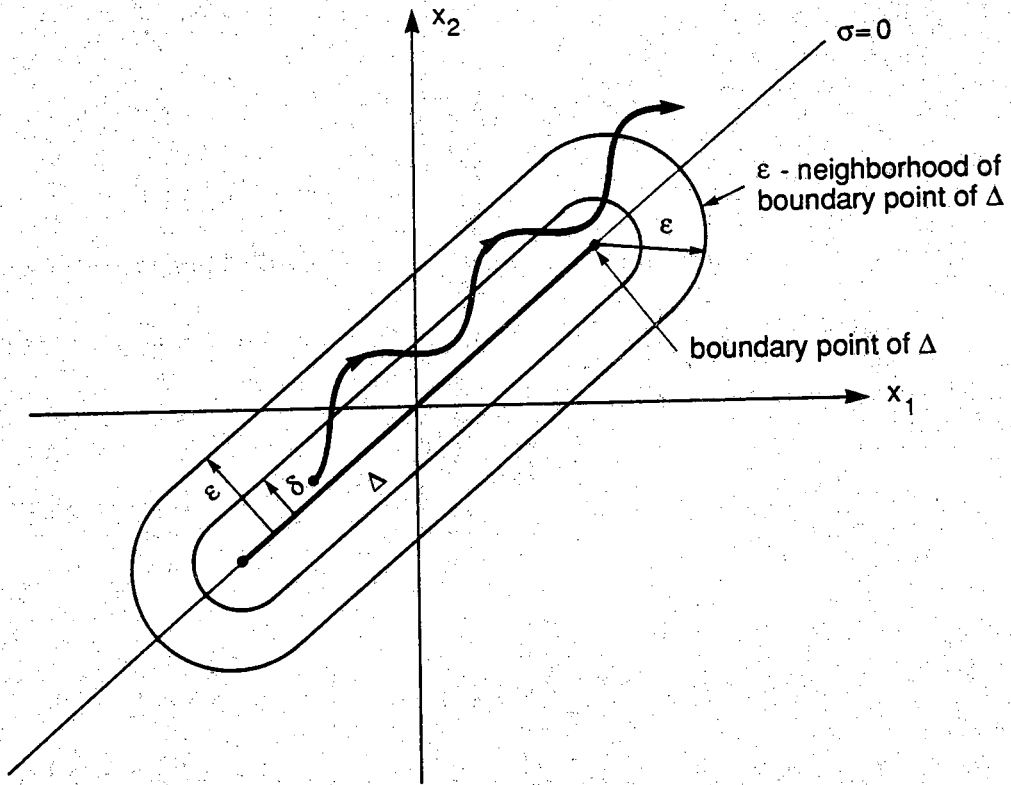
Fig. 6. Two-dimensional illustration of sliding mode domain.

We next describe the manifold which is used in this paper. Suppose

$$S = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} \in \mathbb{R}^{m \times n} \ ,$$

where

$$s_i \in \mathbb{R}^{1 \times n} \ .$$

We assume that S is of full rank.

Let

$$\sigma(x) = \begin{bmatrix} \sigma_1(x) \\ \vdots \\ \sigma_m(x) \end{bmatrix} = \begin{bmatrix} s_1 x \\ \vdots \\ s_m x \end{bmatrix} = Sx \ ,$$

where

$$x \in \mathbb{R}^n ,$$

and let

$$\Omega = \{x \mid \sigma(x) = 0\} .$$

Consider the system

$$\dot{x} = Ax + Bu , \tag{4.1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $u \in \mathbb{R}^m$. We make the following assumptions:

Assumption 1: The matrix SB is nonsingular.

Assumption 2: The pair (A,B) is completely controllable.

Definition 4.2. ([25]) The solution of the alebraic equation in u of

$$S\dot{x} = SAx + SBu = 0$$

is called the equivalent control and denoted by u_{eq}, that is,

$$u_{eq} = - (SB)^{-1} SAx .$$

Definition 4.3. The equivalent system is the system that is obtained when the original control u is replaced by the equivalent control u_{eq}, that is,

$$\dot{x} = [I_n - B(SB)^{-1}S]A x .$$

We assume that the control u in the system (4.1) is bounded and that

$$|u_i| \leq \mu_i , \quad i = 1,...,m , \tag{4.2}$$

where $\mu_i > 0$.

Our goal is to design a controller which satisfies the bounds (4.2) and which induces the sliding mode on Ω in the sense of Definition 4.1.

In general, the controller in VSC varies its structure depending on the position relative to the switching surface and has the form:

$$u_i = \begin{cases} u_i^+(x) & \text{if} \quad \sigma_i(x) > 0 \\ u_i^-(x) & \text{if} \quad \sigma_i(x) < 0 . \end{cases}$$

It is easy to see that if $\sigma^T \dot{\sigma} < 0$, then the trajectory is tending towards the switching surface. Hence if $\sigma^T \dot{\sigma} < 0$ in a neighborhood of a region Δ of the switching surface, then Δ is a sliding domain ([25]). For example, let

$$\tilde{u} = \begin{bmatrix} -k_1 \operatorname{sgn} \sigma_1 \\ \vdots \\ -k_m \operatorname{sgn} \sigma_m \end{bmatrix}$$

where

$$\operatorname{sgn} \sigma_i = \begin{cases} 1 & \text{if} \quad \sigma_i > 0 \\ 0 & \text{if} \quad \sigma_i = 0 \\ -1 & \text{if} \quad \sigma_i < 0 , \end{cases}$$

and $k_i > 0$ for $i = 1,...,m$. One can easily check that if

$$u = u_{eq} + (SB)^{-1}\tilde{u} ,$$

then

$$\sigma^T \dot{\sigma} = \sigma^T \tilde{u} < 0 .$$

Hence with the above control u, we have a sliding mode.

Let us assume that the switching surface is chosen so that $SB = I_m$; we will see how this can be accomplished Section 6 and 8. With this assumption, $u_{eq} = -SAx$. We next give a sufficient condition for $\sigma^T \dot{\sigma} < 0$ to hold. Note that

$$\sigma^T \dot\sigma = \sigma^T(SAx + SBu)$$

$$= \sigma^T(-u_{eq} + u)$$

$$= \sum_{i=1}^{m} \sigma_i\big(-(u_{eq})_i + u_i\big) \, .$$

Hence if

$$\begin{cases} u_i^+(x) < (u_{eq})_i & \text{for} \quad \sigma_i(x) > 0 \\ u_i^-(x) > (u_{eq})_i & \text{for} \quad \sigma_i(x) < 0 \, , \end{cases} \tag{4.3}$$

then $\sigma^T \dot\sigma < 0$. In this report, we use the control law

$$u = \begin{bmatrix} -\mu_1 \text{sgn}\,\sigma_1 \\ \vdots \\ -\mu_m \text{sgn}\,\sigma_m \end{bmatrix} \, .$$

With this control law, we have $\sigma^T \dot\sigma < 0$ in the region

$$\Omega^* = \bigcap_{i=1}^{m} \{x \mid |(u_{eq})_i| < \mu_i\}$$

$$= \bigcap_{i=1}^{m} \{x \mid |s_i Ax| < \mu_i\} \, .$$

Note that Ω^* is an open neighborhood of the origin. If $\Delta = \Omega \cap \Omega^*$, where $\Omega = \{x \mid \sigma(x) = 0\}$, then Δ is a sliding domain.

Observe that a sliding domain is a region of asymptotic stability (RAS) for the system.

