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TABLE OF CONTENTS

Page

LIST OF FIGURES

ABSTRACT

G/M/1 and M/G/1-type Markov processes provide natural models for widely differing stochastic phenomena. Efficient recursive solutions for the equilibrium and transient analysis of these processes are therefore of considerable interest. In this direction, a new class of recursive solutions are proposed for the analysis of M/G/1 and G/M/1 type processes.

In this report, the notion of when a process is *LEDI-complete,* which means it has *complete Level Entrance Direction Information,* is introduced for G/M/1-type Markov processes. This notion leads to a new class of recursive solutions, called *finite-memory recursive solutions,* for the equilibrium probabilities of a class of G/M/1-type Markov processes. A finite-memory recursive solution of order k has the form

$$\overline{\pi}_{n+k} = \overline{\pi}_n \mathbf{W}_1 + \overline{\pi}_{n+1} \mathbf{W}_2 + \cdots + \overline{\pi}_{n+k-1} \mathbf{W}_k,$$

where $\overline{\pi}_n$ is the vector of limiting probabilities of the states on level n of the process and $\mathbf{W}_i$, $1 \leq i \leq k$, are square matrices.

It is also shown that the concept of LEDI-completeness leads to a finite-memory recursive solution for the transient behavior of this class of G/M/1-type processes. Such a recursive solution has the form

$$\overline{\pi}_{n+k}(s) = \overline{\pi}_n(s)\mathbf{W}_1(s) + \overline{\pi}_{n+1}(s)\mathbf{W}_2(s) + \cdots + \overline{\pi}_{n+k-1}(s)\mathbf{W}_k(s).$$

where $\overline{\pi}_n(s)$ is the Laplace transform of $\overline{\pi}_n(t)$, the vector of state occupancy probabilities at time t for the states on level n of the process.

The relationship between these finite-memory recursive solutions and matrix geometric solutions is also explored. The results are extended to the case where the transition rates are level dependent.

It is also briefly explained how a finite memory recursion for the equilibrium and transient probabilities of M/G/1 type Markov processes can be obtained.

CHAPTER 1
INTRODUCTION

In the algorithmic analysis of various stochastic models which arise in a wide range of stochastic phenomena, efficient computational forms for both the equilibrium and transient probabilities of the underlying stochastic process are desirable. Recursive forms, such as the matrix geometric solutions developed in [Neu1],[Neu2] for the equilibrium probabilities of a large class of processes, are particularly desirable since they often greatly reduce the dimensionality of the computational problem.

One very large class of processes for which these recursive solutions exist, at least for the equilibrium probabilities of the process, is the class of Markov processes of G/M/1-type. This is fortunate since these processes provide good stochastic models for problems arising in computer communications, queueing theory, and inventory theory.

The recursive solution developed in [Neu1],[Neu2] for the equilibrium probabilities of these processes arises in a natural fashion from the structure of the state space and the types of transitions allowed in these processes. The state space E of a G/M/1-type Markov process is usually assumed to have the following form:

$$E = \{ (i,j): i \geq 0,\ 1 \leq j \leq N \} \tag{1.1}$$

in which N is finite but otherwise arbitrary. This state space can be clearly broken up into levels by performing a lexicographic partitioning on the first state variable [Neu1]. For each level k, a probability vector $\overline{\pi}_k(t)$ can then be defined. It contains the probabilities of state occupancy at time t for each of N states on that level. Thus, for level k,

$$\overline{\pi}_k(t) = \left[\pi_{k,1}(t)\ \ \pi_{k,2}(t)\ \ \cdots\ \ \pi_{k,N}(t) \right] \tag{1.2}$$

where $\pi_{k,j}(t)$ is the probability the process is state (k,j) at time t. The entire set of occupancy probabilities at time t for the process is then specified by the infinite-dimensional vector

$$\overline{\pi}(t) = \left[\overline{\pi}_0(t)\ \ \overline{\pi}_1(t)\ \ \overline{\pi}_2(t)\ \ \cdots\ \ \overline{\pi}_n(t)\ \ \cdots \right]. \tag{1.3}$$

In a G/M/1-type process, upward transitions from level k can only reach level k+1; downward transitions from level k can reach, in one transition, any level j for $j < k$. This fact, plus the above partitioning and ordering of the state space into levels, implies that the generator $\mathbf{Q}$ of the process has the following form:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{C}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{B}_3 & \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \mathbf{B}_4 & \mathbf{A}_4 & \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{1.4}$$

in which all matrices except possibly some of the boundary matrices are N×N.

The probability vector $\overline{\pi}(t)$ is then the unique solution to

$$\frac{d\,\overline{\pi}(t)}{dt} = \overline{\pi}(t)\mathbf{Q}, \qquad \overline{\pi}(t)\big|_{t=0} = \overline{\pi}(0). \tag{1.5}$$

and the vector of limiting probabilities is given by

$$\lim_{t\to\infty} \overline{\pi}(t) = \overline{\pi}. \tag{1.6}$$

When the process is recurrent nonnull, the vector $\overline{\pi}$ is the unique solution to

$$\overline{\pi}\mathbf{Q} = \mathbf{0} \tag{1.7}$$

whose entries sum to one.

Solving (1.5) and (1.7) directly is very cumbersome without some type of recursive structure. Ideally, we would like to find an expression for $\overline{\pi}_k$ in terms of $\overline{\pi}_j$ for $j < k$, and an expression for $\overline{\pi}_k(t)$ in terms of $\overline{\pi}_j(t)$ for $j<k$.

In [Neu1],[Neu2] it is shown that $\overline{\pi}_k$ can be found in terms of $\overline{\pi}_{k-1}$ as follows:

$$\overline{\pi}_k = \overline{\pi}_{k-1}\mathbf{R} \tag{1.8}$$

where the matrix $\mathbf{R}$ is the minimal nonnegative solution of

$$\sum_{i=0}^{\infty} \mathbf{X}^i \mathbf{A}_i = \mathbf{0}. \tag{1.9}$$

The $\mathbf{A}_i$'s in (1.9) are the submatrices in the generator in (1.4). Iterative techniques for computing $\mathbf{R}$ are provided in, for instance, [Neu1]. Methods of directly computing the minimal nonnegative solution of (1.9) have recently been developed in [MuC1].

This chapter introduces new techniques for the solution of (1.5) and (1.7) for G/M/1-type Markov processes. These techniques rely on the notion of *Complete Level Entrance Direction Information* or, more concisely, *LEDI-completeness,* and the fact that every G/M/1-type Markov process can be modified to be LEDI-complete by an appropriate expansion of its state space.

Specifically, the notion of LEDI-completeness is used to develop a new class of recursions, called *Finite-Memory Recursive (FMR) Solutions,* for those G/M/1-type Markov processes in which there is exactly one state on each level which accepts downward transitions. A finite-memory recursive solution of order r for the equilibrium probabilities of such processes has the form

$$\bar{\pi}_{k+r} = \bar{\pi}_{n}\mathbf{W}_1 + \bar{\pi}_{k+1}\mathbf{W}_2 + \cdots + \bar{\pi}_{k+r-1}\mathbf{W}_r, \tag{1.10}$$

where $\bar{\pi}_k$ is the vector of limiting probabilities of states on level k of the process and $\mathbf{W}_i$, $1 \leq i \leq r$, are N×N matrices. An algorithm for computing the equilibrium probabilities through finite memory recursions is provided. Also, the relationship between finite memory recursions and matrix geometric recursions for the equilibrium probabilities is explored. These results are proven in Chapter 2.

A finite-memory recursive solution of order r for the transient probabilities is then shown to exist in the transform domain for this class of G/M/1-type processes. With $\bar{\pi}_k(s)$ representing the Laplace transform of $\bar{\pi}_k(t)$, it will be shown that

$$\bar{\pi}_{n+k}(s) = \bar{\pi}_n(s)\mathbf{W}_1(s) + \bar{\pi}_{n+1}(s)\mathbf{W}_2(s) + \cdots + \bar{\pi}_{n+k-1}(s)\mathbf{W}_k(s). \tag{1.11}$$

On inverting the transform, the vector of state occupancy probabilities at time t can be found. These results are explained in Chapter 3. Also, the results are extended to the level dependent G/M/1-type Markov processes.

Thus, finite memory recursive solutions provide a tractable method for both the equilibrium *and* transient analysis of the class of G/M/1-type Markov processes considered in this chapter.

In Chapter 4, it is briefly described how the notion of LEDI-completeness leads to a finite memory recursion for the computation of the equilibrium and time dependent probability distribution of M/G/1-type Markov processes.

The notion of LEDI-completeness and the finite memory recursions developed in this chapter are related to the notion of LCI-completeness and the matrix geometric solutions that have been developed in [BeC] and [ZhC1] for both the equilibrium and transient behavior of Quasi-Birth-and-Death (QBD) processes. For QBD-processes, which are a special class of G/M/1-type processes, the matrix geometric solution

$$\overline{\pi}_k = \overline{\pi}_{k-1}\mathbf{W} \tag{1.12}$$

has been shown to exist [BeC], where $\mathbf{W}$ is not equal to the matrix $\mathbf{R}$. For the transient case, it has been shown in [ZhC1] that

$$\overline{\pi}_k(s) = \overline{\pi}_{k-1}(s)\mathbf{W}(s). \tag{1.13}$$

The significance of the approach in this chapter and the related work in [ZhC1] for transient analysis is now explained by contrasting it with other currently known approaches to transient analysis. In general, these other approaches do not lead to recursions of small dimension and do not always reveal the structure of the transient solution.

One other possible approach is that of direct solution of the differential equation in (1.5). This is certainly possible when the state space is finite or has been truncated, since the solution is then the well known matrix exponential:

$$\overline{\pi}(t) = \overline{\pi}(0)\exp(\mathbf{Q}t). \tag{1.14}$$

This solution can be computed using one of the many possible methods in [MoV]. When $\mathbf{Q}$ is infinite-dimensional the only case in which an explicit solution has been found is for the birth-death process with constant transition rates [Coh],[Kle]. Even for this very simple process the expression for the transition function is very complicated and approximations must often be employed just to evaluate it [Ack],[Can],[CaO],[Ste].

For a continuous-time Markov chain with an infinite-dimensional infinitesimal generator, $\mathbf{Q}$, the technique of uniformization often leads to nice solutions [Gra]. It is based on the reduction of the continuous-time Markov chain to a discrete-time chain subordinated to a Poisson process, and can be applied when all the diagonal entries of $\mathbf{Q}$ are bounded in magnitude by the same real number. If this real number is called q, the transient solution is computed by truncating the following series:

$$\overline{\pi}(t) = \sum_{n=0}^{\infty} \nu^n \frac{(qt)^n}{n!} \tag{1.15}$$

where $\nu^n = \nu^{n-1}\mathbf{P}$, $\nu^0 = \pi(0)$, $\mathbf{P} = (\mathbf{Q}/q) + \mathbf{I}$.

It may be possible that the approach in [Neu1], which results in the matrix geometric solution in (1.8) above, can be extended to the transient case as well. One possible vehicle for this extension might be the matrix $\mathbf{R}(s)$ introduced in [Ram]. It would be very interesting to see if this leads to a result of the form

$$\overline{\pi}_k(s) = \overline{\pi}_{k-1}(s)\mathbf{R}(s) \tag{1.16}$$

for G/M/1-type processes.

Some other recent work on transient analysis consists of studies of the transient behavior of regulated (reflecting) Brownian motion and the M/M/1 queue [AbW1],[AbW2],[AbW3] and transient analysis of the integrated services digital networks [ZhC2] and the time-dependent M/M/1 queue [ZhC3].

Notation

Throughout this report, bold lower-case letters, such as $\mathbf{f}$, denote column vectors; bold lower-case letters with an overbar, such as $\overline{\mathbf{f}}$, denote row vectors and bold capital letters, such as $\mathbf{H}$, denote matrices. Regular type is used for sets and scalars, and the symbols $\mathbf{I}$, $\mathbf{0}$, and $\mathbf{e}$ denote the identity matrix, the matrix of zeros, and a column vector of 1's, respectively.

CHAPTER 2
FINITE MEMORY RECURSIVE SOLUTIONS FOR THE EQUILIBRIUM ANALYSIS OF G/M/1 TYPE MARKOV PROCESSES

2.1 State Space Expansions-Complete Level Entrance Direction Information:

Consider a G/M/1-type Markov process with the generator $\mathbf{Q}$ in the canonical form given in (1.4). The state space of the Markov process is

$$E = \{(i,j) : i \geq 0 ; j \in T \text{ for } i=0, j \in S \text{ for } i \geq 1\},$$

where $S = \{s_1, s_2, \cdots, s_N\}$ and $T = \{t_1, t_2, \cdots, t_k\}$. Thus, the matrices $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{B}_i$, $i \geq 1$ are k×k, k×N and N×k, respectively. The matrices $\mathbf{A}_i$, $i \geq 0$, are all square matrices of dimension N.

It is assumed throughout this chapter that the entries of all of the matrices defined above are finite, and that there is a fixed integer m such that $\mathbf{A}_k = \mathbf{0}$, $\mathbf{B}_k = \mathbf{0}$ $\forall$ $k \geq m+1$. Consequently, the G/M/1-type Markov process is regular [Cin, pg 251] since all transitions rates are then bounded by a common constant. It is also assumed throughput this paper that the process is irreducible.

The state space of the Markov process can be partitioned in a natural manner as explained in the following definition.

Definition 2.1: The *Lexicographic partition* of the state space E, of a G/M/1-type Markov process consists of levels L_i such that

$$L_i \triangleq \{(i,j) : i \text{ fixed}; j \in S\} \quad \forall \ i \geq 1$$

and the boundary level

$$L_0 \triangleq \{ (0,j) : j \in T \}.$$

■

The state space E can be expanded to help understand the probabilistic behavior of the original process and to simplify the computation of the equilibrium probabilities. In the following, the considerations which arise in the expansion of the state space of an arbitrary Markov process are discussed and a specific expansion procedure for G/M/1-

type Markov processes is developed.

Consider a Markov process $\mathbf{X}_t'$ with state space E′ such that $\mathbf{X}_t$ is its aggregated process with state space E. Equivalently, under state space expansion $\mathbf{X}_t$ is embedded into $\mathbf{X}_t'$. For any superset E′ of E, there exists a partition δ of E′ with index set E. Let $\mathbf{Q}' = (q'_{ij})_{i,j \in E'}$ be the infinitesimal generator of $\mathbf{X}_t'$ and assume that every coset E_i is aggregated into a state i ($i \in E$). It is shown in [FeL], [MeY] that $\mathbf{X}_t$ is Markovian if and only if $\sum_{l \in E_j} q'_{kl}$ is independent of k for all $k \in E_i$, and is equal to q_{ij}. Thus the total transition rate from any of the states in E_i into E_j must be equal. There is, however, freedom in how to direct these rates into the states of the coset E_j. This freedom can be exploited to arrive at an efficient computational form for the limiting probabilities.

Now G/M/1-type Markov processes are considered and a state space expansion technique is described which will eventually lead to a recursive computational form for the equilibrium probabilities. The following definition provides the criterion for state space expansion.

Definition 2.2: The state space of a canonical, irreducible Markov process of G/M/1-type is said to have *Complete Level Entrance Direction Information,* or to be *LEDI-Complete,* if for every positive integer i the set of states L_i on level i can be partitioned into two sets U and D such that none of the states in U can be reached in one transition from any of the states on levels L_k, $k > i$ and none of the states in D can be reached in one transition from any of the states in level L_{i-1}.
■

Clearly, the states in U accept upward transitions, but not downward transitions; those in D accept downward transitions, but not upward transitions. States which accept neither upward or downward transitions may be placed in either D or U.

The G/M/1-type process described by the generator $\mathbf{Q}$ in (1.4) is LEDI-complete if and only if the following condition on the columns of the matrices, $\mathbf{A}_i$, $i \geq 0$, $i \neq 1$ is satisfied. *If column j in* $\mathbf{A}_0$ *has any nonzero entries, all entries in column j of each matrix* $\mathbf{A}_i$, $i \geq 2$ *must be exactly zero; and, if column k in some matrix* $\mathbf{A}_i$, $i \geq 2$ *has any non zero entry, all entries in column k of* $\mathbf{A}_0$ *must be exactly zero.*

It is shown in the following theorem that any G/M/1-type process can be modified so that it is LEDI-complete without changing the probabilistic behavior of the process.

Theorem 2.1: Any G/M/1-type process can be modified into a second G/M/1-type process which is LEDI-complete by an expansion of its state space.

