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ABSTRACT

This paper presents a generalized approach for controlling various cases of the constrained robot system. To accomplish specific tasks successfully by a constrained robot system, both the constraint forces/torques and the motion of the manipulator end-effector must be specified and controlled. Using the Jacobian matrix of the constraint function, the generalized coordinates of the constrained robot system can be partitioned into two sets; this leads to partitioning the constrained robot system into two subsystems. The constraint forces/torques in each subsystem can be decomposed into two components: the motion-independent and the motion-dependent forces/torques. Using the constraint function in the Cartesian space, the motion-independent forces/torques can be expressed by a generalized multiplier vector and the Jacobian matrix of the constraint function. The motion-dependent forces/torques can be determined by the motion of the manipulator end-effector, the motion-independent forces/torques, and other known quantities. This decomposition of the constraint robot system into subsystems leads to the design of a nonlinear decoupled controller with a simple structure, which takes the constraints into consideration for controlling the constrained robot system. Applying the proposed nonlinear decoupled controller to each subsystem and using the relation between the motion-independent forces/torques in the subsystems, we can show that both the errors in the manipulator end-effector motion and the constraint forces/torques approach zero asymptotically. Typical examples of the constrained robot systems are analyzed and discussed.

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TABLE OF CONTENTS

1. Introduction

Depending on the task performed by a robot system, the robot system can be classified into two categories: unconstrained and constrained robot systems. In the unconstrained robot system, the manipulator end-effector moves freely in the workspace and does not interact with the environment to complete the task, and there is no force/torque which cannot be determined by the motion of the manipulator. These tasks, in general, can be specified in terms of the preplanned motion trajectory of the manipulator end-effector. Since the motion trajectory contains only position/orientation information, existing positional controllers can be used successfully to control the unconstrained robot system [1]. In the constrained robot system, the manipulator end-effector is actively interacting with its environment to complete the task. Due to the constraints imposed by the task geometry and the environment, the manipulator end-effector motion is confined by the constraints and the motion has to comply with the constraints. Typical examples are the peg-and-hole insertion task, a robot with its end-effector contacting a rigid surface, and two robots manipulating a common object. For these tasks, the contact exists between the manipulator end-effector and the environment. In other cases of the constrained robot system, the contact may be at points other than the manipulator end-effector. In this paper, a generalized approach for controlling various cases of the constrained robot system in which the manipulator end-effector is interacting with its environment will be addressed.

Since the constraint forces/torques resulting from the contact between the manipulator end-effector and the environment cannot be determined by the motion of the manipulator end-effector, the tasks performed by the constrained robot system cannot be specified only in terms of the motion trajectory. To accomplish the task successfully, the constraint forces/torques as well as the motion of the manipulator end-effector must be specified and controlled. At present, several strategies/architectures for controlling these constraint forces/torques have been proposed, that include, for example, the stiffness control [2], the hybrid control [3], the damping control [4], and the operational space approach [5]. However, none of these controls considers the dynamic model of the constrained robot system incorporating the constraint effects. Recently, significant progress has been made on the dynamic model of the constrained robot system [6-11]. In particular, the dynamic model of the constrained robot system in [8, 11] was used to control both the motion of the manipulator end-effector and the constraint forces/torques in tasks which are characterized by physical contacts between the manipulator end-effector and a constraint surface. Using the constraint function, nonlinear transformations were introduced to develop the equations of motion for the constrained robot system [8, 11]. These equations of motion are written in a reduced model consisting of two sets of equations. One set of the equations of motion contains no constraint forces/torques and characterizes the motion of the manipulator end-effector on the constraint surface. The other set of the equations is used to calculate the constraint forces/torques which are caused by the interaction between the manipulator end-effector and the environment. Nonlinear decoupled controllers are then proposed to track the preplanned motion and the constraint forces/torques of the manipulator end-effector.

Other cases of the constrained robot system involving two cooperating robot systems have also been studied. Most of the research on two cooperating robot systems focuses on the load distribution [12-14], the master-slave scheme [15], and the extended hybrid control [16, 17]. Some of the research considered the constraint effects of the internal forces/torques in two cooperating robot systems. Since these internal forces/torques cause stress (compression, tension, and shear) in the load and do not contribute to the motion of the load because the summation of these forces/torques is zero [14, 18, 19], the internal forces/torques as well as the motion of the manipulator end-effector must be controlled for the general manipulation of the load. Although the above methods considered the constraint effects in two cooperating robots, no constraint equations were derived explicitly to show that the relation of the internal forces/torques can be obtained from the constraint equations. In this paper, considering a two cooperating robot system as a constrained robot system, we will show that the relation of the internal forces/torques in two cooperating robots can be derived from the constraint equations.

This paper presents a generalized approach for controlling various cases of the constrained robot system. To accomplish specific tasks successfully by a constrained robot system, both the constraint forces/torques and the motion of the manipulator end-effector must be specified and controlled. Using the Jacobian matrix of the constraint function, the generalized coordinates of the constrained robot system can be partitioned into two sets, and an influence coefficient matrix can be constructed to relate the time derivatives of the dependent variables to the independent variables [20, 21]. The equations of motion of the constrained robot system are partitioned according to the partitioned coordinates, and the dependent variables are replaced by functions of the independent variables. This results in partitioning the constrained robot system into two subsystems, which are different from the two subsystems generated from the nonlinear transformation technique [8, 11]. Although this two subsystem representation is in a redundant form from the mathematical point of view, the decomposition of the constraint forces/torques can be easily performed and a nonlinear decoupled controller for the constrained robot system, which has a much simpler structure than other controllers, can be constructed and realized. The constraint forces/torques resulting from the interaction of the manipulator end-effector and the environment can be decomposed into two components: the motion-independent and the motion-dependent forces/torques. Using the constraint function in the Cartesian space, the motion-independent forces/torques can be expressed by a generalized multiplier vector and the Jacobian matrix of the constraint function. The motion-dependent forces/torques can be expressed by the motion of the manipulator end-effector, the motion-independent forces/torques, and other known quantities. Thus, the control problem of the constrained robot system reduces to controlling only the motion of the manipulator end-effector and the motion-independent forces/torques. Applying the proposed nonlinear decoupled controller to each subsystem and using the relation between the motion-independent forces/torques in the subsystems, we can show that both the errors in the manipulator end-effector motion and the constraint forces/torques approach zero asymptotically. Utilizing this approach, typical examples of the constrained robot system are analyzed and discussed.

2. A Generalized Approach for the Control of Constrained Robots

Let us consider a constrained robot system whose equations of motion in the joint-variable space are expressed as [8, 10, 11]

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}}(t) + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{\Gamma}_a(t) + \mathbf{J}(\mathbf{q})^T \mathbf{f}_c(t) \tag{1}$$

and its associated constraint function is expressed in a vector algebraic equation of the form

$$\mathbf{\Theta}(\mathbf{q}) = \mathbf{0} \tag{2}$$

where $\mathbf{D}(\mathbf{q}) \in R^{n \times n}$ is a positive definite matrix function, $\mathbf{q}(t) \in R^n$ is the vector of generalized coordinates (or joint variables), $\mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) \in R^n$ and $\mathbf{G}(\mathbf{q}) \in R^n$ are vector functions, $\mathbf{\Gamma}_a(t) \in R^n$ is the vector function consisting of actuator joint forces/torques, $\mathbf{f}_c(t) \in R^n$ is the vector function consisting of generalized forces/torques due to constraints in the Cartesian space [22], $\mathbf{J}(\mathbf{q}) \in R^{n \times n}$ is the matrix function consisting of robot Jacobian matrices, the superscript "T" denotes vector or matrix transpose, and the constraint function $\mathbf{\Theta} : R^n \to R^m$ with $m < n$. As discussed in [11], if the constraint function $\mathbf{\Theta}(\mathbf{q}) = \mathbf{0}$ is identically satisfied, then also $\left[\frac{\partial \mathbf{\Theta}(\mathbf{q})}{\partial \mathbf{q}}\right] \dot{\mathbf{q}} = \mathbf{0}$. Thus, $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ are restricted to the manifold S_1 in R^{2n} defined by $S_1 = \left\{ (\mathbf{q}, \dot{\mathbf{q}}) : \mathbf{\Theta}(\mathbf{q}) = \mathbf{0},\ \left[\frac{\partial \mathbf{\Theta}(\mathbf{q})}{\partial \mathbf{q}}\right] \dot{\mathbf{q}} = \mathbf{0} \right\}$ rather than to the space R^{2n}.

The above dynamic model is used in most of previous work on constrained robot systems. However, since most of the task specifications for robot motion and constrained forces/torques can be easily specified in the Cartesian space, the dynamic model of the constrained robot system in the Cartesian space must be considered. For the dynamic model in the Cartesian space†, we shall assume that the robot is nonredundant and always at a nonsingular configuration. Extensions of the proposed approach to the redundant robot and to the robot in a singular configuration require further research. Let us consider a constrained robot system whose equations of motion in the Cartesian space are expressed as [5, 7]

$$\mathbf{\Lambda}(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{c}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{p}(\mathbf{x}) = \mathbf{F}_a(t) + \mathbf{f}_c(t) \tag{3}$$

and its associated constraint function is expressed in a vector algebraic equation of the form

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{0} \tag{4}$$

where $\mathbf{x}(t) \in R^n$ is the vector of generalized coordinates (or Cartesian variables) of the constrained robot system†, $\mathbf{\Lambda}(\mathbf{x}) \in R^{n \times n}$ is a positive definite matrix function, $\mathbf{c}(\mathbf{x}, \dot{\mathbf{x}}) \in R^n$ and $\mathbf{p}(\mathbf{x}) \in R^n$ are vector functions, $\mathbf{F}_a(t) \in R^n$ is the vector of generalized forces/torques, and the

† All the quantities in the Cartesian space are referenced to the global reference coordinate frame.

†Throughout this paper, the instantaneous angular rotations are used for the description of orientation error of the manipulator end-effector.

constraint function $\mathbf{\Phi}: R^n \rightarrow R^m$ with $m < n$. As in the joint-variable space, if the constraints are identically satisfied, $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ are restricted to the manifold S_2 in R^{2n} defined by

$$S_2 = \left\{ (\mathbf{x},\dot{\mathbf{x}}) : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0},\ \left[\frac{\partial \mathbf{\Phi}(\mathbf{x})}{\partial \mathbf{x}} \right] \dot{\mathbf{x}} = \mathbf{0} \right\}$$ rather than to the space R^{2n}.

2.1. Partitioning of Generalized Coordinates

One of the major differences between the constrained and the unconstrained robot systems is the existence of the constraint function in (4). Using this constraint function (4), the order of the dynamic model of the constrained robot system can be reduced via nonlinear transformation [8] or the dynamic model can be transformed into two equivalent subsystems, one containing motion component and the other containing force/torque and motion components [11]. In our approach, we use the constraint function and the implicit function theorem to partition the constrained robot system into two subsystems, both containing motion and force/torque components. The result of this partitioning will yield a nonlinear decoupled controller with a much simpler structure. To partition the Cartesian coordinates into two sets of coordinates [21], let us use the implicit function theorem [23]. Suppose that a constant vector $\mathbf{x}_c \in R^n$ satisfies the following properties:

(i) The constraint function $\mathbf{\Phi}(\mathbf{x})$ is twice continuously differentiable in some neighborhood of $\mathbf{x}_c$.

(ii) $\mathbf{\Phi}(\mathbf{x}_c) = \mathbf{0}$.

(iii) Rank $\left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} \right]_{\mathbf{x}=\mathbf{x}_c} = m$.

After a reordering of the variables and using the implicit function theorem, an open set $V \subset R^{n-m}$ and a twice continuously differentiable function $\mathbf{\Omega}: V \rightarrow R^m$ can be obtained such that

$$\mathbf{\Phi}(\,\mathbf{\Omega}(\mathbf{v}),\ \mathbf{v}\,) = \mathbf{0} \quad \text{for all } \mathbf{v} \in V. \tag{5}$$

Assuming that (5) holds with $V = R^{n-m}$, a set of generalized coordinates $\mathbf{x}$ can be partitioned into two sets of coordinates,

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \quad \text{where } \mathbf{u} \in R^m \text{ and } \mathbf{v} \in R^{n-m}. \tag{6}$$

Using the partitioned coordinates, the equations of motion in (3) can be partitioned

$$\begin{bmatrix} \mathbf{\Lambda}_{uu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{uv}(\mathbf{u},\mathbf{v}) \\ \mathbf{\Lambda}_{vu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{vv}(\mathbf{u},\mathbf{v}) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{c}_u(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \\ \mathbf{c}_v(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \end{bmatrix} + \begin{bmatrix} \mathbf{p}_u(\mathbf{u},\mathbf{v}) \\ \mathbf{p}_v(\mathbf{u},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{au} \\ \mathbf{F}_{av} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_{cu} \\ \mathbf{f}_{cv} \end{bmatrix} \tag{7}$$

where $\mathbf{F}_{au} \in R^m$ and $\mathbf{F}_{av} \in R^{n-m}$. Differentiating the constraint function (4) with respect to time and expressing the resultant equations in terms of the partitioned coordinates, we have

$$\dot{\boldsymbol{\Phi}}(\mathbf{x}) = \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}}\right] \dot{\mathbf{x}} = \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}} \; \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right] \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{bmatrix}$$

$$= \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right] \dot{\mathbf{u}} + \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right] \dot{\mathbf{v}} = \mathbf{0} .$$

Since $\left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]$ is nonsingular, the time derivative of $\mathbf{u}$ can be expressed as

$$\dot{\mathbf{u}} = - \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]^{-1} \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right] \dot{\mathbf{v}} .$$

