9-1-1987

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TR-EE 87-32
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ACKNOWLEDGMENTS

Special thanks go to Mary Schultz for her invaluable help in typing. We also wish to thank Mickey Krebs for her assistance in putting this document together.

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ABSTRACT

In this project we use the second method of Lyapunov to develop several controllers to stabilize discrete-time dynamical systems with or without parameter uncertainties and/or external disturbances. We also use the notion of a sliding mode on a preferred hyperplane, previously developed for continuous-time variable structure control systems, to stabilize discrete-time dynamical systems.

In particular, feedback controllers are proposed that: (i) stabilize discrete systems with no uncertainties by forcing their state trajectories onto prespecified hyperplanes; (ii) provide a needed level of stability robustness to discrete systems with uncertainties which are modeled by cone bounded functions; (iii) robustly stabilize discrete uncertain systems.

CHAPTER I
INTRODUCTION

1.1. MOTIVATION

Recent advances in digital signal processing technology brought about by digital computers have open the way to the implementation of broad classes of controllers conceived thus far. Guided by this fact, we try in this work to solve the problem of control and stability of uncertain dynamical systems purely from the discrete-time systems point of view.

We first briefly review the results on the subject which have provided the motivation behind the various developments in this project.

In an attempt at driving the state trajectory of a linear discrete-time dynamical system toward a desired hyperplane, Milosavljevic′ [26] tries to extend the results obtained by Utkin [12] and Itkis [11] for continuous-time variable structure systems, i.e., he tries to show that a sliding mode can also be achieved with discrete-time dynamical systems; however, a closer look at this problems will reveal that a sliding mode does not exist for such systems in the strict sense.

In order to gain more insight into solving the problem of forcing the state trajectory of a discrete-time dynamical system onto a desired hyperplane, we found that the idea of a continuous-time system with high feedback gain proposed by Utkin [31] and Marino [32] offered some possibilities, since it has been shown that a high feedback gain continuous-time system behaves as a variable structure control system in the limit.

So far we have made no mention of the system uncertainties that the designer is faced with in real life when designing a controller. Corless and Leitmann [7] propose a deterministic treatment of uncertainties for continuous-time systems which are constrained to meet the so-called matching conditions [33]. Manela [20] and Corless and Manela [23] provide a possible solution to the discrete-time problem with matched uncertainties using the minimum-maximum approach.

Finally, realizing that implementation is a very important facet of a control system, we looked at ways of how one could solve the above problem using output information only. Walcott and Żak [27] and Steinberg and Corless [28] suggest possible solutions to the problem of stabilizing continuous-time uncertain dynamical systems through output feedback whenever certain algebraic constraints are met.

1.2. OBJECTIVE OF THE PROJECT

The topic of this project is the control and stabilization of discrete-time uncertain dynamical systems via the second method of Lyapunov.

We shall first show that by applying Lyapunov's second method to linear time-invariant discrete-time dynamical systems with no uncertainties, we can drive the state trajectory of such system onto a desired linear hyperplane, where the system possesses certain desirable characteristics such as stability and reduced dimension. Next, we shall show that under certain conditions, we can stabilize a class of discrete-time uncertain dynamical systems where the "nominal" system is linear and the uncertainties do not depend on the control input through the direct application of Lyapunov's second method.

Finally, we shall show that a controller which steers the state trajectory of the class of discrete-time uncertain dynamical systems with linear "nominal" system toward the vicinity of a linear hyperplane.

1.3. OVERVIEW OF THE REPORT

The report is organized as follows:

Chapter 2 gives a fairly complete explication of the application of the second method of Lyapunov to determine the stability properties of discrete-time dynamical systems modeled by ordinary difference equations. This review is necessary in order to have a clear and thorough understanding of the method in order to use it effectively to develop controllers that stabilize the class of systems that we shall deal with in the following chapters. The information presented in this chapter is organized in the following fashion. First, the most well known definitions that describe discrete-time dynamical systems are introduced. Second, several well accepted notions of stability are stated and discussed. Third, since the second method of Lyapunov stability relies on the existence of a positive definite function, definitions of time-invariant and time dependent positive definite and positive semidefinite functions are presented along with specific examples to clarify the concepts. Next, six main theorems on Lyapunov stability, which constitute the heart of the chapter, are stated and their proofs included. Finally, the important notions of uniform boundedness and uniform ultimate boundedness are introduced, as they are extensions of Lyapunov stability.

In Chapter 3 we develop several control strategies which steer the state trajectory of a linear time-invariant discrete-time dynamical system without uncertainties onto a hyperplane where the given system has certain desirable characteristics such as stability and reduced dimension. The controller design strategies are based on the idea of a sliding mode of continuous-time variable structure control systems on a switching hyperplane. Additionally, we present a recent and effective hyperplane design methodology in order to facilitate the design of these types of controllers.

In Chapter 4 we propose a solution to the problem of stabilization of a class of discrete-time uncertain dynamical systems where the "nominal" system is linear and the uncertainties do not depend on the control input. The approach used to solve this problem is of a deterministic nature, i.e., no knowledge of the statistical behavior of the uncertain elements is assumed, except the bounded sets that they belong to. The type of controller proposed in this development utilizes full state feedback and at least guarantees uniform boundedness and uniform ultimate boundedness of the solution of the closed loop system.

In Chapter 5 we extend the results obtained in Chapter 4 and propose an output feedback controller, which under some not very restrictive assumptions solves the same problem posed in the previous chapter.

In Chapter 6 we make an attempt to unify the theories developed in Chapters 3 and 4.

Finally, in Chapter 5, we present a summary along with the open problems that still remain to be solved.

CHAPTER II

DISCRETE-TIME CONTROL SYSTEMS STABILITY ANALYSIS VIA THE "SECOND METHOD" OF LYAPUNOV

2.1. INTRODUCTION

The purpose of the chapter is a review of the application of the second method of Lyapunov to determine the stability properties of discrete-time dynamic systems described ordinary difference equations.

The essence of Lyapunov's second method lies on the fact that the stability of a discrete-time dynamical system governed by a difference equation can be determined without actually having to solve such an equation [1,2,3,4,5,6].

2.2. DESCRIPTION OF DISCRETE-TIME DYNAMICAL SYSTEMS

Throughout this chapter, we shall study systems that are governed by the vector difference equation

$$x(t_{k+1}) = f(t_k, x(t_k), u(t_k)) \tag{2.1}$$

where t_k is a discrete value of time, $k \in Z$; $x(t_k) \in \mathbb{R}^n$ is the state vector; $u(t_k) \in \mathbb{R}^m$ is the input (control) vector and $f \in \mathbb{R}^n$ is a vector-valued function, and Z denotes the set of integers.

We now introduce the following definitions

Definition 2.2.1. The discrete-time dynamic system (2.1) is said to be free (unforced), if $u(t_k) = 0$, $\forall t_k$, $k \in Z$, that is,

$$x(t_{k+1}) = f(t_k, x(t_k)) \tag{2.2}$$

Definition 2.2.2. The discrete-time dynamic system (2.1) is stationary if f does not explicitly depend on t_k, i.e.,

$$x(t_{k+1}) = f(x(t_k), u(t_k)) \tag{2.3}$$

Definition 2.2.3. If a discrete-time dynamic system is both free and stationary, it is autonomous, namely,

$$x(t_{k+1}) = f(x(t_k)) \tag{2.4}$$

Definition 2.2.4. The state x_e is an equilibrium state of the free discrete-time dynamic system (2.2) if

$$x_e = f(t_k, x_e), \qquad \forall\, t_k , \tag{2.5}$$

in other words, the solution to (2.2) starting in state x_e at time t_0 is (2.5) for all $t_k \geq t_0$, where the symbol $\forall$ means "for all".

2.3. DISCRETE-TIME DYNAMICAL SYSTEMS STABILITY DEFINITIONS

Although many stability definitions have been proposed for continuous-time systems, only the ones, as applied to discrete-time systems, in this report shall be discussed in this section.

Definition 2.3.1. An equilibrium state x_e of a free discrete-time dynamic system is stable if, given any $\epsilon > 0$, $\epsilon \in \mathbb{R}$, there exists a $\delta(t_0,\epsilon) > 0$ such that $\|x_0 - x_e\| \leq \delta(t_0,\epsilon)$ implies $\|x(t_k) - x_e\| \leq \epsilon$, $\forall\, t_k \geq t_0$, where $x_0 = x(t_0)$ and $x(t_k)$ is the solution $\phi(t_k;x_0,t_0)$ to (2.2). In the above inequalities, $\|\cdot\|$ refers to the standard Euclidean norm. This concept of stability is illustrated in Figure 2.1.

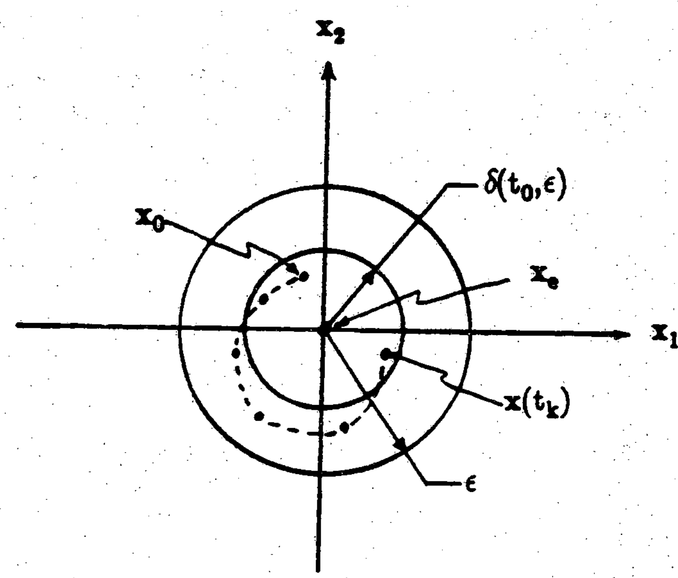

Figure 2.1. Definition of stability (second order case)

As shown in the above Figure 2.1, this notion of stability (also known as stability in the sense of Lyapunov or i.s.L.) is of the local type, namely, it states that if the equilibrium state x_e is stable, then every solution $x(t_k) = \phi(t_k;x_0,t_0)$ to (2.2), starting in the neighborhood of x_e must stay arbitrarily close to x_e for all t_k's, $t_k \geq t_0$.

Definition 2.3.2. An equilibrium state x_e of a free discrete-time dynamic system is asymptotically stable if

(i) it is stable (i.s.L.) and

(ii) every trajectory $x(t_k) = \phi(t_k;x_0,t_0)$ starting sufficiently close to x_e converges to x_e as $t_k \rightarrow \infty$. In other words, for a given $\mu > 0$, $\mu \in \mathbb{R}$, there exist real numbers $\gamma(t_0) > 0$ and $T(\mu,x_0,t_0)$ such that $\|x_0 - x_e\| \leq \gamma(t_0)$ implies that $\|x(t_k) - x_e\| \leq \mu$, $\forall$ $t_k \geq t_0 + T(\mu,x_0,t_0)$.

As seen in Figure 2.2, asymptotic stability is also a local concept, since it is only known that there exists some region in the state space around the equilibrium state such that all motions starting from within that region are asymptotically stable, however, one does not know a priori how small $\delta(t_0)$ may have to be.

The definition of asymptotic stability also implies that all motions that start at the same distance from x_e shall remain at a distance no larger than μ from x_e at arbitrarily large values of time.

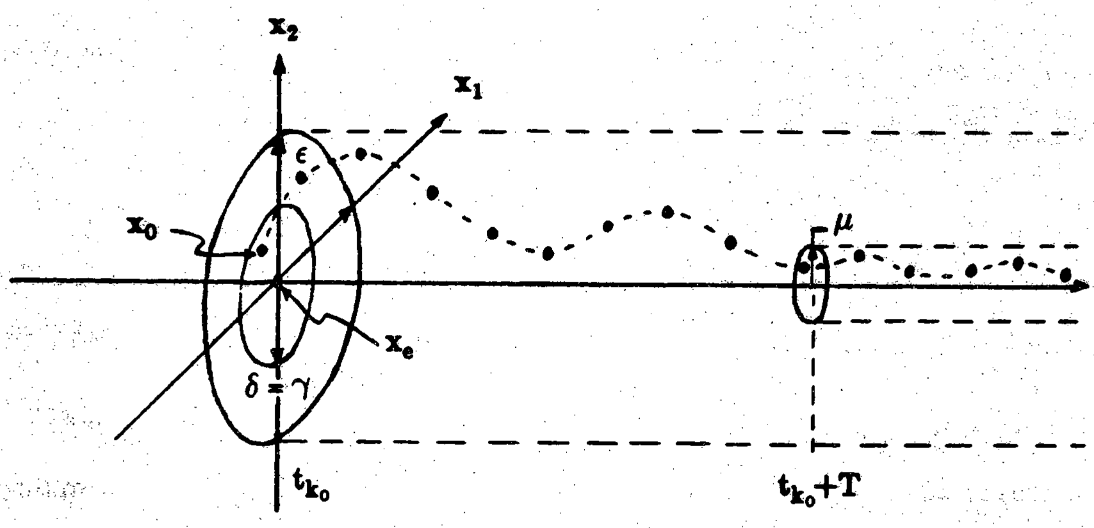

Figure 2.2. Illustration of asymptotic stability (second order case)

Definition 2.3.3. An equilibrium state x_e of a free discrete-time dynamic system is asymptotically stable in the large or globally asymptotically stable, if

(i) it is stable and

(ii) every motion converges to x_e as $k \rightarrow \infty$, namely, $x(t_k) \rightarrow x_e$ as $k \rightarrow \infty$.

Asymptotic stability in the large results if all the trajectories of the system converge to the equilibrium state x_e as $k \rightarrow \infty$, that is, the region of attraction is the entire state space $\mathbb{R}^n$, where the region of attraction is defined by $B_{\delta(t_0)} = \{x \in \mathbb{R}^n : \|x(t_k) - x_e\| < \delta(t_0)\}$.

Note that if a discrete-time system is autonomous (free and stationary), then δ and T in the above definitions do not depend on t_0.

The concept of equiasymptotic stability of x_e is now introduced. It is a stronger concept than asymptotic stability, in fact, the former implies the latter.

Definition 2.3.4. An equilibrium state x_e of a free discrete-time dynamic system is equiasymptotically stable if

(i) it is stable

(ii) given $\mu > 0$, $\mu \in \mathbb{R}$, there exists a number $T(\mu,r,t_0)$ such that $\|\phi(t_k;x_0,t_0)\| = \|x(t_k)\| \leq \mu \quad \forall\, t_k \geq t_0 + T(\mu,r,t_0)$ whenever $\|x_0 - x_e\| \leq r(t_0)$, with $r(t_0) > 0$ a fixed constant that does not depend on μ or x_0. In other words, every motion starting sufficiently close to x_e converges to x_e as $t_k \rightarrow \infty$ uniformly in x_0.

Definition 2.3.5. An equilibrium state x_e of a free discrete-time dynamic system is equiasymptotically stable in the large if

(i) it is stable,

(ii) all motions are bounded, and

(iii) all motions $\phi(t_k;x_0,t_0) = x(t_k)$, with x_0 and t_0 arbitrary, converge to x_e as t_k increases, i.e., $\|x(t_k) - x_e\| \rightarrow 0$ as $t_k \rightarrow \infty$.

Definition 2.3.6. An equilibrium state x_e of a free discrete-time dynamic system is uniformly stable if given any $\epsilon > 0$, $\epsilon \in \mathbb{R}$, there exists a number $\delta(\epsilon) > 0$, $\delta(\epsilon) \in \mathbb{R}$, such that if $\|x_0 - x_e\| \leq \delta(\epsilon)$ then $\|\phi(t_k;x_0,t_0) - x_e\| \leq \epsilon$ for all $t_k \geq t_0$.

The difference between the concepts of stability and uniform stability is that the real number δ can be chosen independently of the initial time t_0 in the case of uniform stability. Therefore, one should bear in mind that while a system may be stable (i.s.L.), it may not be uniformly stable because δ

may always depend on t_0.

Definition 2.3.7. An equilibrium state x_e of a free discrete-time dynamic system is uniformly asymptotically stable if

(i) it is uniformly stable and

(ii) given $\mu > 0$, $\mu \in \mathbb{R}$, there exists a number $T(\mu)$ such that $\|\phi(t_k;x_0,t_0) - x_e\| \leq \mu$ for all $t_k \geq t_0 + T(\mu)$ whenever $\|x_0 - x_e\| \leq \gamma$, $\gamma > 0$ being a real number which does not depend on μ or x_0.

Definition 2.3.8. An equilibrium state x_e of a free discrete-time dynamic system is uniformly asymptotically stable in the large (uniformly globally asymptotically stable) if

(i) it is uniformly stable,

(ii) all motions are uniformly bounded, that is, given any $\gamma > 0$, $\gamma \in \mathbb{R}$, there exists some $B(\gamma)$ such that $\|x_0 - x_e\| \leq \gamma$ implies that $\|\phi(t_k;x_0,t_0) - x_e\| \leq B$ for all $t_k \geq t_0$, and

(iii) every motion $\phi(t_k;x_0,t_0)$, with x_0 and t_0 arbitrary, converges uniformly in $\|x_0\| \leq \gamma$; $\gamma > 0$ is fixed but arbitrarily large, to x_e with increasing t_k (as $k \rightarrow \infty$).