Example 4.1. Consider the dynamical system modeled by:

$$\ddot{x} = u ,$$

where $u = - \text{sgn } \sigma(x)$, and $\sigma(x) = x_1 + x_2 = [1 \ \ 1]x$. In the state-space representation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{sgn } \sigma .$$

In this case

$$\Omega = \{(x_1, x_2) \mid x_1 + x_2 = 0\} ,$$
$$\Omega^* = \{(x_1, x_2) \mid |x_2| < 1\} ,$$

and $\Delta = \Omega^* \cap \Omega$ is the segment of the line $x_1 + x_2 = 0$ with $|x_2| < 1$. (See Fig. 7)
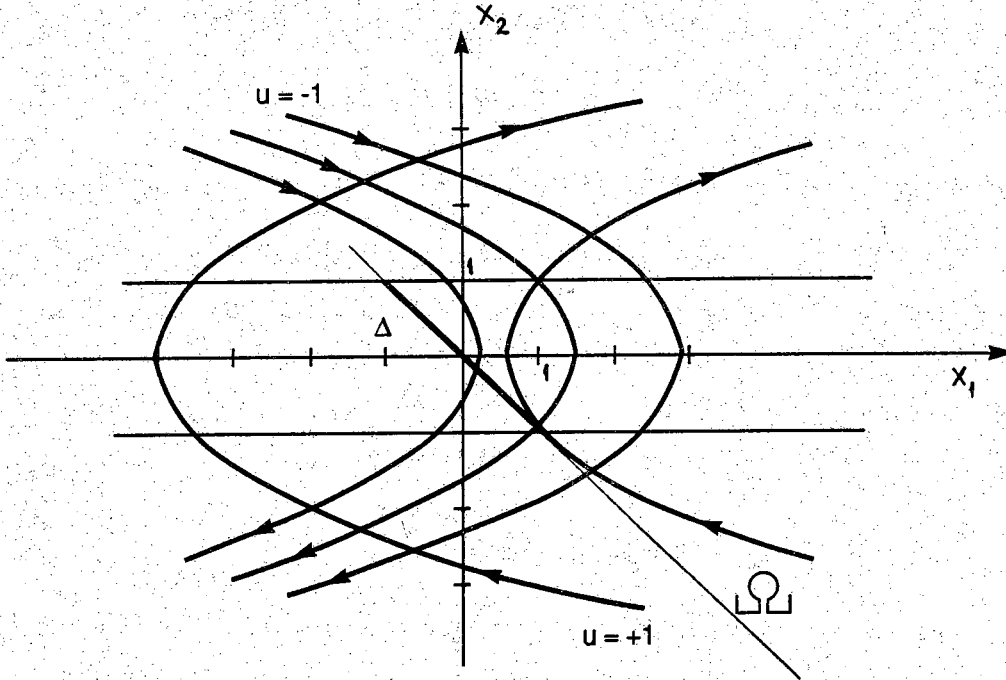
Fig. 7. Illustration of a sliding domain in Example 4.1.

5. CONTROL OF SINGLE-INPUT SYSTEMS

There are two basic steps in the design of VSC:

(1) The design of the switching surface (manifold) so that the behavior of the system has certain prescribed properties on the surface. For example, the switching surface will be designed so the system is asymptotically stable on the surface.

(2) The design of the control strategy to steer the system to the switching surface and to maintain it there.

In this Section we consider a class of single-input dynamic system modeled by

$$\dot{x} = Ax + bu \qquad (5.1)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $u \in \mathbb{R}$. We assume that the pair (A,b) is completely controllable and hence (5.1) is equivalent to the controller canonical form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & & 0 & 1 \\ \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1} & \alpha_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u . \tag{5.2}$$

The mainfold we use has the form

$$\Omega = \{x \mid sx = 0\} , \quad s \in \mathbb{R}^{1 \times n} . \tag{5.3}$$

We can assume that $s = [s_1, \ldots, s_{n-1}, 1]$. Observe that if the system is in the controller canonical form then $sb = 1$. When the system dynamics is given by the controller canonical form then the equivalent system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & -s_1 & -s_2 & \cdots & -s_{n-2} & -s_{n-1} \end{bmatrix} x . \tag{5.4}$$

The controller on which we will concentrate is

$$u = -\mu \operatorname{sgn}(sx) , \tag{5.5}$$

where μ is a positive real number and

$$\operatorname{sgn}(sx) = \begin{cases} 1 & \text{if} \quad sx > 0 \\ 0 & \text{if} \quad sx = 0 \\ -1 & \text{if} \quad sx < 0 . \end{cases}$$

Note that this controller is bounded by μ.

We now choose the switching surface so that the system restricted to the surface has prescribed distinct negative eigenvalues $-\lambda_1, \ldots, -\lambda_{n-1}$. If the system is in the controller canonical form then in sliding mode the system is described by (5.4) and $sx = 0$. The order of the system in sliding is n-1 and its characteristic equation is given by

$$\lambda^{n-1} + s_{n-1}\lambda^{n-2} + \dots + s_1 = 0 . \tag{5.6}$$

The prescribed eigenvalues $-\lambda_1, \dots, -\lambda_{n-1}$ must satisfy (5.6) and hence we have the linear equations

$$\begin{bmatrix} 1 & -\lambda_1 & (-\lambda_1)^2 & \dots & (-\lambda_1)^{n-1} \\ & \vdots & & & \\ 1 & -\lambda_{n-1} & (-\lambda_{n-1})^2 & \dots & (-\lambda_{n-1})^{n-1} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_{n-1} \\ 1 \end{bmatrix} = 0 . \tag{5.7}$$

Since $\lambda_1, \dots \lambda_{n-1}$ are distinct, the coefficient matrix has full rank and $s_1, \dots, s_{n-1}$ are uniquely determined. This completes the design of the switching surface $\Omega = \{x \mid sx = 0\}$. Note that one does not have to have a model of a dynamic system to be controlled in the controller canonical form. We have used the canonical form to facilitate the analysis. From this point one can assume that our process to be controlled does not have any particular form.

To proceed further we introduce a state-variable transformation. For ℓ, k positive integers, we let

$$V_\ell(\beta_1, \dots, \beta_k) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \beta_1 & \beta_2 & & \beta_k \\ \vdots & \vdots & & \vdots \\ \beta_1^\ell & \beta_2^\ell & & \beta_k^\ell \end{bmatrix} \in \mathbb{R}^{(\ell+1)\times k} .$$

Let

$$W = V_{n-1}(-\lambda_1, \dots, -\lambda_{n-1}) \ \text{diag} \ (p_1, \dots, p_{n-1}) \in \mathbb{R}^{n\times(n-1)} ,$$

and

$$W^g = [\{V_{n-2}(-\lambda_1, \dots, -\lambda_{n-1}) \ \text{diag} \ (p_1, \dots, p_{n-1})\}^{-1} \mid 0] \in \mathbb{R}^{(n-1)\times n} .$$

Note that $W^g W = I_{n-1}$. The p_i's are to be chosen so that the system matrix will have a desired form to be given.

Let, as in [19],

$$M = \begin{bmatrix} W^g \\ s \end{bmatrix}.$$

Observe that $M^{-1} = [W \; b]$ (see (5.7)). We introduce the new coordinates

$$\begin{bmatrix} z \\ y \end{bmatrix} = Mx = \begin{bmatrix} W^g x \\ sx \end{bmatrix}, \tag{5.8}$$

where $z \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. In these coordinates, the system (5.2) has the form

$$\dot{z} = A_{11} z + A_{12} y$$
$$\dot{y} = A_{21} z + A_{22} y + u , \tag{5.9}$$

where $A_{11} = \text{diag} \; (-\lambda_1, ..., -\lambda_{n-1})$. If we use the controller

$$u = - \mu \, \text{sgn}(sx) = - \mu \, \text{sgn} \, y ,$$

then the system is described by

$$\dot{z} = A_{11} z + A_{12} y$$
$$\dot{y} = A_{21} z + A_{22} y - \mu \, \text{sgn} \, y . \tag{5.10}$$

Remark 5.1.

One can interpret equations (5.10) as follows. We are given a dynamic system

$$\dot{z} = A_{11} z + A_{12} y \tag{5.11}$$

driven by a signal neuron type controller

$$\dot{y} = A_{22} y - \mu \, \text{sgn} y + A_{21} z. \tag{5.12}$$

This observation follows from the comparison of (2.3) and (5.12), where $I = A_{21} z$.

Note that in order to arrive at the above conclusion we had to perform a state-space transformation (5.8) to reveal the implicit presence of the neural controller in the

closed-loop system (5.2), (5.5).