Because $\mathbf{u}$ is expressed by $\boldsymbol{\Omega}(\mathbf{v})$ on the constraint manifold, we can define a function of $\mathbf{v}$, $\mathbf{W}(\mathbf{v}) \in R^{m \times (n-m)}$, called the influence coefficient matrix by some authors [21], as

$$\mathbf{W}(\mathbf{v}) \triangleq - \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]^{-1} \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right] . \tag{8}$$

Using the influence coefficient matrix, the first and second time derivatives of $\mathbf{u}$ can be expressed as

$$\dot{\mathbf{u}} = \mathbf{W}(\mathbf{v})\dot{\mathbf{v}} \quad \text{and} \quad \ddot{\mathbf{u}} = \dot{\mathbf{W}}(\mathbf{v}, \dot{\mathbf{v}})\dot{\mathbf{v}} + \mathbf{W}(\mathbf{v})\ddot{\mathbf{v}} .$$

If there is no confusion, we shall omit $\mathbf{v}$ and $\dot{\mathbf{v}}$ in all the functions of $\mathbf{v}$ and $\dot{\mathbf{v}}$ for clarity and ease of notation. Substituting $\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ with functions of $\mathbf{v}$, $\dot{\mathbf{v}}$ and $\ddot{\mathbf{v}}$, the partitioned equations of motion (7) become

$$\begin{bmatrix} \boldsymbol{\Lambda}_{uu} & \boldsymbol{\Lambda}_{uv} \\ \boldsymbol{\Lambda}_{vu} & \boldsymbol{\Lambda}_{vv} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{W}\ddot{\mathbf{v}} \\ \ddot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{c}_u \\ \mathbf{c}_v \end{bmatrix} + \begin{bmatrix} \mathbf{p}_u \\ \mathbf{p}_v \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{au} \\ \mathbf{F}_{av} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_{cu} \\ \mathbf{f}_{cv} \end{bmatrix} .$$

This results in partitioning the constrained robot system into two subsystems according to the partitioned coordinates $\mathbf{u}$ and $\mathbf{v}$

$$(\boldsymbol{\Lambda}_{uu}\mathbf{W} + \boldsymbol{\Lambda}_{uv})\ddot{\mathbf{v}} + \mathbf{c}_u + \boldsymbol{\Lambda}_{uu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_u = \mathbf{F}_{au} + \mathbf{f}_{cu} \qquad \text{(subsystem 1)} ,$$

$$(\boldsymbol{\Lambda}_{vu}\mathbf{W} + \boldsymbol{\Lambda}_{vv})\ddot{\mathbf{v}} + \mathbf{c}_v + \boldsymbol{\Lambda}_{vu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_v = \mathbf{F}_{av} + \mathbf{f}_{cv} \qquad \text{(subsystem 2)} . \tag{9}$$

These two subsystems as in (9) are equivalent to the constrained robot system as in (3) and (4). Considering that the constrained robot system as in (3) and (4) has $(n-m)$ degrees of freedom, the above two subsystems with n equations of motion is in a *redundant* form. Thus, as in [8, 21], the number of equations of motion can be reduced from n to $(n-m)$. However, as will be explained later, this *redundant* representation as in (9) together with the decomposition of the constraint forces/torques will yield a very simple nonlinear decoupled controller structure for the constrained robot system.

In (9), the constraint forces/torques $\mathbf{f}_c(t)$ have been partitioned into $\mathbf{f}_{cu}(t)$ and $\mathbf{f}_{cv}(t)$. Due to the interactions between the manipulator end-effector and the environment, some of these constraint forces/torques are independent of the manipulator end-effector motion, while the others are dependent. In the next subsection, the constraint forces/torques in (9) will be decomposed into two components depending on their charteristics.

2.2. Decomposition of Constraint Forces/Torques

To accomplish specific tasks successfully by a constrained robot system, the constraint forces/torques as well as the motion of the manipulator end-effector must be specified and controlled. The constraint forces/torques resulting from the interaction of the manipulator end-effector and the environment can be decomposed into two components: the *motion-independent* and the *motion-dependent* forces/torques. That is,

$$\mathbf{f}_c(t) = \mathbf{f}_n(t) + \mathbf{f}_m(t) \tag{10}$$

where $\mathbf{f}_c \in R^n$ is the constraint forces/torques, $\mathbf{f}_n \in R^n$ is the motion-independent forces/torques, and $\mathbf{f}_m \in R^n$ is the motion-dependent forces/torques. These two types of constraint forces/torques are characterized by the following:

(i) The motion-independent forces/torques are the forces/torques which are independent of the motion of the manipulator end-effector. The motion-dependent forces/torques are the forces/torques which are equal to $\mathbf{f}_c - \mathbf{f}_n$. In other words, the motion-independent forces/torques do not contribute to the motion of the manipulator end-effector, while the motion-dependent forces/torques contribute to the motion of the manipulator end-effector.

(ii) As a result of (i), the motion-independent forces/torques perform no virtual work, while the motion-dependent forces/torques perform work.

(iii) In the case of a robot with its end-effector contacting a rigid friction surface, the motion-independent forces/torques cause the normal forces on the surface and the motion-dependent forces/torques react to the surface friction force during the motion of the manipulator end-effector on a surface.

(iv) In the case of two robots manipulating a common object, the motion-independent forces/torques play the same role as the internal forces/torques [18] and the motion-dependent forces/torques contribute directly to the motion of the common object.

(v) Using the constraint function in the Cartesian space, the motion-independent forces/torques can be determined by a generalized multiplier vector and the Jacobian matrix of the constraint function. The motion-dependent forces/torques can be expressed by the motion of the manipulator end-effector, the motion-independent forces/torques, and other known quantities. Thus, the control problem of the constrained robot system reduces to controlling only the motion of the manipulator end-effector and the motion-independent forces/torques.

Using the partitioned coordinates $\mathbf{u}$ and $\mathbf{v}$ in (6), the motion-independent and the motion-dependent forces/torques in (10) can be partitioned as

$$\begin{bmatrix} \mathbf{f}_{cu} \\ \mathbf{f}_{cv} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{nu} \\ \mathbf{f}_{nv} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_{mu} \\ \mathbf{f}_{mv} \end{bmatrix} . \tag{11}$$

To express the motion-independent forces/torques in another form, let us use the Lagrange multiplier method [24]. Since the motion-independent forces/torques are independent of the manipulator motion, they perform no work in the virtual displacement $\delta\mathbf{x}$. Hence we have

$$\mathbf{f}_n^T\,\delta\mathbf{x} = 0\,. \tag{12}$$

From the constraint equation in (4), the virtual displacement $\delta\mathbf{x}$ satisfies

$$\boldsymbol{\lambda}^T \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{x}}\right] \delta\mathbf{x} = 0 \quad \text{for } \boldsymbol{\lambda} \in R^m\,. \tag{13}$$

Subtracting (13) from (12) and expressing the resultant equations in terms of virtual displacements of partitioned sets, $\delta\mathbf{u}$ and $\delta\mathbf{v}$, we have

$$[\,\mathbf{f}_{nu}^T - \boldsymbol{\lambda}^T \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]]\,\delta\mathbf{u} + [\,\mathbf{f}_{nv}^T - \boldsymbol{\lambda}^T \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right]]\,\delta\mathbf{v} = 0\,. \tag{14}$$

Since $\mathbf{u}$ is expressed by $\boldsymbol{\Omega}(\mathbf{v})$, we treat $\mathbf{u}$ as dependent variables and $\mathbf{v}$ as independent variables. By appropriately choosing the values of the multiplier $\boldsymbol{\lambda}$, the coefficient of $\delta\mathbf{u}$, $\mathbf{f}_{nu}^T - \boldsymbol{\lambda}^T \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]$, can be set to zero. As a result, the coefficient of $\delta\mathbf{v}$, $\mathbf{f}_{nv}^T - \boldsymbol{\lambda}^T \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right]$, also becomes zero because the components in $\delta\mathbf{v}$ are independent of each other. Thus, the partitioned motion-independent forces/torques can be expressed as follows:

$$\mathbf{f}_{nu} = \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]^T \boldsymbol{\lambda} \;\text{ and }\; \mathbf{f}_{nv} = \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right]^T \boldsymbol{\lambda}\,. \tag{15}$$

From these relations and the fact that the constraint forces/torques consist of the motion-independent and the motion-dependent forces/torques, the two subsystems as in (9) can be rewritten as (i.e., use (11) and (15))

$$(\boldsymbol{\Lambda}_{uu}\mathbf{W} + \boldsymbol{\Lambda}_{uv})\ddot{\mathbf{v}} + \mathbf{c}_u + \boldsymbol{\Lambda}_{uu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_u = \mathbf{F}_{au} + \mathbf{f}_{mu} + \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}}\right]^T \boldsymbol{\lambda} \qquad \text{(subsystem 1)}\,,$$

$$(\boldsymbol{\Lambda}_{vu}\mathbf{W} + \boldsymbol{\Lambda}_{vv})\ddot{\mathbf{v}} + \mathbf{c}_v + \boldsymbol{\Lambda}_{vu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_v = \mathbf{F}_{av} + \mathbf{f}_{mv} + \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}}\right]^T \boldsymbol{\lambda} \qquad \text{(subsystem 2)}\,. \tag{16}$$

The above two subsystems as in (16) are equivalent to the constrained robot system as in (3) and (4). In addition, the constraint forces/torques have been decomposed into the motion-independent and the motion-dependent components. Although these two subsystems are in a redundant representation, this representation leads to the design of a simple nonlinear decoupled controller, which takes the constraints into consideration for controlling the constrained robot system.

2.3. Design of Nonlinear Decoupled Controller

In this section, a nonlinear decoupled controller is proposed for controlling the constrained robot system as in (3) and (4). The desired generalized coordinates, $\mathbf{x}_d$, and the desired constraint forces/torques, $\mathbf{f}_c^d$,† are assumed to be consistent with the constraints of the robot [11]. Then, the control problem can be stated as: Given (i) a constrained robot system as in (3) and (4), and its equivalent subsystems as in (16), (ii) the initial conditions of $\mathbf{x}$ and $\dot{\mathbf{x}}$ satisfying the constraint equation, (iii) the desired generalized coordinates, $\mathbf{x}_d$, and (iv) the desired constraint forces/torques, $\mathbf{f}_c^d$, find a feedback control, $\mathbf{F}_a$, based on $\mathbf{x}_d$, $\dot{\mathbf{x}}_d$, $\ddot{\mathbf{x}}_d$, $\mathbf{f}_c^d$, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\mathbf{f}_c$ such that both the generalized coordinates of the manipulator and the constraint forces/torques approach the desired values asymptotically.

Since the motion-dependent forces/torques can be determined from the motion-independent forces/torques and the motion of the manipulator end-effector, the above control problem can be restated as finding a feedback control, $\mathbf{F}_a$, based on $\mathbf{x}_d$, $\dot{\mathbf{x}}_d$, $\ddot{\mathbf{x}}_d$, $\mathbf{f}_n^d$, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\mathbf{f}_c$ such that both the generalized coordinates of the manipulator and the motion-independent forces/torques approach the desired values asymptotically.

Since the motion-independent and the motion-dependent forces/torques, $\mathbf{f}_n$ and $\mathbf{f}_m$, can be determined from the measured values of $\mathbf{f}_c$, $\mathbf{x}$, and $\dot{\mathbf{x}}$, the partitioned motion-independent and motion-dependent forces/torques, $\mathbf{f}_{nu}$, $\mathbf{f}_{nv}$, $\mathbf{f}_{mu}$, and $\mathbf{f}_{mv}$, are measurable quantities. Thus, $\boldsymbol{\lambda} = \left[\left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}} \right]^T \right]^{-1} \mathbf{f}_{nu}$ is also a measurable quantity. With these measurable quantities, a nonlinear decoupled controller using (16) can be computed as

$$\mathbf{F}_{au}(t) = (\boldsymbol{\Lambda}_{uu}\mathbf{W} + \boldsymbol{\Lambda}_{uv})\,(\,\ddot{\mathbf{v}}_d(t) + \mathbf{K}_v\,\dot{\mathbf{e}}_v(t) + \mathbf{K}_p\,\mathbf{e}_v(t)\,) + \mathbf{c}_u + \boldsymbol{\Lambda}_{uu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_u - \mathbf{f}_{mu}$$

$$- \mathbf{K}_{fu} \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}} \right]^T (\,\boldsymbol{\lambda}_d - \boldsymbol{\lambda}\,) - \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{u}} \right]^T \boldsymbol{\lambda}_d \,,$$

$$\mathbf{F}_{av}(t) = (\boldsymbol{\Lambda}_{vu}\mathbf{W} + \boldsymbol{\Lambda}_{vv})\,(\,\ddot{\mathbf{v}}_d(t) + \mathbf{K}_v\,\dot{\mathbf{e}}_v(t) + \mathbf{K}_p\,\mathbf{e}_v(t)\,) + \mathbf{c}_v + \boldsymbol{\Lambda}_{vu}\dot{\mathbf{W}}\dot{\mathbf{v}} + \mathbf{p}_v - \mathbf{f}_{mv} \,,$$

$$- \mathbf{K}_{fv} \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}} \right]^T (\,\boldsymbol{\lambda}_d - \boldsymbol{\lambda}\,) - \left[\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{v}} \right]^T \boldsymbol{\lambda}_d \qquad (17)$$

where $\mathbf{e}_v = (\mathbf{v}_d - \mathbf{v}) \in R^{n-m}$, $\mathbf{K}_{fu} \in R^{m \times m}$ and $\mathbf{K}_{fv} \in R^{(n-m) \times (n-m)}$ are the force/torque feedback gain matrices, $\mathbf{K}_v, \mathbf{K}_p \in R^{(n-m) \times (n-m)}$ are, respectively, the velocity and position feedback gain matrices.