2.4. POSITIVE DEFINITE FUNCTIONS

This section reviews the concepts of positive definite and of positive semidefinite functions, since they are central to the development of the Lyapunov stability theory. [5,6].

2.4.1. Time-invariant Positive Definite Functions

Let V(x) be a real scalar function of the vector x, i.e., $V:\mathbb{R}^n \to \mathbb{R}$, and let S be a closed bouned region in the x space which conains the origin.

Definition 2.4.1.1. The function V(x) is locally positive semidefinite in S if, for all x and S

(i) $V(0) = 0$ and

(ii) $V(x) \geq 0$.

Definition 2.4.1.2. The function V(x) is locally positive definite in S, if for all x in S

(i) $V(0) = 0$ and

(ii) $V(x) > 0$, for all $x \neq 0$, $x \in S$

Definition 2.4.1.3. The function V(x) is positive definite if

(i) $V(0) = 0$,

(ii) $V(x) > 0$, for all $x \neq 0$, $x \in \mathbb{R}^n$, and

(iii) $V(x) \to \infty$ as $\|x\| \to \infty$, uniformly in x.

Notice that the difference between the last two definitions is that the latter is a global type of concept.

Example 2.4.1.1. Let $V_1(x) = x_1^2$, $x^T = [x_1\ x_2]$, then $V_1(x)$ is a positive semidefinite function because while $V_1(x) = 0$, the vector x may not be identically zero.

Example 2.4.1.2. Let $V_2(x) = x_1^2 + x_2^2$, $x^T = [x_1\ x_2]$, then $V_2(x)$ is positive definite function since (i) and (ii) in definition 2.4.1.3 are clearly satisfied. Moreover, (iii) is satisfied because $V_2(\mathbf{x}) = \|x\|^2$ where $\|x\|$ is the Euclidean norm in $\mathbb{R}^2$.

2.4.2. Time Dependent Positive Definite Functions

Let $W(t_k,x)$ be a real scalar function of time t_k and of the vector x, that is, $W : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$, and let S be a closed bouned region in the x space which contains the origin.

Definition 2.4.2.1. The function $W(t_k,x)$ is locally positive semidefinite in S if, for all **x** in S and t_k

(i) $W(t_k,0) = \mathbf{0}$, $\forall\ t_k$ and

(ii) $W(t_k,x) \geq \mathbf{0}$, $\forall\ t_k$ and $x \in S$.

Definition 2.4.2.2. The function $W(t_k,x)$ is locally positive definite in S, if for all x in S

(i) there exists a continuous scalar functin α such that $\alpha(0) = 0$, $\alpha(\gamma) > 0$,

(ii) $W(t_k,0) = 0$, $\forall\, t_k$, and

(iii) for all t_k and all $x \neq 0$, $x \in S$, $W(t_k,x) \geq \alpha(\|x\|)$.

Definition 2.4.2.3. The function $W(t_k,x)$ is positive definite if (i)-(ii) same as definition 3.22, and

(iii) for all t_k and all $x \neq 0$ $x \in \mathbb{R}^n$, $W(t_k,x) \geq \alpha(\|x\|)$.

Definition 2.4.2.2 (2.4.2.3) shows that a function of t_k and x is locally positive definite (positive definite) if and only if it *dominates,* at each instant of time t_k, $k \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers and over some closed bouned region S in the space of x which includes the orgin (the entire space $\mathbb{R}^n$), a continuous real scalar function $\alpha(\|x\|)$. Condition (iii) in the last two definitions is often replaced with (iiia) there exists a positive definite function $V(x)$, $V : \mathbb{R}^n \rightarrow \mathbb{R}$ (time-invariant), such that $W(t_k,x) \geq V(x)$, $\forall\, t_k \geq 0$, $\forall\, x \in S$ ($x \in \mathbb{R}^n$).

Definition 2.4.2.4. A function $W : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be decrescent in S if there exists a function $\beta(\cdot)$ such that $W(t_k,x) \leq \beta(\|x\|)$, $\forall\, t_k \geq 0$ and $\forall\, x \in S$.

Example 2.4.2.1. Let $W_1(t_k,x) = (x_1^2 + x_2^2)\ e^{-t_k}$, $x^T = [x_1\ x_2]$, then W_1 is positive semidefinite since $W_1(t_k,x) \rightarrow 0$ as $t_k \rightarrow \infty$ for all $x \neq 0$.

Example 2.4.2.2. Let $W_2(t_k,x) = (x_1^2 + x_2^2)\ (t_k^2 + 1)$, $x^T = [x_1\ x_2]$, then W_2 is positive definite because it dominates the positive definite, time-invariant function $W_2(x) = x_1^2 + x_2^2$.

Example 2.4.2.3. Let $W_3(t_k,x) = (x_1^2 + x_2^2)/(t_k^2 + 1)$, $x^T = [x_1\ x_2]$, then W_3 is positive definite and decrescent.

2.5. LYAPUNOV STABILITY THEOREMS FOR DISCRETE-TIME DYNAMICAL SYSTEMS

Consider the discrete-time free dynamic systems

$$x(t_{k+1}) = f(t_k,x(t_k))\ , \tag{2.6}$$

which has the origin as an equilibrium state, i.e., $x_e = \theta$. Furthermore, we assume that

$$f(t_k,0) = 0, \quad \forall\ t_k\ . \tag{2.7}$$

Let the solution of (2.6) be denoted by

$$\phi(t_k;x_0,t_0) = x(t_k) \tag{2.8}$$

such that

$$\phi(t_0;x_0,t_0) = x_0 \quad \forall\ x_0,\ t_0 \tag{2.9}$$

$$\phi(t_{k+1};x(t_k),t_k) = x(t_{k+1}) = f(t_k,x(t_k)),\ \forall\ x(t_k),\ t_k\ , \tag{2.10}$$

for any initial state x_0, any initial time t_0, and any time t_k.

Theorem 2.1. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is stable if there exists a positive definite function $W(t_k,x)$ in some neighborhood S_s of the origin such that

$$\Delta W(t_k,x) = \text{rate of increase of W along motion starting at x, } t_k$$
$$= [W(t_{k+1},\phi(t_{k+1};x,t_k) - W(t_k,x)]/(t_{k+1} - t_k)$$
$$\leq 0, \forall\, t_k \geq t_0, \forall\, x \in S_s = \{x : \|x\| \leq s\} \tag{2.11}$$

Proof: To show that θ is a stable equilibrium point at time t_0, we have to show that, given any $\epsilon > 0$, we can find a $\delta(t_0,\epsilon) > 0$ such that $\|x_0\| \leq \delta(t_0,\epsilon)$ implies $\|x(t_k)\| \leq \epsilon$, $\forall\, t_k \geq t_0$. Now, given $\epsilon > 0$, pick $\delta > 0$ such that

$$\beta(t_0,\delta) = \sup_{\|x\|<\delta} \{W(t_0,x)\} \leq \alpha(\epsilon) \tag{2.12}$$

hence, $\alpha(\delta) \leq \beta(t_0,\delta)$.

Notice that such a δ can always be found, since $\alpha(\epsilon) > 0$ for $\epsilon > 0$ and $\beta(\delta,t_0) \longrightarrow 0$ as $\delta \longrightarrow 0$.

Suppose $\|x_0\| \leq \delta$, then $W(t_0,x_0) \leq \beta(t_0,\delta) \leq \alpha(\epsilon)$. But $\Delta W(t_k,x) \leq 0$, $\forall$ $t_k \geq t_0$ and $\forall\, x \in S_s$ implies that

$$W(t_k,x) \leq W(t_0,x_0) \leq \alpha(\epsilon), \forall\, t_k \geq t_0 \text{ whenever } \|x\| \leq \delta, \tag{2.13}$$

now, since $W(t_k,x(t_k)) \geq \alpha(\|x(t_k)\|)$, we have that

$$\alpha(\|x(t_k)\|) \leq W(t_k,x(t_k)) \leq W(t_0,x_0) \leq \alpha(\epsilon), \tag{2.14}$$

which implies that $\|x(t_k)\| \leq \epsilon$, since α is a scalar nondecreasing and positive function.

□

Theorem 2.2. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is uniformly stable if in addition to the conditions of Theorem 2.1, $W(t_k,x)$ is descrescent in S_s.

Proof: We want to show that given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ such that $\|x_0\| \leq \delta(\epsilon)$ implies $\|x(t_k)\| \leq \epsilon$, $\forall\, t_k \geq t_0$. Because $W(t_k,x)$ is decrescent, there exists a nondecreasing function $\beta(\gamma)$, with $\beta(0) = 0$ and such that $W(t_k,x) \leq \beta(\|x\|)$, $\forall\, x \in S_s = \{x : \|x\| \leq 1\}$ and $\forall\, t_k$. If we pick $\delta > 0$ such that

$$\beta(\delta) = \sup_{\|x\| \leq \delta} \{\sup_{t_k \geq t_0} \{W(t_k,x)\}\} \leq \alpha(\epsilon)\ , \tag{2.15}$$

then δ only depends on ϵ. Moreover, suppose that $\|x_0\| \leq \delta$, with arbitrary t_0. Then

$$W(t_0,x_0) \leq \beta(\delta) \leq \alpha(\epsilon)\ . \tag{2.16}$$

Now, $\Delta W(t_k,x) \leq 0$, $\forall\, t_k \geq t_0$ and $\forall\, x \in S_s$ implies that

$$W(t_k,x) \leq W(t_0,x_0)\ ,\ \ \forall\, t_k \geq t_0\ ,\ \forall\, x \in S_s\ . \tag{2.17}$$

Therefore, noting that $\alpha(\|x(t_k)\|) \leq W(t_k,x(t_k))$, we get

$$\alpha(\|x(t_k)\|) \leq W(t_k,x(t_k)) \leq W(t_0,x_0) \leq \beta(\delta) \leq \alpha(\epsilon) \tag{2.18}$$

from which we conclude that $\|x(t_k)\| \leq \epsilon$ whenever $\|x_0\| \leq \delta(\epsilon)$, since α is a scalar nondecreasing and positive function.

□

A stronger stability concept of the eqiuilbrium point $x_e = 0$ is now presented, namely, equiasymptotic stability since it implies asymptotic stability.

Theorem 2.3. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is equiasymptotically stable if

(i) it is stable (in the sense of Theorem 2.1) and

(ii) there exists a continuous scalar function γ such that $\gamma(0) = 0$ and, for all t_k and $x \neq 0$, $x \in S_s$

$$\Delta W(t_k,x) \leq -\gamma(\|x\|) < 0 \,. \tag{2.19}$$

Proof: Since the stability of $x_e = \theta$ has already been proved in Theorem 2.1, it only has to be shown that $\|\phi(t_k;x_0,t_0)\| = \|x(t_k)\| \to 0$ as $t_k \to \infty$ uniformly in x_0.

From assumption (i), there exists a continuous scalar nondecreasing function α such that $\alpha(0) = 0$ and $\forall\, x \neq 0$, $x \in S_s$, $\alpha(\|x\|) \leq W(t_k,x)$. Now, given $\mu > 0$, $\mu \in \mathbb{R}$, pick $r(t_0) > 0$ such that

$$\beta(t_0,r) = \sup_{\|x\| \leq r} \{W(t_0,x)\} \leq \alpha(s) \tag{2.20}$$

The choice of $r(t_0)$ is illustrated in Figure 2.3. Thus, if $\|x_0\| \leq r(t_0)$, then

$$W(t_0,x_0) \leq \beta(t_0,r) \leq \alpha(s) \,, \tag{2.21}$$

pick $r_1 > 0$ such that

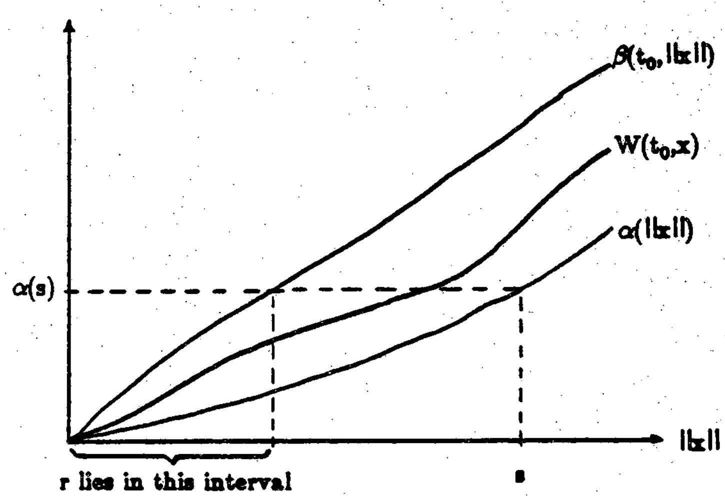

Figure 2.3. Selection of r.

$$\beta(t_0,r_1) = \min\{\alpha(\mu),\ \beta(t_0,r)\}\ , \tag{2.22}$$

and define

$$T(t_0,\mu,r) = \frac{\alpha(s)}{\gamma(r_1)} \tag{2.23}$$

Assume $\|\phi(t_n;x_0,t_0)\| = \|x(t_n)\| > r_1$ for some $t_0 \le t_n \le t_0 + T$. Assume further that $T = t_m - t_0$ for some integer $m > 0$. Then for $\|x_0\| \le r(t_0)$, $0 < \alpha(r_1) \le W(t_0 + T, \phi(t_0 + T;x_0,t_0))$, by hypothesis (i). But

$$\begin{aligned} W(t_0 + T,\phi(t_0 + T;x_0,t_0)) &= W(t_0,x_0) + \sum_{n=0}^{m-1} \Delta W(t_n,x)\,(t_{n+1} - t_n) \\ &\le W(t_0,x_0) - \sum_{n=0}^{m-1} \gamma(\|x\|)\,(t_{n+1} - t_n)\ , \text{ by (ii)} \\ &\le W(t_0,x_0) - \sum_{n=0}^{m-1} \gamma(r_1)\,(t_{n+1} - t_n)\ , \end{aligned}$$

since $\|x(t_n)\| > r_1 \Rightarrow \gamma(\|x\|) > \gamma(r_1) \Rightarrow -\gamma(\|x\|) < -\gamma(r_1)$, thus

$$W(t_0 + T,\phi(t_0 + T;x_0),t_0) \le \beta(t_0,r) - \gamma(r_1) \sum_{n=0}^{m-1} (t_{n+1} - t_n), \text{ from (2.19)}$$

$$W(t_0 + T,\phi(t_0 + T;x_0,t_0)) \le \beta(t_0,r) - \gamma(r_1)(t_m - t_0) = \beta(t_0,r) - \gamma(r_1)T$$

$$\le \beta(t_0,r) - \alpha(s) \le 0 \text{ , using (2.22) .}$$

Clearly, $0 < \alpha(r_1) \le \beta(r,t_0) - \alpha(s) \le 0$ is a contradiction. Therefore $\|\phi(t_n;x_0,t_0)\| = \|x(t_n)\| \le r_1$ for some $t_0 \le t_n \le t_0 + T$. We then conclude that for $t_k \ge t_n$,

$$\alpha(\|\phi(t_k;x_0,t_0)\|) \le W(t_k,\phi(t_k;x_0,t_0)) \le W(t_n,\phi(t_n;x_0,t_0)) \le \beta(t_0,r_1) \text{ ,}$$

using (2.21) we see that $\beta(t_0,r_1) \le \alpha(\mu)$, hence $\alpha(\|\phi(t_k;x_0,t_0)\|) \le \alpha(\mu)$, which implies that $\|\phi(t_k;x_0,t_0)\| \le \mu$ for $t_k \ge t_0 + T$, whenever $\|x_0\| \le r(t_0)$.

□

Theorem 2.4. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is uniformly asymptotically stable if

(i) it is uniformly stable (in the sense of Theorem 2.2) and

(ii) there exists a continuous scalar function γ such that $\gamma(0) = 0$ and, for all t_k and $x \ne \theta$, $x \in S_s$

$$\Delta W(t_k,x) \le -\gamma(\|x\|) < 0 \text{ .} \tag{2.24}$$

Proof: Here again, we only need to show the uniform convergence of the motions of (2.6) to the equilibrium point $x_e = 0$, that is, we have to show that $\|\phi(t_k;x_0,t_0)\| \to 0$ as $t_k \to \infty$ uniformly in t_0 whenever $\|x_0\| \leq r$ (r is independent of t_0 and x_0), since uniform stability has already been proved in Theorem 2.2.