We now explicitly employ the neural type of controller to a given dynamic system. In particular suppose we are given a dynamic system modeled by (5.1). We propose a controller of the form (2.3)

$$\dot{u} = -\beta u - \mu \operatorname{sgn} \sigma(x, u) + U, \tag{5.13}$$

where $U = c^T x$, and $\sigma(x, u)$ is a switching surface to be chosen. The equations of the closed-loop system can be represented as

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & b \\ c^T & -\beta \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-\mu \operatorname{sgn} \sigma(x, u)) . \tag{5.14}$$

Observe that $[x^T, u]^T \in \mathbb{R}^{n \times 1}$. In order to proceed further one has to decide what kind of dynamic behavior is to be imposed on the closed-loop system. This then should be expressed in the form of n prescribed eigenvalues which will correspond to the eigenvalues of the system while in sliding along $\sigma(x, u) = 0$. Having chosen desired eigenvalues we can determine the switching surface $\sigma(x, u)$ using (5.7). If one then transforms (5.14) into the new coordinates utilizing (5.8) then the resulting system will be in the form (5.10).

Example 5.1. Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{5.15}$$

where $u = -\mu \operatorname{sgn} \sigma(x)$. We choose the switching surface so that the system restricted to the switching surface has eigenvalue $-\lambda_1$. From (5.6) and (5.7) the switching surface is

$$s_1x_1 + x_2 = \lambda_1x_1 + x_2 = 0 .$$

We use the controller

$$u = - \mu \operatorname{sgn}(\lambda_1x_1 + x_2) .$$ (5.16)

A block diagram representation of the closed-loop system is given in Fig. 8.
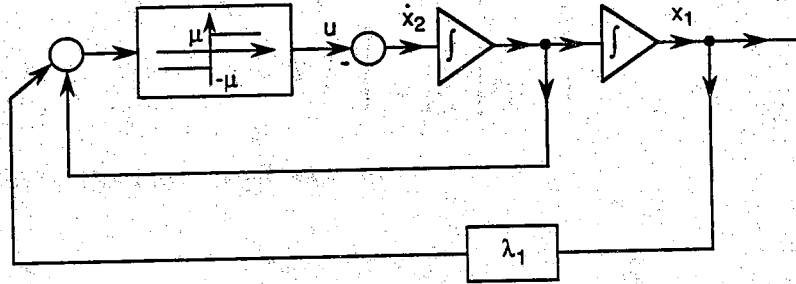
Fig. 8.   Block diagram of the closed-loop system (5.15), (5.16) in Example 5.1 in the old coordinates.

Using the method described above we have

$$M = \begin{bmatrix} W^g \\ s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda_1 & 1 \end{bmatrix} ,$$

and the system in the new coordinates is

$$\dot{z} = - \lambda_1 z + y$$
$$\dot{y} = - \lambda_1^2 z + \lambda_1 y - \mu \operatorname{sgn} y .$$

Consider again the system (5.15). This time we choose an explict neural controller

$$\dot{u} = - \beta u + c^T x - \mu \operatorname{sgn} \sigma(x,u) ,$$ (5.17)

where $c^T = [c_1, c_2]$. The closed-loop system now has the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-\mu \operatorname{sgn} \sigma(x,u)) \qquad (5.18)$$

A block diagram of this closed-loop system is depicted in Fig. 9.
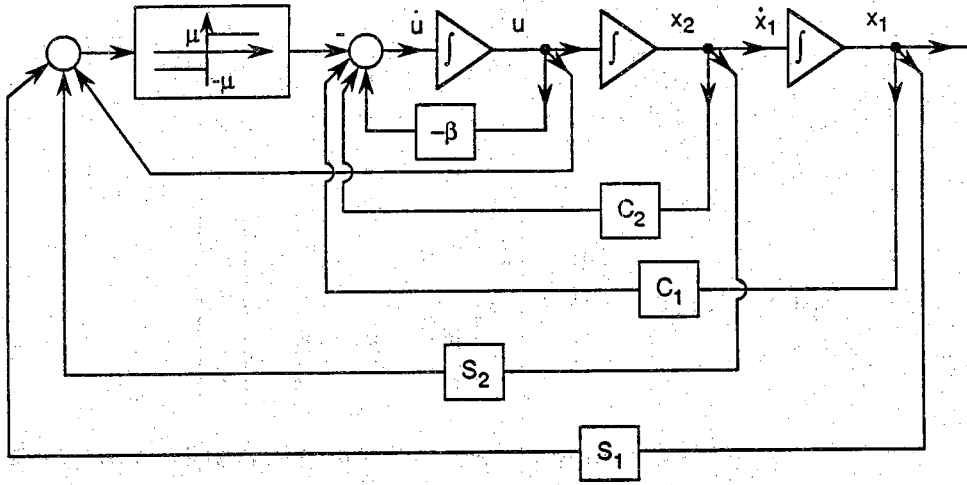
Fig. 9.   Block diagram of the closed-loop system (5.18).

The switching surface is found from equation (5.7)

$$\begin{bmatrix} 1 & -\lambda_1 & (-\lambda_1)^2 \\ 1 & -\lambda_2 & (-\lambda_2)^2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ 1 \end{bmatrix} = 0 , \qquad (5.19)$$

where $-\lambda_1$ and $-\lambda_2$ are the desired (distinct) eigenvalues. Solving (5.19) yields

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} -\lambda_2 & \lambda_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \lambda_2 \\ \lambda_2 + \lambda_1 \end{bmatrix} . \qquad (5.20)$$

We now can construct the state-space transformation (5.8). We have

$$M = \begin{bmatrix} V_1^{-1} & | & 0 \\ & | & 0 \\ - & - & - \\ & s & \end{bmatrix} = \begin{bmatrix} W^g \\ s \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda_2}{\lambda_2 - \lambda_1} & \dfrac{1}{\lambda_2 - \lambda_1} & 0 \\[2mm] \dfrac{\lambda_1}{\lambda_1 - \lambda_2} & \dfrac{1}{\lambda_1 - \lambda_2} & 0 \\[2mm] \lambda_1 \lambda_2 & \lambda_2 + \lambda_1 & 1 \end{bmatrix}.$$

Note that

$$M^{-1} = [W \ \ b] = [V_2 \ \vdots \ b]$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -\lambda_1 & -\lambda_2 & 0 \\ (-\lambda_1)^2 & (-\lambda_2)^2 & 1 \end{bmatrix}.$$

In the new coordinates (5.18) has the form (5.10) where

$$A_{11} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \quad \text{and} \quad y = [s_1 \ \ s_2 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}.$$

The results of Section 4 imply that the closed-loop system described by (5.10) is locally asymptotically stable. The goal of the next Section is to investigate regions of asymptotic stability for dynamic systems driven by a single neuron type controller. The closed-loop system then is modeled by (5.10).

6. A FIRST APPROXIMATION OF THE REGION OF ASYMPTOTIC STABILITY WITH SLIDING

In this Section we give a first approximation of the region of asymptotic stability (RAS) with sliding. The results of this Section are based on the paper by Madani-Esfahani et al. [19].

Better approximation will be given in the following Section. We start with the following lemma.

The system under consideration is modeled by (5.10).

Lemma 6.1. ([19]). For $0 < \epsilon < \mu$, the region

$$\Delta = \{(z,0) \mid |A_{21}z| < \mu-\epsilon\} \subset \Omega$$

is a sliding mode domain.

Proof: Let $\Delta_\epsilon = \{(z,y) \mid |A_{21}z| < \mu-\epsilon, \; |y| < \frac{\epsilon}{|A_{22}|}\}$. Then in $\Delta_\epsilon \setminus \{(z,y) \mid y = 0\}$

$$\frac{1}{2} \frac{d}{dt}(y^2) = y\,\dot{y} = y(A_{21}z + A_{22}y) - \mu\,|y|$$

$$< |y|\,(\mu - \epsilon + \epsilon - \mu)$$

$$\leq 0.$$

Therefore a trajectory starting in Δ_ϵ can leave Δ_ϵ only through the $\epsilon/|A_{22}|$-neighborhood of the boundary of Δ in Ω.