Substituting the controller from (17) into (16), we obtain the error equations of the subsystems

† Throughout this paper, the notations $(\cdot)^d$ or $(\cdot)_d$ are used to represent the desired value of $(\cdot)$.

$$(\mathbf{\Lambda}_{uu}\mathbf{W}+\mathbf{\Lambda}_{uv})\,(\,\ddot{\mathbf{e}}_v(t)+\mathbf{K}_v\,\dot{\mathbf{e}}_v(t)+\mathbf{K}_p\,\mathbf{e}_v(t)\,)=(\mathbf{K}_{fu}+\mathbf{I}_m)\left[\frac{\partial\mathbf{\Phi}}{\partial\mathbf{u}}\right]^T(\,\boldsymbol{\lambda}_d-\boldsymbol{\lambda}\,) \tag{18}$$

$$(\mathbf{\Lambda}_{vu}\mathbf{W}+\mathbf{\Lambda}_{vv})\,(\,\ddot{\mathbf{e}}_v(t)+\mathbf{K}_v\,\dot{\mathbf{e}}_v(t)+\mathbf{K}_p\,\mathbf{e}_v(t)\,)=(\mathbf{K}_{fv}+\mathbf{I}_{n-m})\left[\frac{\partial\mathbf{\Phi}}{\partial\mathbf{v}}\right]^T(\,\boldsymbol{\lambda}_d-\boldsymbol{\lambda}\,) \tag{19}$$

where $\mathbf{I}_m \in R^{m\times m}$ is the identity matrix. Choosing $\mathbf{K}_{fu}$ and $\mathbf{K}_{fv}$ to be positive semidefinite, then the matrices, $(\mathbf{K}_{fu}+\mathbf{I}_m)$ and $(\mathbf{K}_{fv}+\mathbf{I}_{n-m})$, are positive definite and therefore nonsingular. Premultiplying (18) and (19) with $\mathbf{W}^T(\mathbf{K}_{fu}+\mathbf{I}_m)^{-1}$ and $(\mathbf{K}_{fv}+\mathbf{I}_{n-m})^{-1}$, respectively, and adding the resultant equations together, and using (8), we have

$$\mathbf{A}\,(\,\ddot{\mathbf{e}}_v(t)+\mathbf{K}_v\,\dot{\mathbf{e}}_v(t)+\mathbf{K}_p\,\mathbf{e}_v(t)\,)=\mathbf{0} \tag{20}$$

where the matrix $\mathbf{A} \in R^{(n-m)\times(n-m)}$ is given as

$$\begin{aligned}\mathbf{A} &= \mathbf{W}^T(\mathbf{K}_{fu}+\mathbf{I}_m)^{-1}(\mathbf{\Lambda}_{uu}\mathbf{W}+\mathbf{\Lambda}_{uv})+(\mathbf{K}_{fv}+\mathbf{I}_{n-m})^{-1}(\mathbf{\Lambda}_{vu}\mathbf{W}+\mathbf{\Lambda}_{vv})\\ &=[\,\mathbf{W}^T\ \mathbf{I}_{n-m}\,]\begin{bmatrix}(\,\mathbf{K}_{fu}+\mathbf{I}_m\,)^{-1} & \mathbf{0}\\ \mathbf{0} & (\,\mathbf{K}_{fv}+\mathbf{I}_{n-m}\,)^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}_{uu} & \mathbf{\Lambda}_{uv}\\ \mathbf{\Lambda}_{vu} & \mathbf{\Lambda}_{vv}\end{bmatrix}\begin{bmatrix}\mathbf{W}\\ \mathbf{I}_{n-m}\end{bmatrix}\end{aligned}$$

and the matrix $\mathbf{A}$ can be proved to be nonsingular. If $\mathbf{K}_{fu}$ and $\mathbf{K}_{fv}$ are positive semidefinite, it can be easily shown that the matrix

$$\begin{bmatrix}(\,\mathbf{K}_{fu}+\mathbf{I}_m\,)^{-1} & \mathbf{0}\\ \mathbf{0} & (\,\mathbf{K}_{fv}+\mathbf{I}_{n-m}\,)^{-1}\end{bmatrix}$$

is positive definite and therefore nonsingular. The matrix

$$\begin{bmatrix}\mathbf{\Lambda}_{uu} & \mathbf{\Lambda}_{uv}\\ \mathbf{\Lambda}_{vu} & \mathbf{\Lambda}_{vv}\end{bmatrix}$$

is nonsingular because it is equivalent to the matrix $\mathbf{\Lambda}$ which is positive definite.

Furthermore, Rank $\begin{bmatrix}\mathbf{W}^T & \mathbf{I}_{n-m}\end{bmatrix}$ = Rank $\begin{bmatrix}\mathbf{W}\\ \mathbf{I}_{n-m}\end{bmatrix}$ = $n-m$. Thus, Rank $\mathbf{A}=n-m$, and matrix $\mathbf{A}$ is nonsingular.

Since matrix $\mathbf{A}$ is nonsingular, (20) reduces to

$$\ddot{\mathbf{e}}_v(t)+\mathbf{K}_v\,\dot{\mathbf{e}}_v(t)+\mathbf{K}_p\,\mathbf{e}_v(t)\;=\mathbf{0}\,.$$

Choosing the feedback gain matrices $\mathbf{K}_v$ and $\mathbf{K}_p$ to be positive definite, $\mathbf{e}_v(t)$ will approach zero asymptotically. From the given assumption, the error in the partitioned Cartesian coordinate $\mathbf{u}$ also approaches zero asymptotically. Thus, the generalized coordinates $\mathbf{x}$ of the constrained robot system in the Cartesian space will approach the desired value $\mathbf{x}_d$ asymptotically,

$$\lim_{t\to\infty}\mathbf{x}(t)\to\mathbf{x}_d(t)\,.$$

Since the matrices, $(\mathbf{K}_{fu}+\mathbf{I}_m)$ and $\left[\frac{\partial\mathbf{\Phi}}{\partial\mathbf{u}}\right]$, are nonsingular, from (18) and (19), we know that the multiplier $\boldsymbol{\lambda}$ approaches the desired value asymptotically,

$$\lim_{t \to \infty} \boldsymbol{\lambda} \to \boldsymbol{\lambda}_d \,.$$

Since the desired motion-independent forces/torques also satisfy the constraint equation (4), they can be expressed as

$$\mathbf{f}_{nu}^d = \left[\frac{\partial \Phi}{\partial \mathbf{u}} \right]^T_{\substack{\mathbf{u}=\mathbf{u}_d \\ \mathbf{v}=\mathbf{v}_d}} \boldsymbol{\lambda}_d \quad \text{and} \quad \mathbf{f}_{nv}^d = \left[\frac{\partial \Phi}{\partial \mathbf{v}} \right]^T_{\substack{\mathbf{u}=\mathbf{u}_d \\ \mathbf{v}=\mathbf{v}_d}} \boldsymbol{\lambda}_d \,.$$

Thus, the motion-independent forces/torques approach the desired value asymptotically,

$$\lim_{t \to \infty} \mathbf{f}_n(t) \to \mathbf{f}_n^d(t) \,.$$

In the above discussion, a generalized approach for controlling the constrained robot system as in (3) and (4) has been presented. In the previous work for controlling the constrained robot system [3, 8, 11], their analyses are restricted to the case of the constrained robot system in which the direction of the constraint forces/torques $\mathbf{f}_c$ is orthogonal to the direction of the motion of the manipulator end-effector $\mathbf{x}$. However, our proposed generalized approach is not restricted to this case and can be applied to other cases of the constrained robot system in which the direction of the constraint forces/torques $\mathbf{f}_c$ is not orthogonal to the direction of the motion of the manipulator end-effector $\mathbf{x}$. In other words, the proposed approach can be applied to controlling general cases of the constrained robot system.

Utilizing this approach, typical examples of the constrained robot systems are analyzed and discussed. Section 3 considers the case of one robot with its end-effector contacting a rigid friction surface; section 4 considers the case of two cooperating robots handling a common rigid load; sections 5 and 6 will consider the cases of two cooperating robots handling two rigid bodies connected by a revolute joint and a spherical (ball-and-socket) joint, respectively.

3. One Robot with Its End-Effector Contacting a Rigid Friction Surface

In this section, the proposed generalized approach for controlling the constrained robot system will be applied to controlling a robot with its end-effector contacting a rigid surface. A review of the previous work shows that the friction of the constraint surface was not included in the control of this constrained robot system [8, 11]. And [6, 7] included the frictional effects in the dynamic model of this constraint robot system, while no controller was designed to control this constraint robot system with the frictional effects taken into consideration. However, [6] provides a detailed account of the frictional effects in the dynamic model of this constraint robot system. For ease of discussion, we assume that:

(i) There is no effect of the static friction between the manipulator end-effector and the constraint surface.

(ii) The manipulator end-effector is in a frictional point contact with the surface. This assumption implies that the manipulator end-effector does not exert any torque to the surface, and the torque component in the constraint forces/torques is always zero.

(iii) Initially, the manipulator end-effector is in contact with the constraint surface and its velocity is zero or tangent to the constraint surface.

In this constrained robot system, the constraint force is the contact force between the manipulator end-effector and the constraint surface, the motion-independent force is the normal force on the constraint surface, and the motion-dependent force reacts to the surface sliding friction. Thus, the equations of motion of this constrained robot system in the Cartesian space are described as

$$\Lambda(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{p}(\mathbf{x}) = \mathbf{F}_a(t) + \mathbf{f}_c(t) \tag{21}$$

$$\mathbf{f}_c(t) = \mathbf{f}_m(t) + \mathbf{f}_n(t) \tag{22}$$

$$\mathbf{f}_{mp}(t) = -\mu \,\|\, \mathbf{f}_{np}(t) \,\| \frac{\dot{\mathbf{x}}_p(t)}{\|\, \dot{\mathbf{x}}_p(t) \,\|} \tag{23}$$

where $\mathbf{x} \in R^6$ denotes the vector of generalized coordinates (or Cartesian variables) describing the position/orientation (i.e. pose) of the manipulator end-effector, $\Lambda(\mathbf{x}) \in R^{6\times6}$ is the robot inertia matrix in the Cartesian space, $\mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) \in R^6$ is the vector of Coriolis and centrifugal forces/torques in the Cartesian space, $\mathbf{p}(\mathbf{x}) \in R^6$ is the vector of gravitational forces/torques in the Cartesian space, and $\mathbf{F}_a(t) \in R^6$ is the vector of generalized forces/torques. $\mathbf{f}_{mp}(t)$, $\mathbf{f}_{np}(t) \in R^3$ are the vectors of the force components in $\mathbf{f}_m(t)$, $\mathbf{f}_n(t)$, respectively, $\mathbf{x}_p(t) \in R^3$ is the vector of the Cartesian variables describing the position of the manipulator end-effector, μ is the sliding friction coefficient, and $\|\cdot\|$ denotes the Euclidean norm. The motion constraints of this constrained robot system can be described by the constraint surface equation

$$\Phi(\mathbf{x}) = \mathbf{0} \tag{24}$$

where the constraint surface function $\Phi : R^6 \to R^m$. If $m = 1$, the motion of the manipulator end-effector is constrained on a surface. If $m = 2$, it is constrained on a curve. Let us assume that the constraint surface function $\Phi(\mathbf{x})$ and a function $\Omega(\mathbf{v}) : R^{6-m} \to R^m$ satisfy the following condition,

$$\Phi(\,\Omega(\mathbf{v}), \mathbf{v}\,) = \mathbf{0} \quad \text{for all } \mathbf{v} \in R^{6-m} .$$

Then the proposed generalized approach can be applied to control this constrained robot system. Using the constraint function in (24) and following the similar procedures as discussed and outlined in sections 2.1 and 2.2, this constrained robot system can be partitioned into two subsystems whose equations have the similar form as in (16).

The control problem of this constrained robot system can be stated as: Given (i) the constrained robot system as in (21)-(24), (ii) the initial conditions that the manipulator end-effector is in contact with the constraint surface and its end-effector velocity is zero or tangent to the constraint surface, (iii) the desired position/orientation of the manipulator end-effector, $\mathbf{x}_d$, and the desired contact force/torque on the constraint surface, $\mathbf{f}_c^d$, which are consistent with the constraints, find a feedback control, $\mathbf{F}_a$, based on $\mathbf{x}_d$, $\dot{\mathbf{x}}_d$, $\ddot{\mathbf{x}}_d$, $\mathbf{f}_c^d$, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\mathbf{f}_c$ such that both the position/orientation of the manipulator end-effector and the contact forces/torques approach the desired values asymptotically.

Since the motion-dependent forces/torques can be determined by the motion-independent forces/torques and the manipulator motion as in (23), the above problem can be restated as finding a feedback control, $\mathbf{F}_a$, based on $\mathbf{x}_d$, $\dot{\mathbf{x}}_d$, $\ddot{\mathbf{x}}_d$, $\mathbf{f}_n^d$, $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\mathbf{f}_c$ such that both the position/orientation of the manipulator end-effector and the normal force/torque on the constraint surface force/torque approach the desired values asymptotically.

Taking the similar steps as in section 2.3, the nonlinear decoupled controller which has a similar structure as in (17) can be derived. Applying this nonlinear decoupled controller, the motion of the manipulator end-effector in the Cartesian space approaches the desired motion asymptotically,

$$\lim_{t \to \infty} \mathbf{x}(t) \to \mathbf{x}_d(t)$$

and the motion-independent force/torque approaches the desired value asymptotically,

$$\lim_{t \to \infty} \mathbf{f}_n(t) \to \mathbf{f}_n^d(t) \,.$$

4. Two Cooperating Robots Handling a Common Rigid Load

The constrained robot system also arises when two robots are cooperatively manipulating a load. In this section, the proposed generalized approach for controlling the constrained robot system will be applied to controlling two cooperating robots handling a rigid load. For this constrained robot system (see Figure 1), we assume that:

(i) The gripping pose of each manipulator end-effector with respect to a rigid load is fixed. This assumption is valid for a broad range of tasks.