From the hypotheses of the theorem, there exists three scalar continuous nondecreasing functions α, β, and γ such that

$$\alpha(0) = \beta(0) = \gamma(0) = 0 \quad \text{and} \quad \forall\, t_k \quad \text{and} \quad \forall\, x \neq \theta,\ x \in S_s$$

$$\alpha(\|x\|) \leq W(t_k,x) \leq \beta(\|x\|) \tag{2.25}$$

$$\gamma(\|x\|) \leq -\Delta W(t_k,x)\ . \tag{2.26}$$

Pick r and r_1 such that

$$\beta(r) = \sup_{\|x\| \leq r} \{\sup_{t_k \geq t_0} \{W(t_k,x)\}\} \leq \alpha(s) \tag{2.27}$$

$$\beta(r_1) = \min\{\alpha(\mu),\beta(r)\}\ . \tag{2.28}$$

Define

$$T = T(\mu) = \frac{\alpha(s)}{\beta(r_1)} > 0 \tag{2.29}$$

As in the case of the proof of the previous theorem, we find that $\|\phi(t_n;x_0,t_0)\| = \|x(t_n)\| \leq \gamma_1$ for some $t_0 \leq t_n \leq t_0 + T$. The difference here is that r is independent of t_0 and T only depends on μ. We therefore have that for $\|x_0\| \leq r$ and $t_k \geq t_n$

$$\alpha(\|\phi(t_k;x_0,t_0)\|) \le W(t_k,\phi(t_k;x_0,t_0)) \le W(t_n,\phi(t_n;x_0,t_0)) \text{ , by(2.24) and (2.25)}$$

$$\le \beta(r_1) \text{ , since } \|x(t_n)\| \le r_1$$

$$\le \alpha(\mu) < \text{ by (2.28)}$$

We conclude that $\|\phi(t_k;x_0,t_0)\| = \|x(t_k)\| \le \mu$ for $t_k \ge t_0 + T(\mu)$, whenever $\|x_0) \le r$, since α is a nondecreasing scalar function, and that $\|\phi(t_k;x_0,t_0)\| \to 0$ as $t_k \to \infty$ uniformly in t_0 when $\|x_0\| \le r$.

□

Theorem 2.5. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is equiasymptotically stable in the large if there exists a scalar function $W(t_k,x)$ which is positive definite for all $x \in \mathbb{R}^n$, radially unbounded, i.e., $\alpha(\|x\|) \le W(t_k,x)$ with $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$, and the rate of increase of W along the motion starting at x, t_k, $\Delta W(t_k,x)$, is negative definite for all $x \ne \theta$, $x \in \mathbb{R}^n$, i.e., $\Delta W(t_k,x) \le -\gamma(\|x\|) < 0$.

Proof: Stability of $x_e = \theta$ was already proved in Theorem 2.1. We therefore proceed as follows. Because $W(t_k,x)$ is radially unbounded, for any constant $B > 0$, $B \in \mathbb{R}$, there exists a $B' > 0$, $B' \in \mathbb{R}$ such that $\alpha(B') > \beta(t_0,B)$. Such a B' can be picked as follows:

$$\text{Let} \qquad \alpha(B') = \min_{\|x\| > B'} \{W(t_0,x)\} \ge \beta(t_0,B) \text{ ,} \tag{2.30}$$

this procedure is illustrated in Figure 2.4.

Now, for $\|x_0\| \le B$ and $t_k \ge t_0$, we have

$$\alpha(B') \geq \beta(t_0,B) \geq W(t_0,x_0) \geq W(t_k,\phi(t_k;x_0,t_0)) \geq \alpha(\|\phi(t_k;x_0,t_0)\|) ,$$

since a negative definite $\Delta W(t_k,x)$ implies that for $t_k \geq t_0$, $W(t_k,\phi(t_k;x_0,t_0)) \leq W(t_0,x_0)$, and the positive definiteness of W implies that $W(t_k,\phi(t_k;x_0,t_0)) \geq \alpha(\|\phi(t_k;x_0,t_0)\|)$. Therefore, $\|\phi(t_k;x_0,t_0)\| \leq B'$ for $t_k \geq t_0$ when every $\|x_0\| \leq B$, in other words, all motions of the system described by (2.6) are bounded.

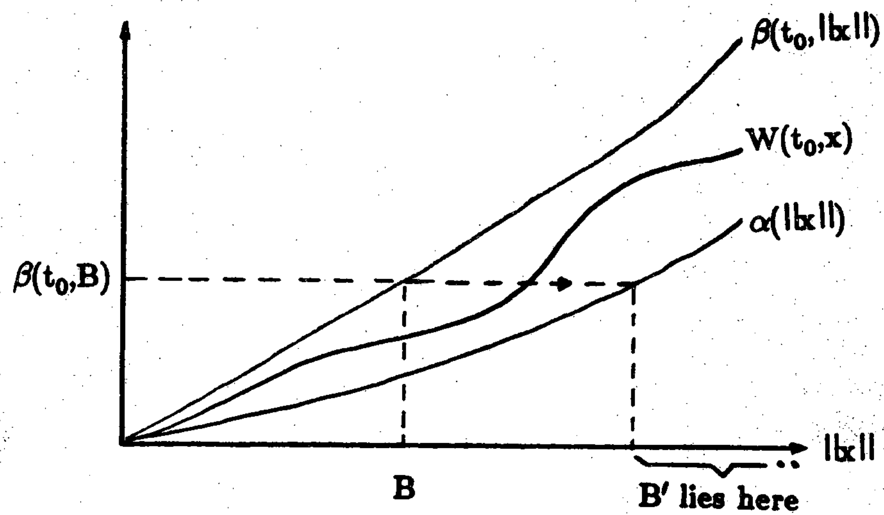

Figure 2.4. The choice of B′.

For any given $\mu > 0$, $\mu \in \mathbb{R}$, choose δ such that

$$\beta(t_0,\delta) \leq \alpha(\mu), \tag{2.31}$$

and define

$$T = \frac{\alpha(B')}{\gamma(\delta)} > 0. \tag{2.32}$$

Using an argument similar to the one used in the last two theorems, we find that if we assume that $\|\phi(t_n;x_0,t_0)\| > \delta$ for some $t_0 \leq t_n \leq t_0 + T$, and $\|x_0\| \leq B$, we get

$$0 < \alpha(\delta) \leq W(t_0 + T, \phi(t_0 + T; x_0, t_0)) \leq W(t_0, x_0) - \gamma(\delta)T \leq \beta(t_0, B) - \alpha(B') \leq 0 ,$$

a contradiction, which implies that $||\phi(t_n;x_0,t_0)|| = ||x(t_n)|| \leq \delta$ for $t_0 \leq t_n \leq t_0 + T$. Now, for $||x_0|| \leq B$ and for $t_k \geq t_n$, we get

$$\alpha(||\phi(t_k;x_0,t_0)||) \leq W(t_k, \phi(t_k;x_0,t_0)) \leq W(t_n, \phi(t_n;x_0,t_0)) \leq \beta(t_0, \delta) \leq \alpha(\mu) ,$$

or that $||\phi(t_k;x_0,t_0)|| = ||x(t_k)|| \leq \mu$ for $t_k \geq t_0 + T$ whenever $||x_0|| \leq B$.

□

Theorem 2.6. The equilibrium point $x_e = \theta$ at time t_0 of (2.6) is uniformly asymptotically stable in the large if in addition to the hypotheses of the previous theorem, $W(t_k,x)$ is decrescent for all $t_k \geq t_0$ and $x \in S_s$.

Proof: Since uniform stability of $x_e = \theta$ has already been proved in Theorem 2.2, we can show that every motion of (2.6) converges to $x_e = \theta$ uniformly in $||x_0|| \leq B$ and t_0, with B fixed but arbitrarily large, as $t_k \longrightarrow \infty$ in the same manner as in the preceding theorem once we choose $B' > 0$ and $\delta > 0$, given $B > 0$ and $\mu > 0$, $B, \mu \in \mathbb{R}$, that is, once we pick B' and δ such that

$$\alpha(B') \geq \beta(B) , \tag{2.33}$$

and

$$\beta(\delta) \leq \alpha(\mu) , \tag{2.34}$$

since the assumptions of the theorem imply the existence of three scalar, continuous, nondecreasing functions α, β and γ such that for $x \neq \theta$, $x \in \mathbb{R}^n$ and $\forall\, t_k$,

$$\alpha(\|x\|) \leq W(t_k,x) \leq \beta(\|x\|)\ , \tag{2.35}$$

$$\gamma(\|x\|) \leq -\Delta W(t_k,x)\ , \tag{2.36}$$

and

$$\alpha(\|x\|) \to \infty \ \text{ as } \ \|x\| \to \infty\ . \tag{2.37}$$

□

2.6. EXTENSIONS OF LYAPUNOV STABILITY THEORY OF DISCRETE-TIME DYNAMICAL SYSTEMS

We now adapt to discrete-time dynamic systems the notions of uniform boundedness and uniform ultimate boundedness of uncertain continuous-time systems which were utilized by Corless and Leitmann [7] in the context of continuous-time dynamical systems.

Definition 2.6.1. The solution of (2.6) are uniformly bounded if and only if given any compact subset S of the state space $\mathbb{R}^n$, there exists $d(S) \in \mathbb{R}_+$ such that if $x(\cdot) : [t_{k_0}, t_{k_1}) \to \mathbb{R}^n$ is any solution of (2.6) with $x_0 = x(t_{k_0}) \in S$, then $\|x(t_k)\| \leq d(S)$ for all $t_k \in [t_{k_0}, t_{k_1})$.

Definition 2.6.2. Given any subset B of the state space $\mathbb{R}^n$, the solutions of (2.6) are uniformly ultimately bounded within B if and only if given any compact subset S of $\mathbb{R}^n$, there exists $T(S,B) \in \mathbb{R}_+$ such that if $x(\cdot) : [t_{k_0}, \infty) \to \mathbb{R}^n$ is any solution of (2.6) with $x_0 = x(t_{k_0}) \in S$, $x(t_k) \in B\ \forall$ $t_k \geq t_{k_0} + T(S,B)$.

2.7. CONCLUSIONS

The application of the second method of Lyapunov to the study of the stability of discrete-time dynamic systems modeled by difference equations clearly shows that uniform asymptotic stability in the large implies equiasymptotic stability in the large and uniform asymptotic stability; uniform asymptotic stability implies equiasymptotic stability and uniform stability. Finally, either uniform stability or equiasymptotic stability implies stability.

As made evident in the above development, Lyapunov's second method has been applied to systems described by the time-varying, generally nonlinear difference equation (2.6). In so far as discrete-time linear time invariant systems are concerned, other well-known tests exist which determine their stability properties in a rather straight forward manner [8,9,10].

CHAPTER III

STABILIZATION OF DISCRETE-TIME DYNAMICAL SYSTEMS VIA PROJECTION METHODS

3.1. INTRODUCTION

We shall look at the problem of stabilizing linear time-invariant discrete dynamical systems and provide a solution based on a nonclassical approach. More precisely, we shall solve the stability problem by steering the state trajectory of the system towards a desired hyperplane and keep it on it until it reaches the origin. The idea behind constraining the system to a particular hyperplane is to reduce the system's dimension and to tailor its stability properties.

The method we shall utilize is based on ideas used in continuous-time variable structure control systems [11,12,13,14,15] and specially from the results on continuous-time dynamical systems with high feedback gain obtained by Utkin [31] and by Marino [32], since these types of systems behave like variable structure systems as the feedback gain becomes large.

We shall first find a solution to the single-input system case and then generalize it to the multiple-input case.

3.2. CONTROLLER DESIGN I

We first consider a single-input linear time-invariant discrete-time dynamical system described by the following difference equation

$$x_{k+1} = Ax_k + Bu_k \ , \ \ x_0 = x_{k_0} \tag{3.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}$, A and B are constant matrices of appropriate dimensions.

Assumption A1. The pair (A,B) is completely controllable, i.e., we can transform (3.1) into the controllable canonical form

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & 1 \\ -a_1 & -a_2 & \dots & \dots & -a_n \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u_k \ . \tag{3.2}$$

Define

$$\sigma_k \triangleq \sigma(x_k) = S\,x_k \ , \tag{3.3}$$

where S is a 1xn matrix whose components are yet to be determined.

Our goal is to drive system (3.1) to the hyperplane $\sigma_k = 0$ as fast as possible and to have it slide on it towards the origin.

Theorem 3.1: If system (3.2) is constrained to the hyperplane $\sigma_k = 0$, then the equivalent system has (n-1)-dimension.

Proof: Without loss of generality, assume that the n^{th} component of S is equal to one, i.e., $s_n = 1$. Then if $\sigma_k = 0$, that is, when the trajectory x_k of system (3.2) reaches the hyperplane $Sx_k = 0$ at the k^{th} step,

$$s_1 x_1(k) + s_2 x_2(k) + \ldots + x_n(k) = 0 ,$$

from which we get

$$x_n(k) = -\, s_1 x_1(k) - s_2 x_2(k) \tag{3.4}$$

Moreover, if system (3.2) remains on $\sigma_k = 0$, then it is also true that $\sigma_{k+1} = 0$, namely,

$$\sigma_{k+1} = Sx_{k+1} = SAx_k + SBu_k^* = 0 ,$$

or

$$u_k^* = -\,(SB)^{-1}SAx_k = -\sum_{i=1}^{n} (a_i + s_{i-1})\, x_i(k) \, , \; s_0 = 0 \, . \tag{3.5}$$

Substituting $u_k = u_k^*$ into (3.2), we get

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & 1 \\ 0 & -s_1 & \ldots & \ldots & -s_{n-1} \end{bmatrix} x_k ,$$

but the n^{th} component of the state vector x_k is given by (3.4), thus

$$x_{k+1}^* = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & 1 \\ -s_1 & -s_2 & \ldots & \ldots & -s_{n-1} \end{bmatrix} x_k^* , \tag{3.6}$$

where

$$x_k^* = [x_1(k) \ldots x_{n-1}(k)]^T .$$

Therefore, the system (3.6), which we shall designate as the equivalent system, is (n-1)-dimensional.

□

Let the function $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be given by

$$V(x_k) \triangleq \sigma^2(x_k) \tag{3.7}$$

where

$$\mathbb{R}_+ = [0,\infty)$$

and $\sigma(x_k)$ is given by (3.3).

Assumption A2. The matrix S is such that its components are chosen to yield an asymptotically stable equivalent system.

We now state the following theorem:

Theorem 3.2: If the matrix $S \in \mathbb{R}^{1\times n}$ is chosen according to assumption (A2), and if the controller

$$u_k = \sum_{i=1}^{n} (\lambda^{k+1} s_i - s_{i-1} - a_i)\, x_i(k) \ , \ s_0 = 0 \ , \tag{3.8}$$

where $\lambda \in (0,1)$, s_i is the i^{th} component of the $1\times n$ matrix S and a_i is the i^{th} element of the last row of the A matrix in (3.2); is applied to system (3.2), then the closed-loop system is asymptotically stable for all $x_k \in \mathbb{R}^n$ and the hyperplane $\sigma(x_k) = 0$ is approached asymptotically for any initial condition $x_0 \notin \mathrm{Ker}(S)$.

Proof: Let $V(x_k)$ as defined above be a generalized Lyapunov function candidate. A sufficient condition for the closed-loop system to be asymptotically stable is that the first forward difference of the generalized Lyapunov function candidate, $\Delta V(x_k)$, be negative for all $x_k \in \mathbb{R}^n$, i.e., we require that (see Chapter 2)

$$\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) < 0 \quad , \quad \forall \, x_k \in \mathbb{R}^n \, .$$

Now,

$$V(x_{k+1}) = \sigma^2(x_{k+1}) \, ,$$

but

$$\begin{aligned} \sigma(x_{k+1}) &= Sx_{k+1} \\ &= SAx_k + SBu_k \, . \end{aligned}$$

Substituting the A and B matrices of (3.2) into the above equation yields

$$\sigma(x_{k+1}) = \sum_{i=1}^{n} (s_{i-1} + a_i)x_k + u_k \tag{3.9}$$

Utilizing the proposed controller (3.8) in (3.9) produces

$$\begin{aligned} \sigma(x_{k+1}) &= \lambda^{k+1} \sum_{i=1}^{n} s_i x_i(k) \\ &= \lambda^{k+1} Sx_k \\ &= \lambda^{k+1} \sigma(x_k) \, . \end{aligned} \tag{3.10}$$

Hence,

$$V(x_{k+1}) = \lambda^{2k+2}\sigma^2(x_k)\ ,$$

and

$$\Delta V(x_k) = (\lambda^{2k+2} - 1)\sigma^2(x_k)\ . \tag{3.11}$$

For $x_k \notin Ker(S)$, namely, when the representative point x_k lies outside the hyperplane $\sigma(x_k) = 0$ or $\sigma(x_k) \neq 0$, then $\Delta V(x_k) < 0$ since $\lambda \in (0,1)$.

For $x_k \in Ker(S)$, i.e., when the representative point x_k lies on the hyperplane $\sigma(x_k) = 0$, we proceed as follows. We first note that (3.8) can be rewritten as

$$u_k = \lambda^{k+1}\sigma(x_k) - \sum_{i=1}^{n} (s_{i-1} + a_i)x_i(k)\ , \tag{3.12}$$

Thus, if $\sigma(x_k) = 0$, then u_k is equal to the equivalent control u_k^*, which is given by (3.5). Additionally, if the components of S are picked according to assumption (A2), then the (n-1)-dimensional equivalent system is asymptotically stable, which implies that the closed-loop system is asymptotically stable for $x_k \in Ker(S)$.

We therefore conclude that if we apply (3.8) to (3.2), the resulting closed-loop system is asymptotically stable for all $x_k \in \mathbb{R}^n$.

To show that the trajectory of the closed-loop system approaches the hyperplane $\sigma(x_k) = 0$ asymptotically for $x_0 \notin Ker(S)$ we note that

$$\sigma(x_{k+1}) = \lambda^{k+1}\sigma(x_k)\ ,$$

which implies that

$$\sigma(x_k) = (\prod_{i=1}^{k} \lambda^i)\sigma(x_0)$$

$$= \lambda^{\sum_{i=1}^{k} i} \sigma(x_0)$$

$$= \lambda^{k(k+1)/2} \sigma(x_0) . \qquad (3.13)$$

Clearly, $\sigma(x_k) \to 0$ as $k \to \infty$ for all $\sigma(x_0) \neq 0$ since $\lambda \in (0,1)$.