$\square$

Observe that if the initial point is in Δ, then the system will be in sliding for some positive time. However, there is no guarantee that we stay in Δ for subsequent times. From the fact that $A_{11} = \text{diag}(-\lambda_1,...,-\lambda_{n-1})$ we have the following. Let B_r denote the ball centered at 0 with radius r.

Proposition 6.2. ([19]). Let

$$R = \sup \{r \mid B_r \cap \Omega \subset \Delta\},$$

and

$$\Sigma = B_R \cap \Omega.$$

Then Σ is a RAS with sliding.

Proof: While in sliding the system is governed by $\dot{z} = \mathrm{diag}\,(-\lambda_1,...,-\lambda_{n-1})z$. Hence $z_i = z_i(0)e^{-\lambda_i t}$ for $i = 1,...,n-1$. Thus if $\sum_{i=1}^{n-1} z_i^2(0) < R^2$, then $\sum_{i=1}^{n-1} z_i^2(t) < R^2$ and by Lemma 6.1 $z(t) \in \Sigma$ for $t \geq 0$.

$\square$

Note that Σ is the largest circular region that is contained in Δ. We can easily see that $R = \mu/a_{21}$, where $a_{21} = \|A_{21}\|$, the Euclidean norm of A_{21}.

7. IMPROVED ESTIMATES OF RAS WITH SLIDING

In the previous Section we obtained a RAS with sliding contained in the switching surface. We now use this information to obtain RAS's that are not constrained to the switching surface. The method we use is that of finding RAS's whose restriction to the switching surface is contained in Σ, and hence will be a RAS with sliding. Our main tool is a Lure-like Lyapunov function candidate

$$V(z,y;\beta,\eta,h) = (a_{21}\|z\|)^2 + 2\beta(A_{21}z)y + hy^2 + \mu\eta\,|y|\,, \tag{7.1}$$

where β, h, and η are positive constants. When there is no ambiguity, we will write $V(z,y)$ for $V(z,y;\beta,\eta,h)$. Observe that

$$V(z,y) \geq (a_{21}\|z\| - \beta |y|)^2 + (h - \beta^2)y^2 + \mu\eta |y| . \qquad (7.2)$$

Hence if $h - \beta^2 \geq 0$, then V is positive definite. If $\beta^2 - h > 0$ then V is positive in $\{(z,y) \mid |y| < \dfrac{\mu\eta}{\beta^2 - h}\}$. Since V contains a multiple of $|y|$, the Lyapunov derivative $\dot{V}$ may not exist on a trajectory which intersects $\Omega = \{(z,y) \mid y = 0\}$. However, when restricted to Ω the system takes the form $\dot{z} = A_{11}z = \text{diag}(-\lambda_1, ..., -\lambda_{n-1})z$. Therefore if the trajectory $(z(t),y(t))$ is in Ω for $t_1 \leq t \leq t_2$, we must have $\|z(t_2)\| \leq \|z(t_1)\|$. Since the restriction of level sets of V to Ω are circular regions, the trajectories of the system cannot leave a sublevel set $\{(z,y) \mid V(z,y) < a^2\}$ of V through Ω. Therefore if Γ is a region such that

(i)   V is positive in Γ,

(ii) $\dot{V}$ is negative in $\Gamma \backslash \Omega$,

then the largest sublevel set of V contained in Γ is a RAS. If in addition we have

(iii)   the restriction of Γ to Ω is contained in Σ,

then the largest sublevel set of V contained in Γ is a RAS with sliding. Note that we do not need to consider $\dot{V}$ on Ω.

Theorem 7.1. ([19]). Suppose $\eta = 2\beta$. Then for each $0 < \delta < \mu$ there is an $\epsilon > 0$ so that for $h \geq \dfrac{\mu^2}{\epsilon^2}$ the region

$$\{(z,y) \mid V(z,y;\beta,\eta,h) < (\mu - \delta)^2\}$$

is a RAS with sliding.

Proof: After some manipulations we obtain

$$\dot{V} = 2a_{21}^2 z^T A_{11} z + [2\beta(A_{21}z)^2 - \eta\mu^2]$$

$$+ 2h\,|y|\,[A_{21}z\,\text{sgn}\,y - \mu] + \mu A_{21}z[\eta - 2\beta]\,\text{sgn}\,y$$

$$+ A_{22}\,|y|\,[2h\,|y| + 2\beta(A_{21}z)\,\text{sgn}\,y + \mu\eta]$$

$$+ 2y\,[\beta A_{21}A_{11}z + \beta y A_{21}A_{12} + a_{21}^2 z^T A_{12}]\,.$$

We choose $K_1 > 0$ so that Σ is contained in $\{(z,y)\mid \|A_{11}z\| < K_1,\ |z^T A_{12}| < K_1,\ y = 0\}$. Therefore there is a fixed constant $K > 0$ so that in $\Theta = \{(z,y)\mid \|A_{11}z\| < K_1,\ |z^T A_{12}| < K_1,\ |y| < \epsilon,\ \epsilon > 0\}$

$$\dot{V} \le 2\,a_{21}^2 z^T A_{11} z + 2\beta\,[(A_{21}z)^2 - \mu^2]$$

$$+ 2h\,|y|[\,|A_{21}z| - \mu] + K\,|y|\,.$$

In the above we used the assumption that $2\beta = \eta$. If $|y| < \epsilon$ and $h \ge \dfrac{\mu^2}{\epsilon^2}$ then

$$\dot{V} \le 2a_{21}^2 z^T A_{11} z + 2\beta[(A_{21}z)^2 - \mu^2]$$

$$+ 2\frac{\mu^2}{\epsilon}\,[\,|A_{21}z| - \mu] + K\,\epsilon\,.$$

Let $0 < \delta_1 < \delta$. We can find $\epsilon_1 > 0$ small so that $\delta_1 + \beta\epsilon_1 < \delta$ and $\dot{V} < 0$ in

$$\Theta \cap \{(z,y)\mid |A_{21}z| < \mu - \delta_1,\ |y| < \epsilon_1\}\,.$$

Consider the sublevel set

$$S = \{(z,y)\mid V(z,y) < (\mu - \delta)^2\}\,.$$

By choosing ϵ small, hence $h \ge \mu^2/\epsilon^2$ large, we can make $|y| < \epsilon_1$ using (7.2). Thus for $(z,y)\in S$ we have

$$a_{21}\|z\| < \mu - \delta + \beta\epsilon_1 < \mu - \delta_1 \, .$$

Since $S \cap \Omega$ is contained in Σ, we conclude that $S \subset \Theta$. Hence S is a RAS with sliding.

$\square$

In the following sections we will extend the obtained results to multi-input systems, that is to dynamic systems driven by controllers whose structure is modeled by the additive neural network models. As in the single neuron type controllers we shall utilize ideas from the variable structure control. To proceed with the analysis we will need a method for designing a switching surface (hyperplane) for multi-input dynamic systems. This is the subject of the next Section.

It is important to observe that by using the variable structure systems approach we are able to circumvent the problems which arise from discontinuity of the nonlinearities which characterize neurons.

8. DESIGN OF THE SWITCHING HYPERPLANE FOR MULTI-INPUT SYSTEMS

In this Section we will briefly discuss a method for designing of the switching surface for multi-input systems. The method is based on that of El-Ghezawi et al. [8]. Certain relations which come out during the analysis of this method are instrumental in the construction of the state transformation discussed in the following Section.