Proof (by Construction): Let the original G/M/1-type process be $\mathbf{Z}_t = (X_t, Y_t)$ and the new process be $\tilde{\mathbf{Z}}_t = (\tilde{X}_t, \tilde{Y}_t, \tilde{W}_t)$. The new process can be defined by specifying its state transition diagram or, equivalently, its generator. Since the state space expansion technique can be better visualized through the state transition diagram, it is chosen.

The state transition diagram for $\tilde{\mathbf{Z}}_t$ is obtained from the state transition diagram of $\mathbf{Z}_t$ by following the procedure outlined below for the modification of the state transition diagram of $\mathbf{Z}_t$.

The states in the original state transition diagram of $\mathbf{Z}_t$ are considered one at a time. Suppose the state currently being considered is (i,y).

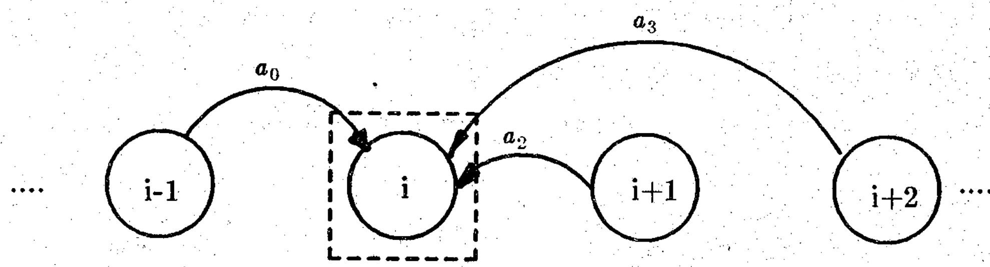
(a) If (i,y) accepts inward transitions only from level i, relabel it as (i,y,0). The transitions into and out of (i,y,0) are the same as those into and out of (i,y).

(b) If (i,y) accepts one or more inward transitions originating in any of the levels L_k, $k \geq i+1$, and accepts no inward transitions originating in L_{i-1}, then relabel the state as (i,y,2). The transitions into and out of (i,y,2) are the same as those into and out of (i,y).

(c) If (i,y) accepts one or more inward transitions from level L_{i-1}, and no inward transitions originating from any level L_k with $k > i$, then relabel it as (i,y,1). The transitions into and out of (i,y,1) are the same as those into and out of (i,y).

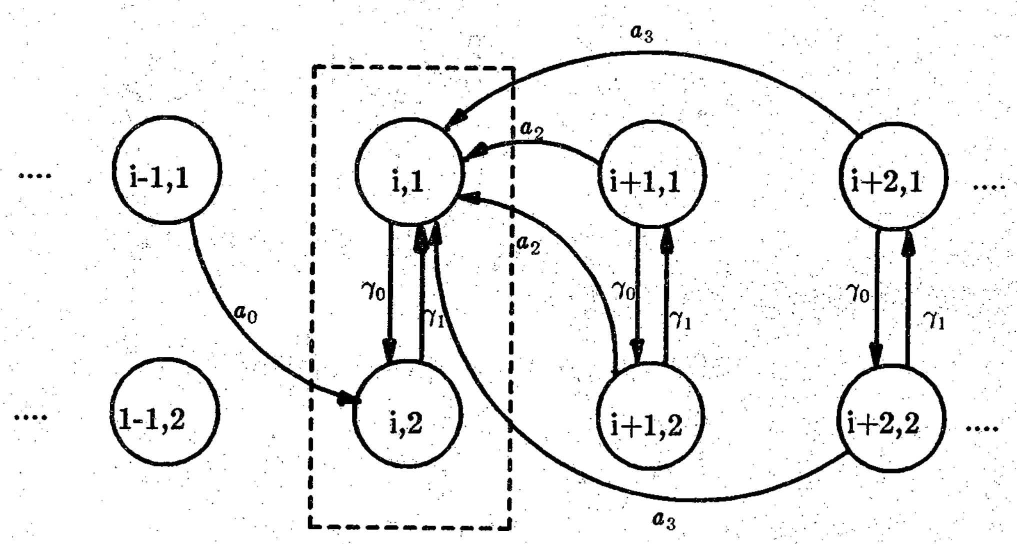
(d) If (i,y) accepts inward transitions from both level L_{i-1} and one or more of the levels L_k for $k \geq i+1$, split it into two states, labeled as (i,y,1) and (i,y,2). All transitions from L_{i-1} which used to end in (i,y) are now transferred to (i,y,1). All transitions from every L_k, $k \geq i+1$ which used to end in (i,y) are now transferred to (i,y,2). Any transition from L_i which used to end in (i,y) may now be sent to either (i,y,1) or (i,y,2). The rates of all these inward transitions remain the same, even if their destination has been changed. The destination and rates of all transitions leaving both (i,y,1) and (i,y,2) are the same as the destination and rates of all transitions leaving state (i,y). This state space expansion technique is illustrated in Figure 2.1.

Once steps (a) through (d) have been applied to each state in the state transition diagram for $\mathbf{Z}_t$, the new state transition diagram specifies the generator of the process



LEDI Incomplete G/M/1-Type Markov Process

Fig 1-a. Before Splitting



LEDI Complete G/M/1-Type Markov Process

Fig 1-b. After Splitting

2.1 State Space Expansion Technique applied to a G/M/1 type queue.

$\tilde{Z}_t = (\tilde{X}_t,\ \tilde{Y}_t,\ \tilde{W}_t)$ with state space

$$\tilde{E} = \left\{ (i,j,k) : i \geq 0;\ j \in T \text{ for } i=0,\ j \in S \text{ for } i \geq 1;\ k \in \{0,1,2\} \right\}.$$

From the construction of $\tilde{Z}_t$, the following facts are immediate:

(1) The process $\tilde{Z} = (\tilde{X}_t,\ \tilde{Y}_t,\ \tilde{W}_t)$ is an LEDI-complete G/M/1-type Markov process since, by construction, none of the states at any level receives both upward and downward transitions.

(2) $\tilde{Z}_t$ is irreducible if and only if Z_t is irreducible.

(3) For any w, the distribution of the sojourn time of the new process in the state (x,y,w) is the same as the sojourn time of the original process in the state (x,y). This follows since these states have the same outward transition rates.

(4) Let the initial distribution of the original process be $\pi_0(i,j)$ which is defined for all the states $(i,j) \in E$. Define the initial distribution $\hat{\pi}_0(i,j,k)$ of the new process to be any distribution for which $\sum_k \hat{\pi}_0(i,j,k) = \pi_0(i,j)$. Then, for any time t, the probability that the original process is in the state (i,j) at time t is the sum, over all values of k of the probability that the new process is in the state (i,j,k).

(5) Suppose that $\tilde{Z}_t(\omega) = (\tilde{X}_t(\omega), \tilde{Y}_t(\omega), \tilde{W}_t(\omega))$ is a sample function of $\tilde{Z}_t$. Then the projection $(\tilde{X}_t(\omega), \tilde{Y}_t(\omega))$ is a valid sample function of the original process, Z_t.

■

Example 2.1: The effect of the above state splitting algorithm on the state transition diagram of a G/M/1-type Markov process with a single state on each level is illustrated in Figures 1.a and 1.b.

Without loss of generality, we now restrict consideration to LEDI-complete G/M/1-type Markov processes. This allows the set of states on each level to be partitioned into two disjoint groups as described below

Consider the N states $L_i = \{s_1, s_2, \cdots, s_N\}$ on level i of an LEDI-complete process. Group the states in L_i in such a way that the last r elements correspond to the states which accept downward transitions. These last r elements in L_i are all the states (i,j,k) where k=2. They are all the states in level L_i which accept downward transitions. This partitioning leads, in the next section, to a canonical form for the generator of an LEDI-complete, G/M/1-type Markov process and consequently to *finite memory recursive* solutions.

2.2 Existence of Finite Memory Recursive Solutions for the Equilibrium Probabilities:

The submatrices $\mathbf{A}_k$, $k \geq 0$ of the generator of an LEDI-complete G/M/1-type Markov process may each be blocked into right and left halves by grouping the states at each level as described above. Let $\mathbf{A}_{kL}$, for all $k \geq 0$, consist of the leftmost m-r columns of the matrices $\mathbf{A}_k$, $k \geq 0$. Let $\mathbf{A}_{kR}$, for all $k \geq 0$, be the rightmost r columns of $\mathbf{A}_k$, $k \geq 0$.

Since the process is LEDI-complete and the set of states at each level is partitioned so that the last r states are those that accept downward transitions, the rightmost r columns of $\mathbf{A}_0$, which are called $\mathbf{A}_{0R}$, are all zero. Similarly the leftmost m-r columns of $\mathbf{A}_k$, $k \geq 2$ which are called $\mathbf{A}_{kL}$, are all zero.

The above discussion shows that the generator of an LEDI-complete G/M/1-type Markov process can be written as

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \cdots \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{A}_{3R} & \mathbf{0} & \mathbf{A}_{2R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{A}_{mR} & \mathbf{0} & \mathbf{A}_{m-1R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.1}$$

It can be noted that the states at the level zero need not be split since they receive only downward transitions. Recall that in this chapter we are considering G/M/1-type Markov processes for which $\mathbf{A}_k = \mathbf{0}$ and $\mathbf{B}_k = \mathbf{0}$, for $k \geq m+1$ or equivalently $[\mathbf{A}_{kL} : \mathbf{A}_{kR}] = \mathbf{0}$ for $k \geq m+1$.

Perform the following column operations on the generator. Throughout the generator, add to the set of N-r columns with the following block structure

$$\begin{bmatrix} \mathbf{A}_{0L} \\ \mathbf{A}_{1L} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{bmatrix} = \mathbf{G}_1, \text{ the set of N--r columns } \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{A}_{0L} \\ \mathbf{A}_{1L} \\ \vdots \end{bmatrix} = \mathbf{G}_2. \tag{2.2}$$

More precisely, for all $k \geq 2$, add to the N-r columns, $\{kN+s\ ,\ 1 \leq s \leq N-r\}$, the N-r columns $\{kN + N(N-2) + t\ ,\ 1 \leq t \leq N-r\}$.

After these column operations, a new matrix $\hat{\mathbf{Q}}$ is obtained, where

$$\hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{m-2} & \mathbf{0} & \mathbf{A}_{(m-2)R} & \mathbf{0} & \mathbf{A}_{(m-3)R} & \cdots \\ \mathbf{B}_{m-1} & \mathbf{0} & \mathbf{A}_{(m-1)R} & \mathbf{A}_{0L} & \mathbf{A}_{(m-2)R} & \cdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{A}_{mR} & \mathbf{A}_{1L} & \mathbf{A}_{(m-1)R} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{mR} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.3}$$

When the Markov process is recurrent nonnull, there exists a unique solution to the infinite system of linear equations $\overline{\pi}\tilde{\mathbf{Q}} = \mathbf{0}$, or, equivalently, $\overline{\pi}\hat{\mathbf{Q}} = \mathbf{0}$. On utilizing the structure of $\hat{\mathbf{Q}}$, the system of equations $\overline{\pi}\hat{\mathbf{Q}} = \mathbf{0}$ leads to

$$\begin{bmatrix} \overline{\pi}_0 & \overline{\pi}_1 & \cdots & \overline{\pi}_m \end{bmatrix} \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} \\ \mathbf{B}_1 & \mathbf{A}_{1L} \\ \mathbf{B}_2 & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} \end{bmatrix} = \mathbf{0} \tag{2.4}$$

and

$$\left[\overline{\pi}_n \ \ \overline{\pi}_{n+1} \ \ \cdots \ \ \overline{\pi}_{n+m-1}\right]\begin{bmatrix} \mathbf{A}_{1R} & \mathbf{A}_{0L} \\ \mathbf{A}_{2R} & \mathbf{A}_{1L} \\ \mathbf{A}_{3R} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{A}_{(m-2)R} & \mathbf{0} \\ \mathbf{A}_{(m-1)R} & \mathbf{A}_{0L} \\ \mathbf{A}_{mR} & \mathbf{A}_{1L} \end{bmatrix} = \mathbf{0}, \ \text{for all} \ n \geq 1. \tag{2.5}$$

When $[\mathbf{A}_{mR} : \mathbf{A}_{1L}]$ is invertible, (2.5) is equivalent to

$$\overline{\pi}_{n+m-1} = \overline{\pi}_n \mathbf{W}_1 + \overline{\pi}_{n+1} \mathbf{W}_2 + \cdots + \overline{\pi}_{n+m-2} \mathbf{W}_{m-1},$$

where, for instance, $\mathbf{W}_{m-1} = \left[\mathbf{A}_{(m-1)R} : \mathbf{A}_{0L}\right]\left[\mathbf{A}_{mL} : \mathbf{A}_{1L}\right]^{-1}$.

Thus, the notion of LEDI-completeness provides a new class of recursive solutions for the equilibrium probabilities of G/M/1-type processes.

Definition 2.3. A *Finite memory recursive solution of order k* for the equilibrium probabilities has the following structure:

$$\overline{\pi}_{n+k} = \overline{\pi}_n \mathbf{W}_1 + \overline{\pi}_{n+1} \mathbf{W}_2 + \cdots + \overline{\pi}_{n+k-1} \mathbf{W}_k, \tag{2.6}$$

where $\overline{\pi}_n$ is the vector of limiting probabilities of states on level n of the process and each $\mathbf{W}_i$, $1 \leq i \leq k$ is an $N \times N$ matrix.

■

Remark: Note that a finite memory recursive solution of order 1 is a matrix geometric solution for the equilibrium probabilities.

It is now shown that when there is only one state at each level receiving a downward transition, then a finite memory recursive solution of order k, for some $1 \leq k \leq m-1$ exists. First, some useful details are developed.

Lemma 2.1: If the G/M/1-type Markov process is irreducible, then the $(N-1)\times(N-1)$ matrix $\mathbf{A}_{1LU}$ formed by removing the last row and last column of $\mathbf{A}_1$ has a nonpositive inverse.

Proof. Refer [BeC].

■

Lemma 2.2: If the G/M/1-type Markov process is irreducible, then the space spanned by any proper subset of successive columns of $\mathbf{A}_1$ is disjoint from the strictly positive orthant.

Proof. Consider a set of k successive columns of $\mathbf{A}_1$ starting from the i'th column. Let this N×k matrix be denoted by **L**. Since the G/M/1-type Markov process is irreducible, $\mathbf{A}_1$ is nonsingular [Neu1]. Consequently, **L** is a full rank matrix. Let

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_U \\ \mathbf{L}_M \\ \mathbf{L}_D \end{bmatrix},$$

where $\mathbf{L}_U$ is an i−1×k matrix, $\mathbf{L}_M$ is an ktimek matrix and $\mathbf{L}_D$ is a N−i−k+1×k matrix. Since **L** is a sub-matrix of $\mathbf{A}_1$, $\mathbf{L}_M$ is a square sub-matrix on the diagonal of the generator, and it is nonsingular. Proof of this fact is avoided for the sake of brevity.

Suppose the column space of **L** is not disjoint from the strictly positive orthant. This implies that there exists an N×1 positive vector **y** and an k×1 vector **x** such that

$$\mathbf{Lx} = \mathbf{y},$$

which, after proper partitioning, implies

$$\mathbf{L}_M\mathbf{x} = \mathbf{y}_M$$

where $\mathbf{y}_M$ is k×1. Hence, $\mathbf{x} = \mathbf{L}_M^{-1}\mathbf{y}_M$. Since $\mathbf{y}_M$ is strictly positive and $\mathbf{L}_M^{-1}$ is nonpositive, **x** is strictly negative. Hence $\mathbf{y}_u = \mathbf{L}_u\mathbf{x}$ is non-positive. But this contradicts the hypothesis that **y** is strictly positive, so we have the desired result.
■

The above lemmas provide the following useful algebraic result on the generator of a G/M/1-type process when there is only one state at each level receiving a downward transition; i.e., r=1. Throughout the remaining chapter we assume that r=1.

Theorem 2.2: If the LEDI-complete G/M/1-type Markov process is irreducible, there exists an i, with $2 \le i \le m$, such that $[\mathbf{A}_{iR}:\mathbf{A}_{1L}]$ is nonsingular.