□

To shed more light on the claim that the closed-loop system is asymptotically stable for $x_k \in \mathrm{Ker}(S)$, we note that when $u_k = u_k^*$ is applied to (3.2), the resulting system is given by

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & 1 \\ 0 & -s_1 & \dots & \dots & -s_{n-1} \end{bmatrix} x_k , \qquad (3.14)$$

whose characteristics polynomial is

$$p(z) = z(z^{n-1} + s_{n-1}z^{n-2} + \dots + s_2 z + s_1) = zp^*(z) , \qquad (3.15)$$

where $p^*(z)$ is the characteristic polynomial of the equivalent system (3.6).

Therefore, if the s_i's are such that the equivalent system is asymptotically stable, then the closed-loop system (3.14) is asymptotically stable for $x_k \in K(S)$, since $p(z)$ has one extra root at zero, which is clearly inside the unit circle.

Example 3.1: Let system (3.1) be given by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) , \tag{3.16}$$

with open-loop eigenvalues 1+j and 1-j.

When constrained to the desired hyperplane, we would like the first order equivalent system to have an eigenvalue at 0.5.

On the hyperplane $s_1x_1 + s_2x_2 = 0$, we have that

$$x_1^*(k+1) = -\frac{s_1}{s_2} x_1^*(k) , \tag{3.17}$$

By assumption, $s_2 = 1$. Thus, if we choose $s_1 = -0.5$, then the first order equivalent system is given by

$$x_1^*(k+1) = 0.5\ x_1^*(k) , \tag{3.18}$$

which has the desired eigenvalue at 0.5.

We have thus designed the hyperplane to be

$$-0.5x_1 + x_2 = 0 . \tag{3.19}$$

For simulation purposes, we let $\lambda = 0.5$, the controller (3.8) is then given by

$$u(k) = (2 - 0.5(0.5)^{k+1})x_1(k) + (-1.5 + (0.5)^{k+1})x_2(k) , \tag{3.20}$$

and the closed-loop system by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.5(0.5)^{k+1} & 0.5 + (0.5)^{k+1} \end{bmatrix} x(k) . \tag{3.21}$$

Choose $x_0 = [25\ \ 10]^T$. Clearly, $x_0 \notin \text{Ker}\ ([-0.5\ \ 1])$.

Figure 3.1 shows that the hyperplane (3.19) is reached asymptotically as the time index k increases. Note that because of computer word size limitations, the hyperplane (3.19) appears to be reached in a finite number of steps. Fig. 3.2 illustrates the resulting phase plane plot of x_1 and x_2. Finally, Fig. 3.3 shows the time history of the control effort given by eq. (3.20).

We now choose $x_0 = [20 \;\; 10]^T$, $x_0 \in \mathrm{Ker}([-0.5 \;\; 1])$.

Figure 3.4 makes it evident that the representative point x_k slides on the hyperplane $-0.5x_1 + x_2 = 0$ toward the origin. Figure 3.5 shows that the trajectory of the closed-loop system stays on the Kernel of S, $S = [-0.5 \;\; 1]$ for all $k \in \mathbb{N}$. The control effort $u_k = u_k^*$ is shown in Figure 3.6.

Example 3.2: Let system (3.1) now be given by

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 8 & -3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_k \qquad (3.22)$$

with open-loop eigenvalues located at -5, $1 + j$ and $1 - j$.

Again, when constrained to the desired hyperplane, we would like the second order equivalent system to have its two eigenvalues located at 0.2 + j0.5 and 0.2 - j0.5.

On the desired hyperplane $s_1x_1 + s_2x_2 + x_3 = 0$, we have that the equivalent second order system is given by

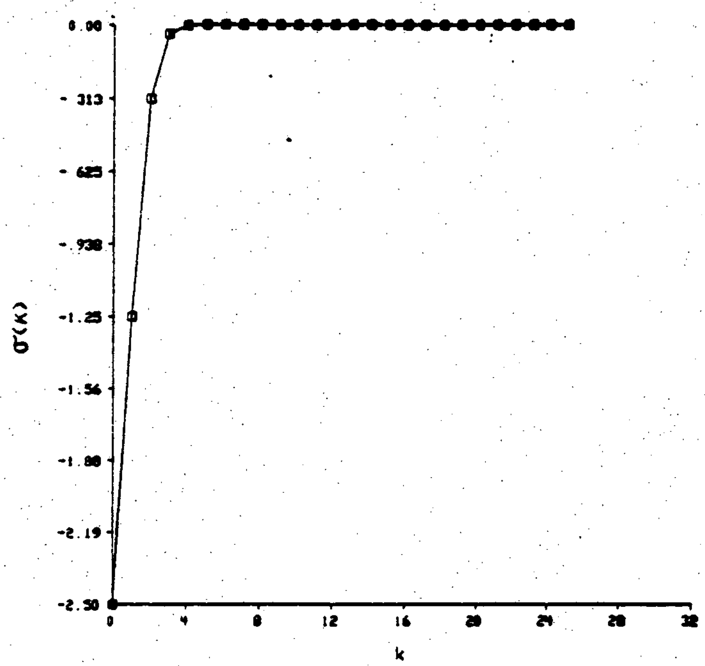

Fig. 3.1. Time history of σ, $\sigma(x_0) = -2.5$.

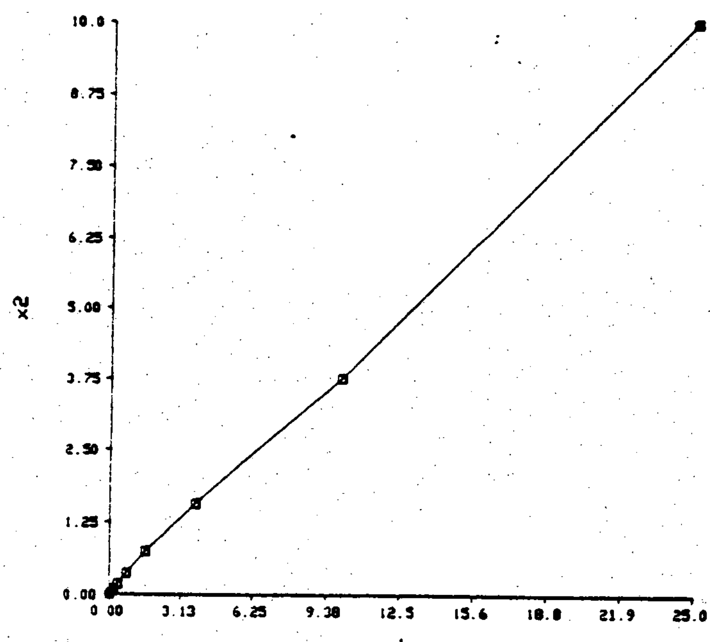

Fig. 3.2. Phase-plane plot of x_1 and x_2, $x_0 \notin \mathrm{Ker}(S)$.

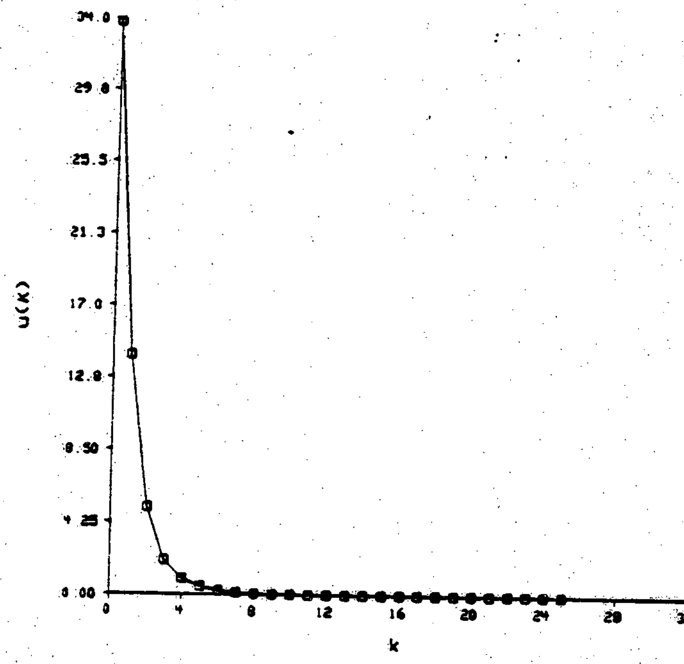

Fig. 3.3. Time history of control effort u_k.

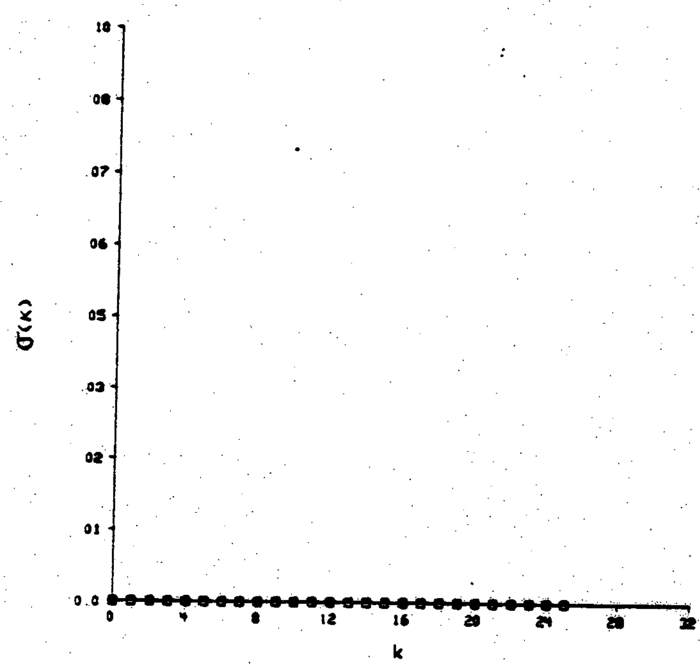

Fig. 3.4. Time history of $\sigma(x_k)$, $x_k \in Ker(S)$.

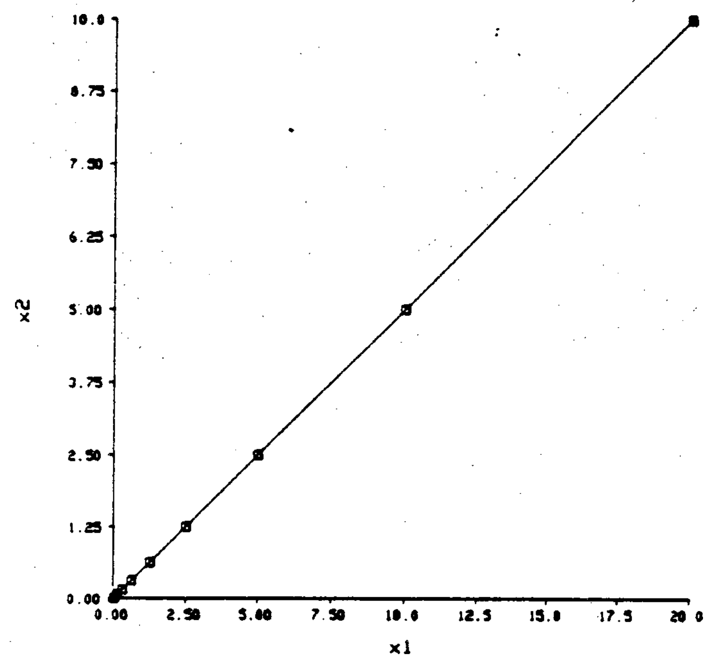

Fig. 3.5. Phase-plane plot of x_1 and x_2.

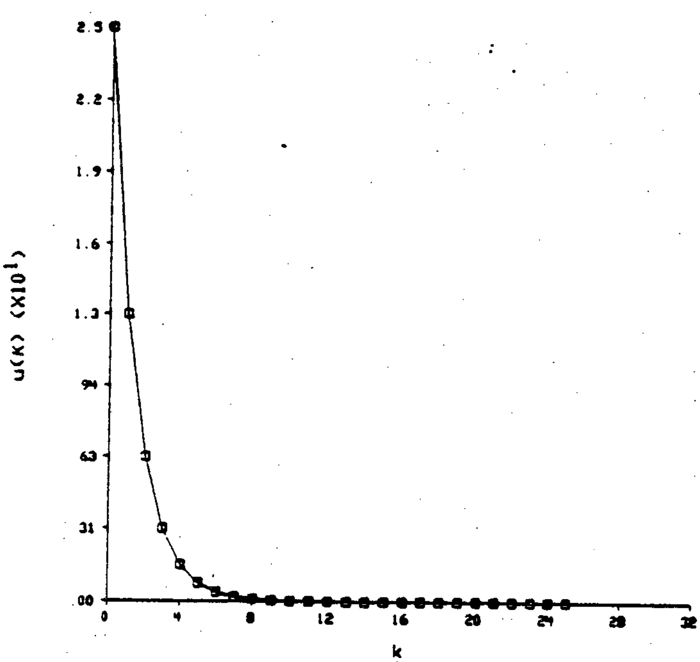

Fig. 3.6. Control effort.

$$x^*_{k+1} = \begin{bmatrix} 0 & 1 \\ -s_1 & -s_2 \end{bmatrix} x^*_k \tag{3.23}$$

with the characteristic polynomial

$$p(z) = z^2 + s_2 z + s_1 \tag{3.24}$$

We can easily show that if we choose $s_1 = 0.29$ and $s_2 = -0.4$, then

$$x^*_{k+1} = \begin{bmatrix} 0 & 1 \\ -0.29 & 0.4 \end{bmatrix} x^*_k \tag{3.25}$$

has the desired eigenvalues at 0.2 + j0.5 and 0.2 - j0.5. Moreover, the desired hyperplane is finally determined to be

$$0.29x_1 - 0.4x_2 + x_3 = 0 \ . \tag{3.26}$$

Again, for simulation purposes, let $\lambda = 0.5$, the controller (3.8) then becomes

$$u_k = (10 + 0.29(0.5)^{k+1})x_1(k) + (-8.29 - 0.4(0.5)^{k+1})x_2(k)$$

$$+ (3.4 + (0.5)^{k+1})x_3(k) \ , \tag{3.27}$$

and the closed-loop system is given by

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.29(0.95)^{k+1} & -0.29 - 0.4(0.95)^{k+1} & 0.4 + (0.95)^{k+1} \end{bmatrix} x_k \ . \tag{3.28}$$

With $x_0 = [25 \ \ 15 \ \ 10]^T$, $x_0 \notin Ker(S)$, $S = [0.29 \ \ -0.4 \ \ 1]$, Figure 3.7 shows that the system trajectory reaches the hyperplane (3.26) asymptotically as the time index k increases. Figures 3.8 and 3.9 show the time history of x_1, x_2, x_3 and u.

3.2.1 Multi-input System Case

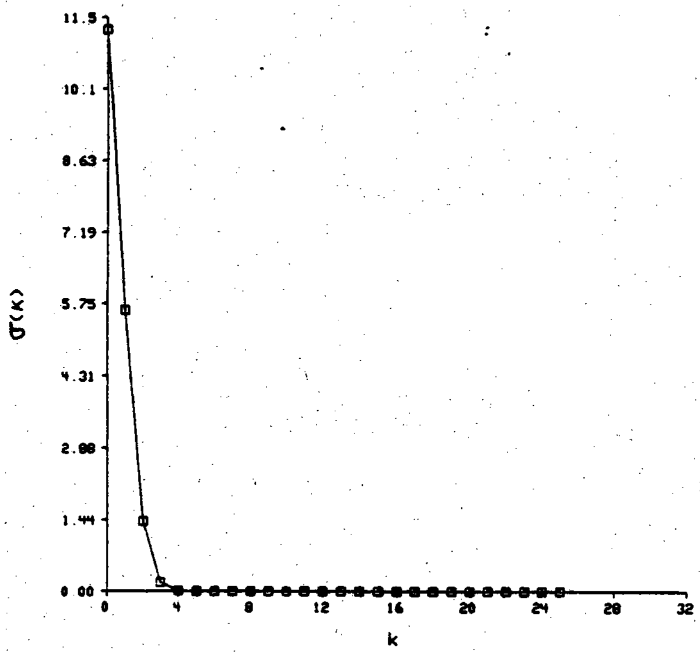

Fig. 3.7. Time history of σ.