Consider the equivalent system

$$\dot{x} = [I_n - B(SB)^{-1}S]Ax \, .$$

It is easy to see that $B(SB)^{-1}S$ is a projector and has rank m. Hence $I_n - B(SB)^{-1}S$ is also a projector with rank n−m. Therefore the matrix $A_{eq} = [I_n - B(SB)^{-1}S]A$ in the equivalent system can have at most n−m nonzero eigenvalues. Our goal is to choose S

so that the nonzero eigenvalues of A_{eq} are prescribed negative real numbers and the corresponding eigenvectors $\{w_1,...,w_{n-m}\}$ are to be chosen. Let $W = [w_1...w_{n-m}]$; note that $W \in \mathbb{R}^{n \times (n-m)}$. In sliding mode, the system is described by

$$\dot{x} = A_{eq}x$$
$$\sigma(x) = Sx = 0 \ .$$

The order of the system is $n-m$ and the solution must be in the null space of S, that is, $SW = 0$. It is well known that complete controllability of the pair (A,B) is equivalent to the existence of a controller of the form $u = -Kx$ so that the eigenvalues of $A - BK$ can be arbitrarily assigned [6]. Our equivalent system has the form

$$\dot{x} = Ax - B[(SB)^{-1}SA]x \ .$$

If we let $K = (SB)^{-1}SA$, we need $A - BK$ to have $n-m$ prescribed negative eigenvalues $\{\lambda_1,...,\lambda_{n-m}\}$ and $n-m$ corresponding eigenvectors $\{w_1,...,w_{n-m}\}$. This is equivalent to

$$(A - BK)W = WJ \tag{8.1}$$

where $J = \text{diag}[\lambda_1,...,\lambda_{n-m}]$.

Denote by R(T) the range of the operator T. Since we requires SB to be nonsingular and $SW = 0$, we must have

$$R(B) \cap R(W) = \{0\} \ . \tag{8.2}$$

It then follows that we should choose the generalized inverses B^g, W^g of B, W so that

$$B^g W = 0 \tag{8.3a}$$

and

$$W^g B = 0 \ . \tag{8.3b}$$

The above relations follow from the following identity $\begin{bmatrix} W^g \\ S \end{bmatrix} [W \ B] = \begin{bmatrix} I_{n-m} & 0 \\ 0 & I_m \end{bmatrix}$. We choose $\{w_1,...,w_{n-m}\}$ so that (8.3b) holds. We can now construct S. Let $W^{\perp} \in \mathbb{R}^{m \times n}$ be

any full rank annihalator of W, that is $W^\perp W = 0$. Since a necessary condition for $Sx = 0$ to be a switching surface is $SW = 0$, we see that $\Gamma W^\perp$ for any nonsingular $\Gamma \in \mathbb{R}^{m \times m}$ is a candidate. We also require that $SB = I_m$. Note that since $R(W) \cap R(B) = \{0\}$, $W^\perp B$ is invertible. We let $\Gamma = (W^\perp B)^{-1}$ and let $S = \Gamma W^\perp$. It is easy to see that $SB = I_m$ and hence $(W^\perp B)^{-1} W^\perp$ is a generalized inverse of B. If we let $B^g = S$ in (8.3a), the condition is satisfied.

We will utilize the results of the Section to construct a state-space transformation decoupling the neural controller from the rest of the system.

9. DECOUPLING THE NEURAL CONTROLLER FROM THE REST OF THE SYSTEM

In this Section we introduce a transformation which brings the closed-loop system into the new coordinates in which the neural structure of the controller is revealed. This transformation will also facilitate the task of estimating stability regions. The results of this Section are based on the paper by Madani-Esfahani et al. [18].

Let $M \in \mathbb{R}^{n \times n}$ be defined by

$$M = \begin{bmatrix} W^g \\ S \end{bmatrix},$$

where W^g is defined by (8.3b). Note that M is invertible with $M^{-1} = [W \ B]$. Introduce the new coordinates

$$\hat{x} = Mx.$$

Let $z = W^g x$ and $y = Sx$. Then $\hat{x} = \begin{bmatrix} z \\ y \end{bmatrix}$. In the new coordinates, the system becomes

$$\dot{\hat{x}} = MAM^{-1}\hat{x} + MBu \ .$$

We write

$$MAM^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$, $A_{22} \in \mathbb{R}^{m \times m}$. Note that

$$MB = \begin{bmatrix} 0 \\ I_m \end{bmatrix} .$$

hence

$$\dot{z} = A_{11}z + A_{12}y$$
$$\dot{y} = A_{21}z + A_{22}y + u \ . \qquad (9.1)$$

Observe that $y = \sigma$ and that $A_{21}z + A_{22}y = SAM^{-1}\hat{x} = SAx = -u_{eq}$. Thus (9.1) can be rewritten as

$$\begin{cases} \dot{z} = A_{11}z + A_{12}\sigma \\ \dot{\sigma} = -u_{eq} + u \ . \end{cases} \qquad (9.2)$$

From (8.1) we have

$$(A - BK)W = WJ \ .$$

Hence

$$W^g AW = J$$

since $W^g B = 0$ and $W^g W = I_{n-m}$. We known that $A_{11} = W^g AW$, and therefore $A_{11} = J$.

Example 9.1. Consider the following system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ .$$

We would like to design the switching surface so that the system restricted to the surface is stable and has eigenvalue -1. Suppose we choose the corresponding eigenvector for the equivalent system to be $W = w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. One can easily check that

$$W^g = [1 \ \ 0] \ , \ \ B^g = S = [1 \ \ 1]$$

satisfy (8.3). Hence the switching surface is $\sigma(x) = x_1 + x_2 = 0$. The transformation matrix is

$$M = \begin{bmatrix} W^g \\ S \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The system in the new coordinates is:

$$\dot{z} = -z + \sigma$$
$$\dot{\sigma} = 3\sigma + u \ .$$

The system restricted to the switching surface is governed by

$$\dot{z} = -z \ .$$

We analyze the closed-looped system (9.1) with the controller

$$u = - \begin{bmatrix} \mu_1 \text{sgn}\sigma_1 \\ \vdots \\ \mu_m \text{sgn}\sigma_m \end{bmatrix}. \qquad (9.3)$$

where

$$\text{sgn } \sigma_i = \begin{cases} 1 & \text{if} \quad \sigma_i > 0 \\ 0 & \text{if} \quad \sigma_i = 0 \\ -1 & \text{if} \quad \sigma < 0 . \end{cases}$$

For convenience, we let

$$D = \text{diag}[\mu_1, ..., \mu_m]$$

and

$$\text{sgn}\sigma = \begin{bmatrix} \text{sgn}\sigma_1 \\ \vdots \\ \text{sgn}\sigma_m \end{bmatrix}.$$

We can now write (9.3) as

$$u = - D \text{ sgn}\sigma . \tag{9.4}$$

Combining (9.1) and (9.4) yields

$$\left. \begin{aligned} \dot{z} &= A_{11}z + A_{12}y \\ \dot{y} &= A_{21}z + A_{22}y - D \text{ sgn } \sigma \end{aligned} \right\} \tag{9.5}$$

Note that the subsystem

$$\dot{y} = A_{21}z + A_{22}y - D \text{ sgn } \sigma$$

which can be interpreted as a dynamic controller driving the dynamic system

$$\dot{z} = A_{11}z + A_{12}y$$

has a structure of an additive neural network model. Although we arrived at (9.5) starting with the controller (9.4) whose structure does not correspond to an additive neural network model, we can utilize the above analysis in the case when we explicitly apply a neural control strategy. We proceed as follows. Suppose we are given a dynamic system model

$$\dot{x} = Ax + Bu$$

We apply an additive neural network control law

$$\dot{u} = \beta u - D \text{ sgn } \sigma(x,u) + c^T x .$$ (9.6)

The closed-loop system is

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ c^T & \beta \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} (-D \text{ sgn } \sigma(x,y)),$$ (9.7)

where $[x^T, u^T]^T \in \mathbb{R}^{n+m}$, and $\sigma(x,u)$ is a switching surface to be chosen. Using the approach presented in Section 8 we design the switching hyperplane $\sigma(x,u)$ and then construct the transformation M following the development in Section 9. In the new coordinates (9.7) will have the form (9.5), where now $A_{11} \in \mathbb{R}^{n \times n}$, $A_{22} \in \mathbb{R}^{m \times m}$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. We know that the above procedure yields a stable closed-loop system (see Section 4). However, we are also interested in the extent of the stability properties of the closed-loop system. The next Section deals with this issue.

10. ESTIMATION OF STABILITY REGIONS OF DYNAMIC SYSTEMS DRIVEN BY THE NEURAL NETWORK CONTROLLERS

This Section is devoted to the problem of estimating sliding domains of a class of systems modeled by (9.5). The development of this Section follows closely the arguments of Madani-Esfahani et al. [18]. In the analysis we shall use the following notation.