Proof. Assume that $[\mathbf{A}_{iR}:\mathbf{A}_{1L}]$ is singular for all i such that $2 \le i \le m$. Equivalently, there exist N-1 dimensional column vectors $\mathbf{y}_i$, $2 \le i \le m$, such that

$$\mathbf{A}_{1L}\,\mathbf{y}_i = \mathbf{A}_{iR} \quad \text{for} \quad 2 \le i \le m.$$

For each i, this is a system of N linear equations in N-1 unknowns. Separating the N-th equation from the rest, we have

$$\begin{bmatrix} \mathbf{A}_{1LU} \\ \mathbf{A}_{1LD} \end{bmatrix} \mathbf{y}_i = \begin{bmatrix} \mathbf{A}_{iRU} \\ \mathbf{A}_{iRD} \end{bmatrix}. \tag{2.7}$$

Equivalently

$$\mathbf{A}_{1LU}\, \mathbf{y}_i = \mathbf{A}_{iRU}$$

and

$$\mathbf{A}_{1LD}\, \mathbf{y}_i = \mathbf{A}_{iRD}. \tag{2.8}$$

Since $\mathbf{A}_{1LU}$ is nonsingular by Lemma 2.1,

$$\mathbf{y}_i = \mathbf{A}_{1LU}^{-1}\mathbf{A}_{iRU}. \tag{2.9}$$

Because of their position in the generator, all the components of $\mathbf{A}_{1LD}$ and $\mathbf{A}_{iRU}$ are nonnegative. Since, by Lemma 2.1, $\mathbf{A}_{1LU}$ has a nonpositive inverse, $\mathbf{y}_i$ is nonpositive. Hence if $\mathbf{y}_i$ satisfies (2.8), then $\mathbf{A}_{iRD}$ is nonpositive. But, from its position in the generator, $\mathbf{A}_{iRD}$ must be nonnegative. Therefore, if $[\mathbf{A}_{iR} : \mathbf{A}_{1L}]$ is singular, then $\mathbf{A}_{iRD} = \mathbf{0}$. So, by hypothesis, for each i, $2 \leq i \leq m$, $\mathbf{A}_{iRD} = \mathbf{0}$. This is possible if $\mathbf{A}_{1LD} = \mathbf{0}$ or $\mathbf{A}_{1LD}\mathbf{y}_i = \mathbf{0}$. We consider these two cases separately.

<u>Case 1:</u> $\mathbf{A}_{1LD} = \mathbf{0}$.

The last state, (i, N, 2), at each level i is the only state which receives a downward transition from any of the higher levels. Hence if $\mathbf{A}_{iRD} = \mathbf{0}$ for all $2 \leq i \leq m$, then (i, N, 2) cannot be reached in one step from any of the states (i+k, N, 2), $1 \leq k \leq m-1$. Thus, in order to reach the states at level i from (i+m-1, N, 2), there should be a sequence of transitions from (i+m-1, N, 2) through the states on level i+m-1 to some state (i+m-1, j, .), from which there is a transition to some state at a lower level and ultimately to (i, N, 2). If however $\mathbf{A}_{1LD} = \mathbf{0}$, then no other state on level i+m-1 can be reached from the state (i+m-1, N, 2) and the process is reducible.

<u>Case 2:</u> $\mathbf{A}_{1LD}\mathbf{y}_i = \mathbf{0}$.

Now consider the more interesting case, where $\mathbf{A}_{1LD} \neq \mathbf{0}$ but $\mathbf{A}_{1LD}\, \mathbf{y}_i = \mathbf{0}$. For some fixed i, let $\mathbf{y}_i = \mathbf{z}$ where $\mathbf{z} = [z_1\ z_2\ \cdots\ z_{N-1}]^T$ and $\mathbf{A}_{1LD} = [c_1\ c_2\ \cdots\ c_{N-1}]$. Also let $c_{j_1}, c_{j_2}, \ldots, c_{j_l}$ be strictly positive, where $1 \leq j_l < N-1$. By hypothesis

$$\mathbf{A}_{1LD}\,\mathbf{z} \;=\; \sum_{k=1}^{N-1} c_k z_k = 0.$$

Since all the terms in the sum are nonpositive, z_{j_1}, z_{j_2}, ..., z_{j_l} must equal zero. Let the other zero components of $\mathbf{z}$ be labeled $z_{j_{l+s}}$, $0 \leq s \leq t$, where $t \geq 0$. Let the set of indices corresponding to the zero components of $\mathbf{z}$ be denoted by H; i.e., $H = \{ j_1, j_2, \cdots, j_{l+t} \}$.

Since

$$\mathbf{A}_{1LU}\,\mathbf{z} = \mathbf{A}_{iRU},$$

the j'th equation yields

$$\sum_{k=1}^{N-1} \mathbf{A}_{1L_{(j,k)}} z_k \;=\; \mathbf{A}_{iR_j} \geq 0 \;\text{ for }\; j = 1,\, 2,\; \cdots\; N-1.$$

Since $\mathbf{A}_{1L_{(j,j)}}\, z_j$ is the only possible positive term in the above sum, if $z_j = 0$, then $\mathbf{A}_{iR_j}$ must be zero. Hence

$$\mathbf{A}_{iR_k} \;=\; 0 \;\text{ for }\; k \in H$$

which implies that

$$\mathbf{A}_{1L_{(j,p)}} \;=\; 0 \quad \text{for} \quad j \in H \quad \text{and} \quad p \notin H\,.$$

In the above discussion a fixed value of i is considered. But the results hold for any value of i, since $[\,\mathbf{A}_{iR} : \mathbf{A}_{1L}\,]$ is singular, by hypothesis, for all $2 \leq i \leq m$. But the set H can depend on i. Let

$$G \;=\; \bigcap_{i=2}^{m} H_i.$$

Since c_{j_1}, c_{j_2}, ..., c_{j_l} are strictly positive components of $\mathbf{A}_{1LD}$, $G \supseteq \{ j_1, j_2, \cdots, j_l \}$.

Let the state $(i,N,2) = v$ be the one that accepts downward transitions into level i, and reorder the set S of states on each level of the process arriving at

$$\tilde{S} \;=\; \left\{ \{\, k : k \notin G \cup \{v\} \,\},\; G \cup \{v\} \right\}$$

With the above reordering of the states on each level of the process, $\mathbf{A}_{1L}$ and $\mathbf{A}_{iR}$, $i \geq 2$ will be transformed to

$$\hat{\mathbf{A}}_{1L} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{0} & \mathbf{P}_3 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{A}}_{iR} = \begin{bmatrix} \vec{\mathbf{A}}_{iR} \\ \mathbf{0} \end{bmatrix}.$$

Hence, from the state v at level i, the only states at level i that are reachable are the ones which do not have an outward transition to any state on a lower level. Thus there is no path from any state on level i to any state on level i-m (or any lower level). Therefore the new generator $\hat{\mathbf{Q}}$ obtained by reordering the states at each level is reducible.

Thus, if the G/M/1-type Markov process is irreducible, then $[\mathbf{A}_{iR} : \mathbf{A}_{1L}]$ is nonsingular for some $2 \leq i \leq m$.

■

Corollary: Let $\tilde{\mathbf{A}}_{iR} = \mathbf{A}_{iR} + \mathbf{v}_i$, where $\mathbf{v}_i$ is nonnegative. If $[\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{iL} \,]$ is singular for all $2 \leq i \leq m$, then the G/M/1-type process is reducible.

Proof. By the above theorem, if $[\, \mathbf{A}_{iR} : \mathbf{A}_{iL} \,]$ is singular, then $\mathbf{A}_{iRD} = \mathbf{0}$. By an identical argument if $[\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{iL} \,]$ is singular, then $\tilde{\mathbf{A}}_{iRD} = 0$. Since $\mathbf{v}_i$ is nonnegative, this in turn implies that $\mathbf{A}_{iRD} = 0$. As in the case of the theorem, this can happen if $\mathbf{A}_{1LD} \equiv \mathbf{0}$. Since $\mathbf{A}_{iRD} = 0$ for all $2 \leq i \leq m$, in this case, the process is reducible. Even in the case where $\mathbf{A}_{1LD} \neq \mathbf{0}$ and $\tilde{\mathbf{A}}_{iRD} = 0$, by using the same reasoning as in the theorem, it can be seen that the process is reducible.

■

Theorem 2.3: If the G/M/1-type Markov process is irreducible, then a finite memory recursive solution for the equilibrium probabilities exists.

Proof (by construction). Consider the generator obtained after splitting the states. This is shown in (2.1) and is repeated here for the sake of convenience.

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \cdots \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{A}_{3R} & \mathbf{0} & \mathbf{A}_{2R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{A}_{mR} & \mathbf{0} & \mathbf{A}_{m-1R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

The following procedure provides a method of finding a finite memory recursive solution for the equilibrium probabilities.

Initialization. $i = m$, $\tilde{\mathbf{A}}_{mR} = \mathbf{A}_{mR}$, $\mathbf{Q}_0 = \tilde{\mathbf{Q}}$.

Step 1. If $[\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{1L} \,]$ is singular

then go to step 2

else go to step 3.

Step 2. If $[\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{1L} \,]$ is singular, then the column $\tilde{\mathbf{A}}_{iR}$ belongs to the column space of $\mathbf{A}_{1L}$. Thus there exists a vector $\mathbf{x}_i$ such that $\tilde{\mathbf{A}}_{iR} = \mathbf{A}_{1L}\,\mathbf{x}_i$. Let $-\mathbf{x}_i = [\, \alpha_1^i, \alpha_2^i, \cdots, \alpha_{N-1}^i \,]^T$. From the proof of Theorem 2.2, it is clear that $\mathbf{x}_i$ is nonpositive and hence the components of $-\mathbf{x}_i$ are nonnegative.

Zero out the column $\tilde{\mathbf{A}}_{iR}$ by performing column operations on the generator. Add to the column kN, the linear combination of N-1 columns $kN + N(N-2) + t$, $1 \leq t \leq N-1$, with the α^i's as the coefficients of the linear combination, for all $k \geq 2$.

After these column operations, the matrix $\mathbf{Q}_{m-i}$ becomes

$$Q_{m-i+1} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \cdots \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{A}_{3R} & \mathbf{0} & \mathbf{A}_{2R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{i-1} & \mathbf{0} & \tilde{\mathbf{A}}_{(i-1)R} & \mathbf{0} & \mathbf{A}_{(i-2)R} & \cdots \\ \mathbf{B}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{A}}_{(i-1)R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

It can be noted that $\tilde{\mathbf{A}}_{(i-1)R} = \mathbf{A}_{(i-1)R} +$ a linear combination of the columns of $\mathbf{A}_{0L}$ with the coefficients determined by the $\alpha^{i\prime}$s. Equivalently, $\tilde{\mathbf{A}}_{i-1R} = \mathbf{A}_{i-1R} + \mathbf{v}_i$, where $v_i \geq \mathbf{0}$ (since $\mathbf{A}_{0L} \geq \mathbf{0}$).

If i=3, from the global balance equations

$$\overline{\pi}_j \left[\mathbf{A}_{1R} : \mathbf{A}_{0L} \right] + \overline{\pi}_{j+1} \left[\tilde{\mathbf{A}}_{2R} : \mathbf{A}_{1L} \right] \equiv \mathbf{0} \;\; \forall \;\; j \geq 1.$$

Hence

$$\overline{\pi}_{j+1} = -\overline{\pi}_j \left[\mathbf{A}_{1R} : \mathbf{A}_{0L} \right] \left[\tilde{\mathbf{A}}_{2R} : \mathbf{A}_{1L} \right]^{-1}.$$

else

set $i \leftarrow i-1$ and go to step 1.

Step 3. Perform the following column operations on the matrix $\mathbf{Q}_{m-i}$:

Add to the set of N-r columns with the block structure as in $\mathbf{G}_1$ in (2.2), the set of N-r columns $\mathbf{G}_2$. Formally for all $k \geq 2$ add to the N-1 columns, $kN + s$, $1 \leq s \leq N-1$, the N-1 columns $kN + N(N-2) + t$, $1 \leq t \leq N-1$ respectively.

After these column operations, the matrix $\mathbf{Q}_{m-i}$ becomes

$$Q_{m-i+1} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L} & \mathbf{A}_{1R} & \cdots \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{A}_{3R} & \mathbf{0} & \mathbf{A}_{2R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{i-1} & \mathbf{0} & \mathbf{A}_{(i-1)R} & \mathbf{A}_{0L} & \mathbf{A}_{(i-2)R} & \cdots \\ \mathbf{B}_i & \mathbf{0} & \tilde{\mathbf{A}}_{iR} & \mathbf{A}_{1L} & \mathbf{A}_{(i-1)R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \qquad (2.10)$$

From the global balance equations, which state that $\overline{\pi}\, \mathbf{Q}_{m-i+1} \equiv \mathbf{0}$, we have

$$\overline{\pi}_j \mathbf{E}_1 + \overline{\pi}_{j+1} \mathbf{E}_2 + \cdots + \overline{\pi}_{j+i-1} \mathbf{E}_i \equiv \mathbf{0}, \quad \text{for all } j \geq 1,$$

where $\mathbf{E}_1 = [\mathbf{A}_{1R} : \mathbf{A}_{0L}]$, $\mathbf{E}_2 = [\mathbf{A}_{2R} : \mathbf{A}_{1L}]$, $\mathbf{E}_k = [\mathbf{A}_{kR} : \mathbf{0}]$ for $3 \leq k \leq i-2$, $\mathbf{E}_{i-1} = [\, \mathbf{A}_{i-1R} : \mathbf{A}_{0L} \,]$ and $\mathbf{E}_i = [\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{1L} \,]$. Hence

$$\overline{\pi}_{j+i-1} = \overline{\pi}_j \mathbf{W}_1 + \overline{\pi}_{j+1} \mathbf{W}_2 + \cdots + \overline{\pi}_{j+i-2} \mathbf{W}_{i-1} \qquad \forall \quad j \geq 1,$$

where $\mathbf{W}_k = \mathbf{E}_k\, \mathbf{E}_i^{-1}$ for $1 \leq k \leq i-1$. Thus a finite memory recursive solution of order i-1 for the equilibrium probabilities exists. stop (or exit).

It remains to verify that the above procedure terminates in a finite number of iterations and a finite memory recursive solution of order k ($1 \leq k \leq m-1$) is found.

If $[\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{1L} \,]$ is nonsingular for some k, $3 \leq k \leq m$, then it is clear that the procedure terminates in m-k+1 steps and that a finite memory recursive solution of order k-1 for the equilibrium probabilities is found. Otherwise, by the corollary to the previous theorem, $[\, \tilde{\mathbf{A}}_{2R} : \mathbf{A}_{1L} \,]$ is nonsingular and a matrix geometric solution for the equilibrium probabilities is found in m-2 steps. Thus, the procedure terminates in at most m-2 steps.

■

Even though the above algorithm provides a finite memory recursive solution for the equilibrium probabilities, it leads to a boundary value problem, which is discussed in the next section.

2.3. Resolution of the Associated Boundary Value Problem:

From the algorithm considered in the previous section, let a finite memory recursive solution of order l result for the equilibrium probabilities. Then from the system of equations

$$\overline{\pi}\mathbf{Q} = \mathbf{0}, \tag{2.11}$$

we have the following two systems of linear equations.

Boundary System:

$$\begin{bmatrix} \overline{\pi}_0 & \overline{\pi}_1 & \cdots & \overline{\pi}_m \end{bmatrix} \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{E}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{E}_2 & \mathbf{E}_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{l-1} & \mathbf{0} & \mathbf{E}_{l-1} & \mathbf{E}_{l-2} & \cdots & \mathbf{0} \\ \mathbf{B}_l & \mathbf{0} & \mathbf{E}_l & \mathbf{E}_{l-1} & \cdots & \mathbf{0} \\ \mathbf{B}_{l+1} & \mathbf{0} & \mathbf{E}_{l+1} & \mathbf{E}_l & \cdots & \mathbf{E}_1 \\ \mathbf{B}_{l+2} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{l+1} & \cdots & \mathbf{E}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_{l+1} \end{bmatrix} = \mathbf{0}. \tag{2.12}$$

Relabeling the matrix in the above system as $\mathbf{Z}$, we have

$$\begin{bmatrix} \overline{\pi}_0 & \overline{\pi}_1 & \cdots & \overline{\pi}_m \end{bmatrix} \mathbf{Z} = \mathbf{0}.$$

Finite Memory Recursion System:

$$\overline{\pi}_{j+l} = \overline{\pi}_j \mathbf{W}_1 + \overline{\pi}_{j+1} \mathbf{W}_2 + \cdots + \overline{\pi}_{j+l-1} \mathbf{W}_l \quad \forall\ j \geq m-l+1 . \tag{2.13}$$

Since the G/M/1-type Markov process is recurrent nonnull, the solution to $\overline{\pi}\mathbf{Q} = \mathbf{0}$ is unique up to a multiplicative constant. But (2.12) is a system of k+(m+1-l)N-r equations in k+mN unknowns. Thus, to uniquely determine a solution to (2.12), we need (l-1)N+r-1 more equations. It is not clear where the missing equations come from. This boundary value problem is now resolved by relating the finite memory recursive solution in (2.13) to an associated matrix geometric solution.