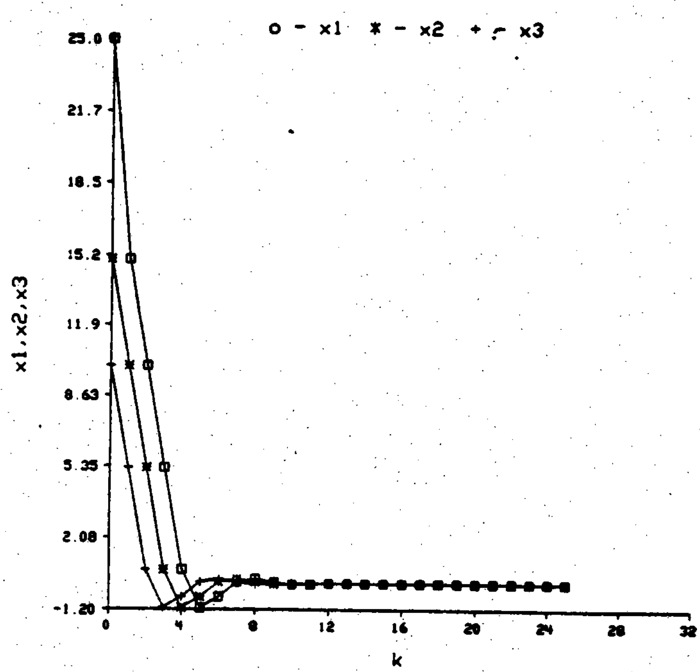

Fig. 3.8. Time history of x_1, x_2, x_3.

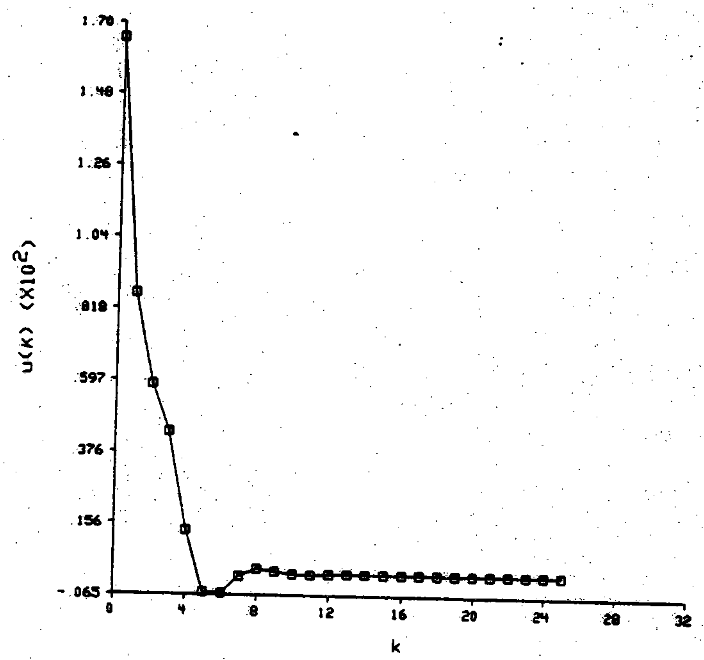

Fig. 3.9. Control effort u_k.

We now consider the case when $u_k \in \mathbb{R}^m$, i.e., when the discrete-time dynamical system is described by

$$x_{k+1} = Ax_k + Bu_k \ , \ \ x_0 = x_{k_0} \ , \tag{3.29}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, A and B are constant matrices, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

We will show in this subsection that the results we obtained for single-input systems can be extended to multi-input systems. Let the generalized Lyapunov function candidate V be given by

$$V(x_k) = \sigma^T(x_k)\sigma(x_k) \ , \tag{3.30}$$

where $\sigma(x_k)$ is given by (3.3), except that $S \in \mathbb{R}^{m \times n}$.

Theorem 3.3: If the pair (A,B) is completely controllable and the matrix $S \in \mathbb{R}^{m \times n}$ is chosen such that when the trajectory of the system is constrained to lie on Ker(S), the $(n-m)^{st}$ order equivalent system is asymptotically stable and $\det(SB) \neq 0$ then the controller

$$u_k = (SB)^{-1}[\Lambda^{k+1}S - SA]x_k , \tag{3.31}$$

where Λ is an m×m real symmetric positive definite convergent matrix (see Appendix A for the definition of a convergent matrix), yields an asymptotically stable closed-loop system whose trajectory reaches the hyperplane Ker(S) asymptotically whenever $x_0 \notin \mathrm{Ker}(S)$.

Proof: Using the same type of reasoning as in the proof of Theorem 3.2, we can show that

$$\Delta V(x_k) = \sigma^T(x_k)\,(\Lambda^{2k+2} - I)\sigma(x_k) . \tag{3.32}$$

where $I = I_n$ is the n×n identity matrix. Clearly, if $x_k \notin \mathrm{Ker}(S)$, i.e., $\sigma(x_k) \neq 0$, then $\Delta V(x_k) < 0$ because $\Lambda^{2k+2} - I$ is a negative definite symmetric matrix, $\forall\ k \in \mathbb{N}$ If, on the other hand, $x_k \in \mathrm{Ker}(S)$, then the $(n-m)^{st}$ order equivalent system is asymptotically stable by assumption.

To show that the hyperplane $\sigma(x_k) = 0$ is reached asymptotically for all $x_0 \notin \mathrm{Ker}(S)$, we have that

$$\sigma(x_{k+1}) = \Lambda^{k+1}\sigma(x_k) ,$$

which yields

$$\sigma(x_k) = \Lambda^{k(k+1)/2}\sigma(x_0) . \tag{3.33}$$

It is evident that if $x_0 \notin \mathrm{Ker}(S)$, then $\sigma(x_k) \to 0$ as $k \to \infty$ since $\sigma(x_0) \neq 0$.

Remark: It is evident that the controller given by equation (3.31) requires the computation of the $(k+1)^{th}$ power of the matrix Λ; however, if we assume that Λ has distinct eigenvalues, it can be easily diagonalized, i.e.,

$$\Lambda = NDN^{-1} , \tag{3.34}$$

where N is a nonsingular similarity transformation and D is a diagonal matrix whose nonzero entries are the eigenvalues of Λ. Furthermore,

$$\Lambda^k = ND^kN^{-1} , \quad k = 0,1,2... \tag{3.35}$$

where

$$D^k = \begin{bmatrix} \lambda_1^k & & & & \\ & \lambda_2^k & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \lambda_m^k \end{bmatrix} . \tag{3.36}$$

Hence, it is not difficult to compute the k^{th} power of Λ in principle (see Appendix B).

Example 3.3: Let us consider the discrete-time dynamical system given by

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 9 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_k \tag{3.37}$$

with eigenvalues -1, 1, 5 and 10.

We would like the second order equivalent system to have eigenvalues at 0.1 and 0.2.

When the trajectory of (3.37) is constrained to lie on Ker(S), we have

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \ ,$$

therefore, we can determine any two variables in terms of the other two. Expressing x_2 and x_4 in terms of x_1 and x_3, we get

$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \frac{1}{s_{12}s_{24} - s_{14}s_{22}} \begin{bmatrix} s_{14}s_{21} - s_{11}s_{24} & s_{14}s_{23} - s_{13}s_{24} \\ s_{11}s_{22} - s_{12}s_{21} & s_{13}s_{22} - s_{12}s_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

and the second order equivalent system is given by

$$\begin{bmatrix} x_1^*(k+1) \\ x_3^*(k+1) \end{bmatrix} = \begin{bmatrix} \dfrac{s_{14}s_{21} - s_{11}s_{24}}{\Delta} & \dfrac{s_{14}s_{23} - s_{13}s_{24}}{\Delta} \\ \dfrac{s_{11}s_{22} - s_{12}s_{21}}{\Delta} & \dfrac{s_{13}s_{22} - s_{12}s_{23}}{\Delta} \end{bmatrix} \begin{bmatrix} x_1^*(k) \\ x_3^*(k) \end{bmatrix} \tag{3.38}$$

where $\Delta \triangleq s_{12}s_{24} - s_{14}s_{22}$.

If we are to place the eigenvalues of the second order equivalent system at 0.1 and 0.2, the following choice of S will yield such eigenvalues

$$S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1.32 & 0 & -1.3 & 1 \end{bmatrix} . \tag{3.39}$$

The equivalent system (3.38) becomes

$$x_{k+1}^* = \begin{bmatrix} -1 & -1 \\ 1.32 & 1.3 \end{bmatrix} x_k^* . \tag{3.40}$$

We note that with the above choice of S, SB = I_2 implies det(SB) = 1.

Let

$$\Lambda \triangleq \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in (0,1)$$

then controller (3.31) is explicitly given by

$$u_k = \begin{bmatrix} \lambda_1^k + 5 & \lambda_1^k - 7 & \lambda_1^k - 1 & -2 \\ -1.32\lambda_2^k & 1.32 & -1.3\lambda_2^k - 10 & \lambda_2^k - 7.7 \end{bmatrix} x_k \cdot$$

substituting the above controller into (3.37) yields the following closed-loop system

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_1^k & \lambda_1^k - 1 & \lambda_1^k & -1 \\ 0 & 0 & 0 & 1 \\ -1.32\lambda_2^k & 1.32 & -1.3\lambda_2^k & \lambda_2^k + 1.3 \end{bmatrix}$$

Figures 3.10, 3.11 and 3.12 show the results of the discrete-time domain simulation when $x_0 = [5\ -1\ 2\ 1]^T$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.4$.

Although controllers (3.8) and (3.31) drive systems (3.1) and (3.29) toward the desired hyperplanes asymptotically and in the direction of the origin, they have the drawback that they are dependent on the time index k, thus presenting practical limitations when implemented on a digital computer with finite word size (which is the case in real life). This problem is made evident by the fact that after a finite number of iterations λ^k and the entries of Λ^k can no longer be represented by a finite word size computer because they become very small numbers.

We now introduce a controller which is a variation of the one just discussed, but one that can be easily implemented on a finite word size

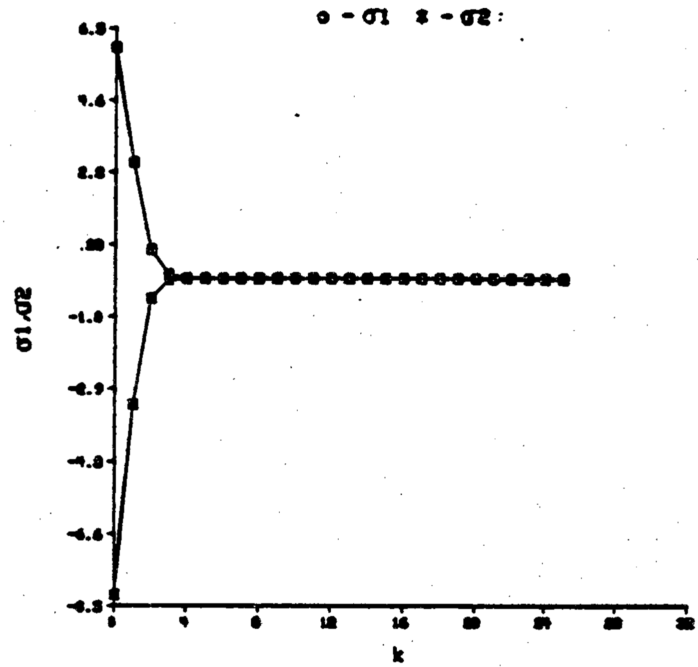

Fig. 3.10. Time history of σ_1 and σ_2.

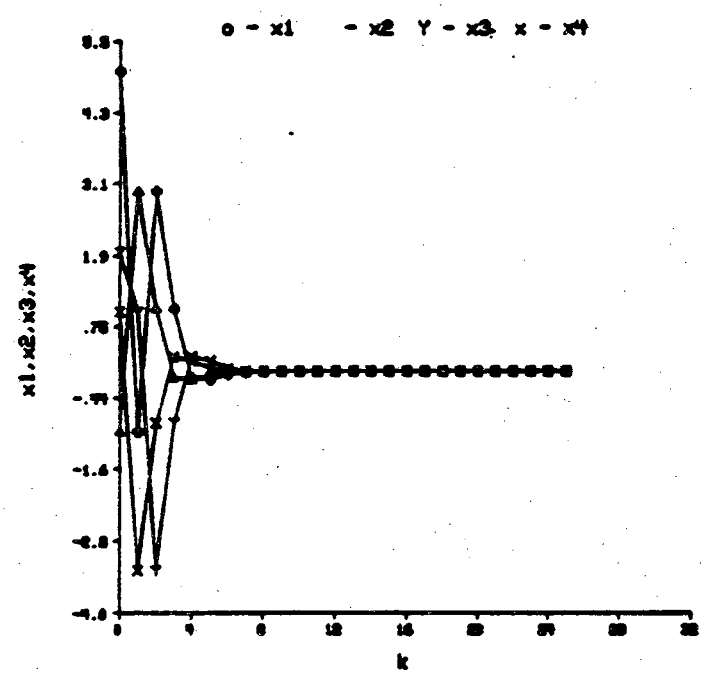

Fig. 3.11. Time history of states x_1, x_2, x_3 and x_4.

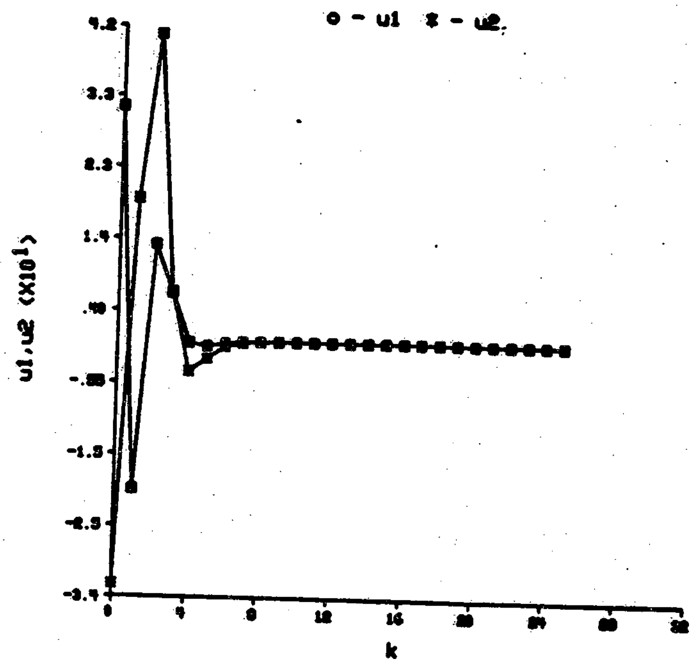

Fig. 3.12. Control efforts u_1 and u_2.

digital computer.

3.3. CONTROLLER DESIGN II

We again consider a single-input linear time invariant discrete-time dynamical system described by (3.1), and assume that (A1) is true, i.e., the pair (A,B) in (3.1) is completely controllable.

Our goal here is to design an alternative controller that does not depend explicitly on the time index k, and which yields a closed-loop system whose characteristics are similar to the one that resulted when controller (3.8) was used.

Theorem 4.4: If the matrix $S \in \mathbb{R}^{1\times n}$ is chosen in accordance with assumption (A2) and if the controller

$$u_k = \lambda\sigma(x_k) - \sum_{i=1}^{n} (s_{i-1} + a_i)x_i(k)\ , \quad s_0 = \mathbf{0}\ , \tag{3.41}$$

where $\lambda \in (0,1)$, s_i is the i^{th} component of the $1\times n$ matrix s and a_i is the i^{th} element of the last row of the A matrix in (3.2); is applied to system (3.2), then the closed-loop system is asymptotically stable for all $x_k \in \mathbb{R}^n$ and the hyperplane $\sigma(x_k) = 0$ is approached asymptotically for any initial condition $x_0 \notin Ker(S)$.

Proof: To prove the above theorem, we proceed in the same manner as in the proof of Theorem 3.2.

Let the generalized Lyapunov function candidate be

$$V(x_k) \triangleq \sigma^2(x_k)\ ,$$

and

$$\Delta V(x_k) = \sigma^2(x_{k+1}) - \sigma^2(x_k) .$$

Now, it can be easily shown that

$$\sigma(x_{k+1}) = \lambda\sigma(x_k) , \tag{3.42}$$

thus

$$\Delta V(x_k) = (\lambda^2 - 1)\, \sigma^2(x_k) . \tag{3.43}$$

Again, if $x_k \notin \mathrm{Ker}(S)$, i.e, $\sigma(x_k) \neq 0$, then $\Delta V(x_k) < 0$, because $\lambda^2 < 1$. Thus the closed-loop system is asymptotically stable for $x_k \notin \mathrm{Ker}(S)$.

If, on the other hand, $x_k \in \mathrm{Ker}(S)$, that is, $\sigma(x_k) = 0$, then the controller given by (3.41) becomes the equivalent control u_k^*, which when applied to system (3.2) results in the closed-loop system given by (3.14), which is asymptotically stable, provided that S is chosen according to assumption (A2).

Finally, if the initial condition x_0 does not lie on the hyperplane $\sigma(x_k) = 0$, then the representative point of the closed-loop system approaches such a hyperplane asymptotically as the time index k increases because

$$\sigma(x_k) = \lambda^k \sigma(x_o) . \tag{3.44}$$

We can see that $\sigma(x_k) \longrightarrow 0$ as $k \longrightarrow \infty$, because $\lambda \in (0,1)$, for all $x_0 \notin \mathrm{Ker}(S)$.

If we now compare (3.44) with (3.13) we notice that controller (3.8) yields a closed-loop system whose trajectory reaches the hyperplane $\sigma(x_k) = 0$ faster than when controller (3.41) is applied to the same system, however, the latter does not depend on the time index k, thus making it more amenable to implement.

Example 3.4: Let us look at the same system we considered in Example 3.1, i.e.,

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) ,$$

with open-loop eigenvalues located at 1 + j and 1 - j.

It is straightforward to show that if we wish the first order equivalent system constrained to the subspace Ker(S), $S = [s_1 \;\; 1]$, to have its eigenvalue at 0.5, then $s_1 = -5$.