For $i = 1,2$, $j = 1,2$, we let

$$a_{ij} = \|A_{ij}\| ,$$

where the A_{ij}'s are from (9.5) and $\|A_{ij}\| = \max_x \{\|A_{ij}x\|_2 \mid \|x\|_2 \leq 1\}$.

Let

$$\mu = \min\{\mu_1,...,\mu_m\} ,$$
$$\lambda = \min\{\, |\lambda_1|,..., |\lambda_{n-m}|\, \} .$$

Note that the controllability of the pair (A,B) implies that A_{12} is not zero and hence $a_{12} \neq 0$.

We will consider the cases A_{21} is 0 and A_{21} not necessarily 0 separately in Subsections 10.1 and 10.2. The case of $A_{21} = 0$ is simpler and gives a flavor of the argument used in the general case. We obtain explicit bounds on the time it takes to reach the switching surface for both cases. Here again, as in the single-input case, we utilize a variable structure approach. This will guard us against problems caused by the discontinuous nature of the nonlinearities which characterize neural network models.

10.1. $A_{21} = 0$.

We need the following lemma.

Lemma 10.1. Suppose $\phi(t)$ is real-valued and $k \neq 0$. If

$$\dot{\phi} - k\,\phi \leq -\mu ,$$

then for $t \geq t_0$,

$$\phi(t) \leq \frac{\mu}{k} + (\phi(t_0) - \frac{\mu}{k})\, e^{k(t-t_0)} .$$

Proof. Note that $\dot{\phi} - k\,\phi \leq -\mu$ is equivalent to $\dfrac{d}{dt}\,(e^{-kt}\phi) \leq -\mu e^{-kt}$. The conclusion is obtained by integration.

□

The system to be considered in this Subsection has the form:

$$\dot{z} = A_{11} z + A_{12} \sigma$$

$$\dot{\sigma} = A_{22} \sigma - D \operatorname{sgn} \sigma .$$

(10.1)

Suppose $\sigma(t_0) = \sigma_0 \in \mathbb{R}^m$. If $\sigma \neq 0$, then

$$\frac{d \|\sigma\|}{dt} = \sigma^T A_{22} \frac{\sigma}{\|\sigma\|} - \frac{1}{\|\sigma\|} \sum_{i=1}^{m} \mu_i \, |\sigma_i| \, .$$

Since $\sum_{i=1}^{m} |\sigma_i| \geq \|\sigma\|$, we have

$$\frac{d \|\sigma\|}{dt} \leq \|A_{22}\| \, \|\sigma\| - \mu = a_{22} \|\sigma\| - \mu \, .$$

Note that if $0 < \|\sigma\| < \dfrac{\mu}{a_{22}}$, then $\dfrac{d \|\sigma\|}{dt} < 0$. Hence $\sigma(t_1) = 0$ implies that $\sigma(t) = 0$

for $t \geq t_1$. Also $\dfrac{d \|\sigma\|}{dt} < 0$ is equivalent to the condition $\sigma^T \dot{\sigma} < 0$ we have in Sec-

tion 4. By Lemma 10.1,

$$\|\sigma(t)\| \leq \frac{\mu}{a_{22}} + \left(\|\sigma_0\| - \frac{\mu}{a_{22}} \right) e^{a_{22}(t-t_0)} \, .$$

Therefore if $\|\sigma_0\| < \dfrac{\mu}{a_{22}}$, then $\sigma(t)$ is 0 for some finite t with

$$t \leq t_0 + \frac{1}{a_{22}} \left[\log \mu - \log(\mu - a_{22} \|\sigma_0\|) \right].$$

(10.2)

Observe that if A_{22} is stable, then $\mathbb{R}^n$ is a region of asymptotic stability and the switching surface is reached in finite time. Otherwise, a region of asymptotic stability is given by $\{x \in \mathbb{R}^n : \|Sx\| < \dfrac{\mu}{a_{22}}\}$. In both cases, $\sigma = 0$ is a sliding domain.

Example 9.1. (continued). Let $u = -10 \, \mathrm{sgn} \, \sigma$. The closed-loop system in the new coordinates has the form:

$$\dot{z} = -z + \sigma$$

$$\dot{\sigma} = 3\sigma - 10 \, \mathrm{sgn} \sigma .$$

In this case, $A_{22} = 3$ is unstable. The switching surface is $\sigma(x) = x_1 + x_2 = 0$. From the above, a region of asymptotic stability is given by:

$$\mathscr{R} = \{(x_1, x_2) \mid |x_1 + x_2| < \frac{10}{3}\} .$$

Actually $\mathscr{R}$ is the region of asymptotic stability (see Fig. 10).
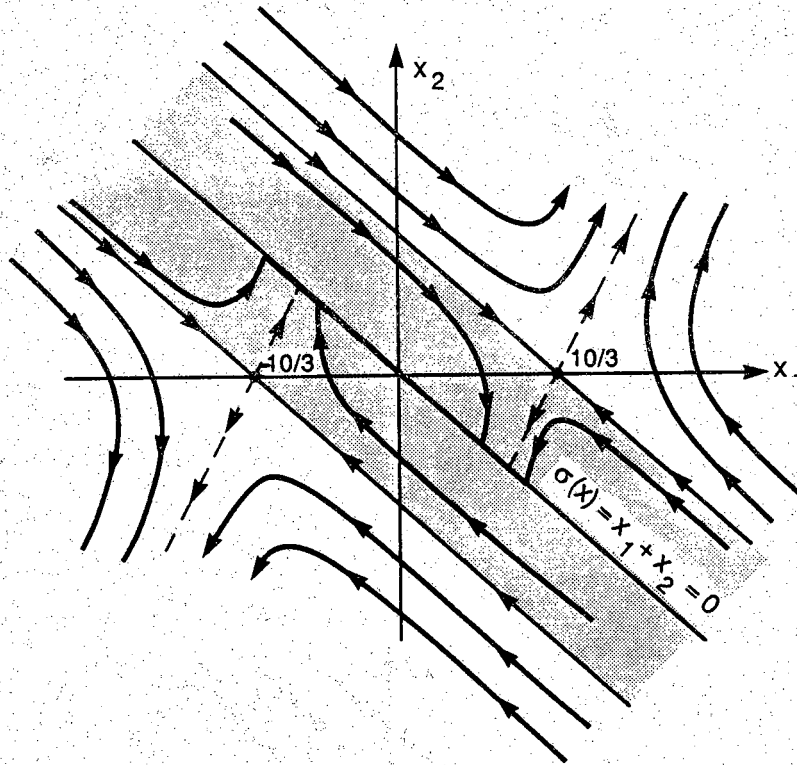
Fig. 10.   Phase-plane portrait for Example 9.1.

10.2. The general case when A_{21} is not necessarily zero

We now have

$$\dot{z} = A_{11}z + A_{12}\sigma$$
$$\dot{\sigma} = A_{21}z + A_{22}\sigma - D\text{sgn}\sigma .$$

Suppose $\|z\| \neq 0, \|\sigma\| \neq 0$. Then

$$\frac{d\|\sigma\|}{dt} = \frac{\sigma^T \dot{\sigma}}{\|\sigma\|} = \frac{\sigma^T}{\|\sigma\|} A_{21}z + \sigma^T A_{22} \frac{\sigma}{\|\sigma\|} - \sum_{i=1}^{m} \mu_i \frac{|\sigma_i|}{\|\sigma\|} ,$$

and

$$\frac{d\|z\|}{dt} = \frac{z^T \dot{z}}{\|z\|} = z^T A_{11} \frac{z}{\|z\|} + \frac{z^T}{\|z\|} A_{12} \sigma.$$

After some manipulations we obtain

$$\frac{d\|\sigma\|}{dt} \leq a_{21} \|z\| + a_{22} \|\sigma\| - \mu \qquad (10.3)$$

and

$$\frac{d\|z\|}{dt} \leq -\lambda \|z\| + a_{12}\|\sigma\| . \qquad (10.4)$$

Let

$$\Sigma_1 = \left\{ (z,\sigma) \mid \|\sigma\| < \frac{\lambda\mu}{a_{12}a_{22} + \lambda a_{22}} , \quad z \in \mathbb{R}^{n-m} , \quad \sigma \in \mathbb{R}^m \right\} ,$$

$$\Sigma_2 = \left\{ (z,\sigma) \mid a_{21}\|z\| + a_{22}\|\sigma\| < \mu , \quad z \in \mathbb{R}^{n-m} , \quad \sigma \in \mathbb{R}^m \right\} ,$$

and

$$\Sigma = \Sigma_1 \bigcap \Sigma_2 .$$

Theorem 10.2. A trajectory that starts in Σ stays in Σ and reaches the switching surface in finite time, which implies that Σ is a region of asymptotic stability (RAS).