(ii) No relative motion between the load and each manipulator end-effector exists. We do not consider the slipping effect between the load and the manipulator end-effector. Thus, if the mass and the inertia of the load are known, the motion of the load can be determined from the motion of each manipulator end-effector.

(iii) Initially, the common rigid load is firmly grasped by both manipulator end-effectors.

In this constrained robot system, the motion-independent forces/torques play the same role as the internal forces/torques [18] and the motion-dependent forces/torques contribute directly to the motion of the common load. To form the equations of motion for this constrained robot system, we select a set of generalized coordinates by augmenting the vectors of generalized coordinates (or Cartesian variable) of both manipulators, $\mathbf{x}_1 \in R^6$ and $\mathbf{x}_2 \in R^6$, into an $\mathbf{x}$ vector as

$$\mathbf{x}(t) \triangleq \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} \quad , \quad \mathbf{x} \in R^{12} \,. \tag{25}$$

Then, from the equations of motion of each robot manipulator as in (21), the equations of motion of this constrained robot system in the Cartesian space can be augmented and written as

$$\Lambda(\mathbf{x})\,\ddot{\mathbf{x}} + \mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{p}(\mathbf{x}) = \mathbf{F}_a + \mathbf{f}_c \tag{26}$$

where $\Lambda(\mathbf{x}) = \begin{bmatrix} \Lambda_1(\mathbf{x}_1) & \mathbf{0} \\ \mathbf{0} & \Lambda_2(\mathbf{x}_2) \end{bmatrix}$, $\mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) = \begin{bmatrix} \mathbf{c}_1(\mathbf{x}_1,\dot{\mathbf{x}}_1) \\ \mathbf{c}_2(\mathbf{x}_2,\dot{\mathbf{x}}_2) \end{bmatrix}$,

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} \mathbf{p}_1(\mathbf{x}_1) \\ \mathbf{p}_2(\mathbf{x}_2) \end{bmatrix}, \quad \mathbf{F}_a = \begin{bmatrix} \mathbf{F}_{1a} \\ \mathbf{F}_{2a} \end{bmatrix}, \quad \mathbf{f}_c = \begin{bmatrix} \mathbf{f}_{1c} \\ \mathbf{f}_{2c} \end{bmatrix},$$

$\Lambda_1(\mathbf{x}_1)$, $\mathbf{c}_1(\mathbf{x}_1, \dot{\mathbf{x}}_1)$, $\mathbf{p}_1(\mathbf{x}_1)$, $\mathbf{F}_{1a}$, and $\mathbf{f}_{1c}$ have corresponding meanings as in (21) for the first robot's end-effector, $\Lambda_2(\mathbf{x}_2)$, $\mathbf{c}_2(\mathbf{x}_2, \dot{\mathbf{x}}_2)$, $\mathbf{p}_2(\mathbf{x}_2)$, $\mathbf{F}_{2a}$, and $\mathbf{f}_{2c}$ have corresponding meanings as in (21) for the second robot's end-effector, $\mathbf{x}_1(t) \in R^6$ and $\mathbf{x}_2(t) \in R^6$ are the respective vectors of generalized coordinates in the Cartesian space of each robot manipulator, and these two vectors describe the pose of both manipulator end-effectors.

To partition this constrained robot system into two subsystems, let us discuss the constraints of this constrained robot system. The constraint equation for this robot system is described as

$$\Phi(\mathbf{x}) \equiv \Phi\left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}\right) \equiv \begin{bmatrix} \mathbf{x}_{1p} - \mathbf{x}_{2p} - \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{x}_{1r} - \mathbf{x}_{2r} - \phi \end{bmatrix} = \mathbf{0} \tag{27}$$

where the constraint function $\Phi : R^{12} \to R^6$, $\mathbf{x}_{1p}, \mathbf{x}_{2p} \in R^3$ are the respective vectors of the Cartesian variables describing the position of each manipulator end-effector, $\mathbf{x}_{1r}, \mathbf{x}_{2r} \in R^3$ are the respective vectors of the Cartesian variables describing the rotation of each manipulator end-effector, $\mathbf{r}_1, \mathbf{r}_2 \in R^3$ are the respective Cartesian position vectors from the center of mass of the load to the gripping position of each manipulator end-effector (see Figure 1), and $\phi \in R^3$ is fixed from the initial values of $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$ because the load is rigid and no relative motion exists between the load and each manipulator end-effector. Similarly, $\mathbf{r}_1$ and $\mathbf{r}_2$ can be determined from $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$, respectively, and they can be written as

$$\mathbf{r}_1 = \mathbf{r}_1(\mathbf{x}_{1r}) \quad \text{and} \quad \mathbf{r}_2 = \mathbf{r}_2(\mathbf{x}_{2r}).$$

Thus, for a function $\Omega : R^6 \to R^6$ such that

$$\Omega(\mathbf{x}_2) = \Omega\left(\begin{bmatrix} \mathbf{x}_{2p} \\ \mathbf{x}_{2r} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{x}_{2p} + \mathbf{r}_1(\mathbf{x}_{2r} + \phi) - \mathbf{r}_2(\mathbf{x}_{2r}) \\ \mathbf{x}_{2r} + \phi \end{bmatrix},$$

we have

$$\Phi(\Omega(\mathbf{x}_2), \mathbf{x}_2) = \mathbf{0} \qquad \text{for all } \mathbf{x}_2 \in R^6.$$

Therefore, the pose vectors of both manipulator end-effectors, $\mathbf{x}_1$ and $\mathbf{x}_2$, partition the set of generalized coordinates of this constrained robot system, $\mathbf{x}$, into the dependent set and the independent set. As in section 2.1, letting $\mathbf{u}$ be the dependent set and $\mathbf{v}$ be the independent set, we have

$$\mathbf{u} = \mathbf{x}_1 \quad \text{and} \quad \mathbf{v} = \mathbf{x}_2. \tag{28}$$

Taking the similar steps as in sections 2.1 and 2.2, the constrained robot system is partitioned into two subsystems.

Let us discuss the effects of the motion-independent and the motion-dependent forces/torques on this constrained robot system. Taking the similar steps as in section 2.2 and using (28), we obtain

$$\mathbf{f}_n = \left[\frac{\partial \Phi}{\partial \mathbf{x}}\right]^T \boldsymbol{\lambda} = \begin{bmatrix} \left[\frac{\partial \Phi}{\partial \mathbf{u}}\right]^T \boldsymbol{\lambda} \\ \left[\frac{\partial \Phi}{\partial \mathbf{v}}\right]^T \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial \Phi}{\partial \mathbf{x}_1}\right]^T \boldsymbol{\lambda} \\ \left[\frac{\partial \Phi}{\partial \mathbf{x}_2}\right]^T \boldsymbol{\lambda} \end{bmatrix} \quad \text{for } \boldsymbol{\lambda} \in R^6 . \tag{29}$$

The above motion-independent forces/torques $\mathbf{f}_n$ are contributed by each manipulator end-effector as

$$\mathbf{f}_n = \begin{bmatrix} \mathbf{f}_{1n} \\ \mathbf{f}_{2n} \end{bmatrix} \tag{30}$$

where $\mathbf{f}_{1n}, \mathbf{f}_{2n} \in R^6$ are the respective motion-independent forces/torques of each manipulator end-effector. Then using (27), (29), and (30), $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$ can be expressed as

$$\mathbf{f}_{1n} = \left[\frac{\partial \Phi}{\partial \mathbf{x}_1}\right]^T \boldsymbol{\lambda} = \begin{bmatrix} \mathbf{I}_3 & -\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix}^T \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ -(\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}})^T \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \end{bmatrix},$$

$$\mathbf{f}_{2n} = \left[\frac{\partial \Phi}{\partial \mathbf{x}_2}\right]^T \boldsymbol{\lambda} = \begin{bmatrix} -\mathbf{I}_3 & \frac{\partial \mathbf{r}_2}{\partial \mathbf{x}_{2r}} \\ \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}^T \boldsymbol{\lambda} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ (\frac{\partial \mathbf{r}_2}{\partial \mathbf{x}_{2r}})^T \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \end{bmatrix}, \tag{31}$$

where $\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix}, \quad \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in R^3 .$

Since the following relations hold (see Appendix A),

$$\left[\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}}\right]^T \boldsymbol{\lambda}_1 = \mathbf{r}_1 \times \boldsymbol{\lambda}_1 , \quad \left[\frac{\partial \mathbf{r}_2}{\partial \mathbf{x}_{2r}}\right]^T \boldsymbol{\lambda}_1 = \mathbf{r}_2 \times \boldsymbol{\lambda}_1 , \tag{32}$$

equation (31) can be expressed, respectively, as

$$\mathbf{f}_{1n} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ -\mathbf{r}_1 \times \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2n} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ \mathbf{r}_2 \times \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \end{bmatrix}.$$

Let $\mathbf{f}_T^{1n}$ be the force/torque, generated by $\mathbf{f}_{1n}$, at the center of mass of the load. Then $\mathbf{f}_T^{1n}$ is found to be

$$\mathbf{f}_T^{1n} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \mathbf{r}_1 \times \boldsymbol{\lambda}_1 + (-\mathbf{r}_1 \times \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix}.$$

Similarly, the force/torque $\mathbf{f}_T^{2n}$, generated by $\mathbf{f}_{2n}$, at the center of mass of the load can be expressed as

$$\mathbf{f}_T^{2n} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ \mathbf{r}_2 \times (-\boldsymbol{\lambda}_1) + (\mathbf{r}_2 \times \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ -\boldsymbol{\lambda}_2 \end{bmatrix} .$$

Thus, the summation of the forces/torques at the center of mass of the load, which are generated by $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$, is zero,

$$\mathbf{f}_T^{1n} + \mathbf{f}_T^{2n} = \mathbf{0} . \tag{33}$$

This result is expected because the net effect of the forces/torques, which are not contributing to the motion of the load, is null at the center of mass of the load [14, 18, 19]. Also, (33) shows that the motion-independent forces/torques play the same role as the internal forces/torques [18].

For the constraint forces/torques, $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$, let us express them as

$$\mathbf{f}_{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{f}_{1cr} \end{bmatrix} , \quad \mathbf{f}_{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{f}_{2cr} \end{bmatrix} \tag{34}$$

where $\mathbf{f}_{1cp} \in R^3$ and $\mathbf{f}_{1cr} \in R^3$ are, respectively, the force and torque vector components of the constraint forces/torques of the first robot, and $\mathbf{f}_{2cp} \in R^3$ and $\mathbf{f}_{2cr} \in R^3$ are, respectively, the force and torque vector components of the constraint forces/torques of the second robot. Then the forces/torques, $\mathbf{f}_T^{1c} \in R^6$ and $\mathbf{f}_T^{2c} \in R^6$, at the center of mass of the load, which are generated by these constraint forces/torques, can be written as

$$\mathbf{f}_T^{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{r}_1 \times \mathbf{f}_{1cp} + \mathbf{f}_{1cr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_T^{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{r}_2 \times \mathbf{f}_{2cp} + \mathbf{f}_{2cr} \end{bmatrix} .$$

Thus, the equations of motion of the load can be written as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{z}}(t) \\ \dot{\boldsymbol{\omega}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega} \times \mathbf{L} \boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} m\mathbf{g} \\ \mathbf{0} \end{bmatrix} = -\mathbf{f}_T^{1c} - \mathbf{f}_T^{2c} \tag{35}$$

where $\mathbf{M} \in R^{3\times3}$ is the diagonal matrix whose non-zero elements denote the mass of the load, $m \in R^1$ is the mass of the load, $\mathbf{L} \in R^{3\times3}$ is the diagonal matrix whose non-zero elements denote the principal moments of inertia of the load about its center of mass, $\mathbf{g} \in R^3$ is the gravity vector, $\mathbf{z}(t) \in R^3$ and $\boldsymbol{\omega}(t) \in R^3$ are the vectors of the Cartesian coordinates describing the position of the center of mass and the angular velocity of the load, respectively. For ease of notation, let us denote the left hand side of equation (35) as $\mathbf{F}_{load}$, then (35) becomes

$$\mathbf{F}_{load} = -\mathbf{f}_T^{1c} - \mathbf{f}_T^{2c} .$$

To discuss the effect of the motion-dependent forces/torques on the motion of the load, let us express the motion-dependent forces/torques of each respective manipulator end-effector as $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$, respectively. Since the constraint forces/torques are decomposed into the motion-dependent and the motion-independent forces/torques, $\mathbf{f}_T^{1c}$ and $\mathbf{f}_T^{2c}$ can also be decomposed into two components which are contributed by the motion-dependent forces/torques and the motion-independent forces/torques,

$$\mathbf{f}_T^{1c} = \mathbf{f}_T^{1m} + \mathbf{f}_T^{1n} \quad \text{and} \quad \mathbf{f}_T^{2c} = \mathbf{f}_T^{2m} + \mathbf{f}_T^{2n} \tag{36}$$

where $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ are, respectively, the forces/torques at the center of mass of the load, which are generated by $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$, and $\mathbf{f}_T^{1n}$ and $\mathbf{f}_T^{2n}$ are the forces/torques at the center of mass of the load, which are generated by $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$, respectively. Since the motion-independent forces/torques do not have any effect on the motion of the load as in (33), the equations of motion of the load can be rewritten as

$$\mathbf{F}_{load} = -\mathbf{f}_T^{1m} - \mathbf{f}_T^{2m} .$$

To distribute the forces/torques, the minimum exerted force/torque criterion [13], which makes $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ pointing to the same direction as $\mathbf{F}_{load}$, is used. Then, the $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ can be computed, respectively, as

$$\mathbf{f}_T^{1m} = -\eta\, \mathbf{F}_{load} \quad \text{and} \quad \mathbf{f}_T^{2m} = -(\,1-\eta\,)\, \mathbf{F}_{load}$$

for some $0 \le \eta \le 1$. The optimal η can be determined by minimizing the energy consumption.