Consider the set of states at levels j, j+1, ... , j+l-1 and form the vector $[\,\overline{\pi}_j, \overline{\pi}_{j+1}, \cdots, \overline{\pi}_{j+l-1}\,]$. The finite memory recursive solution of order l described in

(2.13) implies and is implied by the following associated matrix geometric solution.

$$[\, \bar{\pi}_{j+1}, \bar{\pi}_{j+2}, \cdots, \bar{\pi}_{j+l} \,] = [\, \bar{\pi}_j, \bar{\pi}_{j+1}, \ldots, \bar{\pi}_{j+l-1} \,]\, \mathbf{T}, \tag{2.14}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_1 \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_2 \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{W}_3 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{W}_l \end{bmatrix}.$$

Defining $\bar{\nu}_j = [\, \bar{\pi}_j, \bar{\pi}_{j+1}, \ldots, \bar{\pi}_{j+l-1} \,]$, from (2.14) we have

$$\bar{\nu}_{j+1} = \bar{\nu}_j\, \mathbf{T} \quad \forall\ j \geq m-l+1. \tag{2.15}$$

In the following, we assume for the sake of simplicity that $\mathbf{T}$ is diagonalizable. Let S be the index set of the eigenvalues of $\mathbf{T}$ which are on or outside the unit circle and let $\bar{p}_i$ be a right eigenvector corresponding to an eigenvalue λ_i such that $|\lambda_i| \geq 1$. The following lemma provides a necessary and sufficient condition for the G/M/1-type process to be recurrent nonnull.

Theorem 2.4. The irreducible G/M/1-type Markov process with generator $\mathbf{Q}$ is recurrent nonnull if and only if $\bar{\nu}_{m-l+1}\mathbf{p}_i = \mathbf{0} \quad \forall\ i \in S$.

Proof. If $\mathbf{Q}$ is recurrent nonnull, then the solution to $\bar{\pi}\mathbf{Q} = \mathbf{0}$ is unique and is such that

$$\sum_{n=0}^{\infty} \bar{\pi}_n \mathbf{e} = 1\,. \tag{2.16}$$

But

$$\sum_{n=m-l+1}^{\infty} \bar{\nu}_n \mathbf{e} = \bar{\pi}_{m-l+1}\mathbf{e} + 2\bar{\pi}_{m-l+2}\mathbf{e} + 3\bar{\pi}_{m-l+3}\mathbf{e} + \cdots + (l-1)\bar{\pi}_{m-1}\mathbf{e} + l \sum_{n=m}^{\infty} \bar{\pi}_n \mathbf{e}.$$

Hence (2.16) implies

$$\sum_{n=m-l}^{\infty} \bar{\nu}_n \mathbf{e} < \infty . \tag{2.17}$$

Now let $\mathbf{p}$ be the right eigenvector corresponding to an eigenvalue λ such that $|\lambda| \geq 1$. Then

$$\bar{\nu}_{n+1}\mathbf{p} = \bar{\nu}_{m-l+1}\mathbf{T}^{n-m+l}\mathbf{p} = \lambda^{n-m+l}\,\bar{\nu}_{m-l+1}\mathbf{p}.$$

If $\bar{\nu}_{m-l+1}\mathbf{p} = c \neq 0$, then $\lim_{n\to\infty} \lambda^{n+1+l-m} c \neq 0$. Hence

$$\sum_{n=m-l}^{\infty} \bar{\nu}_n \mathbf{e} = \infty.$$

But this contradicts (2.17). Thus necessity holds.

Suppose $\bar{\nu}_{m-l+1}\mathbf{p}_i = 0 \;\; \forall \;\; i \in S$. It is sufficient to show that $\bar{\pi}$ which is related to $\bar{\nu}$ is the unique solution to $\bar{\pi}\mathbf{Q} = \mathbf{0}$ and is also a probability distribution.

Let $\tilde{\mathbf{T}}$ be the matrix obtained from $\mathbf{T}$ by removing all these modes corresponding to the eigenvalues on or outside the unit circle. After removing these unexcited modes, a recursion of the form

$$\bar{\nu}_{n+1} = \bar{\nu}_n \tilde{\mathbf{T}} \quad \forall \;\; n \geq m-l+1$$

holds. This recursion is stable since $\tilde{\mathbf{T}}^n$ converges to a zero matrix as n approaches infinity. Since the solution to this recursion also satisfies the finite memory recursion system,

$$\sum_{n=m-l+1}^{\infty} \bar{\pi}_n \mathbf{e} < \infty$$

Hence the process $\mathbf{Q}$ is recurrent nonnull.

■

Remark 1: The above Lemma provides a necessary condition for an unstable mode to disappear from the recursion.

Remark 2: Utilizing the results of Theorem 2.3, let $\tilde{\mathbf{T}}$ denote the matrix obtained from $\mathbf{T}$ by subtracting out all the unstable modes.

Now the above Lemma is utilized to find the missing equations in (2.12).

Theorem 2.5: If $\{ p_i, \bar{q}_i \; ; 1 \leq i \leq s \}$ are the right and left eigenvectors of $\mathbf{T}$

respectively corresponding to the eigenvalues on or outside the unit circle, then the matrix

$$
\mathbf{U} = \begin{bmatrix}
\mathbf{B}_0 & \mathbf{A}_{0L} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{B}_1 & \mathbf{A}_{1L} & \mathbf{E}_1 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{B}_2 & \mathbf{0} & \mathbf{E}_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{B}_{l-1} & \mathbf{0} & \mathbf{E}_{l-1} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{B}_l & \mathbf{0} & \mathbf{E}_l & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{B}_{l+1} & \mathbf{0} & \mathbf{E}_{l+1} & \cdots & \mathbf{E}_1 & \mathbf{p}_1 & \cdots & \mathbf{p}_s \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\mathbf{B}_m & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_{l+1} & & &
\end{bmatrix}
$$

has rank k+mN-1 and the initial probability vector can be determined up to a scalar constant from the system

$$\begin{bmatrix}\overline{\pi}_0 & \overline{\pi}_1 & \ldots\ldots & \overline{\pi}_m\end{bmatrix}\mathbf{U} = \mathbf{0} \tag{2.18}$$

Proof. From Theorem 2.4,

$$\begin{bmatrix}\overline{\pi}_0 & \cdots & \overline{\pi}_{m-l-1} & \cdots & \overline{\pi}_{m-1}\end{bmatrix}\begin{bmatrix}\mathbf{0} \\ \mathbf{p}_i\end{bmatrix} = \mathbf{0} \quad \forall \ 1 \leq i \leq s.$$

Now by appending the columns of the form $\begin{bmatrix}\mathbf{0} \\ \mathbf{p}_i\end{bmatrix}$ to the boundary system (2.12), its column space may be expanded. After tagging these equations, the rank of the matrix

$$\begin{bmatrix}\mathbf{Z} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_t\end{bmatrix}$$

is k+(m+1-l)N-r+s. Now suppose k+(m+1-l)N-r+s $<$ k+mN-1. Then the dimension of the left null space, $\mathbf{V}$ of $\mathbf{U}$ is at least two. Consider any vector α in the solution space $\mathbf{V}$. Since $\overline{\alpha}\begin{bmatrix}\mathbf{0} \\ \mathbf{p}_i\end{bmatrix} = 0 \ \forall \ 1 \leq i \leq s$, from the recursion in (2.15), all the unstable modes are eliminated. Thus the recursion

$$\overline{\nu}_{n+1} = \overline{\nu}_{m-l}\mathbf{T}^{n+1+l-m},$$

initialized by α is stable. Since the solution generated by this recursion also satisfies the finite memory recursion system, the solution space of $\overline{\pi}\mathbf{Q} = \mathbf{0}$, has a dimension of at least two. This contradicts the hypothesis that the G/M/1-type Markov process $\mathbf{Q}$ is recurrent nonnull, so (2.18) and (2.13) must determine the equilibrium probabilities up to a normalizing constant.

■

Remark: When a finite memory recursive solution of order one, or, equivalently, a matrix geometric solution results from the algorithm considered in the previous section, there will be r missing equations in the boundary system and similar results can be derived to solve the boundary value problem. For these results for QBD-processes, see [Zha].

Now we summarize the results of the previous two sections by providing a procedure for computing the equilibrium probabilities of G/M/1-type Markov processes.

(a) Utilizing the algorithm provided in the proof of Theorem 2.3, determine the order of the finite memory recursion system and the $\mathbf{W}_i$'s. Compute the spectral representation of the associated matrix geometric solution and find the matrix $\tilde{\mathbf{T}}$ by removing the unexcited modes.

(b) A scalar multiple of the initial probability vector can be computed by finding a vector in the left null space of $\mathbf{U}$ described in Theorem 2.5. Utilizing this vector and $\tilde{\mathbf{T}}$ computed in step (a), find a solution $\overline{\pi}$ to $\overline{\pi}\mathbf{Q} = \mathbf{0}$. Since the G/M/1-type Markov process is recurrent nonnull, $\overline{\pi}$ can be normalized to arrive at a probability distribution, $\overline{\mu}$. The normalization method is illustrated below. From (2.15), using the definition of $\overline{\nu}_n$

$$k\left(\sum_{n=0}^{\infty}\overline{\pi}_n\right)\mathbf{e} = k\left(\sum_{n=0}^{m-k}\overline{\pi}_n\right)\mathbf{e} + \left(\sum_{n=1}^{k-1}(k-n)\overline{\pi}_{m-k+n}\right)\mathbf{e}.$$

Thus if a solution $\overline{\pi}$ for which $k\left(\sum_{n=0}^{\infty}\overline{\pi}_n\right)\mathbf{e} = \alpha$ is found, then the equilibrium probability vector $\overline{\mu}$ of the G/M/1-type Markov process can be found by multiplying $\overline{\pi}$ by $\frac{k}{\alpha}$.

Now we illustrate the results of the above procedure with the following example.

Example. 2.2: Consider the LEDI-*in*complete G/M/1-type Markov process shown in Fig. 1-a, where $a_0 = 2$, $a_2 = 4$, $a_3 = 4$. By the state space expansion technique described in section 2, an LEDI-complete G/M/1-type Markov process can be obtained. The generator of the G/M/1-type Markov process has the form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{C}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{C}_1 & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{C}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots \\ \mathbf{B}_3 & \mathbf{C}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \cdots \\ \mathbf{B}_4 & \mathbf{C}_4 & \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2.19}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} \text{ and } \mathbf{A}_k = 0 \text{ for } k \geq 4.$$

$$\mathbf{B}_0 = -2, \mathbf{B}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \text{ and } \mathbf{B}_k = \mathbf{0} \text{ for } k \geq 3.$$

$$\mathbf{C}_0 = 2, \mathbf{C}_1 = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}, \mathbf{C}_3 = \begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}, \mathbf{C}_k = \mathbf{0} \text{ for } k \geq 4.$$

From the global balance equations $\overline{\pi}\mathbf{Q} = \mathbf{0}$, a finite memory recursive solution of order 2 results, since $[\,\mathbf{A}_{3R} : \mathbf{A}_{1L}\,]$ is invertible. In other words,

$$\overline{\pi}_{n+2} = \overline{\pi}_n \mathbf{W}_1 + \overline{\pi}_{n+1} \mathbf{W}_2 \qquad \text{for } n \geq 2,$$

where

$$\mathbf{W}_1 = \begin{bmatrix} 0.2 & -0.2 \\ 0.2 & 2.3 \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} -0.8 & -0.2 \\ 0.2 & -1.2 \end{bmatrix}.$$

The boundary system of linear equations is given by

$$[\,\overline{\pi}_0 : \overline{\pi}_1 : \overline{\pi}_2 : \overline{\pi}_3\,]\begin{bmatrix} -2 & 2 & 0 & 0 \\ 4 & -6 & 0 & 2 \\ 4 & 0 & -6 & 2 \\ 4 & 0 & 4 & -10 \\ 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \mathbf{0}.$$

This is a system of 4 equations in 7 unknowns. Thus we need 2 more equations to determine the solution uniquely up to a scalar multiple. As shown in the previous section, the conditions for the stability of the associated matrix geometric recursive solution provide the clue to the problem. The associated matrix geometric recursive solution is described by

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0.2 & -0.2 \\ 0 & 0 & 0.2 & 2.3 \\ 1 & 0 & -0.8 & -0.2 \\ 0 & 1 & 0.2 & -1.2 \end{bmatrix}.$$

The eigenvalues of $\mathbf{T}$ are: 1, -1, 0.22474487131586, -2.22474487139159.

By Theorem 2.5, the modes corresponding to the eigenvalues on or outside the unit circle must disappear. Thus, by tagging the corresponding eigenvectors of $\mathbf{T}$ to the boundary system, we can determine the solution uniquely. This is shown below.

$$[\overline{\pi}_0 : \overline{\pi}_1 : \overline{\pi}_2 : \overline{\pi}_3]\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 4 & -6 & 0 & 2 & 0 & 0 \\ 4 & 0 & -6 & 2 & 0 & 0 \\ 4 & 0 & 4 & -10 & 1 & 0.0824844 \\ 4 & 0 & 4 & 0 & -9 & -1.0412626 \\ 0 & 0 & 4 & 0 & 1 & 0.0824844 \\ 0 & 0 & 4 & 0 & -4 & 1 \end{bmatrix} = \mathbf{0}.$$

Since the coefficient matrix in the above system has full rank, a vector in the left null space of this matrix is determined up to a multiplicative constant. One such vector is

$$\begin{bmatrix} 3 \\ 1 \\ 0.25 \\ 0.25 \\ 0.0337116584135978 \\ 0.0837116584135982 \\ 0.00757668317280430 \end{bmatrix}^t .$$

Now, using a solution to the boundary system and the definition of the finite memory recursion system, a solution to $\overline{\pi}\mathbf{Q} = \mathbf{0}$ can be determined. The solution can be normalized to arrive at a probability distribution. The normalization method is illustrated below. From (2.15), using the definition of $\overline{\nu}_n$

$$2(\sum_{n=0}^{\infty} \overline{\pi}_n)\mathbf{e} = 2\overline{\pi}_0\mathbf{e} + 2\overline{\pi}_1\mathbf{e} + \overline{\pi}_2\mathbf{e} + \sum_{n=1}^{\infty} \overline{\nu}_n\hat{\mathbf{e}} \,. \tag{2.20}$$

where $\hat{\mathbf{e}}$ is a column vector whose first n components are 1's and the next n components are all zeros. Thus if a solution $\overline{\pi}$ for which $2(\sum_{n=0}^{\infty} \overline{\pi}_n)\mathbf{e} = \alpha$ is found, then the equilibrium probability vector of the G/M/1-type Markov process can be found by multiplying $\overline{\pi}$ by $\frac{2}{\alpha}$.

■

Now it is shown how various equilibrium performance measures can be computed through the finite memory recursion for the limiting probabilities.

Computation of the Marginal and Conditional Probabilities, Moments:

Whenever a finite memory recursive solution for the equilibrium probabilities exists, easily computable formulas for several marginal and conditonal probabilities can be found. Also, closed form expressions for the factorial moments can be obtained. In many specific models, the following quantities have a useful practical interpretation.

(a) The row vector $\tilde{\pi} = \sum_{k=0}^{\infty} \overline{\pi}_k$ is given by

$$\tilde{\pi} = \overline{\pi}_0 + \overline{\pi}_1 + \cdots + \overline{\pi}_m + \overline{\nu}_{m-l+1}[\,\mathbf{I} - \tilde{\mathbf{T}}\,]^{-1}\hat{\mathbf{e}}, \tag{2.21}$$

where l is the order of the finite memory recursion system and $\hat{\mathbf{e}}$ is a column vector, whose first n components are 1's and the remaining components are zeros. Once the initial probability vector $[\,\overline{\pi}_0 : \overline{\pi}_1 : \cdots : \overline{\pi}_m\,]$ is found from the boundary system, $\tilde{\pi}$ is easily computed from (2.21). The component $\tilde{\pi}_j$ of $\tilde{\pi}$ is the equilibrium probability that the process is in the set $\{(i,j) : i \geq 0\}$.

(b) The marginal distribution $\{\mu_{n+1};\ n \geq m\}$ is given by

$$\mu_{n+1} = \overline{\pi}_{n+1}\mathbf{e} = \overline{\pi}_{n-l+1}\mathbf{W}_1\mathbf{e} + \overline{\pi}_{n-l+2}\mathbf{W}_2\mathbf{e} + \cdots + \overline{\pi}_n\mathbf{W}_l\mathbf{e}. \tag{2.22}$$

(c) Using the results from (a) and (b), the conditional probabilities $\{\,\mu_n(j),\ n \geq m+1\,\}$ for $1 \leq j \leq n$ are found to be

$$\mu_n(j) = (\tilde{\pi}_j)^{-1}\overline{\pi}_{nj}\,, \quad n \geq 0. \tag{2.23}$$

In many practical applications they play a useful role.