Writing (3.41) in an explicit form, we get

$$u(k) = (2 - 0.5\lambda)x_1(k) - (1.5 - \lambda)x_2(k), \; \lambda \in (0,1) . \tag{3.45}$$

The closed-loop system is

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.5\lambda & \lambda + 0.5 \end{bmatrix} x(k) . \tag{3.46}$$

Figures 3.13, 3.14 and 3.15 show the results of the simulation of system (3.46) for $\lambda = 0.5$ and $x_0 = [25 \;\; 10]^T$. Figure 3.13 illustrates how the hyperplane $-5x_1 + x_2 = 0$ is approached by the representative point. Figure 3.14 depicts the progress of $\sigma(x_k)$ towards zero. Finally, Figure 3.15 shows the time history of the control effort.

3.3.1. Multi-input System Case

The results obtained for the single-input case can now be extended to the multiple-input case.

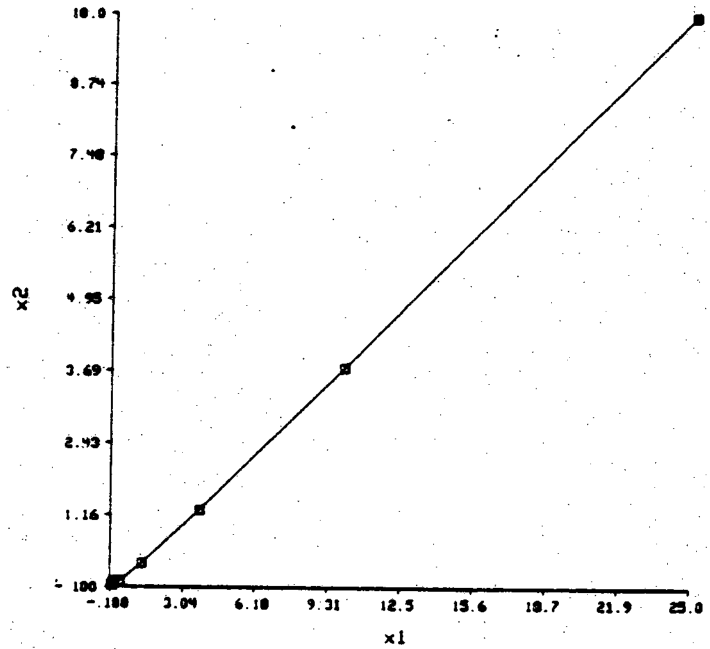

Fig. 3.13. Phase-plane plot of x_1 and x_2.

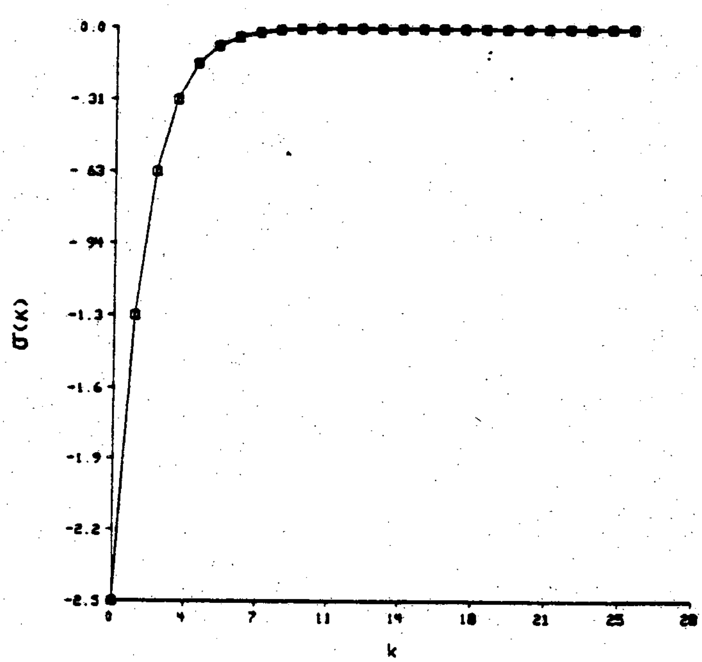

Fig. 3.14. Time history of $\sigma(x_k)$.

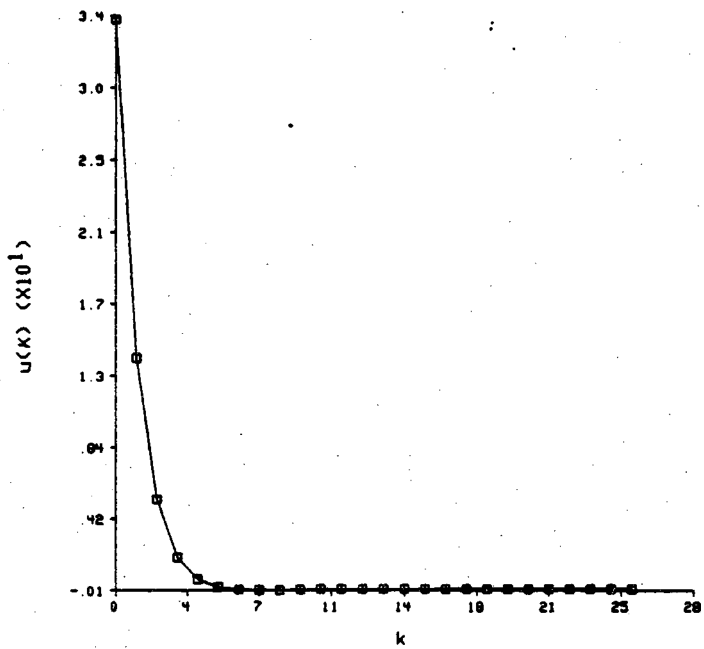

Fig. 3.15. Control effort u_k.

Let $u_k \in \mathbb{R}^m$ and define the generalized Lyapunov function candidate V by

$$V(x_k) \triangleq \sigma^T(x_k)\sigma(x_k) , \tag{3.30}$$

where $\sigma(x_k) \in \mathbb{R}^m$, and

$$\sigma(x_k) \triangleq Sx_k , \tag{3.3}$$

$S \in \mathbb{R}^{m \times n}$ is a constant matrix such that $\det(SB) \neq 0$.

Again, using Lyapunov's second method for stability of discrete-time dynamical systems we prove the following theorem.

Theorem 3.5: Assume there is a controller u_k such that

$$\sigma(x_{k+1}) = \Lambda\sigma(x_k)\ , \tag{3.47}$$

where $\Lambda \in \mathbb{R}^{m\times m}$ is a real symmetric positive definite convergent matrix. Then such a controller when applied to the system

$$x_{k+1} = Ax_k + Bu_k\ , \tag{3.29}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, A and B are constant matrices of appropriate dimensions, yields an asymptotically stable closed-loop system on $\mathbb{R}^n \backslash \mathrm{Ker}(S)$. Moreover, this controller is given by

$$u_k = (SB)^{-1}(\Lambda S - SA)x_k\ , \tag{3.48}$$

provided that $\det(SB) \neq 0$ and S is picked according to assumption A2.

Proof: To show that the application of a controller with the above properties to system (3.29) yields a closed-loop asymptotically stable, it is sufficient to show that $\Delta V(x_k)$, the first forward difference of the Lyapunov function candidate be less than zero. Specifically,

$$\begin{aligned}\Delta V(x_k) &= \sigma^T(x_{k+1})\sigma(x_{k+1}) - \sigma^T(x_k)\sigma(x_k) \\ &= \sigma^T(x_k)\Lambda^2\sigma(x_k) - \sigma^T(x_k)\sigma(x_k) \\ &= \sigma^T(x_k)\,(\Lambda^2 - I_m)\sigma(x_k)\end{aligned} \tag{3.49}$$

Clearly, $\Lambda^2 - I_m < 0$, i.e., $\Lambda^2 - I_m$ is negative definite. Now, for $x_k \notin \mathrm{Ker}(S)$ $\sigma(x_k) \neq 0$ which implies that $\Delta V(x_k) < 0$, $\forall\ x_k \notin \mathrm{Ker}(S)$.

From (3.47),

$$\sigma(x_{k+1}) = Sx_{k+1} = SAx_k + SBu_k = \Lambda Sx_k ,$$

assuming that $\det(SB) \neq 0$, we have

$$u_k = (SB)^{-1}(\Lambda S - SA)\, x_k .$$

Thus, controller (3.48) yields an asymptotically stable closed-loop system for $x_k \in \mathbb{R}^n \backslash Ker(S)$.

□

Theorem 3.6: Assume now that system (3.29) is constrained to the subspace Ker(S), then the $(n-m)^{th}$ order equivalent system is asymptotically stable and the controller (3.48) asymptotically stabilizes (3.29) on Ker(S).

Proof: For $x_k \in Ker(S)$,

$$u_k = -\,(SB)^{-1}SAx_k = u_k^* , \tag{3.50}$$

because $(SB)^{-1}\Lambda Sx_k = 0$.

Therefore,

$$x_{k+1} = [I - B(SB)^{-1}S]Ax_k = A_{eq}x_k , \tag{3.51}$$

for all $x_k \in Ker(S)$.

But according to assumption A2, S is chosen such that the $(n-m)^{th}$ order equivalent system is asymptotically stable. Thus, (3.29) is asymptotically stable on Ker(S) when we apply controller (3.48) to it.

We conclude from Theorems 5 and 6 that controller (3.48) asymptotically stabilizes (3.29) on $\mathbb{R}^n$.

Example 3.5: Let us again consider the discrete-time dynamical system

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 9 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_k , \qquad (3.37)$$

with open-loop eigenvalues located at -1, 1, 5 and 10.

If, as in the case of Example 3.3, we are to place the eigenvalues of the second order equivalent system at 0.1 and 0.2, the following choice of S will yield such eigenvalues

$$S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1.32 & 0 & -1.3 & 1 \end{bmatrix} .$$

The second order equivalent system is again given by

$$\overset{*}{x}_{k+1} = \begin{bmatrix} -1 & -1 \\ 1.32 & 1.3 \end{bmatrix} \overset{*}{x}_k .$$

For simplicity, let

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} , \quad \lambda_1, \lambda_2 \in (0,1) .$$

We then have

$$u_k = \begin{bmatrix} \lambda_1 + 5 & \lambda_1 - 7 & \lambda_1 - 1 & -2 \\ -1.32\lambda_2 & 1.32 & -1.3\lambda_2 - 10 & \lambda_2 - 7.7 \end{bmatrix} x_k .$$

Application of the above controller to system (3.37) yields

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_1 & \lambda_1 - 1 & \lambda_1 & -1 \\ 0 & 0 & 0 & 1 \\ -1.32\lambda_2 & 1.32 & -1.3\lambda_2 & \lambda_2 + 1.3 \end{bmatrix} x_k \, .$$

One can find that the eigenvalues of above closed-loop system are located at 0.1, 0.2, λ_1 and λ_2. Hence it is asymptotically stable since $\lambda_1, \lambda_2 \in (0,1)$.

For the purposes of simulation, let $x_0 = [5 \ \ -1 \ \ 2 \ \ 1]^T$ and $\lambda_1 = 0.5$ and $\lambda_2 = 0.4$. Fig. 3.16 shows that the surfaces $\sigma_1(x_k) = 0$ and $\sigma_2(x_k) = 0$ are reached asymptotically. Figures 3.17 and 3.18 show the time histories of the states and the control effort, respectively.

3.4. CONTROLLER DESIGN III

We now introduce a controller that enables the trajectory of the system given by equation (3.29) to reach the hyperplane $\sigma(x_k) = 0$ in a single step and keeps it on it until the origin is reached.

Theorem 3.7: If $\det(SB) \neq$ and S is chosen according to assumption A2, then the controller

$$u_k = -(SB)^{-1}SAx_k \, , \tag{3.52}$$

yields an asymptotically stable closed-loop system when applied to system (3.29) and the hyperplane $\sigma(x_k) = 0$ is reached in one step for all $x_0 \notin \mathrm{Ker}(S)$ and the trajectory x_k slides toward the origin thereafter.

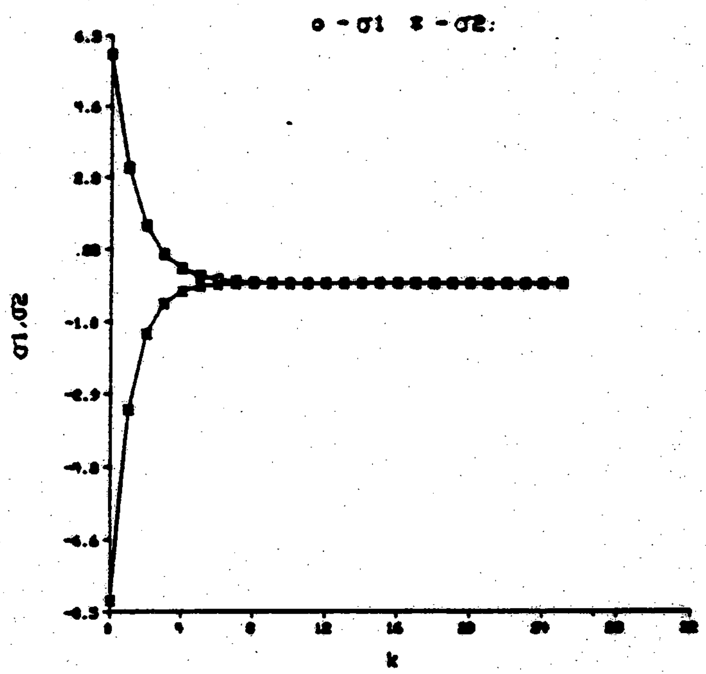

Fig. 3.16. Time history of $\sigma_1(x_k)$ and $\sigma_2(x_k)$.

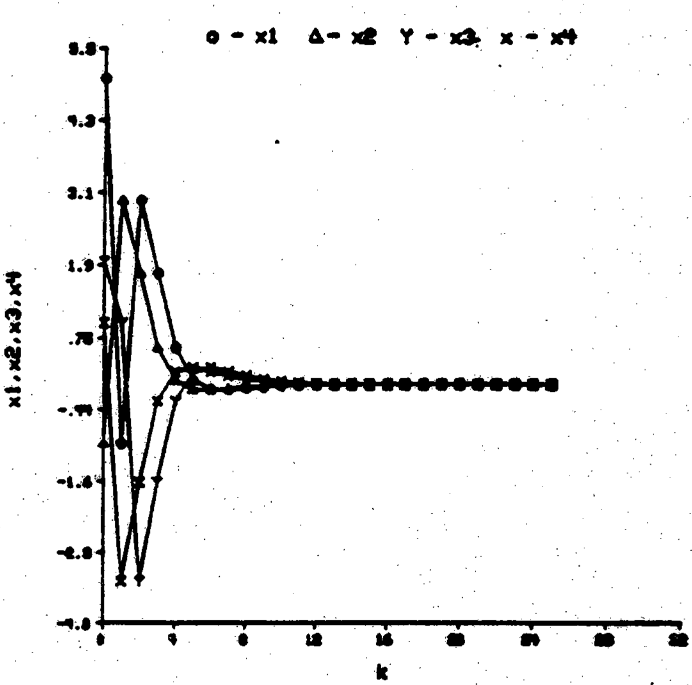

Fig. 3.17. Time history of states x_1, x_2, x_3 and x_4.

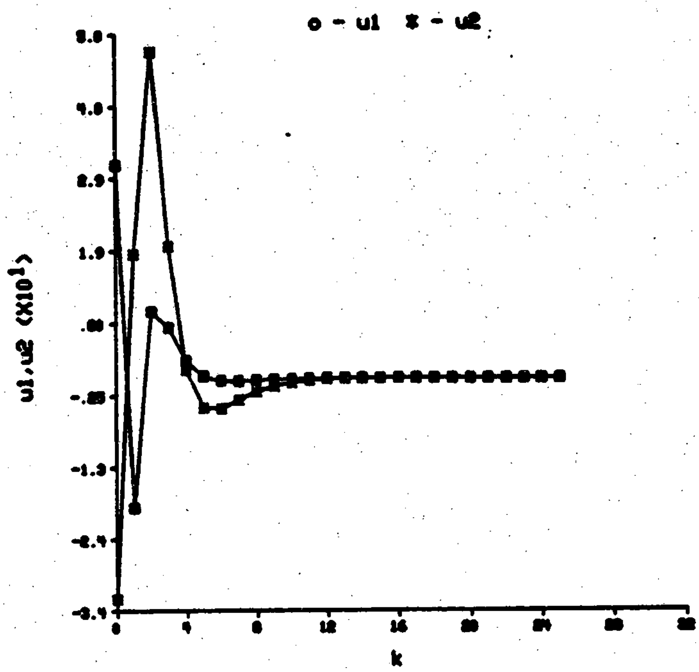

Fig. 3.18. Time history of controls u_1 and u_2.

Proof: Direct substitution of the controller (3.52) into system (3.29) yields a closed-loop system with characteristic polynomial given by

$$p(z) = z^m(z^{n-m} + c_{n-m-1}z^{n-m-1} + \dots + c_1 z + c_0)\ ,$$
$$= z^m p^*(z)\ , \qquad (3.53)$$

where $p^*(z)$ is the characteristic polynomial of the equivalent $(n-m)^{st}$ order system, which by the hypothesis of the theorem, is asymptotically stable. Therefore, p(z) contains m roots at zero and n-m roots located strictly inside the unit circle. Hence, the closed-loop is asymptotically stable.