Instead of using the minimum exerted force/torque criterion, if another optimization criterion such as the minimum energy consumption [12] is used, then $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ may point to a different direction from $\mathbf{F}_{load}$, and thus causing stress in the load as $\mathbf{f}_T^{1n}$ and $\mathbf{f}_T^{2n}$ do. Therefore, if other than the minimum exerted force/torque criterion is used, we have to consider the effect of $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ on the stress of the load before the motion-independent forces/torques are determined. This is necessary to achieve the required stress in the load. Thus, if the effect of the motion-dependent forces/torques on the stress in the load is considered before the determination of the motion-independent forces/torques, it does not make much difference which criterion is used because the motion-dependent forces/torques will be compensated by a feedforward component in the proposed nonlinear decoupled controller.

With the constrained robot system given as in (26) and (27), the control problem can be stated as: Given (i) the initial condition that the common rigid load is firmly grasped by two manipulator end-effector, (ii) the desired pose of both manipulator end-effectors ($\mathbf{x}_1^d$ and $\mathbf{x}_2^d$), and the desired motion-independent forces/torques, $\mathbf{f}_n^d = (\mathbf{f}_{1n}^d\,,\ \mathbf{f}_{2n}^d)^T$, which are consistent with the constraints, find a feedback control, $\mathbf{F}_a = \begin{bmatrix} \mathbf{F}_{1a} \\ \mathbf{F}_{2a} \end{bmatrix}$, based on $\mathbf{x}_1^d$, $\dot{\mathbf{x}}_1^d$, $\ddot{\mathbf{x}}_1^d$, $\mathbf{x}_2^d$, $\dot{\mathbf{x}}_2^d$, $\ddot{\mathbf{x}}_2^d$, $\mathbf{f}_{1n}^d$, $\mathbf{f}_{2n}^d$, $\mathbf{x}_1$, $\dot{\mathbf{x}}_1$, $\mathbf{x}_2$, $\dot{\mathbf{x}}_2$, $\mathbf{f}_{1c}$, and $\mathbf{f}_{2c}$ such that the poses of both the manipulator end-effectors and the motion independent forces/torques approach the desired values asymptotically. Since $\mathbf{f}_n$ and $\mathbf{f}_m$ can be computed from the measured values of $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$ (see Appendix B), $\mathbf{f}_{1n}$, $\mathbf{f}_{2n}$, $\mathbf{f}_{1m}$, and $\mathbf{f}_{2m}$ are measurable quantities. Thus, $\boldsymbol{\lambda} = \left[\left(\dfrac{\partial \Phi}{\partial \mathbf{x}_1} \right)^T \right]^{-1} \mathbf{f}_{1n}$ is also a measurable quantity. Again, taking the similar steps as in section 2.3, this constrained robot system can be controlled by the nonlinear decoupled controller which has a similar structure as in (17). Applying this nonlinear decoupled controller to this constrained robot system, the motion of each manipulator end-effector in the Cartesian space approaches the desired motion asymptotically,

$$\lim_{t \to \infty} \mathbf{x}_1(t) \to \mathbf{x}_1^d(t) \text{ and } \lim_{t \to \infty} \mathbf{x}_2(t) \to \mathbf{x}_2^d(t)$$

and the motion-independent forces/torques also approach the desired value asymptotically,

$$\lim_{t \to \infty} \mathbf{f}_n(t) \to \mathbf{f}_n^d(t) .$$

5. Two Cooperating Robots Handling Two Rigid Bodies Connected by a Revolute Joint

We shall extend the concept and results that we obtained in the last section to the case of controlling two cooperating robots handling two rigid bodies connected by a revolute joint. For this constrained robot system (see Figure 2), in addition to the three assumptions that we made in section 4, we need to add one more assumption (the fourth assumption):

(iv) Except the rotational motion about the axis of motion of the revolute joint of the load, no rotational motion of these two bodies exists. The axis of motion of the revolute joint of the load is parallel to one of the coordinate axes of the global reference coordinate frame.

To partition this constrained robot system into two subsystems, we use the same notations and equations as in (25) and (26). Let us express $\mathbf{x}_1$ and $\mathbf{x}_2$ as

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_{1p} \\ \mathbf{x}_{1r} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{1p} \\ x_{1\alpha} \\ x_{1\beta} \\ x_{1\gamma} \end{bmatrix} , \quad \mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_{2p} \\ \mathbf{x}_{2r} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{2p} \\ x_{2\alpha} \\ x_{2\beta} \\ x_{2\gamma} \end{bmatrix}$$

where $\mathbf{x}_{1p}, \mathbf{x}_{2p} \in R^3$ are the respective Cartesian variable vectors describing the position of each manipulator end-effector, $\mathbf{x}_{1r}, \mathbf{x}_{2r} \in R^3$ are the respective Cartesian variable vectors describing the orientation of each manipulator end-effector, and $x_{1\alpha}, x_{1\beta}, x_{1\gamma}, x_{2\alpha}, x_{2\beta}, x_{2\gamma} \in R^1$ are the Cartesian variables describing the orientation of both manipulator end-effectors,

$$\mathbf{x}_{1r} = \begin{bmatrix} x_{1\alpha} \\ x_{1\beta} \\ x_{1\gamma} \end{bmatrix} , \quad \mathbf{x}_{2r} = \begin{bmatrix} x_{2\alpha} \\ x_{2\beta} \\ x_{2\gamma} \end{bmatrix} .$$

The constraint equation for this constrained robot system is described as

$$\mathbf{\Phi}(\mathbf{x}) \equiv \mathbf{\Phi}\left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \right) \equiv \begin{bmatrix} \mathbf{x}_{1p} - \mathbf{x}_{2p} - \mathbf{r}_1 + \mathbf{r}_2 \\ x_{1\alpha} - x_{2\alpha} - \phi_1 \\ x_{1\beta} - x_{2\beta} - \phi_2 \end{bmatrix} = \mathbf{0} \tag{37}$$

where $\mathbf{\Phi} : R^{12} \to R^5$ is the constraint function, $\mathbf{r}_1, \mathbf{r}_2 \in R^3$ are the respective Cartesian position vectors from the axis of motion of the revolute joint of the load to the gripping position of each manipulator end-effector (see Figure 2), and $\phi_1, \phi_2 \in R^1$ are, respectively, fixed by the initial values of $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$ because each body is rigid and no relative motion exists between each manipulator end-effector and its gripped body. Similarly, $\mathbf{r}_1$ and $\mathbf{r}_2$ can be determined from $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$, respectively, and they can be written as

$$\mathbf{r}_1 = \mathbf{r}_1(\mathbf{x}_{1r}) \quad \text{and} \quad \mathbf{r}_2 = \mathbf{r}_2(\mathbf{x}_{2r}) .$$

Thus, for a function $\mathbf{\Omega} : R^7 \to R^5$ such that

$$\Omega(\begin{bmatrix} x_{1\gamma} \\ \mathbf{x}_2 \end{bmatrix}) = \begin{bmatrix} \mathbf{x}_{2p} + \mathbf{r}_1(\begin{bmatrix} x_{2\alpha}+\phi_1 \\ x_{2\beta}+\phi_2 \\ x_{1\gamma} \end{bmatrix}) - \mathbf{r}_2(\begin{bmatrix} x_{2\alpha} \\ x_{2\beta} \\ x_{2\gamma} \end{bmatrix}) \\ x_{2\alpha}+\phi_1 \\ x_{2\beta}+\phi_2 \end{bmatrix} ,$$

we have

$$\mathbf{\Phi}\,(\Omega(x_{1\gamma},\mathbf{x}_2), x_{1\gamma},\mathbf{x}_2) = \mathbf{0} \qquad \text{for all } x_{1\gamma} \in R^1 ,\ \mathbf{x}_2 \in R^6 .$$

Hence, the sets,

$$\mathbf{u} \triangleq \begin{bmatrix} \mathbf{x}_{1p} \\ x_{1\alpha} \\ x_{1\beta} \end{bmatrix} \in R^5 \ \text{and} \ \mathbf{v} \triangleq \begin{bmatrix} x_{1\gamma} \\ \mathbf{x}_2 \end{bmatrix} \in R^7 ,$$

partition the set of generalized coordinates of this constraint robot system, $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, into the dependent and the independent sets. Let us express

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} , \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} \tag{38}$$

where $\mathbf{u}_1 = \mathbf{x}_{1p} \in R^3$, $\mathbf{u}_2 = \begin{bmatrix} x_{1\alpha} \\ x_{1\beta} \end{bmatrix} \in R^2$,

$$v_1 = x_{1\gamma} \in R^1 , \quad \mathbf{v}_2 = \mathbf{x}_{2p} \in R^3 , \quad \text{and} \ \mathbf{v}_3 = \mathbf{x}_{2r} \in R^3 . \tag{39}$$

Using the partitioned coordinates, $\mathbf{u}$ and $\mathbf{v}$, the equations of motion (26) can be partitioned into

$$\begin{bmatrix} \mathbf{\Lambda}_{uu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{uv}(\mathbf{u},\mathbf{v}) \\ \mathbf{\Lambda}_{vu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{vv}(\mathbf{u},\mathbf{v}) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{c}_u(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \\ \mathbf{c}_v(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \end{bmatrix} + \begin{bmatrix} \mathbf{p}_u(\mathbf{u},\mathbf{v}) \\ \mathbf{p}_v(\mathbf{u},\mathbf{v}) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{au} \\ \mathbf{F}_{av} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_{cu} \\ \mathbf{f}_{cv} \end{bmatrix}$$

where $\mathbf{F}_{au} \in R^5$ and $\mathbf{F}_{av} \in R^7$. Proceeding with the similar steps as in sections 2.1 and 2.2, the two subsystems for this constrained robot system can be obtained as in (16).

As in section 2.2, the motion-independent forces/torques $\mathbf{f}_n \in R^{12}$ can be partitioned and expressed as

$$\mathbf{f}_n = \begin{bmatrix} \mathbf{f}_{nu} \\ \mathbf{f}_{nv} \end{bmatrix} = \left(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} \right)^T \boldsymbol{\lambda} = \begin{bmatrix} \left(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{u}} \right)^T \boldsymbol{\lambda} \\ \left(\frac{\partial \mathbf{\Phi}}{\partial \mathbf{v}} \right)^T \boldsymbol{\lambda} \end{bmatrix} ,$$

where $\mathbf{f}_{nu} \in R^5$ and $\mathbf{f}_{nv} \in R^7$ have similar meaning as in (11), and $\boldsymbol{\lambda} \in R^5$. Using the constraint equation of this constraint robot system as in (37), $\mathbf{f}_{nu}$ and $\mathbf{f}_{nv}$ can be expressed as

$$\mathbf{f}_{nu} = \left\{ \frac{\partial \Phi}{\partial \mathbf{u}} \right\}^T \boldsymbol{\lambda} = \begin{bmatrix} \mathbf{I}_3 & -\frac{\partial \mathbf{r}_1}{\partial \mathbf{u}_2} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}^T \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ -(\frac{\partial \mathbf{r}_1}{\partial \mathbf{u}_2})^T \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \end{bmatrix},$$

$$\mathbf{f}_{nv} = \left\{ \frac{\partial \Phi}{\partial \mathbf{v}} \right\}^T \boldsymbol{\lambda} = \begin{bmatrix} -\frac{\partial \mathbf{r}_1}{\partial v_1} & -\mathbf{I}_3 & \frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I}_2\ \mathbf{0} \end{bmatrix}^T \boldsymbol{\lambda} = \begin{bmatrix} -(\frac{\partial \mathbf{r}_1}{\partial v_1})^T \boldsymbol{\lambda}_1 \\ -\boldsymbol{\lambda}_1 \\ (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda}_1 - \begin{bmatrix} \boldsymbol{\lambda}_2 \\ 0 \end{bmatrix} \end{bmatrix},$$

where $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_1 , \boldsymbol{\lambda}_2]^T$, $\boldsymbol{\lambda}_1 \in R^3$, and $\boldsymbol{\lambda}_2 \in R^2$. As in (30), let us express the respective motion-independent forces/torques of each manipulator end-effector as $\mathbf{f}_{1n}$, $\mathbf{f}_{2n} \in R^6$,

$$\mathbf{f}_n = \begin{bmatrix} \mathbf{f}_{1n} \\ \mathbf{f}_{2n} \end{bmatrix}.$$

Then, using the relation of the partitioned sets and the Cartesian variables of each manipulator end-effector as in (38) and (39), the respective motion-independent forces/torques of each manipulator end-effector can be expressed by the partitioned motion-independent forces/torques $\mathbf{f}_{nu}$ and $\mathbf{f}_{nv}$ as

$$\mathbf{f}_{1n} = \begin{bmatrix} \mathbf{f}_{nu} \\ [1\,0\,0\,0\,0\,0\,0]\, \mathbf{f}_{nv} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2n} = \begin{bmatrix} 0 & \\ 0 & \\ 0 & \\ 0 & \mathbf{I}_6 \\ 0 & \\ 0 & \end{bmatrix} \mathbf{f}_{nv} .$$

Thus, we have

$$\mathbf{f}_{1n} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ -(\frac{\partial \mathbf{r}_1}{\partial \mathbf{u}_2})^T \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 \\ -(\frac{\partial \mathbf{r}_1}{\partial v_1})^T \boldsymbol{\lambda}_1 \end{bmatrix}, \quad \mathbf{f}_{2n} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda}_1 - \begin{bmatrix} \boldsymbol{\lambda}_2 \\ 0 \end{bmatrix} \end{bmatrix}.$$