Also the moments of the marginal and conditional distributions can be expressed in closed form by simple formulae. For example the first moment of the marginal distribution E(N) is given by

$$E(N) = \overline{\pi}_1\mathbf{e} + \cdots + (m-l+1)\overline{\pi}_{m-l+1}\mathbf{e} + [\,(m-l)\mathbf{I} - (m-l-1)\tilde{\mathbf{T}}\,][\mathbf{I} - \tilde{\mathbf{T}}\,]^{-2}\hat{\mathbf{e}}. \tag{2.24}$$

2.4. Relationship between Finite Memory Recursive Solutions and Matrix Geometric Solutions:

The class of finite memory recursive solutions are now related to the matrix geometric solutions through the associated matrix geometric solution described by $\mathbf{T}$.

In the following the characteristic polynomial of $\mathbf{T}$ and some properties of $\mathbf{T}$ are found. Then the nonzero eigenvalues and the corresponding left eigenvectors of $\mathbf{R}$ are related to those of $\mathbf{T}$. It is then explained how this relationship can be used to find the rate matrix $\mathbf{R}$ from $\mathbf{T}$, if desired.

Lemma 2.3. The characteristic polynomial of $\mathbf{T}$ is

$$\mathrm{Det}(\,\lambda^l\mathbf{I} - \lambda^{l-1}\mathbf{W}_l - \lambda^{l-2}\mathbf{W}_{l-1} - \cdots - \lambda\mathbf{W}_2 - \mathbf{W}_1\,). \tag{2.25}$$

Proof (by induction). When a finite memory recursive solution of order one results, the characteristic polynomial of $\mathbf{T}$ is, by definition, $\det(\,\lambda\mathbf{I} - \mathbf{W}_1)$. Hence the claim holds. Now suppose that the claim holds for $l=m$ and consider the case when a finite memory recursive solution of order m+1 results i.e. l=m+1. In this case, the associated matrix geometric solution is described by

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{W}_1 \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{W}_2 \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{W}_3 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{W}_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{W}_m \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{W}_{m+1} \end{bmatrix}.$$

$$\lambda\mathbf{I} - \mathbf{T} = \begin{bmatrix} \lambda\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{W}_1 \\ -\mathbf{I} & \lambda\mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{W}_2 \\ \mathbf{0} & -\mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{W}_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} & \lambda\mathbf{I} & -\mathbf{W}_m \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{I} & (\lambda\mathbf{I} - \mathbf{W}_{m+1}) \end{bmatrix}.$$

Using the formula for the determinant of a block matrix of the above form

$$\mathrm{Det}(\ \lambda\mathbf{I} - \mathbf{T}\ ) = \mathrm{Det}(\ \lambda\mathbf{I}\ )\mathrm{Det}\left(\begin{bmatrix} \lambda\mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{W}_2 \\ -\mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{W}_3 \\ \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & -\mathbf{I} & \lambda\mathbf{I} & -\mathbf{W}_m \\ \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{I} & \lambda\mathbf{I} - \mathbf{W}_{m+1} \end{bmatrix} - \begin{bmatrix} -\mathbf{I} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \frac{1}{\lambda}\mathbf{I} \left[\ \mathbf{0}\ \cdots\ -\mathbf{W}_1\ \right] \right)$$

Simplifying and using the induction hypothesis

$$= \mathrm{Det}(\lambda\mathbf{I})\mathrm{Det}(\ \lambda^m\mathbf{I} - \lambda^{m-1}\mathbf{W}_m - \cdots - \lambda\mathbf{W}_2 - \frac{1}{\lambda}\mathbf{W}_1 - \mathbf{W}_2\ )$$

$$= \mathrm{Det}(\ \lambda^{m+1}\mathbf{I} - \lambda^m\mathbf{W}_m - \cdots - \lambda\mathbf{W}_2 - \mathbf{W}_1\ ).$$

Thus by induction the result in (2.25) holds for all values of l.

■

Lemma 2.4. $\mathbf{T}$ has an eigenvalue at 1 and another at -1.

Proof. From Lemma 2.3, the characteristic polynomial of $\mathbf{T}$ is given by $\mathrm{Det}(\ \lambda^l\mathbf{I} - \lambda^{l-1}\mathbf{W}_l - \lambda^{l-2}\mathbf{W}_{l-1} - \cdots - \lambda\mathbf{W}_2 - \mathbf{W}_1\ )$. Since the matrix $\mathbf{E}_{l+1}$ is invertible, using the definition of the $\mathbf{W}_i$'s, it is sufficient to show that there exists an $\mathbf{f}$ such that

$$(\mathbf{E}_{l+1} + \mathbf{E}_l + \mathbf{E}_{l-1} + \cdots + \mathbf{E}_2 + \mathbf{E}_1)\mathbf{f} = \mathbf{0}.$$

Using the definition of $\mathbf{E}_i$'s and rewriting the above equation,

$$[\, \mathbf{A}_{1R} + \mathbf{A}_{2R} + \cdots + \tilde{\mathbf{A}}_{l+1R} : 2(\mathbf{A}_{0L} + \mathbf{A}_{1L}) \,]\mathbf{f} = \mathbf{0}$$

From the algorithm considered in the previous section, when a finite memory recursive solution of order l results, $[\, \mathbf{A}_{mR} : \mathbf{A}_{1L} \,]$, $\{ [\, \tilde{\mathbf{A}}_{iR} : \mathbf{A}_{1L} \,],\ l+2 \le i \le m-1 \}$ are all singular. Hence there exist $\mathbf{v}_i$'s such that

$$\tilde{\mathbf{A}}_{iR} = \mathbf{A}_{1L}\mathbf{v}_i \ \text{ for } l+2 \le i \le m-1 \ \text{ and } \ \mathbf{A}_{mR} = \mathbf{A}_{1L}\mathbf{v}_m.$$

Also by definition,

$$\tilde{\mathbf{A}}_{iR} = \mathbf{A}_{iR} - \mathbf{A}_{0L}\mathbf{v}_{i+1} \ \text{ for } l+1 \le i \le m-1.$$

Rearranging

$$\mathbf{A}_{iR} = \mathbf{A}_{1L}\mathbf{v}_i + \mathbf{A}_{0L}\mathbf{v}_{i+1} \ \text{ for } l+1 \le i \le m-1. \tag{2.26}$$

Hence

$$\mathbf{A}_{mR} + \cdots + \mathbf{A}_{l+2R} = [\, \mathbf{A}_{1L} + \mathbf{A}_{0L} \,][\sum_{i=l+3}^{m} \mathbf{v}_i] + \mathbf{A}_{1L}\mathbf{v}_{l+2}. \tag{2.27}$$

Also, since $\mathbf{A}_i$'s are the sub matrices of generator $\mathbf{Q}$,

$$\sum_{i=1}^{m} \mathbf{A}_{iR} = -(\, \mathbf{A}_{0L} + \mathbf{A}_{1L} \,)\tilde{\mathbf{e}}, \tag{2.28}$$

where $\tilde{\mathbf{e}}$ is a vector of n-1 ones. From (2.27) and (2.28), we have

$$\sum_{i=1}^{l+1} \mathbf{A}_{iR} = -[\, \mathbf{A}_{0L} + \mathbf{A}_{1L} \,][\, \tilde{\mathbf{e}} + \sum_{i=l+3}^{m} \mathbf{v}_i \,] - \mathbf{A}_{1L}\mathbf{v}_{l+2}$$

Hence

$$\sum_{i=1}^{l} \mathbf{A}_{iR} + \tilde{\mathbf{A}}_{l+1R} = \sum_{i=1}^{l+1} \mathbf{A}_{iR} - \mathbf{A}_{0L}\mathbf{v}_{l+2} = -[\, \mathbf{A}_{0L} + \mathbf{A}_{1L} \,][\, \tilde{\mathbf{e}} + \sum_{i=l+2}^{m} \mathbf{v}_i \,]$$

Thus

$$[\mathbf{A}_{1R} + \mathbf{A}_{2R} + \cdots + \tilde{\mathbf{A}}_{l+1R} : 2(\mathbf{A}_{0L} + \mathbf{A}_{1L})] \begin{bmatrix} 1 \\ \frac{1}{2}(\tilde{\mathbf{e}} + \sum_{i=l+2}^{m} \mathbf{v}_i) \end{bmatrix} = \mathbf{0}$$

and $\mathbf{T}$ has an eigenvalue at 1.

Now it is shown that $\mathbf{T}$ has an eigenvalue of -1. It is sufficient to show that there exists an $\mathbf{f}$ such that

$$[(-1)^l \mathbf{E}_{l+1} + (-1)^{l-1} \mathbf{E}_l + \cdots - \mathbf{E}_2 + \mathbf{E}_1] \mathbf{f} = \mathbf{0}$$

If $(-1)^l = +1$, then using the definition of $\mathbf{E}_i$'s, we have

$$[(-1)^l \mathbf{E}_{l+1} + (-1)^{l-1} \mathbf{E}_l + \cdots - \mathbf{E}_2 + \mathbf{E}_1] = [\tilde{\mathbf{A}}_{l+1R} - \mathbf{A}_{lR} \cdots + \mathbf{A}_{1R} : \mathbf{0}].$$

Hence it is evident that such an $\mathbf{f}$ can be found. Therefore $\mathbf{T}$ has an eigenvalue of -1. The proof in the case where $(-1)^l = -1$ follows along the same lines as in the above case.

■

The following Theorem is useful in computing the nonzero eigenvalues and the corresponding left eigenvectors of the rate matrix from the matrix geometric solution associated with a finite memory recursive solution.

Let

$$\mathbf{B}(\lambda) = \lambda^l \mathbf{I} - \lambda^{l-1} \mathbf{W}_l - \lambda^{l-2} \mathbf{W}_{l-1} - \cdots - \lambda \mathbf{W}_2 - \mathbf{W}_1$$

and

$$\mathbf{A}(\lambda) = \lambda^m \mathbf{A}_m + \lambda^{m-1} \mathbf{A}_{m-1} + \cdots + \lambda \mathbf{A}_1 + \mathbf{A}_0.$$

Theorem 2.6. If μ is a nonzero root of Det($\mathbf{A}(\lambda)$) and $\bar{\mathbf{f}}$ is the corresponding vector in the left null space of $\mathbf{A}(\lambda)$, then μ is a root of Det($\mathbf{B}(\lambda)$) and $\bar{\mathbf{f}}$ is the corresponding vector in the left null space of $\mathbf{B}(\lambda)$. The converse also holds except when μ is an (l-1) th root of -1.

Proof. Suppose μ is a nonzero root of Det($\mathbf{A}(\lambda)$) and $\mathbf{f}$ is the corresponding vector in the left null space of $\mathbf{A}(\lambda)$. Then

$$\bar{\mathbf{f}}(\ \mu^{m}\mathbf{A}_{mR} + \mu^{m-1}\mathbf{A}_{m-1R} + \cdots + \mu\mathbf{A}_{1R}\ ) = \mathbf{0}. \tag{2.29}$$

$$\bar{\mathbf{f}}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ ) = \mathbf{0}. \tag{2.30}$$

Since

$$\mathrm{Det}(\mathbf{B}(\mu)) = \mathrm{Det}(\mu^{l}\mathbf{E}_{l+1} + \mu^{l-1}\mathbf{E}_{l} + \cdots + \mu\mathbf{E}_2 + \mathbf{E}_1)\mathrm{Det}(\mathbf{E}_{l+1}^{-1}), \tag{2.31}$$

the roots of $\mathrm{Det}(\mathbf{B}(\mu))$ and $\mathrm{Det}(\ \mu^{l}\mathbf{E}_{l+1} + \mu^{l-1}\mathbf{E}_{l} + \cdots + \mu\mathbf{E}_2 + \mathbf{E}_1\ )$ coincide. Hence it is sufficient to show that if μ and $\mathbf{f}$ satisfy (2.29) and (2.30),

$$\bar{\mathbf{f}}(\ \mu^{l}\mathbf{E}_{l+1} + \mu^{l-1}\mathbf{E}_{l} + \cdots + \mu\mathbf{E}_2 + \mathbf{E}_1\ ) = \mathbf{0}$$

and that the converse holds when μ is not an l-1 th root of -1. Substituting for $\mathbf{E}_i$'s in the above equation and rewriting,

$$\bar{\mathbf{f}}(\ \mu^{l}\tilde{\mathbf{A}}_{l+1R} + \cdots + \mu\mathbf{A}_{2R} + \mathbf{A}_{1R}\ ) = \mathbf{0}. \tag{2.32}$$

$$\bar{\mathbf{f}}(\ \mu^{l}\mathbf{A}_{1L} + \mu^{l-1}\mathbf{A}_{0L} + \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ ) = \mathbf{0}. \tag{2.33}$$

Now consider a μ and an $\mathbf{f}$ which satisfy (2.29) and (2.30). (2.30) implies

$$\mathbf{f}[\ \mu^{l-1}(\mu\mathbf{A}_{1L} + \mathbf{A}_{0L}) + (\mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ )\ ] = \mathbf{0}$$

Thus (2.33) is satisfied. Now substituting for $\{\ \mathbf{f}\mathbf{A}_{iR};\ l+2 \le i \le m-1\ \}$ from (2.26) in (2.29), we have

$$\mathbf{f}\Big[\ \mu^{m-1}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ )\mathbf{v}_m + \mu^{m-2}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ )\mathbf{v}_{m-1} + \cdots$$

$$+ \mu^{l+2}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ )\mathbf{v}_{l+3}\ \Big] + \mu^{l+2}\mathbf{f}\mathbf{A}_{1L}\mathbf{v}_{l+2} + \mu^{l+1}\mathbf{f}\mathbf{A}_{l+1R} + \mu^{l}\mathbf{f}\mathbf{A}_{lR} + \cdots + \mu\mathbf{f}\mathbf{A}_{1R} = \mathbf{0}.$$

Since $\mathbf{f}[\mu\mathbf{A}_{1L} + \mathbf{A}_{0L}] = \mathbf{0}$, from the above we have

$$\{\ \mu^{l+1}\mathbf{f}[\mu\mathbf{A}_{1L}\mathbf{v}_{l+2} + \mathbf{A}_{l+1R}] + \mathbf{f}[\ \mu^{l}\mathbf{A}_{lR} + \mu^{l-1}\mathbf{A}_{l-1R} + \cdots + \mu\mathbf{A}_{1R}\ ]\ \} = \mathbf{0}.$$

$$\{\mu^{l+1}(\ \mathbf{f}\tilde{\mathbf{A}}_{l+1R} + \mathbf{f}\mathbf{A}_{0L}\mathbf{v}_{l+2} + \mu\mathbf{f}\mathbf{A}_{1L}\mathbf{v}_{l+2}\ ) + \mathbf{f}[\ \mu^{l}\mathbf{A}_{lR} + \mu^{l-1}\mathbf{A}_{l-1R} + \cdots + \mu\mathbf{A}_{1R}\ ]\ \} = \mathbf{0}.$$

Since $\mathbf{f}(\mu\mathbf{A}_{1L} + \mathbf{A}_{0L}) = \mathbf{0}$, the above becomes

$$\mathbf{f}(\ \mu^{l+1}\tilde{\mathbf{A}}_{l+1R} + \mu^{l}\mathbf{A}_{lR} + \cdots + \mu\mathbf{A}_{1R}\ ) = \mathbf{0}$$

Since $\mu \neq 0$,

$$\mathbf{f}(\ \mu^{l}\tilde{\mathbf{A}}_{l+1R} + \mu^{l-1}\mathbf{A}_{lR} + \cdots + \mathbf{A}_{1R}\ ) = \mathbf{0}.$$

Thus if μ and **f** satisfy (2.29) and (2.30), then they also satisfy (2.32) and (2.33). Now let μ and **f** satisfy (2.32) and (2.33) and let μ be not an (l-1) th root of -1. From (2.32) and (2.33),

$$\mathbf{f}\{\ \mu^{l-1}(\mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ ) + (\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L})\ \} = \mathbf{0}$$

Let $\mathbf{f}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ ) = \mathbf{G}$. The above equation implies

$$(\ \mu^{l-1} + 1\ )\mathbf{G} = \mathbf{0}.$$

Since μ is not an (l-1) th root of -1, $\mathbf{G} = \mathbf{f}(\ \mu\mathbf{A}_{1L} + \mathbf{A}_{0L}\ ) = \mathbf{0}$. Thus (2.30) is satisfied. By reversing the argument in the above case, it is easy to see that if μ and **f** satisfy (2.32), they also satisfy (2.29).