Proof. Let

$$N_1 = \left\{ (z,\sigma) \mid a_{12}\|\sigma\| \geq \lambda \|z\| \right\}$$

and

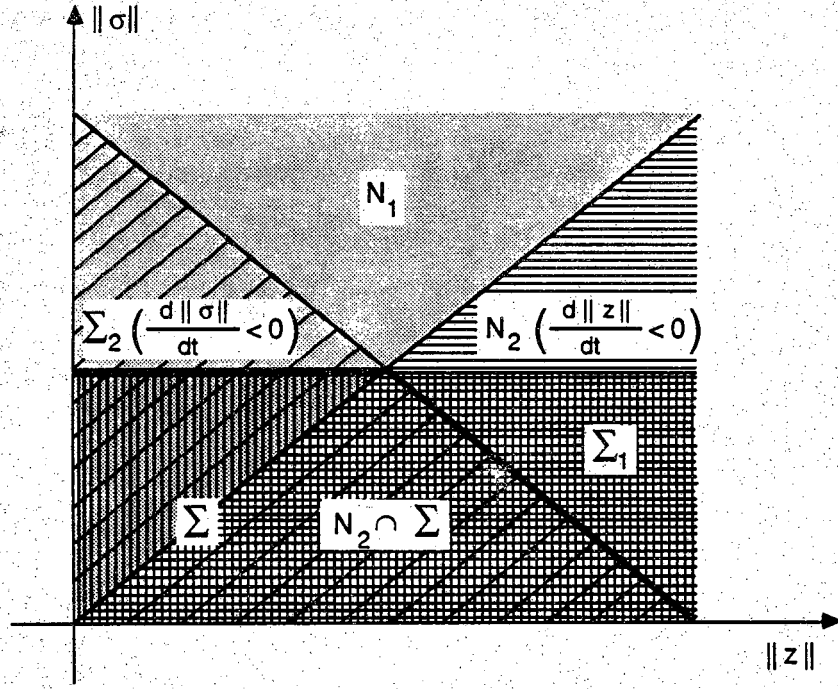$$N_2 = \left\{ (z,\sigma) \mid a_{12}\|\sigma\| < \lambda\|z\| \right\} .$$

(see Fig. 11)

Fig. 11. Illustration of regions used in the proof of Theorem 10.2.

By (10.3) we have $\dfrac{d\|\sigma\|}{dt} < 0$ in Σ. Hence if $(z(t_0)),\ \sigma(t_0))\in\Sigma$, we have $(z(t),\ \sigma(t))\in\Sigma_1$

for $t \geq t_0$. For $(z(t),\sigma(t))\in N_1 \cap \Sigma \subset N_1 \cap \Sigma_1$, we have

$$a_{21}\|z\| + a_{22}\|\sigma\|$$

$$\leq \frac{a_{21}a_{12}}{\lambda}\|\sigma\| + a_{22}\|\sigma\|$$

$$\leq \frac{a_{22}a_{12}}{\lambda}\|\sigma(t_0)\| + a_{22}\|\sigma(t_0)\|$$

$$< \mu-\epsilon \quad \text{for some } \epsilon > 0 .$$

Thus we can conclude that a trajectory $(z(t),\ \sigma(t))$ can leave Σ only through $N_2 \cap \Sigma$.

However we have $\dfrac{d\|\sigma\|}{dt} < 0$ in Σ and $\dfrac{d\|z\|}{dt} < 0$ in N_2, and hence $a_{21}\|z\| + a_{22}\|\sigma\|$ is

a decreasing function in $N_2 \cap \Sigma$. Therefore a trajectory cannot leave $N_2 \cap \Sigma$. Hence a trajectory which starts in Σ stays in Σ.

Suppose $(z(t),\ \sigma(t)) \in N_1 \cap \Sigma$ for $t_1 \le t \le t_2$. Then we have

$$\frac{d\|\sigma\|}{dt} \le a_{21}\|z\| + a_{22}\|\sigma\| - \mu$$

$$\le \left(\frac{a_{12}a_{21}}{\lambda} + a_{22}\right)\|\sigma\| - \mu\ .$$

Let $k = \dfrac{a_{12}a_{21}}{\lambda} + a_{22}$. Then by Lemma 10.1, we have

$$\|\sigma(t_2)\| \le \|\sigma(t_1)\|\ e^{k(t_2-t_1)} - \frac{\mu}{k}\left\{e^{k(t_2-t_1)} - 1\right\}.$$

Therefore

$$\|\sigma(t_1)\| - \|\sigma(t_2)\| \ge \left(\frac{\mu}{k} - \|\sigma(t_1)\|\right)\left(e^{k(t_2-t_1)} - 1\right)$$

$$\ge \left(\frac{\mu}{k} - \|\sigma(t_1)\|\right)k(t_2 - t_1)$$

$$= (\mu - k\|\sigma(t_1)\|)\,(t_2 - t_1)\ .$$

Suppose $\sigma(t_0) = \sigma_0$.

Since $\|\sigma(t)\|$ is decreasing in Σ, we have

$$\|\sigma(t_1)\| - \|\sigma(t_2)\| \ge (\mu - k\,\|\sigma_0\|)\,(t_2 - t_1)\ .$$

We conclude that a trajectory cannot spend an infinite amount of time in $N_1 \cap \Sigma$ with $\|\sigma(t)\| > 0$.

We claim that if $(z(t_1),\ \sigma(t_1)) \in N_2 \cap \Sigma$, $(z(t_2),\ \sigma(t_2)) \in N_2 \cap \Sigma$, and $t_1 < t_2$, then $\|z(t_2)\| < \|z(t_1)\|$. The claim is clear if $(z(t),\ \sigma(t)) \in N_2 \cap \Sigma$ for $t_1 \le t \le t_2$.

Otherwise, suppose $(z(t), \sigma(t)) \in N_2 \cap \Sigma$ for $t_1 < t \leq T_1$, $T_2 \leq t < t_2$, and $(z(T_1), \sigma(T_1))$, $(z(T_2), \sigma(T_2))$ are on the boundary of $N_2 \cap \Sigma$. Therefore

$$\frac{\|z(T_1)\|}{\|\sigma(T_1)\|} = \frac{\|z(T_2)\|}{\|\sigma(T_2)\|} .$$

Since $\|\sigma(t)\|$ decreases in Σ, we have $\|z(T_2)\| < \|z(T_1)\|$ and we can conclude that $\|z(t_2)\| < \|z(t_1)\|$ as in the case where the whole segment is in $N_2 \cap \Sigma$.

Hence if $(z(t_0), \sigma(t_0)) \in N_2 \cap \Sigma$, we have for $t \geq t_0$,

$$\frac{d\|\sigma\|}{dt} \leq a_{21} \|z(t_0)\| + a_{22}\|\sigma(t_0)\| - \mu < 0 .$$

Therefore, if $(z(t), \sigma(t)) \in N_2 \cap \Sigma$ for $t_1 \leq t \leq t_2$ then

$$\|\sigma(t_1)\| - \|\sigma(t_2)\| \geq (\mu - a_{21}\|z(t_0)\| - a_{22}\|\sigma(t_0)\|)(t_2 - t_1).$$

We can conclude that a trajectory cannot spend an infinite amount of time in $N_2 \cap \Sigma$ with $\sigma(t) \neq 0$.