Again using (38) and (39), we have

$$\begin{bmatrix} (\frac{\partial \mathbf{r}_1}{\partial \mathbf{u}_2})^T \boldsymbol{\lambda}_1 \\ (\frac{\partial \mathbf{r}_1}{\partial v_1})^T \boldsymbol{\lambda}_1 \end{bmatrix} = (\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}})^T \boldsymbol{\lambda}_1 \quad \text{and} \quad (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda}_1 = (\frac{\partial \mathbf{r}_2}{\partial \mathbf{x}_{2r}})^T \boldsymbol{\lambda}_1 .$$

Using the identity in Appendix A, we have

$$\begin{bmatrix} (\frac{\partial \mathbf{r}_1}{\partial \mathbf{u}_2})^T \boldsymbol{\lambda}_1 \\ (\frac{\partial \mathbf{r}_1}{\partial v_1})^T \boldsymbol{\lambda}_1 \end{bmatrix} = \mathbf{r}_1 \times \boldsymbol{\lambda}_1 \quad \text{and} \quad (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda}_1 = \mathbf{r}_2 \times \boldsymbol{\lambda}_1 \ .$$

Thus, $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$ are expressed, respectively, as

$$\mathbf{f}_{1n} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ -\mathbf{r}_1 \times \boldsymbol{\lambda}_1 + \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2n} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ \mathbf{r}_2 \times \boldsymbol{\lambda}_1 - \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix} \end{bmatrix} .$$

Let $\mathbf{f}_T^{1n}$ be the force/torque generated by $\mathbf{f}_{1n}$ at the axis of motion of the load. Then $\mathbf{f}_T^{1n}$ can be expressed as

$$\mathbf{f}_T^{1n} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \mathbf{r}_1 \times \boldsymbol{\lambda}_1 + (-\mathbf{r}_1 \times \boldsymbol{\lambda}_1 + \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \lambda_2 \\ 0 \end{bmatrix} .$$

Similarly, the force/torque $\mathbf{f}_T^{2n}$ generated by force/torque $\mathbf{f}_{2n}$ at the axis of motion of the load can be expressed as

$$\mathbf{f}_T^{2n} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ \mathbf{r}_2 \times (-\boldsymbol{\lambda}_1) + (\mathbf{r}_2 \times \boldsymbol{\lambda}_1 - \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix}) \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\lambda}_1 \\ -\lambda_2 \\ 0 \end{bmatrix} .$$

Thus, we obtain the same result as in (33),

$$\mathbf{f}_T^{1n} + \mathbf{f}_T^{2n} = \mathbf{0} ,$$

which indicates that the net effect of the motion-independent forces/torques on the motion of the load is zero.

For the constraint forces/torques $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$, they can be expressed as

$$\mathbf{f}_{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{f}_{1cr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{f}_{2cr} \end{bmatrix}$$

where $\mathbf{f}_{1cp}, \mathbf{f}_{1cr}, \mathbf{f}_{2cp}, \mathbf{f}_{2cr} \in R^3$ have the same meaning as in (34). The forces/torques at the axis of motion of the load, $\mathbf{f}_T^{1c} \in R^6$ and $\mathbf{f}_T^{2c} \in R^6$, which are generated by these constraint forces/torques, can be written respectively as

$$\mathbf{f}_T^{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{r}_1 \times \mathbf{f}_{1cp} + \mathbf{f}_{1cr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_T^{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{r}_2 \times \mathbf{f}_{2cp} + \mathbf{f}_{2cr} \end{bmatrix} .$$

Thus, the equations of motion of the load can be written as

$$\mathbf{M}\,\ddot{\mathbf{z}}(t) + (m_1 + m_2)\mathbf{g} = -[\,\mathbf{I}_3\ \mathbf{0}\,]\,(\,\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c}\,) \ ,$$

$$[100](\mathbf{r}_1^{cm}\times m_1\mathbf{g})+[100](\mathbf{r}_2^{cm}\times m_2\mathbf{g})=-[000100]\mathbf{f}_T^{1c}-[000100]\mathbf{f}_T^{2c}\ ,$$
$$[010](\mathbf{r}_1^{cm}\times m_1\mathbf{g})+[010](\mathbf{r}_2^{cm}\times m_2\mathbf{g})=-[000010]\mathbf{f}_T^{1c}-[000010]\mathbf{f}_T^{2c}\ ,$$
$$L_1\dot{\omega}_{1\gamma}(t)+[001](\mathbf{r}_1^{cm}\times m_1\mathbf{g})=-[000001]\mathbf{f}_T^{1c}\ ,$$
$$L_2\dot{\omega}_{2\gamma}(t)+[001](\mathbf{r}_2^{cm}\times m_2\mathbf{g})=-[000001]\mathbf{f}_T^{2c}\ , \tag{40}$$

where $\mathbf{M}\in R^{3\times3}$ is the diagonal matrix whose non-zero elements denote the mass of the load, m_1 , $m_2\in R^1$ are the respective masses of the rigid bodies, L_1 , $L_2\in R^1$ are the respective moments of inertia of each body about the axis of motion of the load, $\mathbf{r}_1^{cm}$, $\mathbf{r}_2^{cm}\in R^3$ are the respective Cartesian position vectors from the axis of motion of the load to the center of mass of each body, $\mathbf{g}\in R^3$ is the gravity vector, $\mathbf{z}(t)\in R^3$ is the Cartesian vector describing the position of the axis of motion of the load, $\omega_{1\gamma}(t)$, $\omega_{2\gamma}(t)\in R^1$ are the respective angular velocities of each body about the axis of motion of the load.

To discuss the effect of the motion-dependent forces/torques on the motion of the load, let us express the motion-dependent forces/torques of each manipulator end-effector as $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$. Using the relation of the partitioned sets and the Cartesian variables of each manipulator end-effector as in (38) and (39), the respective motion-dependent forces/torques of each manipulator end-effector can be expressed by the partitioned motion-dependent forces/torques $\mathbf{f}_{mu}$ and $\mathbf{f}_{mv}$ as

$$\mathbf{f}_{1m}=\begin{bmatrix}\mathbf{f}_{mu}\\ [1000000]\,\mathbf{f}_{mv}\end{bmatrix}\quad\text{and}\quad \mathbf{f}_{2m}=\begin{bmatrix}0 & \\ 0 & \\ 0 & \\ 0 & \mathbf{I}_6\\ 0 & \\ 0 & \end{bmatrix}\mathbf{f}_{mv}\ .$$

Decomposing $\mathbf{f}_T^{1c}$ and $\mathbf{f}_T^{2c}$ into the motion-dependent and the motion-independent components as in (36), we have

$$\mathbf{f}_T^{1c}=\mathbf{f}_T^{1m}+\mathbf{f}_T^{1n}\quad\text{and}\quad\mathbf{f}_T^{2c}=\mathbf{f}_T^{2m}+\mathbf{f}_T^{2n}$$

where $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ are, respectively, the forces/torques at the axis of motion of the load, which are generated by $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$, and $\mathbf{f}_T^{1n}$ and $\mathbf{f}_T^{2n}$ are, respectively, the forces/torques at the axis of motion of the load, which are generated by $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$. Since the motion-independent forces/torques do not have any effect on the motion of the load, $\mathbf{f}_T^{1c}$ and $\mathbf{f}_T^{2c}$ in the right hand side of equation (40) can be replaced by $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$, respectively. Using the minimum exerted force/torque criterion, $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ are computed as

$$\mathbf{f}_T^{1m}=-\eta\begin{bmatrix}\mathbf{M}\ddot{\mathbf{z}}+(m_1+m_2)\mathbf{g}\\ \mathbf{r}_1^{cm}\times m_1\mathbf{g}+\mathbf{r}_2^{cm}\times m_2\mathbf{g}\end{bmatrix}$$
$$-\begin{bmatrix}0\\0\\0\\0\\0\\(1-\eta)[001](\mathbf{r}_1^{cm}\times m_1\mathbf{g})-\eta[001](\mathbf{r}_2^{cm}\times m_2\mathbf{g})+L_1\dot{\omega}_{1\gamma}\end{bmatrix},$$

$$\mathbf{f}_T^{2m} = -(1-\eta)\begin{bmatrix} \mathbf{M}\ddot{\mathbf{z}} + (m_1+m_2)\mathbf{g} \\ \mathbf{r}_1^{cm}\times m_1\mathbf{g} + \mathbf{r}_2^{cm}\times m_2\mathbf{g} \end{bmatrix}$$

$$-\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -(1-\eta)\,[0\,0\,1]\,(\mathbf{r}_1^{cm}\times m_1\mathbf{g}) + \eta\,[0\,0\,1]\,(\mathbf{r}_2^{cm}\times m_2\mathbf{g}) + L_2\dot{\omega}_{2\gamma} \end{bmatrix},$$

for some $0 \le \eta \le 1$. The optimal η can be determined by minimizing the energy consumption.

Given (i) the constrained robot system as in (26) and (37), (ii) the initial condition that the two rigid bodies of the load are firmly grasped by both manipulator end-effectors, (iii) the desired poses of both manipulator end-effectors ($\mathbf{x}_1^d$ and $\mathbf{x}_2^d$), and the desired motion-independent forces/torques,

$$\mathbf{f}_n^d = \begin{bmatrix} \mathbf{f}_{1n}^d \\ \mathbf{f}_{2n}^d \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{nu}^d \\ \mathbf{f}_{nv}^d \end{bmatrix},$$

which are consistent with the constraints, the control problem is to find a feedback control,

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{F}_{1a} \\ \mathbf{F}_{2a} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{au} \\ \mathbf{F}_{av} \end{bmatrix},$$

based on $\mathbf{x}_1^d, \dot{\mathbf{x}}_1^d, \ddot{\mathbf{x}}_1^d, \mathbf{x}_2^d, \dot{\mathbf{x}}_2^d, \ddot{\mathbf{x}}_2^d, \mathbf{f}_{nu}^d, \mathbf{f}_{nv}^d, \mathbf{x}_1, \dot{\mathbf{x}}_1, \mathbf{x}_2, \dot{\mathbf{x}}_2, \mathbf{f}_{1c}$, and $\mathbf{f}_{2c}$ such that the poses of both the manipulator end-effectors and the motion independent forces/torques approach the desired values asymptotically. Since $\mathbf{f}_n$ and $\mathbf{f}_m$ can be determined from the measured values of $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$ (see Appendix B), $\mathbf{f}_{nu}, \mathbf{f}_{nv}, \mathbf{f}_{mu}$, and $\mathbf{f}_{mv}$ are measurable quantities, and $\boldsymbol{\lambda} = \left[(\frac{\partial \Phi}{\partial \mathbf{u}})^T \right]^{-1} \mathbf{f}_{nu}$ is also a measurable quantity. With these measurable quantities, a nonlinear decoupled controller which has a similar structure as in (17) can be constructed for this constrained robot system. Applying this nonlinear decoupled controller, the motion of each manipulator end-effector in the Cartesian space approaches the desired motion asymptotically,

$$\lim_{t\to\infty} \mathbf{x}_1(t) \to \mathbf{x}_1^d(t) \quad \text{and} \quad \lim_{t\to\infty} \mathbf{x}_2(t) \to \mathbf{x}_2^d(t),$$

and the motion-independent forces/torques also approach the desired values asymptotically,

$$\lim_{t\to\infty} \mathbf{f}_n(t) \to \mathbf{f}_n^d(t).$$

6. Two Cooperating Robots Handling Two Rigid Bodies Connected by a Spherical Joint

This constrained robot system is similar to the constrained robot system discussed in the previous section (see Figure 3), except that the two rigid bodies of the load are connected by a spherical joint. Similar assumptions in section 5 are made for this constrained robot system, except that rotational motion about the center of the spherical joint of the load is allowed.

Using the notations in (25) and (26), the constraint equation for this constrained robot system is described as

$$\mathbf{\Phi}(\mathbf{x}) \equiv \mathbf{\Phi}\left(\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}\right) \equiv \mathbf{\Phi}\left(\begin{bmatrix}\mathbf{x}_{1p}\\ \mathbf{x}_{1r}\\ \mathbf{x}_{2p}\\ \mathbf{x}_{2r}\end{bmatrix}\right) \equiv \mathbf{x}_{1p} - \mathbf{x}_{2p} - \mathbf{r}_1 + \mathbf{r}_2 = \mathbf{0} \tag{41}$$

where the constraint function, $\mathbf{\Phi} : R^{12} \to R^3$, $\mathbf{x}_{1p}$, $\mathbf{x}_{2p}$, $\mathbf{x}_{1r}$, $\mathbf{x}_{2r} \in R^3$ have similar meaning as in (27), $\mathbf{r}_1$, $\mathbf{r}_2 \in R^3$ are the respective Cartesian position vectors from the center of the spherical joint of the load to the gripping position of each manipulator end-effector (see Figure 3). Because each body is rigid and no relative motion between each manipulator end-effector and its gripped body exists, $\mathbf{r}_1$ and $\mathbf{r}_2$ can be determined from $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$, respectively, and they can be written as

$$\mathbf{r}_1 = \mathbf{r}_1(\mathbf{x}_{1r}), \quad \mathbf{r}_2 = \mathbf{r}_2(\mathbf{x}_{2r}) .$$

Thus, for a function $\mathbf{\Omega} : R^9 \to R^3$ such that

$$\mathbf{\Omega}\left(\begin{bmatrix}\mathbf{x}_{1r}\\ \mathbf{x}_2\end{bmatrix}\right) = \mathbf{x}_{2p} + \mathbf{r}_1(\mathbf{x}_{1r}) - \mathbf{r}_2(\mathbf{x}_{2r}) ,$$

we have $\quad \mathbf{\Phi}(\mathbf{\Omega}(\mathbf{x}_{1r}, \mathbf{x}_2), \mathbf{x}_{1r}, \mathbf{x}_2) = \mathbf{0} \quad$ for all $\mathbf{x}_{1r} \in R^3$, $\mathbf{x}_2 \in R^6$.