■

Corollary: The row vector $\overline{\mathbf{f}}$ is a left eigenvector corresponding to a nonzero eigenvalue λ of **R** if and only if

$$\overline{\mathbf{f}}(\ \lambda^{l}\mathbf{I} - \lambda^{l-1}\mathbf{W}_{l} - \cdots - \lambda\mathbf{W}_{2} - \mathbf{W}_{1}\ ) = \mathbf{0} \tag{2.34}$$

Remark 1: The above Theorem and the corollary show that the nonzero eigenvalues of **R** are a subset of those of **T**.

Remark 2: Once the eigenvalues and left eigenvectors of the rate matrix are computed using the criterion in (2.34), the rate matrix **R** can be found using one of the two methods developed in [MuC1],[MuC2].

It is now shown that if the rate matrix **R** is diagonalizable then it satisfies a matrix equation with $\mathbf{W}_i$'s as the coefficient matrices.

Lemma 2.5. **R** satisfies the following matrix equation.

$$\mathbf{R}^{l} - \mathbf{R}^{l-1}\mathbf{W}_{l} - \cdots - \mathbf{R}\mathbf{W}_{2} - \mathbf{H}\mathbf{W}_{1} = \mathbf{0},$$

where **H** is the sum of the residue matrices of **R** corresponding to the nonzero eigenvalues.

Proof. Let B = { λ_i,i=1,2,...,s } be the set of nonzero eigenvalue of **R** and let $\overline{\mathbf{f}}_i$ and $\overline{\mathbf{g}}_i$ be the corresponding left and right eigenvectors respectively. From the corollary to Theorem 5, $\overline{\mathbf{f}}_i$ satisfies (2.34). Premultiplying by $\mathbf{g}_i$, we have

$$\lambda_i^l \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{I} - \lambda_i^{l-1} \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{W}_l - \cdots \lambda_i \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{W}_2 - \mathbf{W}_1 = \mathbf{0}.$$

Summing over the index set of B,

$$\sum_{i=1}^{s} \lambda_i^l \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{I} - \sum_{i=1}^{s} \lambda_i^{l-1} \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{W}_l - \cdots \sum_{i=1}^{s} \lambda_i \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{W}_2 - \sum_{i=1}^{s} \mathbf{g}_i \overline{\mathbf{f}}_i \mathbf{W}_1 = \mathbf{0}.$$

Using the spectral representation of $\mathbf{R}$, we have

$$\mathbf{R}^l \mathbf{I} - \mathbf{R}^{l-1} \mathbf{W}_l - \cdots - \mathbf{R} \mathbf{W}_2 - \mathbf{H} \mathbf{W}_1 = \mathbf{0}. \tag{2.35}$$

■

2.5. Conclusion:

In this chapter, the notion of an *LEDI Complete State Space* is introduced for the G/M/1-type Markov processes. It is shown that this notion leads to a new class of recursive solutions called finite memory recursive solutions for the equilibrium probabilities. A procedure for the computation of the equilibrium probabilities through these recursions is developed and an associated boundary value problem which arises in this computation is resolved. Furthermore, the relationship between finite memory recursions and matrix geometric recursions is explored.

In the next chapter, a transform domain approach to the transient analysis of G/M/1-type Markov processes is discussed. It is based on the idea of converting a system of differential equations into algebraic equations by Laplace transformation. This approach was employed for Quasi-birth-and-death processes in [ZhC1].

CHAPTER 3

FINITE MEMORY RECURSIVE SOLUTIONS FOR THE TRANSIENT ANALYSIS OF G/M/1 TYPE MARKOV PROCESSES

3.1. Transient Analysis of G/M/1-type Markov Processes:

The transition function of a time-homogeneous, continuous-time Markov chain satisfies the differential equation

$$\dot{\overline{\pi}}(t) = \overline{\pi}(t)\,\mathbf{Q}. \tag{3.1}$$

Taking the transform on both sides of (3.1) and rearranging, we have

$$\overline{\pi}(s)(\,\mathbf{Q} - s\mathbf{I}\,) = -\overline{\pi}(0). \tag{3.2}$$

Assume that the process at time 0 is in one of the states at level 0. Equivalently,

$$\overline{\pi}(0) = [\,\overline{\pi}_0(0) : \mathbf{0} : \mathbf{0} : \cdots\,].$$

Since $\pi(t)$ is a probability distribution, each of its components is nonnegative and bounded by one. It is easy to see that this boundedness implies that there is a single constant bounding every entry of $\pi(s)$ whenever $s \in S_{RHP}$, where $S_{RHP} = \{\, s\!: \mathrm{Real}(s) > 0\,\}$. This boundedness of the solution to (3.2) ensures that there is only one solution to (3.2). This follows from Theorem 4.18 in [Cin], which is rephrased below.

Lemma 3.1. If each component of the solution $\pi(s)$ to (3.2) is bounded by a common constant $c(s)$ for $\mathrm{Re}(s) > 0$, then $\pi(s)$ is unique.

■

In the following, the existence of a finite memory recursive solution for the transient state occupancy probabilities is shown.

Consider the matrix $(\,\mathbf{Q} - s\mathbf{I}\,)$ and perform the same column operations that transformed (3.1) into (3.3), arriving at

$$\begin{bmatrix} \mathbf{B}_0(s) & \mathbf{A}_{0L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1 & \mathbf{A}_{1L}(s) & \mathbf{A}_{1R}(s) & \mathbf{A}_{0L} & \mathbf{0} & \cdots \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{A}_{2R} & \mathbf{A}_{1L}(s) & \mathbf{A}_{1R}(s) & \cdots \\ \mathbf{B}_3 & \mathbf{0} & \mathbf{A}_{3R} & \mathbf{0} & \mathbf{A}_{2R} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{m-2} & \mathbf{0} & \mathbf{A}_{m-2R} & \mathbf{0} & \mathbf{A}_{m-3R} & \cdots \\ \mathbf{B}_{m-1} & \mathbf{0} & \mathbf{A}_{m-1R} & \mathbf{A}_{0L} & \mathbf{A}_{m-2R} & \cdots \\ \mathbf{B}_m & \mathbf{0} & \mathbf{A}_{mR} & \mathbf{A}_{1L}(s) & \mathbf{A}_{m-1R} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{mR} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{3.3}$$

In the above, the matrices with s as an argument, such as $\mathbf{A}_{1L}(s)$, have a -s added to those entries which are on the diagonal of $\mathbf{Q}$. On utilizing the structure of the above matrix obtained by performing column operations on $(\mathbf{Q} - s\mathbf{I})$, $\overline{\pi}(s)(\mathbf{Q} - s\mathbf{I}) = -\overline{\pi}(0)$, leads to

$$\begin{bmatrix} \overline{\pi}_0(s) & \overline{\pi}_1(s) & \cdots & \overline{\pi}_m(s) \end{bmatrix} \begin{bmatrix} \mathbf{B}_0(s) & \mathbf{A}_{0L} \\ \mathbf{B}_1 & \mathbf{A}_{1L}(s) \\ \mathbf{B}_2 & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{B}_{m-1} & \mathbf{0} \\ \mathbf{B}_m & \mathbf{0} \end{bmatrix} = [-\overline{\pi}_0(0) : 0]. \tag{3.4}$$

and

$$\begin{bmatrix} \overline{\pi}_n(s) & \overline{\pi}_{n+1}(s) & \cdots & \overline{\pi}_{n+m}(s) \end{bmatrix} \begin{bmatrix} \mathbf{A}_{1R}(s) & \mathbf{A}_{0L} \\ \mathbf{A}_{2R} & \mathbf{A}_{1L}(s) \\ \vdots & \vdots \\ \mathbf{A}_{m-1R} & \mathbf{A}_{0L} \\ \mathbf{A}_{mR} & \mathbf{A}_{1L}(s) \end{bmatrix} = \mathbf{0}. \tag{3.5}$$

When $[\,\mathbf{A}_{mR} : \mathbf{A}_{1L}(s)\,]$ is invertible, (3.5) leads to

$$\overline{\pi}_{n+m-1}(s) = \overline{\pi}_n(s)\mathbf{W}_1(s) + \overline{\pi}_{n+1}(s)\mathbf{W}_2(s) + \cdots + \overline{\pi}_{n+m-2}(s)\mathbf{W}_{m-1}(s).$$

Thus the notion of LEDI-completeness leads to a finite memory recursive solution for the Laplace transform of the vector of state occupancy probabilities at time t.

Now we consider the case where $[\ \mathbf{A}_{mR} : \mathbf{A}_{1L}(s)\ ]$ is not invertible. As in chapter 2, it is now shown that, even in this case, a finite memory recursive solution exists.

Lemma 3.2. If the G/M/1-type Markov process is irreducible, then the matrix $\mathbf{A}_{1LU}(s)$ formed by removing the last row and last column of $\mathbf{A}_{1L}(s)$ is a nonsingular matrix over the field of rational functions in s and the elements of the inverse are strictly rational functions.

Proof. From Lemma 2.1, $\mathbf{A}_{1LU}$ is nonsingular. Also

$$\mathbf{A}_{1LU}(s) = \mathbf{A}_{1LU} - s\mathbf{I} \tag{3.6}$$

Since the matrix $\mathbf{A}_{1LU}$ has only n eigenvalues, the determinant of $\mathbf{A}_{1LU}(s)$ is not identically zero. Hence it is nonsingular in the field of rational functions in s. Since the elements of the adjoint of $\mathbf{A}_{1LU}(s)$ are polynomials of degree at most N-1 and the characteristic polynomial is of degree n, the elements of the inverse of $\mathbf{A}_{1LU}(s)$ are strictly rational functions in s.

■

Theorem 3.1. If the LEDI-complete G/M/1-type Markov process is irreducible, then there exists an index $2 \leq i \leq m$ such that $[\mathbf{A}_{iR} : \mathbf{A}_{1L}(s)]$ is nonsingular.

Proof. Suppose that $[\ \mathbf{A}_{iR} : \mathbf{A}_{1L}(s)\ ]$ is singular for all $2 \leq i \leq m$. Equivalently, there exists an N-1 dimensional column vector $\mathbf{y}_i$ such that

$$\mathbf{A}_{1L}(s)\mathbf{y}_i(s) = \mathbf{A}_{iR} \quad \text{for} \quad 2 \leq i \leq m. \tag{3.7}$$

Rewriting the above system of equations

$$\mathbf{A}_{1LU}(s)\mathbf{y}_i(s) = \mathbf{A}_{iRU} \tag{3.8}$$

and

$$\mathbf{A}_{1LD}\mathbf{y}_i(s) = \mathbf{A}_{iRD} \tag{3.9}$$

Since $\mathbf{A}_{1LU}(s)$ is nonsingular by Lemma 3.2,

$$\mathbf{y}_i(s) = \mathbf{A}_{1LU}^{-1}(s)\mathbf{A}_{iRU}. \tag{3.10}$$

Since the elements of the inverse of $\mathbf{A}_{1L}(s)$ are strictly rational functions in s and $\mathbf{A}_{iRU}$ is nonnegative, $\mathbf{y}_i(s)$ is a strictly rational function of s. Hence if $\mathbf{y}_i(s)$ satisfies (3.9), then $\mathbf{A}_{iRD}$ is a rational function of s. Therefore, if $[\ \mathbf{A}_{iR} : \mathbf{A}_{1L}\ ]$ is singular, then $\mathbf{A}_{iRD} = 0$.

So by hypothesis, $\mathbf{A}_{iRD} = 0 \;\; 2 \leq i \leq m$. This is possible if $\mathbf{A}_{1LD} = \mathbf{0}$ or $\mathbf{A}_{1LD}\mathbf{y}_i = \mathbf{0}$. In the case when $\mathbf{A}_{1LD} = \mathbf{0}$, a contradiction to irreducibility can be obtained by the same argument as in the proof of Theorem 2.2. Even when $\mathbf{A}_{1LD} \neq \mathbf{0}$ but $\mathbf{A}_{1LD}\mathbf{y}_i = \mathbf{0}$, a similar argument as in Theorem 2.2 can be used. Details are omitted for the sake of brevity.

■

Since the above Theorem ensures that $[\, \mathbf{A}_{iR} \vdots \mathbf{A}_{1L}(s) \,]$ is nonsingular for some $2 \leq i \leq m$, by using a similar procedure as in the proof of Theorem 2.3, a finite memory recursive solution for the transient state occupancy probabilities can be found.

Even though the notion of LEDI-completeness leads to a finite memory recursive solution for the transient probabilities, as in the equilibrium case, a boundary value problem results. The boundary value problem is now considered and its solution is described.

Boundary Value Problem:

By considering the system of equations $\overline{\pi}(s)(\, \mathbf{Q} - s\mathbf{I} \,) = -\, \overline{\pi}(0)$ in (3.2) and breaking it appropriately, we arrive at a boundary system and a finite memory recursion system similar to those in (3.2) and (3.3). Suppose a finite memory recursive solution of order l results. i.e.

$$\overline{\pi}_{j+l}(s) = \overline{\pi}_j(s)\mathbf{W}_1(s) + \cdots + \overline{\pi}_{j+l-1}(s)\mathbf{W}_l(s) \quad \forall \; j \geq \; m-l \,. \qquad (3.11)$$

As in the equilibrium analysis, it is easy to see that the boundary system is underdetermined by (l-1)N+r-1 equations. By relating the finite memory recursive solution to an associated matrix geometric solution, it is now shown, where the missing equations come from.

Consider the set of states at levels j, j+1, ... , j+l-1 and form the vector $[\, \overline{\pi}_j(s), \overline{\pi}_{j+1}(s), \cdots, \overline{\pi}_{j+l-1}(s) \,]$. The finite memory recursive solution of order l described in (3.11) implies and is implied by the following associated matrix geometric solution.

$$[\, \overline{\pi}_{j+1}(s), \overline{\pi}_{j+2}(s), \cdots, \overline{\pi}_{j+l}(s) \,] = [\, \overline{\pi}_j(s), \overline{\pi}_{j+1}(s), ..., \overline{\pi}_{j+l-1}(s) \,]\, \mathbf{T}(s), \qquad (3.12)$$

where

$$\mathbf{T(s)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_1(s) \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_2(s) \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{W}_3(s) \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_4(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{W}_l(s) \end{bmatrix}. \tag{3.13}$$

Defining $\overline{\nu}_j(s) = [\,\overline{\pi}_j(s), \overline{\pi}_{j+1}(s), ..., \overline{\pi}_{j+l-1}(s)\,]$, from (3.11) we have

$$\overline{\nu}_{j+1}(s) = \overline{\nu}_j(s)\,\mathbf{T(s)}. \tag{3.14}$$

Now the fact that the solution to (3.2) is a probability distribution and that it is unique leads to the following Theorem.