Now, for any initial condition x_0 outside the hyperplane $\sigma(x_k) = 0$, i.e., $x_0 \in \mathbb{R}^n \backslash Ker(S)$, we have that when we apply the control $u_0 = -(SB)^{-1}SAx_0$

to system (3.29), we get

$$x_1 = [I - B(SB)^{-1}S]Ax_0 \ ,$$

but

$$\sigma(x_1) = Sx_1 = S[I - B(SB)^{-1}S]Ax_0$$

$$= 0 \ .$$

Hence, $x_1 \in Ker(S)$ means that the hyperplane $\sigma(x) = 0$ is reached in one step when $x_0 \notin Ker(S)$ and controller (3.52) is applied to (3.29).

It is now easy to see that once the trajectory x_k of (3.29) reaches the hyperplane $\sigma(x_k) = 0$, that controller (3.52) maintains it on it as it moves toward the origin since the closed-loop system is asymptotically stable.

□

Example 3.6: Suppose now that system (3.29) is the same as that considered in Examples 3.3 and 3.5, i.e., the system is given by equation (3.37). The simulation below assumes that $x_0 = [5 \ \ -1 \ \ 2 \ \ 0]^T$. Figure 3.19 clearly shows that $\sigma(x_k) = 0$ is reached in one step and that control (3.52) keeps the trajectory of (3.37) on Ker(S) where S is given by eq. (3.39). Figures 3.20 and 3.21 display the time histories of the states and the control effort, respectively.

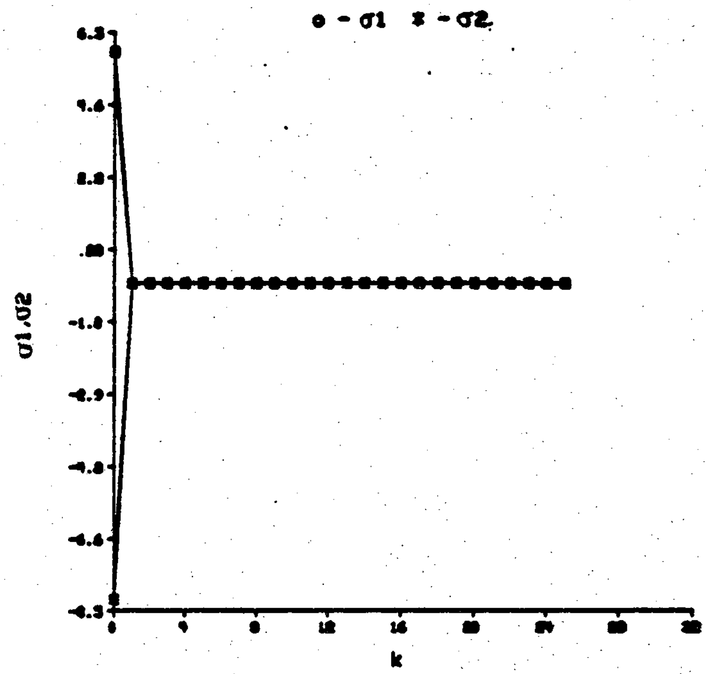

Fig. 3.19. Time history of $\sigma_1(x_k)$ and $\sigma_2(x_k)$.

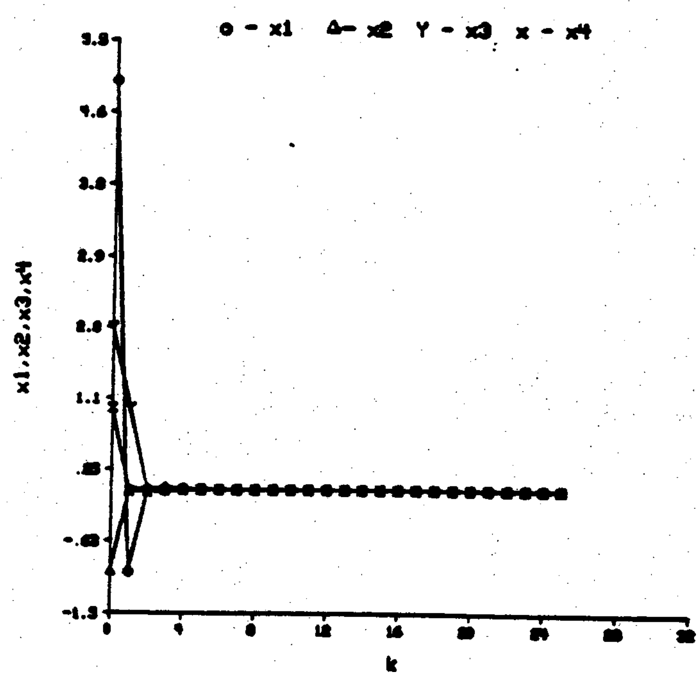

Fig. 3.20. Time history of x_1, x_2, x_3 and x_4.

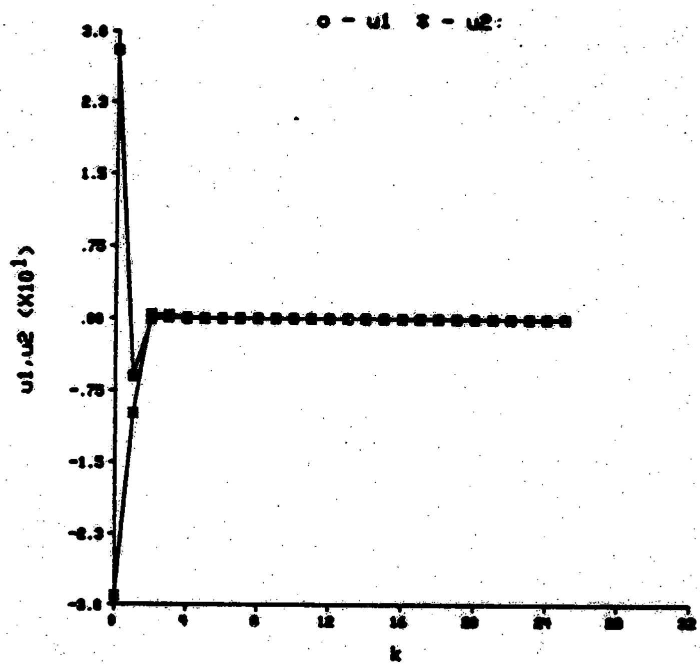

Fig. 3.21. Control efforts u_1 and u_2.

3.5. HYPERPLANE DESIGN

A natural question which arises when using controller (3.1) is: How does one choose the components of S? In other words, how do we design the hyperplane $\sigma(x) = 0$?

3.5.1. Projections

The theory of projections offers an attractive way to design such a hyperplane [17]. We first introduce the definition of a projection and describe its properties [16].

Definition: Given a decomposition of space $\mathcal{D}$ into subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ such that for any $x \in \mathcal{D}$

$$x = x_1 + x_2 \; ; \; x_1 \in \mathcal{D}_1, \, x_2 \in \mathcal{D}_2 \tag{3.54}$$

the linear operator L that maps x into x_1 is called a projection on $\mathcal{D}_1$ along $\mathcal{D}_2$, that is,

$$Lx = x_1 \, , \; Lx_2 = 0 \tag{3.55}$$

2.5.1.1. Properties of projections

(i) A linear operator L is a projection if and only if it is idempotent, i.e., if

$$L^2 = L \tag{3.56}$$

(ii) If L is a projection on $\mathcal{D}_1$ along $\mathcal{D}_2$, then I-L is a projection on $\mathcal{D}_2$ along $\mathcal{D}_1$.

(iii) If L is a projection on Range (L) along Ker(L), then I-L is a projection on Ker(L) along Range (L), where I is an identity matrix.

We therefore have that if $x \in$ Range (L), then

$$Lx = x \tag{3.57}$$

$$(I - L)\,x = x - Lx = x - x = 0 \tag{3.58}$$

Moreover,

$$\text{rank}(L) = \text{trace}(L) \tag{3.59}$$

$$\text{rank}(I - L) = n - \text{rank}(L) \tag{3.60}$$

$$\text{Range}(L) = \text{Ker}(I - L) \tag{3.61}$$

$$\text{Ker}(L) = \text{Range}(I - L) \tag{3.62}$$

Claim [17]: $B(SB)^{-1}S$ and $I - B(SB)^{-1}S$ are projections.

Proof:

We have

$$[B(SB)^{-1}S]^2 = B(SB)^{-1}S\,B(SB)^{-1}S = B(SB)^{-1}S\ ,$$

hence $B(SB)^{-1}$ is idempotent and therefore a projection. Moreover, $B(SB)^{-1}S$ projects $\mathbb{R}^n$ on Range(B) along Ker(S), since

$$\text{range}[B(SB)^{-1}S] = \text{range}(B)\ , \tag{3.63}$$

assuming that B and (SB) are of full rank. Likewise,

$$\text{Ker}[B(SB)^{-1}S] = \text{Ker}(S)\ , \tag{3.64}$$

assuming that $B(SB)^{-1}$ and S are of rull rank.

Now,

$$[I - B(SB)^{-1}S]^2 = I - B(SB)^{-1}S\ ,$$

thus, $I - B(SB)^{-1}S$ is a projection. Furthermore, $I - B(SB)^{-1}S$ projects $\mathbb{R}^n$ on Ker(S) along Range(B).

□

3.5.2. Application of Projections to Systems Constrained to Ker(S)

When the system

$$x(k+1) = Ax(k) + Bu(k)\ , \tag{3.65}$$

$x(k) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are constant matrices of appropriate dimenstions, is constrained to the subspace Ker(S), $S \in \mathbb{R}^{m \times n}$, then

$$u(k) = -\,(SB)^{-1}SAx(k)\ , \tag{3.66}$$

and the dynamics of (3.65) on Ker(S) are governed by

$$x(k+1) = [I - B(SB)^{-1}S]Ax(k) \tag{3.67}$$

Using the results of the previous subsection, we note that $I - B(SB)^{-1}S$ maps the columns of A on Ker(S). The order of system (3.65) has therefore been reduced because $x(k) \in Ker(S)$, which is an $(n-m)^{th}$ dimensional subspace, since $rank(I - B(SB)^{-1}S) = n - rank(B(SB)^{-1}S) = n-m$, which is spanned by the eigenvectors $v_1, v_2, \ldots, v_{n-m}$.

Before we proceed with the computation of the components of S, we will study the relationship between the eigenvector matrix $V = [v_1\ v_2\ \ldots\ v_{n-m}]$ of $[I - B(SB)^{-1}S]A$ the input matrix B and the projection $L = B(SB)^{-1}S$ along with the generalized inverses of V and B.

Theorem 3.8 [17]: The eigenvector matrix V of $[I - B(SB)^{-1}S]A$ is independent of the columns of B, that is, $range(V) \cap range(B) = \{\underset{\sim}{0}\}$, where $\underset{\sim}{0}$ is the zero vector.

Proof: The existence of $(SB)^{-1}$ implies that the columns of B are independent of Ker(S). But, the columns of V are in Ker(S), as Ker(S) is spanned by v_1, v_2, ..., v_{n-m}, hence, range(V)$\cap$range(B) $= \underset{\sim}{0}$.

□

Theorem 3.9 [17]: On the subspace Ker(S), the generalized inverses of the input matrix B and the eigenvector matrix V of $I - B(SB)^{-1}S$ should satisfy the following relations

$$B^g V = 0 \tag{3.68}$$

and

$$V^g B = 0 \tag{3.69}$$

where B^g and V^g are left generalized inverses of B and V, respectively.

Proof: As shown before, range$(B(SB)^{-1}S)$ = range(B) and the columns of V lie in Ker$(B(SB)^{-1}S)$ = Ker(S), thus with $L = B(SB)^{-1}S$

$$L[B \vdots V] = [B \vdots 0] , \tag{3.70}$$

since the columns of B lie in the range space of L and the columns of V lie in the null space of L. Because of the fact that range(V)$\cap$range(B) $= \underset{\sim}{0}$, the inverse of $[B \vdots V]$ always exists, thus

$$L = [B \vdots 0] \, [B \vdots V]^{-1} \tag{3.71}$$

Since $[B \vdots V]$ is an n×n nonsingular matrix (assuming B is of full rank), then

$$[B \vdots V]^{-1} \, [B \vdots V] = I \, . \tag{3.72}$$

Furthermore, it can be shown that

$$[B \vdots V]^{-1} = \begin{bmatrix} B^g \\ \cdots \\ V^g \end{bmatrix}, \tag{3.73}$$

namely,

$$\begin{bmatrix} B^g \\ \cdots \\ V^g \end{bmatrix} [B \vdots V] = \begin{bmatrix} B^gB \vdots B^gV \\ \cdots \vdots \cdots \\ V^gB \vdots V^gV \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{m-m} \end{bmatrix},$$

as B^g and V_g are the left generalized inverses of B and V, respectively. Therefore, conditions (3.68) and (3.69) are satisfied.

□

We infer from (3.70) in the last theorem that

$$L = BB^g, \tag{3.74}$$

subject to $B^gV = 0$, or we could opt to compute the inverse of $[B \vdots V]$ as in (3.71).

3.5.3. Computation of the Eigenvector Matrix V

Although the knowledge of the eigenvector matrix V is presupposed in the previous discussion, nothing has been said as to how to compute it.

When dealing with a linear-time invariant system like (3.65) it is well known that if

$$u(k) = Gx(k) \tag{3.75}$$

then

$$(A + BG)V = VJ \tag{3.76}$$

where G is an m×n matrix chosen such at that A + BG has the desired eigenvalues specified by J [18].

Rewriting (3.76) we have

$$AV - VJ = BGV\ , \tag{3.77}$$

which implies that the columns of AV - VJ are in the range of B provided that the rank of G is m. As a consequence of this we have that [19]

$$AV - VJ = BT \tag{3.78}$$

where T is an arbitrary m×(n−m) matrix that provides linear combinations of the columns of B in such a way as to influence the solution V and provide partial control over the n-m eigenvectors of V. In addition, the columns of V have to satisfy

$$\text{Range(V)} \cap \text{Range(B)} = \{\underset{\sim}{0}\} \tag{3.79}$$

3.5.4. Computation of the Matrix S.

We have now come to the point where the previous lengthy development of projections is more than justified, namely, the computation of S using the theory of projections. In what follows, two methods will be discussed [17].

Method 1:

Let the matrix S satisfy

$$SB = F \tag{3.80}$$

where F is an arbitrary m×m nonsingular matrix and

$$SV = \mathbf{0} \tag{3.81}$$

Clearly, requirement (3.81) is a direct consequence of the fact that we want the columns of V to be in the null space of S.

Recalling that

$$L = B(SB)^{-1}S = BB^{g} \tag{3.74}$$

then

$$BF^{-1}S = BB^{g} \tag{3.82}$$

Premultiplying (3.82) by B^{g}, we get

$$F^{-1}S = B^{g}$$

thus,

$$S = FB^{g} \tag{3.83}$$

3.5.5. Examples

Example 3.7: Suppose we want the system

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & 8 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)\ , \tag{3.84}$$

with open-loop poles at -5, $1 \pm j$, to have closed-loop poles at $0.2 \pm j0.5$

when constrained to the subspace Ker(S),

$$S = [s_1 \;\; s_2 \;\; s_3] \; . \tag{3.85}$$

In other words, we want to find S such that it will assign the eigenvalues specified by J to $[I - B(SB)^{-1}S]A$ according to (3.76).

The matrix J is given by

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{3.86}$$

where $\lambda_1 = 0.2 + j0.5$ and $\lambda_2 = 0.2 - j0.5$.