Hence, the sets,

$$\mathbf{u} \triangleq \mathbf{x}_{1p} \in R^3 \text{ and } \mathbf{v} \triangleq \begin{bmatrix}\mathbf{x}_{1r}\\ \mathbf{x}_2\end{bmatrix} \in R^9 , \tag{42}$$

partition the set of generalized coordinates of this robot system into the dependent and the independent sets. Let us express

$$\mathbf{v} = \begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{bmatrix} \tag{43}$$

where $\quad \mathbf{v}_1 = \mathbf{x}_{1r} \in R^3$, $\mathbf{v}_2 = \mathbf{x}_{2p} \in R^3$, and $\mathbf{v}_3 = \mathbf{x}_{2r} \in R^3$. (44)

Using the partitioned coordinates, **u** and **v**, the equations of motion (26) can be partitioned into

$$\begin{bmatrix}\mathbf{\Lambda}_{uu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{uv}(\mathbf{u},\mathbf{v})\\ \mathbf{\Lambda}_{vu}(\mathbf{u},\mathbf{v}) & \mathbf{\Lambda}_{vv}(\mathbf{u},\mathbf{v})\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}\\ \ddot{\mathbf{v}}\end{bmatrix} + \begin{bmatrix}\mathbf{c}_u(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}})\\ \mathbf{c}_v(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}})\end{bmatrix} + \begin{bmatrix}\mathbf{p}_u(\mathbf{u},\mathbf{v})\\ \mathbf{p}_v(\mathbf{u},\mathbf{v})\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{F}_{au}\\ \mathbf{F}_{av}\end{bmatrix} + \begin{bmatrix}\mathbf{f}_{cu}\\ \mathbf{f}_{cv}\end{bmatrix}$$

where $\mathbf{F}_{au} \in R^3$ and $\mathbf{F}_{av} \in R^9$. Again, proceeding with the similar steps as in sections 2.1 and 2.2, two decomposed subsystems for this constrained robot system can be obtained as in (16).

As in section 2.2, the motion-independent forces/torques $\mathbf{f}_n \in R^{12}$ can be partitioned and expressed as in section 2.2,

$$\mathbf{f}_n = \begin{bmatrix} \mathbf{f}_{nu} \\ \mathbf{f}_{nv} \end{bmatrix} = \left[\frac{\partial \Phi}{\partial \mathbf{x}} \right]^T \boldsymbol{\lambda} = \begin{bmatrix} \left[\frac{\partial \Phi}{\partial \mathbf{u}} \right]^T \boldsymbol{\lambda} \\ \left[\frac{\partial \Phi}{\partial \mathbf{v}} \right]^T \boldsymbol{\lambda} \end{bmatrix},$$

where $\mathbf{f}_{nu} \in R^3$ and $\mathbf{f}_{nv} \in R^9$ have similar meaning as in (11), and $\boldsymbol{\lambda} \in R^3$. Using the constraint equation of this constrained robot system as in (41), $\mathbf{f}_{nu}$ and $\mathbf{f}_{nv}$ can be expressed as

$$\mathbf{f}_{nu} = \left[\frac{\partial \Phi}{\partial \mathbf{u}} \right]^T \boldsymbol{\lambda} = \boldsymbol{\lambda},$$

$$\mathbf{f}_{nv} = \left[\frac{\partial \Phi}{\partial \mathbf{v}} \right]^T \boldsymbol{\lambda} = \begin{bmatrix} -(\frac{\partial \mathbf{r}_1}{\partial \mathbf{v}_1})^T \boldsymbol{\lambda} \\ -\boldsymbol{\lambda} \\ (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda} \end{bmatrix}.$$

As in (30), expressing the respective motion-independent forces/torques of each manipulator end-effector as $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$, and from the relation of the partitioned sets and the Cartesian variables of each manipulator end-effector as in (42)-(44), we have

$$\mathbf{f}_{1n} = \begin{bmatrix} \boldsymbol{\lambda} \\ -(\frac{\partial \mathbf{r}_1}{\partial \mathbf{v}_1})^T \boldsymbol{\lambda} \end{bmatrix}, \quad \mathbf{f}_{2n} = \begin{bmatrix} -\boldsymbol{\lambda} \\ (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda} \end{bmatrix}.$$

From (44) and the identity in Appendix A, we have

$$(\frac{\partial \mathbf{r}_1}{\partial \mathbf{v}_1})^T \boldsymbol{\lambda} = \mathbf{r}_1 \times \boldsymbol{\lambda}, \quad (\frac{\partial \mathbf{r}_2}{\partial \mathbf{v}_3})^T \boldsymbol{\lambda} = \mathbf{r}_2 \times \boldsymbol{\lambda}.$$

Thus, $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$ are, respectively, expressed as

$$\mathbf{f}_{1n} = \begin{bmatrix} \boldsymbol{\lambda} \\ -\mathbf{r}_1 \times \boldsymbol{\lambda} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2n} = \begin{bmatrix} -\boldsymbol{\lambda} \\ \mathbf{r}_2 \times \boldsymbol{\lambda} \end{bmatrix}.$$

Let $\mathbf{f}_T^{1n}$ be the force/torque, generated by $\mathbf{f}_{1n}$, at the center of the spherical joint of the load. Then $\mathbf{f}_T^{1n}$ can be expressed as

$$\mathbf{f}_T^{1n} = \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{r}_1 \times \boldsymbol{\lambda} + (-\mathbf{r}_1 \times \boldsymbol{\lambda}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{0} \end{bmatrix}. \tag{45}$$

Similarly, the force/torque $\mathbf{f}_T^{2n}$, generated by $\mathbf{f}_{2n}$, at the center of the spherical joint of the load can be expressed as

$$\mathbf{f}_T^{2n} = \begin{bmatrix} -\boldsymbol{\lambda} \\ \mathbf{r}_2 \times (-\boldsymbol{\lambda}) + (\mathbf{r}_2 \times \boldsymbol{\lambda}) \end{bmatrix} = \begin{bmatrix} -\boldsymbol{\lambda} \\ \mathbf{0} \end{bmatrix} . \tag{46}$$

Thus, the summation of (45) and (46) leads us to the same result as in (33),

$$\mathbf{f}_T^{1n} + \mathbf{f}_T^{2n} = \mathbf{0} ,$$

which again indicates that the net effect of the motion-independent forces/torques on the motion of the load is null.

For the constraint forces/torques $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$, they can be expressed as

$$\mathbf{f}_{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{f}_{2cr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{f}_{2cr} \end{bmatrix}$$

where $\mathbf{f}_{1cp}$, $\mathbf{f}_{1cr}$, $\mathbf{f}_{2cp}$, and $\mathbf{f}_{2cr} \in R^3$ have the same meaning as in (34). The forces/torques at the center of the spherical joint of the load, $\mathbf{f}_T^{1c} \in R^6$ and $\mathbf{f}_T^{2c} \in R^6$, which are generated by these constraint forces/torques, can be written respectively as

$$\mathbf{f}_T^{1c} = \begin{bmatrix} \mathbf{f}_{1cp} \\ \mathbf{r}_1 \times \mathbf{f}_{1cp} + \mathbf{f}_{1cr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_T^{2c} = \begin{bmatrix} \mathbf{f}_{2cp} \\ \mathbf{r}_2 \times \mathbf{f}_{2cp} + \mathbf{f}_{2cr} \end{bmatrix} .$$

Thus, the equations of motion of the load can be written as

$$\begin{aligned} &\mathbf{M}\,\ddot{\mathbf{z}}(t) + (m_1 + m_2)\mathbf{g} = -\,[\,\mathbf{I}_3\ \mathbf{0}\,]\,(\,\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c}\,) , \\ &\mathbf{L}_1\dot{\boldsymbol{\omega}}_1(t) + \boldsymbol{\omega}_1(t) \times \mathbf{L}_1\boldsymbol{\omega}_1(t) + \mathbf{r}_1^{cm} \times m_1\mathbf{g} = -\,[\,\mathbf{0}\ \mathbf{I}_3\,]\,\mathbf{f}_T^{1c} , \\ &\mathbf{L}_2\dot{\boldsymbol{\omega}}_2(t) + \boldsymbol{\omega}_2(t) \times \mathbf{L}_2\boldsymbol{\omega}_2(t) + \mathbf{r}_2^{cm} \times m_2\mathbf{g} = -\,[\,\mathbf{0}\ \mathbf{I}_3\,]\,\mathbf{f}_T^{2c} , \end{aligned} \tag{47}$$

where $\mathbf{M} \in R^{3\times3}$ is the diagonal matrix whose non-zero elements denote the mass of the load, m_1 , $m_2 \in R^1$ are the respective masses of each body, $\mathbf{L}_1$, $\mathbf{L}_2 \in R^{3\times3}$ are the respective diagonal matrices whose non-zero elements denote the principal moments of inertia of each body about the center of the spherical joint of the load, $\mathbf{r}_1^{cm}$, $\mathbf{r}_2^{cm} \in R^3$ are the respective Cartesian position vectors from the center of the spherical joint of the load to the center of mass of each body, $\mathbf{g} \in R^3$ is the gravity vector, $\mathbf{z}(t) \in R^3$ is the Cartesian vector describing the position of the center of the spherical joint of the load, $\boldsymbol{\omega}_1(t)$, $\boldsymbol{\omega}_2(t) \in R^3$ are the respective angular velocity vectors of each body about the center of the spherical joint of the load.

Expressing the respective motion-dependent forces/torques of each manipulator end-effector as $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$, then from the relation of the partitioned sets and the Cartesian variables of each manipulator end-effector as in (42)-(44), $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$ can be expressed by the partitioned motion-dependent forces/torques $\mathbf{f}_{mu}$ and $\mathbf{f}_{mv}$ as

$$\mathbf{f}_{1m} = \begin{bmatrix} \mathbf{f}_{mu} \\ [\,\mathbf{I}_3\ \mathbf{0}\,]\,\mathbf{f}_{mv} \end{bmatrix} , \quad \mathbf{f}_{2m} = \begin{bmatrix} \mathbf{0}\ \mathbf{I}_6 \end{bmatrix} \mathbf{f}_{mv} .$$

Decomposing $\mathbf{f}_T^{1c}$ and $\mathbf{f}_T^{2c}$ into the motion-dependent and the motion-independent components as in (36), we have

$$\mathbf{f}_T^{1c} = \mathbf{f}_T^{1m} + \mathbf{f}_T^{1n}\ , \quad \mathbf{f}_T^{2c} = \mathbf{f}_T^{2m} + \mathbf{f}_T^{2n}$$

where $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ are, respectively, the forces/torques at the center of the spherical joint of the load generated by $\mathbf{f}_{1m}$ and $\mathbf{f}_{2m}$, $\mathbf{f}_T^{1n}$ and $\mathbf{f}_T^{2n}$ are, respectively, the forces/torques at the center of the spherical joint of the load generated by $\mathbf{f}_{1n}$ and $\mathbf{f}_{2n}$. Again since the motion-independent forces/torques do not have any effect on the motion of the load, $\mathbf{f}_T^{1c}$ and $\mathbf{f}_T^{2c}$ in the right hand side of (47) can be replaced by $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$, respectively. Using the minimum exerted force/torque criterion, $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ are computed as

$$\mathbf{f}_T^{1m} = -\begin{bmatrix} \eta\,(\,\mathbf{M}\,\ddot{\mathbf{z}} + (m_1 + m_2)\mathbf{g}\,) \\ \mathbf{L}_1\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{L}_1\boldsymbol{\omega}_1 + \mathbf{r}_1^{cm} \times m_1\mathbf{g} \end{bmatrix}$$

$$\mathbf{f}_T^{2m} = -\begin{bmatrix} (\,1-\eta\,)\,(\,\mathbf{M}\,\ddot{\mathbf{z}} + (m_1 + m_2)\mathbf{g}\,) \\ \mathbf{L}_2\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{L}_2\boldsymbol{\omega}_2 + \mathbf{r}_2^{cm} \times m_2\mathbf{g} \end{bmatrix}$$

for some $0 \le \eta \le 1$. The optimal η can be determined by minimizing the energy consumption.