Theorem 3.2. If $\{\, p_i(s), \vec{q}_i(s)\,;\, 1 \le i \le s \,\}$ are the right and left eigenvectors of $\mathbf{T(s)}$ respectively corresponding to the eigenvalues on or outside the unit circle, then for every $s \in S_{RHP}$, the matrix

$$\mathbf{U(s)} = \begin{bmatrix} \mathbf{B}_0(s) & \mathbf{A}_{0L} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{A}_{1L}(s) & \mathbf{E}_1 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} & \mathbf{E}_2 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{l-1} & \mathbf{0} & \mathbf{E}_{l-1} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_l & \mathbf{0} & \mathbf{E}_l & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_{l+1} & \mathbf{0} & \mathbf{E}_{l+1}(s) & \cdots & \mathbf{E}_1 & \mathbf{p}_1(s) & \cdots & \mathbf{p}_s(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ \mathbf{B}_m & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_{l+1}(s) & & & \end{bmatrix} \tag{3.15}$$

has full rank and the initial probability vector satisfies the system

$$\left[\overline{\pi}_0(s)\ \ \overline{\pi}_1(s)\ \ ......\ \ \overline{\pi}_{m-1}(s)\right]\mathbf{U}(s) = [\,-\overline{\pi}_0(0) : 0 :...:0\,]. \tag{3.16}$$

Proof. Components of the row vector $\overline{\pi}_n(t)$ are probabilities. Consequently each of them is nonnegative and bounded above by one. Therefore $f_l(t) = \sum_{n=0}^{l} \overline{\pi}_n^{\,i}(t)$ is a monotone sequence of nonnegative functions. Also, since $\sum_{n=0}^{\infty} \overline{\pi}_n(t)\mathbf{e} = 1$, the sequence

$\{ f_l(t) \}$ converges to $\sum_{n=0}^{\infty} \bar{\pi}_n^i(t)$. Now

$$\sum_{n=0}^{\infty} |\bar{\pi}_n^i(s)| \leq \sum_{n=0}^{\infty} \int_0^{\infty} |\bar{\pi}_n^i(t)| \, |e^{-st}| dt.$$

Let $s = \alpha + j\omega$ and $\alpha > 0$. Then

$$\sum_{n=0}^{\infty} |\bar{\pi}_n^i(s)| \leq \sum_{n=0}^{\infty} \int_0^{\infty} \bar{\pi}_n^i(t) e^{-\alpha t} dt.$$

Since $e^{-\alpha t}$ is a nonnegative function, $\sum_{n=0}^{l} \bar{\pi}_n^i(t) e^{-\alpha t}$ is a monotone sequence of nonnegative functions converging to $\sum_{n=0}^{\infty} \bar{\pi}_n(t) e^{-\alpha t}$. Hence by the Monotone Convergence Theorem,

$$\sum_{n=0}^{\infty} \int_0^{\infty} \bar{\pi}_n^i(t) e^{-\alpha t} dt = \int_0^{\infty} \sum_{n=0}^{\infty} \bar{\pi}_n^i(t) e^{-\alpha t} dt.$$

$$< \int_0^{\infty} e^{-\alpha t} dt = \frac{1}{\alpha} < \infty.$$

Hence

$$\sum_{n=0}^{\infty} |\bar{\pi}_n^i(s)| < \infty \text{ for } s \in S_{RHP}.$$

Since the absolute convergence implies convergence

$$\sum_{n=0}^{\infty} \bar{\pi}_n^i(s) < \infty. \quad \text{Hence} \quad \sum_{n=0}^{\infty} \bar{\pi}_n(s) \mathbf{e} < \infty.$$

Now by an argument similar to the one in Theorem 2.4, if $\mathbf{p}$ is the right eigenvector corresponding to an eigenvalue λ such that $|\lambda| \geq 1$, then $\bar{\nu}_{m-l}(s)\mathbf{p} = 0$. Thus these constraints on the vector $\bar{\nu}_{m-l}(s) = [\, \bar{\pi}_{m-l}(s), \bar{\pi}_{m-l+1}(s) \cdots \bar{\pi}_{m-1}(s) \,]$ can be used as additional equations in the boundary system. The remaining part of the proof follows the same argument as in Theorem 2.5.

■

The above results provide a method of finding $[\, \bar{\pi}_0(s) \, \bar{\pi}_1(s) \cdots \bar{\pi}_m(s) \,]$ and a closed form recursive solution for $\bar{\pi}_n(s)$ for $n \geq m+1$. The closed form inversion of the transforms is impossible in all but simple cases, leaving numerical inversion as the only possible alternative. Thus the accuracy of the approach depends on the numerical inversion algorithm.

It is now shown that the above results can be extended in a natural manner to the equilibrium and transient analysis of G/M/1 type Markov processes in which the transition rates between the states on a level and into the states on a level depend on the current level being considered. In other words, in the generator in (1.4), the matrices $\mathbf{A}_i,\ 0 \le i \le m$ are replaced by $\mathbf{A}_i(n),\ 0 \le i \le m \quad \forall\ n \ge 1.$

3.2. Extension to Level Dependent G/M/1-type Markov Processes:

G/M/1-type Markov processes with a canonical, irreducible generator of the following form arise commonly in applications [MuC4].

$$\vec{\mathbf{Q}} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{C}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_1(1) & \mathbf{A}_1(1) & \mathbf{A}_0(1) & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{B}_{2(2)} & \mathbf{A}_2(2) & \mathbf{A}_1(2) & \mathbf{A}_0(2) & \mathbf{0} & \cdots \\ \mathbf{B}_{3(3)} & \mathbf{A}_3(3) & \mathbf{A}_2(3) & \mathbf{A}_1(3) & \mathbf{A}_0(4) & \cdots \\ \mathbf{B}_{4(4)} & \mathbf{A}_4(4) & \mathbf{A}_3(4) & \mathbf{A}_2(4) & \mathbf{A}_1(5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{3.17}$$

Without loss of generality, we assume that the G/M/1-type Markov process described by the generator $\vec{\mathbf{Q}}$ is LEDI-complete. Now by employing a procedure similar to the one described in Theorem 2.3, it can be shown that a finite memory recursive solution of the form

$$\overline{\pi}_{n+k} = \overline{\pi}_n \mathbf{W}_1(n) + \overline{\pi}_{n+1} \mathbf{W}_2(n+1) + \cdots + \overline{\pi}_{n+k-1} \mathbf{W}_k(n+k-1),$$

exists for the equilibrium probabilities of the process $\vec{\mathbf{Q}}$ provided certain conditions are satisfied by the number and type of states at each level: The number of states receiving downward transitions at each level is equal to one and the number of states at each level is the same.

Also, by employing the techniques similar to the one in the previous section, it can be shown that a finite memory recursive solution for the Laplace transform of the state occupancy probabilities of the process $\vec{\mathbf{Q}}$ exists.

3.3. Numerical Considerations:

Equilibrium and transient analysis of G/M/1-type Markov processes by finite memory recursion essentially requires the computation of a stable associated matrix geometric recursion and the inversion of Laplace transform. Numerical considerations arising in the computation are discussed below.

Computation of the $\mathbf{W}_i$ matrices:

$\mathbf{W}_i$'s can be computed using the algorithm described in the Theorem 2.3. This algorithm utilizes routines for checking the singularity of a matrix and for inverting a matrix. Numerically stable routines for performing these computations are available in the IMSL library.

Stable Recursion Matrix Computation:

Consider the matrix geometric recursion matrix $\mathbf{T}$ associated with the finite memory recursion system in (2.13). The spectral representation of $\mathbf{T}$ is

$$\mathbf{T} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{v}_i \tag{3.18}$$

where $\mathbf{u}_i$ is the right eigenvector of $\mathbf{T}$ corresponding to the eigenvalue λ_i and $\bar{\mathbf{v}}_i$ is the left eigenvector. From (2.15)

$$\bar{\nu}_{j+1} = \bar{\nu}_j \mathbf{T} \qquad \forall \geq m-l+1. \tag{3.19}$$

From the proof of Theorem 2.4, if $|\lambda_i| > 1$, then $\bar{\nu}_{m-l+1}\mathbf{p}_i = \mathbf{0}$. Thus the mode corresponding to λ_i is unexcited by the initial probability vector. Let there be k such eigenvalues on or outside the unit circle, denoted by $\lambda_1, \lambda_2, ..., \lambda_n$. Consider the matrix $\tilde{\mathbf{T}}$ obtained from $\mathbf{T}$ by deleting the unexcited modes. This modification of the matrix $\mathbf{T}$ is necessary since unexcited modes can cause instabilities in the computation of equilibrium probabilities. Thus the stable associated matrix geometric recursion matrix has the spectral representation

$$\tilde{\mathbf{T}} = \sum_{i=k+1}^{n} \lambda_i \mathbf{u}_i \mathbf{v}_i. \tag{3.20}$$

It can easily be inferred from (3.18) that $\tilde{\mathbf{T}}$ not only satisfies the same recursion matrix as $\mathbf{T}$, but is also stable in the sense that as $n \rightarrow \infty$, $\tilde{\mathbf{T}}^n$ converges to a zero matrix.

Numerical Inversion of Laplace Transform:

Many numerical inversion algorithms are available [KrS]. We choose the inversion algorithm developed by Honig and Hirdes [HoH] since we have obtained good results with it in the related work in [ZhC1].

The above theoretical results and the numerical procedures for the computation of the transient state occupancy probabilities of a G/M/1-type Markov process are now illustrated with a specific example.

Example 3. Consider a G/M/1-type Markov process described by a generator as in (2.19) in which

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{A}_1 = \begin{bmatrix} -10 & 2 \\ 0 & -8 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 0 & 4 \\ 0 & 6 \end{bmatrix}, \ \mathbf{A}_3 = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \text{and } \mathbf{A}_k = 0 \ \forall \ k \geq 4.$$

$$\mathbf{B}_0 = \begin{bmatrix} -3 & 2 \\ 3 & -3 \end{bmatrix} \ \mathbf{B}_1 = \mathbf{A}_2, \ \mathbf{B}_2 = \mathbf{A}_3 \text{ and } \mathbf{B}_k = 0 \ \forall \ k \geq 3.$$

$$\mathbf{C}_0 = \mathbf{A}_0, \ \mathbf{C}_1 = \begin{bmatrix} -7 & 2 \\ 0 & -6 \end{bmatrix}, \ \mathbf{C}_2 = \mathbf{A}_2, \ \mathbf{C}_3 = \mathbf{A}_3 \text{ and } \mathbf{C}_k = 0 \ \forall \ k \geq 4.$$

From the system of equations in (3.2), since [$\mathbf{A}_{3R}$: $\mathbf{A}_{1L}(s)$ is invertible, a finite memory recursive solution of order 2 results. Equivalently

$$\overline{\pi}_{n+2}(s) = \overline{\pi}_n(s)\mathbf{W}_1(s) + \overline{\pi}_{n+1}(s)\mathbf{W}_2(s) \qquad \text{for } n \geq 2,$$

where

$$\mathbf{W}_1(s) = \begin{bmatrix} \frac{1}{10+s} & -(\frac{23+2s)}{(20+2s)} \\ 0 & \frac{(8+s)}{2} \end{bmatrix} \ \mathbf{W}_2(s) = \begin{bmatrix} \frac{-(9+s)}{10+s} & \frac{-(13+s)}{(20+2s)} \\ 0 & -3 \end{bmatrix}.$$

Furthermore, the boundary system of linear equations is given by

$$[\,\bar{\pi}_0(s) : \bar{\pi}_1(s) : \bar{\pi}_2(s) : \bar{\pi}_3(s)\,] \begin{bmatrix} -3-s & 2 & 1 & 0 & 0 \\ 3 & -3-s & 0 & 0 & 0 \\ 0 & 4 & -7-s & 2 & 1 \\ 0 & 6 & 0 & -6-s & 0 \\ 0 & 3 & 0 & 4 & -10-s \\ 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} = [\,-\bar{\pi}_0 : 0 : \cdots\ 0\,].$$

This is a system of 5 equations in 8 unknowns. To determine the unique solution to the system, 3 more independent equations are needed. Since the G/M/1-type Markov process is recurrent nonnull, these remaining equations should some how be obtained from the finite memory recursion system. As shown in this section, the conditions for the stability of the associated matrix geometric solution provide the clue to the problem. The associated matrix geometric solution is described by

$$\mathbf{T}(s) = \begin{bmatrix} 0 & 0 & \frac{1}{10+s} & \frac{-(23+2s)}{(20+2s)} \\ 0 & 0 & 0 & \frac{(8+s)}{2} \\ 1 & 0 & \frac{-(9+s)}{10+s} & \frac{-(13+s)}{(20+2s)} \\ 0 & 1 & 0 & -3 \end{bmatrix}.$$

The eigenvalues of $\mathbf{T}(s)$ are

$$\lambda_1(s) = \frac{1}{(10+s)},\quad \lambda_2(s) = -1,\quad \lambda_3(s) = -3+\frac{\sqrt{(25+2s)}}{2},\quad \lambda_4(s) = -3-\frac{\sqrt{(25+2s)}}{2}.$$

It is easy to show that the eigenvalue at $\frac{1}{(10+s)}$ is strictly inside the unit circle for $s \in S_{RHP}$ and all the others are on or outside the unit circle. But by Theorem 3.2, all the modes corresponding to the eigenvalues on or outside the unit circle must disappear. Therefore by tagging the corresponding eigenvectors, $p_1(s), p_2(s), p_3(s)$ to the boundary system, we can find the unique solution. This is shown below.

$$[\,\overline{\pi}_0(s) : \overline{\pi}_1(s) : \overline{\pi}_2(s) : \overline{\pi}_3(s)\,] \begin{bmatrix} -3-s & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3-s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -7-s & 2 & 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -6-s & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & -10-s & p_1(s) & p_2(s) & p_3(s) \\ 0 & 2 & 0 & 6 & 0 & & & \\ 0 & 0 & 0 & 3 & 0 & & & \\ 0 & 0 & 0 & 2 & 0 & & & \end{bmatrix} = [\,-\overline{\pi}_0 : 0 : \cdots \; 0\,].$$

Since the coefficient matrix in the above system of linear equations is invertible the boundary vector $[\,\overline{\pi}_0(s) : \overline{\pi}_1(s) : \overline{\pi}_2(s) : \overline{\pi}_3(s)\,]$ can be computed.

To illustrate the utility of the above method various transient performance measures, which are of interest in practical models are computed. These performance measures are

$$T_n(t) = \sum_{i=n}^{\infty} \overline{\pi}_i(t)\mathbf{e}, \tag{3.21}$$

$$M(t) = \sum_{i=1}^{\infty} i\overline{\pi}_i(t)\mathbf{e}, \tag{3.22}$$

$$\mathrm{Var}(t) = L(t) = \sum_{i=1}^{\infty} i^2\overline{\pi}_i(t)\mathbf{e} - M(t)^2. \tag{3.23}$$

Laplace transform of these quantities can be computed in closed form since a finite memory recursion exists. The approach is described below.

Consider the matrix geometric solution associated with the finite memory recursive solution of order 2 and let $\mathbf{u}_i(s)$, $\overline{\mathbf{v}}_i(s)$ be the right and left eigenvectors of $\mathbf{T}(s)$ describing the associated matrix geometric solution. Then the spectral decomposition of $\mathbf{T}(s)$ is given by

$$\mathbf{T}(s) = \lambda_1(s)\mathbf{E}_1(s) + \cdots + \lambda_4\mathbf{E}_4(s),$$

where $\mathbf{E}_i(s) = \mathbf{u}_i(s)\overline{\mathbf{v}}_i(s)$ for $i = 1, 2, 3, 4$. The stable recursion matrix $\tilde{\mathbf{T}}(s)$ is given by

$$\tilde{\mathbf{T}}(s) = \lambda_1(s)\mathbf{E}_1(s). \tag{3.24}$$

Utilizing $\tilde{\mathbf{T}}(s)$, closed form expression for the transient performance measures can easily be found.

Tail Distribution:

$$T_n(s) = \sum_{i \geq n} \overline{\pi}_i(s)\mathbf{e} = \sum_{i \geq n-1} \overline{\nu}_i(s)\hat{\mathbf{e}}, \tag{3.25}$$

$$= \overline{\nu}_{n-1}[\,\mathbf{I} - \tilde{\mathbf{T}}(s)\,]^{-1}\hat{\mathbf{e}} = \overline{\nu}_2 \tilde{\mathbf{T}}^{n-3}[\,\mathbf{I} - \tilde{\mathbf{T}}(s)\,]^{-1}\hat{\mathbf{e}}. \tag{3.26}$$

First Moment of the Transient Probability Distribution:

$$M(s) = \overline{\pi}_1(s)\mathbf{e} + 2\overline{\pi}_2(s)\mathbf{e} + [\,\overline{\pi}_2(s) : \overline{\pi}_3(s)\,][\,3\mathbf{I} - 2\tilde{\mathbf{T}}(s)\,][\mathbf{I} - \tilde{\mathbf{T}}(s)]^{-2}\hat{\mathbf{e}}. \tag{3.27}$$