Thus we must reach the switching surface $\sigma = 0$ in finite time if we start in Σ.

$\square$

From the above, we can give explicit estimates of the time it takes to reach the switching surface starting in Σ.

Corollary 10.3. Let

$$\beta = \min\left\{\mu - (\frac{a_{12}a_{21}}{\lambda} + a_{22})\|\sigma(t_0)\|, \ \mu - a_{21}\|z(t_0)\| - \|\sigma(t_0)\|\right\}$$

Starting at $(z(t_0), \sigma(t_0)) \in \Sigma$, we must reach $\sigma = 0$ in

$$t \leq t_0 + \frac{\|\sigma(t_0)\|}{\beta} .$$

Example 10.1. Consider the following system:

$$\dot{x} = \begin{bmatrix} 3 & 1 & 1 \\ -6 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

with $|u_i| \leq 10$. Suppose the desired eigenvalue of the reduced order system is -3 with

corresponding eigenvector $w_1 = \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}$. One can check that

$$W^g = w_1^g = [1\ 0\ 0], \quad S = B^g = \begin{bmatrix} 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfy (8.3). The transformation matrix is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system in the new coordinates has the form:

$$\dot{z} = -3z + [1\ 2]\ \sigma$$

$$\dot{\sigma} = \begin{bmatrix} -30 \\ 0 \end{bmatrix} z + \begin{bmatrix} 7 & 10 \\ 0 & 3 \end{bmatrix} \sigma - 10\ \text{sgn}\ \sigma\ .$$

In this case, $\mu = 10$, $\lambda = 3$, $a_{12} = 2.24$, $a_{21} = 30$, $a_{22} = 12.46$. Hence

$$\Sigma = \{(z,y) : \|y\| < \frac{30}{104.58},\ 30\|z\| + 12.46\|y\| < 10\}\ .$$

Using a Lyapunov function argument, we can give another region of asymptotic stability.

Theorem 10.4. A region of asymptotic stability of the system is

$$\mathscr{R} = \{(z,\sigma) \, | \, \alpha \|z\| + \beta \|\sigma\| < \mu\}$$

where $\beta = \dfrac{1}{2}\left[(a_{22}-\lambda) + \sqrt{(a_{22}+\lambda)^2 + 4a_{12}a_{21}}\,\right]$ and $\alpha = \beta a_{21}/(\beta+\lambda)$.

Proof. Let V be the positive definite function defined by

$$V(z,\sigma) = \alpha \|z\| + \beta \|\sigma\| \,.$$

The Lyapunov derivative is

$$\dot{V}(z,\sigma) = \alpha \frac{z^T \dot{z}}{\|z\|} + \beta \frac{\sigma^T \dot{\sigma}}{\|\sigma\|} \,.$$

From (10.3), (10.4) we get

$$\dot{V}(z,\sigma) \leq \alpha(-\lambda\|z\| + a_{12}\|\sigma\|) + \beta(a_{21}\|z\| + a_{22}\|\sigma\| - \mu)$$

$$= (-\lambda\alpha + \beta a_{21})\|z\| + (\alpha a_{12} + a_{22}\beta)\|\sigma\| - \mu\beta \,.$$

Using the values of α and β, we have

$$\dot{V}(z,\sigma) \leq \beta[\alpha\|z\| + \beta\|\sigma\| - \mu]$$

The right hand side is less than 0 in $\mathscr{R}$. This finishes the proof if $z(t)\neq 0$ and $\sigma(t)\neq 0$.

Otherwise observe that $\beta \geq a_{22}$ and $\alpha \leq a_{21}$. By (10.3) and (10.4) we have on $\mathscr{R} \cap \{z = 0\}$

$$\frac{d\|\sigma\|}{dt} \leq a_{22}\|\sigma\| - \mu \leq \beta\|\sigma\| - \mu < 0$$

and on $\mathscr{R} \cap \{\sigma = 0\}$

$$\frac{d\|z\|}{dt} \leq -\lambda\|z\| < 0 \,.$$

Hence $V(z,\sigma)$ decreases on the critical surfaces also and we are done.

$\square$

As a consequence of Theorems 10.2 and 10.4 we give a new region of asymptotic stability with sliding.

Theorem 10.5. Let

$$\mathscr{R}_1 = \{(z,\sigma)\,|\,\alpha\|z\| + \beta\|\sigma\| < \mu,\ \|\sigma\| \geq \frac{\lambda}{a_{12}}\|z\|\}$$

$$\cup \{(z,\sigma)\,|\,a_{21}\|z\| + a_{22}\|\sigma\| < \mu,\ \|\sigma\| \leq \frac{\lambda}{a_{12}}\|z\|\}.$$

Then $\mathscr{R}_1$ is a region of asymptotic stability with sliding.

Proof. We use the same notation as in Theorems 10.2 and 10.4. Observe that

$$\mathscr{R}_1 = \Sigma \cup (\mathscr{R} \cap N_1).$$

For a trajectory that starts in $\mathscr{R} \cap N_1$ to reach the switching surface $\{\sigma{=}0\}$, it must pass through Σ, which is a region of asymptotic stability with sliding. See Fig. 12.

$\square$
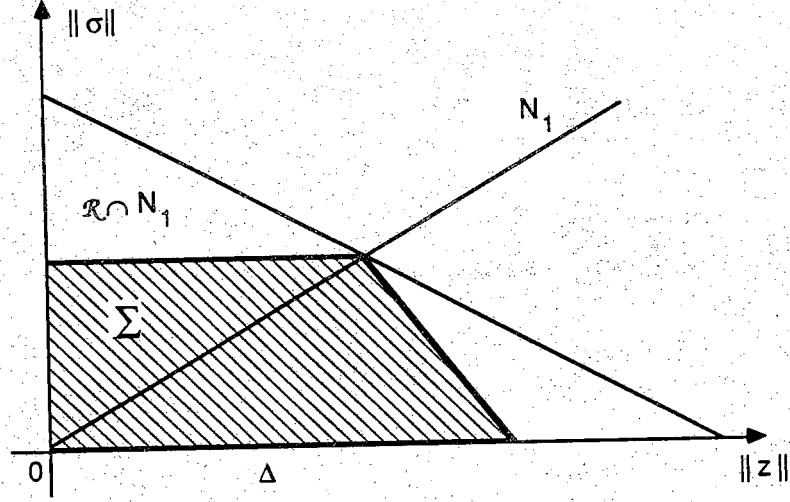
Fig. 12. Illustration of the proof of Theorem 10.5.

Example 10.1. (continued). We have found

$$\Sigma = \{(z,\sigma) \,|\, 30\|z\| + 12.46\|\sigma\| < 10, \,\|\sigma\| < 0.287\} \,.$$

We have $\alpha = 25.263$ and $\beta = 15.997$. Therefore

$$\mathcal{R} = \{(z,\sigma) \,|\, 25.263\|z\| + 15.997\|\sigma\| < 10\}$$

and

$$\mathcal{R}_1 = \Sigma \cup (\mathcal{R} \cap N_1) \,.$$

Recall that $N_1 = \{(z,\sigma) \,|\, \|\sigma\| > 1.34\|z\|\}$.

11. CONCLUDING REMARKS

In this report we investigated viability of employing controllers based on additive neural network models to the problem of stabilization (tracking) of a class of dynamic systems. Two approaches to designing stabilizing controllers were proposed. Elements of the variable structure control theory were utilized to construct such controllers. The proposed controllers are characterized by robustness property which is inherent in the variable structure controllers. An important role in the analysis was played by a special state space transformation. This transformation not only facilitated the stability analysis but also helped to utilize additive neural network models in designing stabilizing controllers. The proposed approach is promising in three ways. First, it results in robust controllers. Second, it has a potential to be employed in constructing fault tolerant controllers. Third, it allowed us to circumvent stability analysis problems caused by the discontinuous nonlinearities which describe neurons. Also generalizations to the control of a more general class of dynamic systems are feasible. The proposed approach in this report and the results of Walcott and Žak [26] constitute a nice starting point to designing neural network based state estimators for dynamic systems.

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