Given the constrained robot system as in (26) and (41), the initial condition that the two rigid bodies of the load are firmly grasped by both manipulator end-effectors, the desired poses of both manipulator end-effectors ($\mathbf{x}_1^d$ and $\mathbf{x}_2^d$), and the desired motion-independent forces/torques,

$$\mathbf{f}_n^d = \begin{bmatrix} \mathbf{f}_{1n}^d \\ \mathbf{f}_{2n}^d \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{nu}^d \\ \mathbf{f}_{nv}^d \end{bmatrix},$$

which are consistent with the constraints, the control problem is to find a feedback control,

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{F}_{1a} \\ \mathbf{F}_{2a} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{au} \\ \mathbf{F}_{av} \end{bmatrix},$$

based on $\mathbf{x}_1^d$, $\dot{\mathbf{x}}_1^d$, $\ddot{\mathbf{x}}_1^d$, $\mathbf{x}_2^d$, $\dot{\mathbf{x}}_2^d$, $\ddot{\mathbf{x}}_2^d$, $\mathbf{f}_{nu}^d$, $\mathbf{f}_{nv}^d$, $\mathbf{x}_1$, $\dot{\mathbf{x}}_1$, $\mathbf{x}_2$, $\dot{\mathbf{x}}_2$, $\mathbf{f}_{1c}$, and $\mathbf{f}_{2c}$ such that both the pose of the manipulator end-effectors and the motion independent forces/torques approach the desired values asymptotically. Since $\mathbf{f}_n$ and $\mathbf{f}_m$ can be determined from the measured values of $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$ (see Appendix B), $\mathbf{f}_{nu}$, $\mathbf{f}_{nv}$, $\mathbf{f}_{mu}$, and $\mathbf{f}_{mv}$ are measurable quantities, and $\boldsymbol{\lambda} = \left[\left[\frac{\partial \Phi}{\partial \mathbf{u}} \right]^T \right]^{-1} \mathbf{f}_{nu}$ is also a measurable quantity. With these measurable quantities, a nonlinear decoupled controller which has a similar structure as in (17) can be constructed for this constrained robot system. Applying this nonlinear decoupled controller, the motion of each manipulator end-effector in the Cartesian space approaches the desired motion asymptotically,

$$\lim_{t \to \infty} \mathbf{x}_1(t) \to \mathbf{x}_1^d(t) \quad \text{and} \quad \lim_{t \to \infty} \mathbf{x}_2(t) \to \mathbf{x}_2^d(t)\,,$$

and the motion-independent forces/torques approach the desired value asymptotically,

$$\lim_{t \to \infty} \mathbf{f}_n(t) \to \mathbf{f}_n^d(t) .$$

7. Conclusion

A generalized approach for controlling various cases of the constrained robot system was developed. The proposed control scheme utilizes the Jacobian matrix of the constraint function to partition the generalized coordinates of the constrained robot system into independent and dependent variables. This leads to partitioning the constrained robot system into two subsystems and yields a much simpler nonlinear decoupled controller than other controllers for the constrained robot system. The constraint forces/torques in each subsystem were decomposed into two components: the motion-independent and the motion-dependent forces/torques. Using the constraint function in the Cartesian space, the motion-independent forces/torques were expressed by a generalized multiplier vector and the Jacobian matrix of the constraint function. The motion-dependent forces/torques were determined by the motion of the manipulator end-effector, the motion-independent forces/torques, and other known quantities. Applying the proposed nonlinear decoupled controller to each subsystem and using the relation between the motion-independent forces/torques in the subsystems, both the errors in the manipulator end-effector motion and the constraint forces/torques were shown to approach zero asymptotically. Finally, four cases of constrained robot systems were carefully analyzed and discussed: one robot with its end-effector contacting a rigid friction surface, two cooperating robots handling a common rigid load, two cooperating robots handling two rigid bodies connected by a revolute joint, and two cooperating robots handling two rigid bodies connected by a spherical (ball-and-socket) joint.

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Appendix A — The Proof of Equation (32)

The vectors $\mathbf{r}_1$ and $\mathbf{x}_{1r}$ are, respectively,

$$\mathbf{r}_1 = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_{1r} = \begin{bmatrix} x_{1\alpha} \\ x_{1\beta} \\ x_{1\gamma} \end{bmatrix}$$

where r_{11} , r_{12} , r_{13} , $x_{1\alpha}$, $x_{1\beta}$, $x_{1\gamma} \in R^1$. When the orientation $\mathbf{x}_{1r}$ is changed by an amount of

$$\delta \mathbf{x}_{1r} = \begin{bmatrix} \delta x_{1\alpha} \\ \delta x_{1\beta} \\ \delta x_{1\gamma} \end{bmatrix} ,$$

the corresponding change of $\mathbf{r}_1$, $\delta \mathbf{r}_1$, is

$$\delta \mathbf{r}_1 = \delta \mathbf{x}_{1r} \times \mathbf{r}_1 = \begin{bmatrix} \delta x_{1\beta} r_{13} - \delta x_{1\gamma} r_{12} \\ \delta x_{1\gamma} r_{11} - \delta x_{1\alpha} r_{13} \\ \delta x_{1\alpha} r_{12} - \delta x_{1\beta} r_{11} \end{bmatrix} .$$

Thus, we have

$$\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}} = \begin{bmatrix} 0 & r_{13} & -r_{12} \\ -r_{13} & 0 & r_{11} \\ r_{12} & -r_{11} & 0 \end{bmatrix}$$

which is a skew symmetric matrix. It can be easily shown that

$$\left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{x}_{1r}} \right)^T \boldsymbol{\lambda}_1 = \mathbf{r}_1 \times \boldsymbol{\lambda}_1 , \quad \text{for any } \boldsymbol{\lambda}_1 \in R^3 .$$

Appendix B — Computation of $\mathbf{f}_n$ and $\mathbf{f}_m$ from $\mathbf{f}_c$

Let us express

$$\mathbf{f}_{1m} = \begin{bmatrix} \mathbf{f}_{1mp} \\ \mathbf{f}_{1mr} \end{bmatrix}, \quad \mathbf{f}_{2m} = \begin{bmatrix} \mathbf{f}_{2mp} \\ \mathbf{f}_{2mr} \end{bmatrix} \tag{B.1}$$

where $\mathbf{f}_{1mp} \in R^3$ and $\mathbf{f}_{1mr} \in R^3$ are the respective vectors of the force and torque components in the motion-dependent forces/torques of the first robot $\mathbf{f}_{1m} \in R^6$, $\mathbf{f}_{2mp} \in R^3$ and $\mathbf{f}_{2mr} \in R^3$ are the respective vectors of the force and torque components in the motion-dependent forces/torques of the second robot $\mathbf{f}_{2m} \in R^6$. Then $\mathbf{f}_T^{1m}$ and $\mathbf{f}_T^{2m}$ can be found to be

$$\mathbf{f}_T^{1m} = \begin{bmatrix} \mathbf{f}_{1mp} \\ \mathbf{r}_1 \times \mathbf{f}_{1mp} + \mathbf{f}_{1mr} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_T^{2m} = \begin{bmatrix} \mathbf{f}_{2mp} \\ \mathbf{r}_2 \times \mathbf{f}_{2mp} + \mathbf{f}_{2mr} \end{bmatrix}. \tag{B.2}$$

Since

$$\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c} = \mathbf{f}_T^{1m} + \mathbf{f}_T^{2m}, \tag{B.3}$$

using the minimum force/torque criterion, we have

$$\mathbf{f}_T^{1m} = \eta\,(\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c}) \tag{B.4}$$

$$\mathbf{f}_T^{2m} = (1-\eta)\,(\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c}). \tag{B.5}$$

Let us denote $(\mathbf{f}_T^{1c} + \mathbf{f}_T^{2c})$ by $\mathbf{F}_C$, and it can be computed from the measured values of $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$. Then

$$\mathbf{F}_C = \mathbf{f}_T^{1c} + \mathbf{f}_T^{2c} = \begin{bmatrix} \mathbf{F}_{Cp} \\ \mathbf{F}_{Cr} \end{bmatrix} \tag{B.6}$$

where $\mathbf{F}_{Cp} \in R^3$ and $\mathbf{F}_{Cr} \in R^3$ are the respective vectors of the force and torque components in $\mathbf{F}_C \in R^6$. From (B.2)-(B.6), we obtain

$$\mathbf{f}_T^{1m} = \begin{bmatrix} \mathbf{f}_{1mp} \\ \mathbf{r}_1 \times \mathbf{f}_{1mp} + \mathbf{f}_{1mr} \end{bmatrix} = \eta \begin{bmatrix} \mathbf{F}_{Cp} \\ \mathbf{F}_{Cr} \end{bmatrix}. \tag{B.7}$$

Thus, $\mathbf{f}_{1m}$ can be obtained as

$$\mathbf{f}_{1m} = \begin{bmatrix} \mathbf{f}_{1mp} \\ \mathbf{f}_{1mr} \end{bmatrix} = \eta \begin{bmatrix} \mathbf{F}_{Cp} \\ \mathbf{F}_{Cr} - \mathbf{r}_1 \times \mathbf{F}_{Cp} \end{bmatrix}. \tag{B.8}$$

Similarly, $\mathbf{f}_{2m}$ can be obtained as

$$\mathbf{f}_{2m} = \begin{bmatrix} \mathbf{f}_{2mp} \\ \mathbf{f}_{2mr} \end{bmatrix} = (1-\eta) \begin{bmatrix} \mathbf{F}_{Cp} \\ \mathbf{F}_{Cr} - \mathbf{r}_2 \times \mathbf{F}_{Cp} \end{bmatrix}. \tag{B.9}$$

Since $\mathbf{f}_n = \mathbf{f}_c - \mathbf{f}_m$, $\mathbf{f}_n$ can be obtained from the measured values of $\mathbf{f}_{1c}$ and $\mathbf{f}_{2c}$.

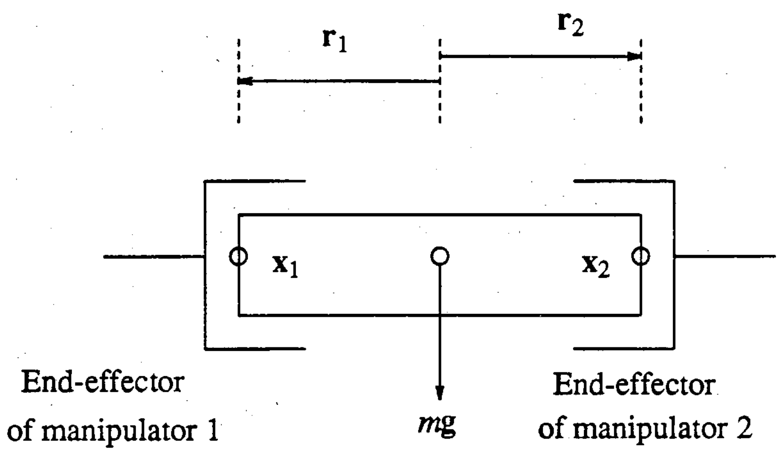

Figure 1. Two cooperating robots handling a common rigid load.

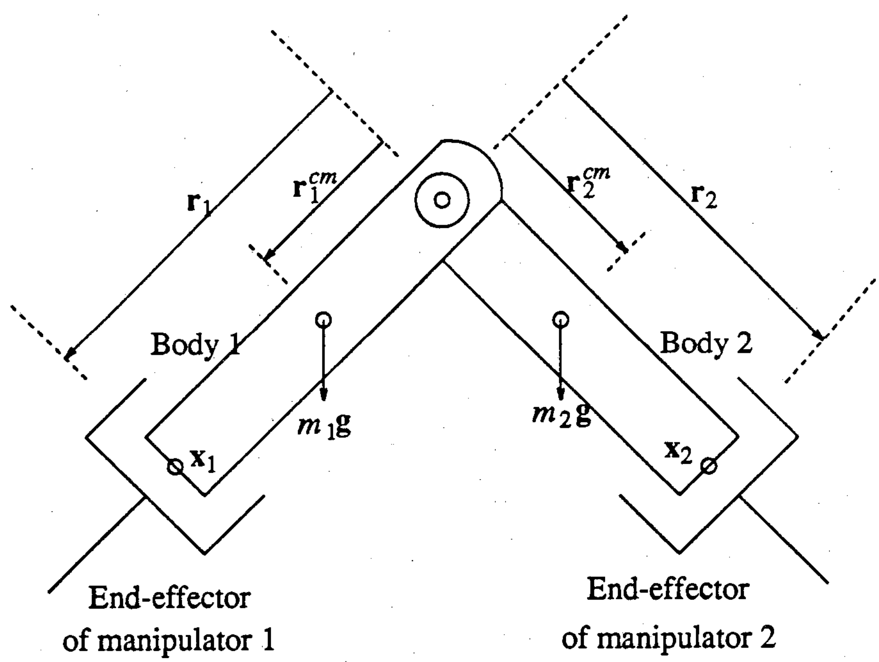

Figure 2. Two cooperating robots handling two rigid bodies connected by a revolute joint.

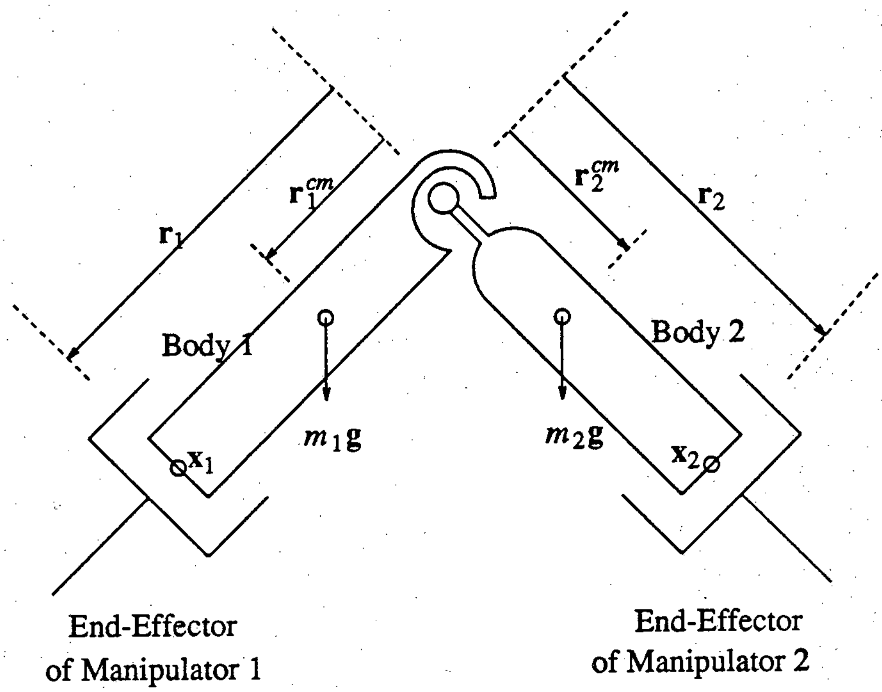

Figure 3. Two cooperating robots handling two rigid bodies connected by a spherical joint.