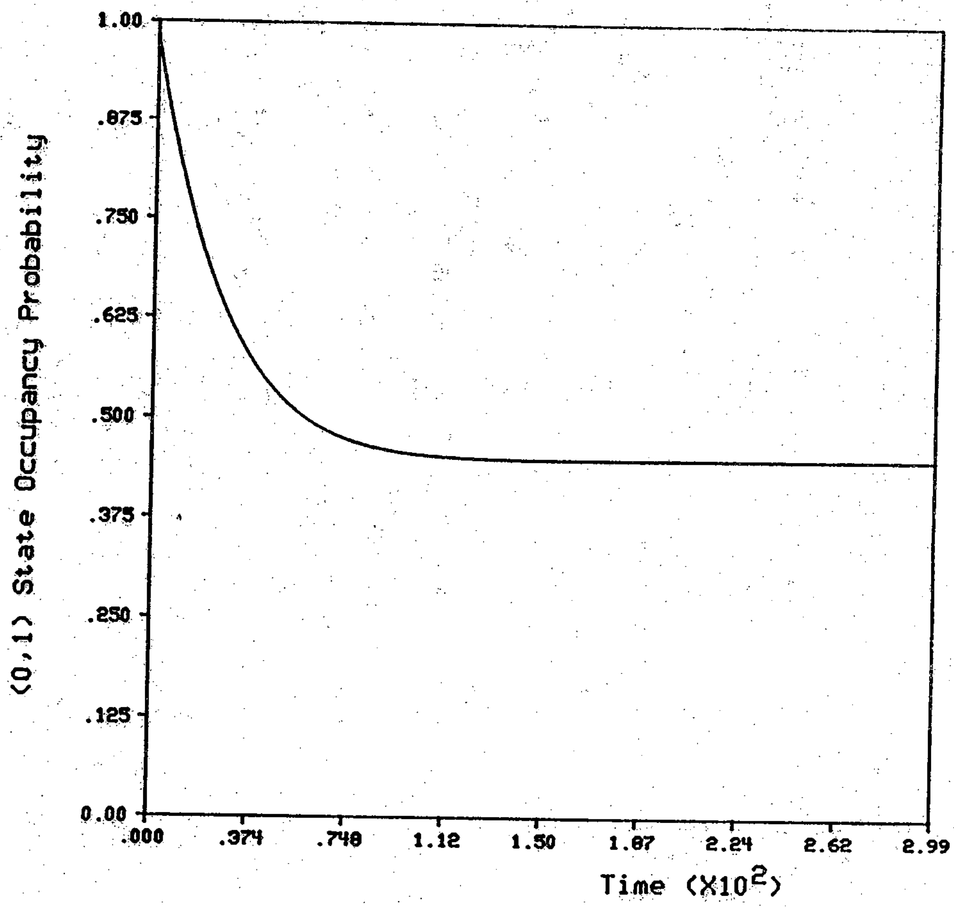
Variance of the Transient Probability Distribution:

$$L(s) = \overline{\pi}_1(s)\mathbf{e} + 4\overline{\pi}_2(s)\mathbf{e} + [\overline{\pi}_2(s) : \overline{\pi}_3(s)][9\mathbf{I} - 11\tilde{\mathbf{T}}(s) + 4\tilde{\mathbf{T}}^2(s)][\mathbf{I} - \tilde{\mathbf{T}}(s)]^{-3}\hat{\mathbf{e}} - [M(s)]^2. \tag{3.28}$$

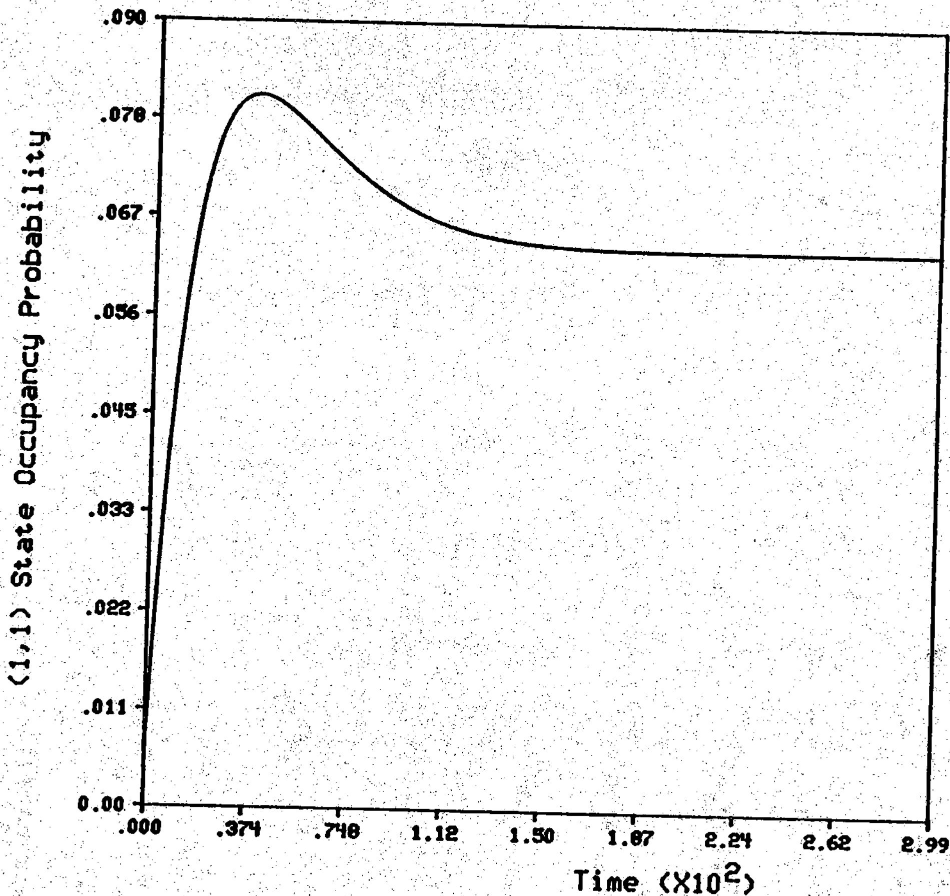
Using the numerical inversion algorithm, $\pi_{(1,1)}(t)$, $\pi_{(1,2)}(t)$, M(t) and the tail distribution $T_n(t)$ can be obtained. In these computations, the formal justification required for interchanging integral and the infinite sum can be provided along the lines of the argument in the proof of Theorem 3.2. The plots of $\pi_{(1,1)}(t)$, $\pi_{(1,2)}(t)$, M(t)and$T_n(t)$ are shown in Figures 3.1-3.5.

3.4 Conclusions:

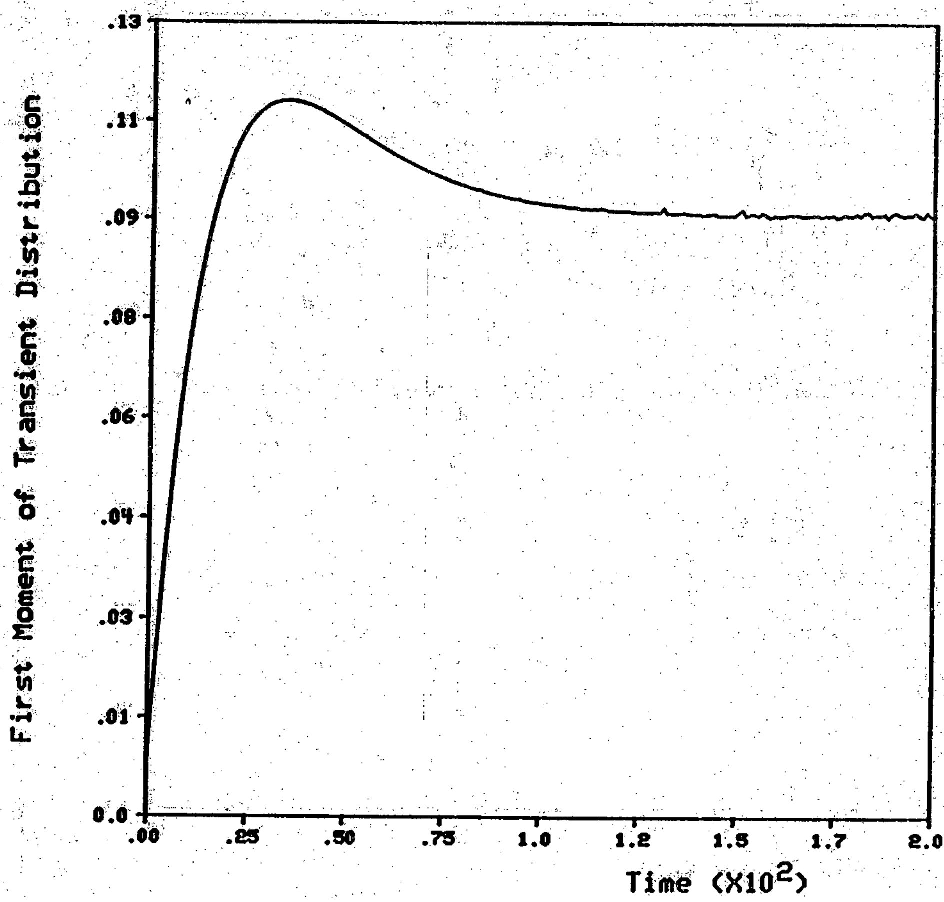
In this chapter, utilizing the notion of LEDI-completeness for G/M/1-type Markov processes, it is shown that a finite memory recursive solution for the Laplace transform of the transient state occupancy probabilities exists. Various transient performance measures are defined. The results are illustrated with a numerical example and the numerical considerations are discussed. The results relating to equilibrium and transient analysis are extended in a natural manner to the level dependent G/M/1-type Markov processes.



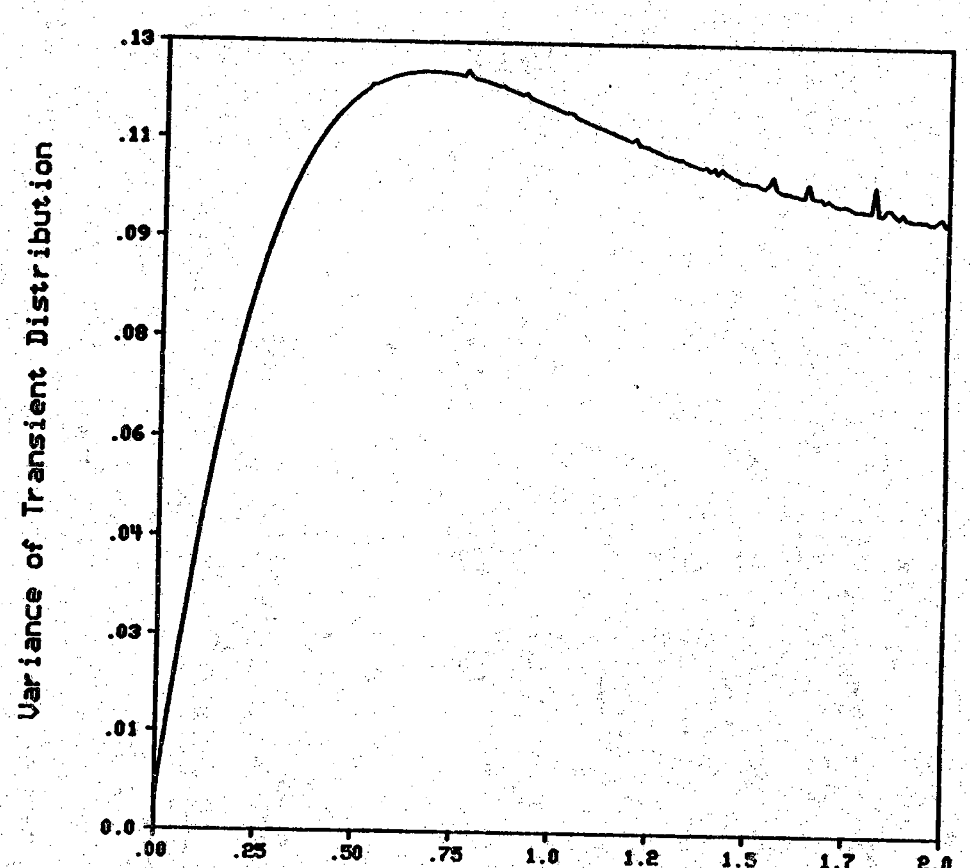
3.1 Time dependent probability that the process is in the state (0,1). The G/M/1 type process starts in this state.



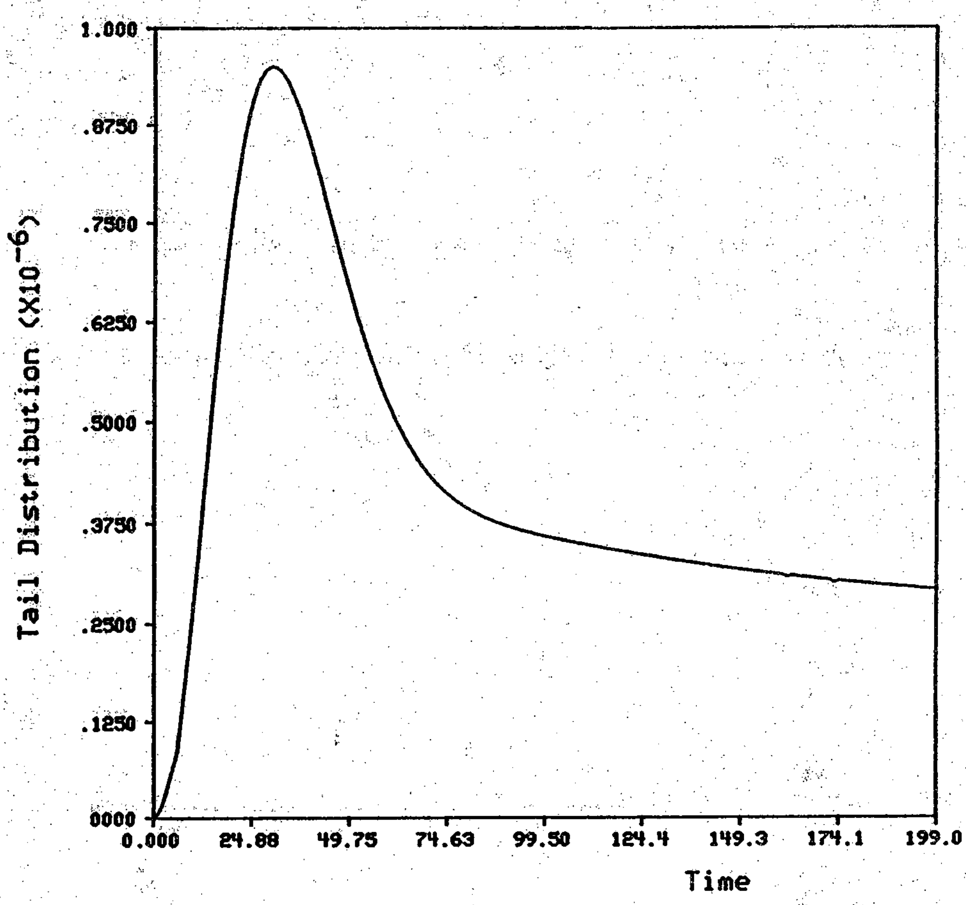
3.2 Probability as a function of time that the process is in the state (1,1).



3.3 First Moment of the time dependent probability distribution.



3.4 Variance of the time dependent probability distribution.



3.5 The tail distribution for the G/M/1 type Markov process.

CHAPTER 4

FINITE MEMORY RECURSIVE SOLUTIONS FOR THE EQUILIBRIUM AND TRANSIENT ANALYSIS OF M/G/1-TYPE MARKOV PROCESSES

4.1 Finite Memory Recursions for the Equilibrium and Transient Probability Distributions:

An M/G/1-type Markov process is a continuous time vector Markov process on the state space $\{(i,j) : i \geq 0, 1 \leq j \leq n\}$, with generator of the form

$$Q = \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \cdots \\ \mathbf{C}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 & \cdots \\ \mathbf{0} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_0 & \mathbf{A}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

M/G/1-type processes provide natural stochastic models for widely differing stochastic phenomena [Neu1].

The notion of *LEDI-Complete* state space, introduced for G/M/1-type Markov processes can be extended in a natural manner to M/G/1-type Markov processes. Any M/G/1-type process can be made LEDI-complete by expanding the state space. Thus there is no loss of generality in restricting consideration to LEDI-complete M/G/1-type Markov processes. By performing column operations on the generator, it can be seen that a finite memory recursive solution for the equilibrium probabilities exists. The boundary value problem which arises in finding the initial probability vector can be resolved by an approach similar to the one described in section 2.3. The results can also be extended for the transient analysis of M/G/1-type Markov processes.

Based on the idea of converting a system of differential equations into algebraic equations by Laplace transformation, a transform domain approach to the transient analysis of M/G/1-type Markov processes can be developed. This method parallels the development in the previous chapter 3. Specifically, it can be shown that a finite memory recursive solution for the Laplace transform of the vector of state occupancy probabilities at time t exists.

It is easy to see that in the same spirit as the above extension, finite memory recursive solutions can be extended to the equilibrium and transient analysis of multi-dimensional M/G/1 and G/M/1-type Markov processes. This extension should be straight forward but tedious.

4.2. Conclusions:

In this chapter, it is briefly described how the notion of LEDI complete state space, introduced for G/M/1-type Markov processes, can be extended to the M/G/1-type Markov processes and how this concept leads to finite memory recursions for the equilibrium and transient analysis of M/G/1-type Markov processes.

CHAPTER 5
CONCLUSION

In this report, the notion of *Level Entrance Direction Information Complete* state space is introduced for the G/M/1-type and M/G/1-type Markov processes. It is then shown that this criterion leads to a finite memory recursive solution for the equilibrium probabilities. Also, it is proven that a finite memory recursive solution for the Laplace transform of the state occupancy probabilities can be found. Thus a tractable method for the transient analysis of these processes is found. The relationship between matrix geometric solutions and finite memory recursive solutions is explored. Thus finite memory recursions in contrast to matrix geometric recursions not only provide closed form recursion matrices but also lead to tractable transient analysis of finite as well as countable state space skip free chains.

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