Let

$$T = [1 \;\; -1] \; , \tag{3.87}$$

then writing (3.78) in an explicit form we get

$$\begin{bmatrix} v_{21} - \lambda_1 v_{11} & v_{22} - \lambda_2 v_{12} \\ v_{31} - \lambda_1 v_{21} & v_{32} - \lambda_2 v_{22} \\ -10v_{11} + 8v_{21} - (3 + \lambda_1)v_{31} & -10v_{12} + 8v_{22} - (3 + \lambda_2)v_{32} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \tag{3.88}$$

Let

$$d_1 = \lambda_1^3 + 3\lambda_1^2 - 8\lambda_1 + 10 \; , \tag{3.89}$$

$$d_2 = \lambda_2^3 + 3\lambda_2^2 - 8\lambda_2 + 10 \; , \tag{3.90}$$

then

$$V = \begin{bmatrix} -1/d_1 & 1/d_2 \\ -\lambda_1/d_1 & \lambda_2/d_2 \\ -\lambda_1^2/d_1 & \lambda_2^2/d_2 \end{bmatrix} \tag{3.91}$$

A systematic way of finding B^g which always satisfies the constraint $B^gV = 0$ is by forming the matrix $[B \vdots V]$ and computing its inverse, since B^g is equal to the first m rows of $[B \vdots V]^{-1}$. Proceding in this manner, we find that

$$[B \vdots V] = \begin{bmatrix} 0 & -1/d_1 & 1/d_2 \\ 0 & -\lambda_1/d_1 & \lambda_2/d_2 \\ 1 & -\lambda_1^2/d_1 & \lambda_2^2/d_2 \end{bmatrix} \tag{3.92}$$

In this particular case, $m = 1$, which means that we only need the first row of $[B \vdots V]^{-1}$. Using the method of cofactors we get

$$\det[B \vdots V] = \frac{\lambda_1 - \lambda_2}{d_1 d_2} = \frac{j}{d_1 d_2}$$

and the first row of the adjoint of $[B \vdots V]$ is found to be

$$\left[\frac{\lambda_1^2\lambda_2 - \lambda_2^2\lambda_1}{d_1 d_2} \quad \frac{\lambda_2^2 - \lambda_1^2}{d_1 d_2} \quad \frac{\lambda_1 - \lambda_2}{d_1 d_2} \right].$$

The generalized left inverse of B is then given by

$$B^g = \frac{1}{j/d_1 d_2} \left[\frac{\lambda_1^2\lambda_2 - \lambda_2^2\lambda_1}{d_1 d_2} \quad \frac{\lambda_2^2 - \lambda_1^2}{d_1 d_2} \quad \frac{\lambda_1 - \lambda_2}{d_1 d_2} \right]$$

$$= [- j(\lambda_1^2\lambda_2 - \lambda_2^2\lambda_1) \quad - j(\lambda_2^2 - \lambda_1^2) \quad - j(\lambda_1 - \lambda_2)] \,.$$

Substituting the values of λ_1 and λ_2 into the above equation, we obtain

$$B^g = [0.29 \quad -0.4 \quad 1] \tag{3.93}$$

Let

$$F = \gamma \neq 0 \tag{3.94}$$

then

$$S = \gamma[0.29 \quad -0.4 \quad 1] \tag{3.95}$$

Method 2: Noting that the columns of V are in the null space of S, it follows that

$$S = \Gamma V^{\perp} \tag{3.96}$$

where $V^{\perp}$ is the annihilator of V, namely, $V^{\perp}V = 0$, and Γ is a nonsingular matrix chosen such that

$$SB = F = \Gamma V^{\perp}B \tag{3.97}$$

or

$$\Gamma = F(V^{\perp}B)^{-1} \tag{3.98}$$

Again, $\det(V^{\perp}B) \neq 0$ since $\mathrm{Range}(V) \cap \mathrm{Range}(B) = \{\theta\}$. Substituting (3.98) into (3.96) we get

$$S = F(V^{\perp}B)^{-1}V^{\perp} \tag{3.99}$$

It is easy to show that $(V^{\perp}B)^{-1}V^{\perp}$ is a generalized left inverse of B and that

$$(V^{\perp}B)^{-1}V^{\perp}V = B^gV = 0 \; . \tag{3.100}$$

Example 3.8: Using method 2 to design S for the system used in the previous example without changing the requirements, and letting $\Gamma = 1$, we get

$$SV = 0 \tag{3.101}$$

explicitly

$$-\frac{1}{d_1}s_1 - \frac{\lambda_1}{d_1}s_2 - \frac{\lambda_1^2}{d_1}s_3 = 0$$

$$\frac{1}{d_2}s_1 + \frac{\lambda_2}{d_2}s_2 + \frac{\lambda_2^2}{d_2}s_3 = 0$$

or

$$-s_1 - \lambda_1 s_2 - \lambda_1^2 s_3 = 0 \tag{3.102a}$$

$$s_1 + \lambda_2 s_2 + \lambda_2^2 s_3 = 0 \tag{3.102b}$$

let $s_3 = 1$, then solving the system of linear equations (3.102) yields

$$s_2 = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_2 - \lambda_1} = -(\lambda_1 + \lambda_2)$$

$$s_1 = \lambda_1\lambda_2$$

but $\lambda_1 = \bar{\lambda}_2 = \frac{1}{5} + j\,\frac{1}{2}$, thus

$$s_1 = 0.29\ ,\quad s_2 = -\,0.4\ ,$$

therefore,

$$S = [0.29 \quad -0.4 \quad 1] \tag{3.103}$$

3.6. CONCLUSIONS

Borrowing ideas from the variable structure control of continuous-time dynamical systems we were able to design several controllers which drove the trajectory of a linear time-invariant discrete-time dynamical system to a linear hyperplane Ker(S), where S was chosen such that when the trajectory of the system in question was constrainted to lie on it, it possessed certain desirable properties, e.g., asymptotic stability. Any of the controllers that we discussed enabled the system to reach the hyperplane Ker(S) at least asymptotically, though the level of complexity decreased as new alternatives were introduced.

To solve the problem of efficiently designing the hyperplane Ker(S), a projection theoretic approach [17] was introduced and illustrated through examples.

It was apparent from the outset that the models which described the kind of systems that we dealt with in this chapter did not possess any uncertainties. Hence, the question of how to drive onto a hyperplane a discrete-time dynamical system which has uncertain elements still remains to be answered.

CHAPTER IV

ROBUST STATE FEEDBACK STABILIZATION OF DISCRETE-TIME UNCERTAIN DYNAMICAL SYSTEMS

4.1. INTRODUCTION AND PROBLEM STATEMENT

The problem of controlling discrete-time dynamical systems has a long history and has been the subject of research activity for many years (see e.g. [3], [24], and [10]). For an account on the history and progress of sampled-data systems see Jury [25].

In the last few years, a considerable amount of work has been done in the field of controlling continuous-time uncertain dynamical systems.

The approach used by many researchers has been of deterministic nature [21,7,23,34], i.e., rather than defining the uncertainties in probabilistic terms, they are defined by known compact sets in which the values of the uncertainties lie.

Recently, Manela [20], and Corless and Manela [23] have proposed possible solutions to this problem as it applies to discrete-time dynamic systems described by difference equations.

In this chapter we consider the problem of robustly stabilizing a class of discrete-time uncertain dynamical systems where the "nominal" system is linear and the uncertainty does not depend on the control input.

The approach used in the following considerations is of deterministic nature, that is, no knowledge of the statistical behavior of the uncertainty is assumed, except its maximum size.

We shall consider linear discrete-time dynamical systems described by the following equation

$$x_{k+1} = (A + \Delta A(r_k))x_k + Bu_k + Ev_k\,, \quad x_0 = x(k_0) \tag{4.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, A and B are constant matrices of appropriate dimensions, and $\Delta A(\cdot)$: $\mathbb{R}^\ell \to \mathbb{R}^{n\times n}$ is a known and continuous function, $E \in \mathbb{R}^{n\times q}$ is a known constant disturbance distribution matrix.

The uncertainties are determined by the variables $r(\cdot)$ and $v(\cdot)$, whose behavior we do not know at any given time index $k \in Z$ (Z is the set of integers). It is assumed, however, that they are Lebesgue measurable and that they are constrained to known compact uncertainty bounding sets, namely,

$$r_k \in \mathcal{R} \subset \mathbb{R}^\ell \quad \text{and} \quad v_k \in \mathcal{V} \subset \mathbb{R}^q\,.$$

Furthermore, we assume the following

Assumption 1: There exists a matrix function $G(\cdot)$: $\mathbb{R}^\ell \to \mathbb{R}^{m\times n}$ which is continuous on $\mathbb{R}^\ell$, and a constant matrix $H \in \mathbb{R}^{m\times q}$ such that

$$\Delta A(r_k) = BG(r_k) \; \forall \; r_k \in \mathcal{R} \tag{4.2}$$

$$E = BH \tag{4.3}$$

that is, $\Delta A(\cdot)$ and E satisfy the matching conditions [21].

Assumption 2: The nominal system

$$x_{k+1} = Ax_k + Bu_k \tag{4.4}$$

is stabilizable.

Assumption 3: The matrix B has rank m.

Making use of Assumption 1, we obtain

$$e(k,x_k) = G(r_k)x_k + Hv_k \ , \tag{4.5}$$

therefore (4.1) can be rewritten in the form

$$x_{k+1} = Ax_k + B(u_k + e(k,x_k)) \ . \tag{4.6}$$

Without loss of generality we assume that the matrix A in (4.6) is stable, i.e., its spectral radius $\rho(A)$ is strictly less than one, where $\rho(A) = \max\{|\lambda|: \ \lambda \text{ is an eigenvalue of } A\}$ (otherwise, by Assumption 2 there exists a constant feedback matrix $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable).

From (4.5) we have

$$\|e(k,x_k)\| = \|G(r_k)x_k + Hv_k\| \leq \max_{r_k \in \mathcal{R}} \{\|G(r_k)\|\} \cdot \|x_k\| + \max_{v_k \in V}\{\|Hv_k\|\} \ .$$

Let

$$\xi(k,x_k) = \max_{r_k \in \mathcal{R}}\{\|G(r_k)\|\} \cdot \|x_k\| + \max_{v_k \in V}\{\|Hv_k\|\} \ , \tag{4.7}$$

then

$$\|e(k,x_k)\| \leq \xi(k,x_k) \ . \tag{4.8}$$

Define

$$\xi(k,x_k) \triangleq \xi_0 + \xi_1 \|x_k\| ,$$

where

$$\xi_0 \triangleq \max_{v_k \in V}\{\|Hv_k\|\} ,$$

$$\xi_1 \triangleq \max_{r_k \in \mathcal{R}}\{\|G(r_k)\|\} ,$$

and $\|\cdot\|$ refers to the Euclidean norm of a vector.

If M is a matrix, then $\|M\|$ denotes the corresponding (induced) norm $\|M\| = (\lambda_{max}(M^T M))^{1/2}$, where $\lambda_{max}(\cdot)$ denotes the largest eigenvalues of a matrix.

The uncertainty $e(k,x_k)$ as defined above is known in the literature as cone bounded [23].

4.2. DERIVATION OF A SATURATION TYPE OF CONTROLLER

Since the free nominal system is asymptotically stable, given a real, symmetric, positive definite (r.s.p.d.) matrix Q, there exists a r.s.p.d. matrix P which uniquely solves the discrete Lyapunov matrix equation

$$A^T P A - P = -Q , \tag{4.9}$$

and

$$V(x_k) = x_k^T P x_k = \langle x_k, P x_k \rangle \triangleq \|x_k\|_P \tag{4.10}$$

is a Lyapunov function for $x_{k+1} = Ax_k$.

Clearly, $V(\cdot)$: $\mathbb{R}^n \rightarrow \mathbb{R}_+$.

Theorem 4.1: Given a discrete-time dynamical system modeled by (4.6)-(4.8). Assume that the nominal system is asymptotically stable. Consider the control law

$$u_k = u_k^* = \begin{cases} -\dfrac{R^{-1}B^TPAx_k}{\|B^TPAx_k\|_{R^{-1}}}\,\gamma(k,x_k)\,, & \text{if } x_k \notin \mathrm{Ker}(B^TPA) \quad (4.11a) \\ 0\,, & \text{if } x_k \in \mathrm{Ker}(B^TPA) \quad (4.11b) \end{cases}$$

where

$R = B^TPB$, $\|B^TPAx_k\|_{R^{-1}} = (x_k^TA^TPBR^{-1}B^TPAx_k)^{1/2}$ and

$$\gamma(k,x_k) = \lambda_{max}^{1/2}(R)\xi(k,x_k)\,.$$

Then the first forward difference of the Lyapunov function (4.10) satisfies the inequalities

$$\Delta V \leq \begin{cases} -\lambda_{min}(Q)\|x_k\|^2 + 4\lambda_{max}(R)\xi^2(k,x_k)\,, & \text{if} \quad x_k \notin \mathrm{Ker}(B^TPA) \\ -\lambda_{min}(Q)\|x_k\|^2 + \lambda_{max}(R)\xi^2(k,x_k)\,, & \text{if} \quad x_k \in \mathrm{Ker}(B^TPA)\,. \end{cases}$$

Proof: The first forward difference of the Lyapunov function is given by $\Delta V(x_k) = V(x_{k+1}) - V(x_k)$.

Using equations (4.6), (4.9) and (4.10), and noting that x_{k+1} depends explicitly on u_k and $e(k,x_k)$, we have

$$\begin{aligned} \Delta V(x_k,u_k,e(k,x_k)) = {} & -x_k^TQx_k + 2u_k^TB^TPAx_k + 2e^T(k,x_k)B^TPAx_k \\ & + 2u_k^TB^TPBe(k,x_k) + u_k^TB^TPBu_k \\ & + e^T(k,x_k)B^TPBe(k,x_k)\,. \end{aligned} \tag{4.12}$$

Notice that the first, second and fifth terms in the above expression correspond to the first forward difference of the Lyapunov function of the nominal system (4.4); we therefore let

$$\Delta V_N(x_k, u_k) \triangleq - x_k^T Q x_k + 2u_k^T B^T P A x_k + u_k^T B^T P B u_k \,. \tag{4.13}$$

Upon substitution of equation (4.11a) into equation (4.12) we get

$$\begin{aligned}\Delta V^*(x_k, e(k,x_k)) &\triangleq \Delta V(x_k, u_k^*, e(k,x_k)) \\ &= - x_k^T Q x_k - 2\frac{x_k^T A^T P B R^{-1} B^T P A x_k}{\|B^T P A x_k\|_{R^{-1}}}\gamma(k,x_k) \\ &+ 2e^T(k,x_k) B^T P A x_k - 2\,\frac{x_k^T A^T P B R^{-1} R e_k}{\|B^T P A x_k\|_{R^{-1}}}\gamma(k,x_k) \\ &+ \frac{x_k^T A^T P B R^{-1} R R^{-1} B^T P A x_k}{\|B^T P A x_k\|^2_{R^{-1}}}\gamma^2(k,x_k) \\ &+ e^T(k,x_k) R e(k,x_k) \,. \end{aligned} \tag{4.14}$$

Hence

$$\begin{aligned}\Delta V^*(x_k, e(k,x_k)) = &- x_k^T Q x_k - 2\|B^T P A x_k\|_{R^{-1}}\gamma(k,x_k) + 2e^T(k,x_k) R R^{-1} B^T P A x_k \\ &- 2\,\frac{x_k^T A^T P B R^{-1} R e(k,x_k)}{\|B^T P A x_k\|_{R^{-1}}}\gamma(k,x_k) + \gamma^2(k,x_k) \\ &+ e^T(k,x_k) R e(k,x_k) \,. \end{aligned} \tag{4.15}$$

We now observe that

$$e^T(k,x_k) R R^{-1} B^T P A x_k \le \|e^T(k,x_k) R R^{-1} B^T P A x_k\| \,.$$

Moreover, we can represent the matrix $R = R^T > 0$ as $R = W^T W$, where

$W \in \mathbb{R}^{m \times m}$ is nonsingular, because R is r.s.p.d. Thus

$$RR^{-1} = W^T(W^T)^{-1} ,$$

and

$$\begin{aligned} e^T(k,x_k)RR^{-1}B^TPAx_k &\leq \| e^T(k,x_k)W^T(W^T)^{-1}B^TPAx_k \| , \\ &\leq \| We(k,x_k) \| \, \| (W^T)^{-1}B^TPAx_k \| , \\ &\leq \| e(k,x_k) \|_R \, \| B^TPAx_k \|_{R^{-1}} . \end{aligned} \quad (4.16)$$

Using the above observation we find that $\Delta V(x_k,e(k,x_k))$ becomes

$$\begin{aligned} \Delta V^*(x_k,e(k,x_k)) \leq & -x_k^TQx_k - 2\| B^TPAx_k \|_{R^{-1}}\gamma(k,x_k) + 2\| B^TPAx_k \|_{R^{-1}} \| e(k,x_k) \|_R \\ & + 2\| e(k,x_k) \|_R \gamma(k,x_k) + \gamma^2(k,x_k) + \| e(k,x_k) \|_R^2 . \end{aligned} \quad (4.17)$$

If we observe further that

$$\| e(k,x_k) \|_R \leq \lambda_{max}^{1/2}(R) \| e(k,x_k) \| , \quad (4.18)$$

then

$$\begin{aligned} \Delta V^*(x_k,e(k,x_k)) \leq & - x_k^TQx_k - 2\| B^TPAx_k \|_{R^{-1}}\gamma(k,x_k) \\ & + 2\| B^TPAx_k \|_{R^{-1}} \lambda_{max}^{1/2}(R) \| e(k,x_k) \| + 2\lambda_{max}^{1/2}(R) \| e(k,x_k) \| \gamma(k,x_k) \\ & + \gamma^2(k,x_k) + \lambda_{max}(R) \| e(k,x_k) \|^2 . \end{aligned} \quad (4.19)$$

From equation (4.8) we see that the norm of $e(k,x_k)$ is bounded from above by $\xi(k,x_k)$. In addition by assumption $\gamma(k,x_k) = \lambda_{max}^{1/2}(R)\xi(k,x_k)$, therefore equation (4.19) simplifies to the following one

$$\Delta V^*(x_k,e(k,x_k) \leq - x_k^TQx_k + 4\lambda_{max}(R)\xi^2(k,x_k) , \text{ if } \| B^TPAx_k \|_{R^{-1}} \neq 0 . \quad (4.20)$$

Lastly, it is well known (see [41], pp. 129) that when Q is symmetric positive definite, then $x_k^T Q x_k \geq \lambda_{min}(Q)||x_k||^2$, $\lambda_{min}(Q) > 0$. Hence if x_k is not in the null space of $B^T PA$, we find that

$$\Delta V^*(x_k, e(k,x_m)) \leq -\lambda_{min}(Q)||x_k||^2 + 4\,\lambda_{max}(R)\xi^2(k,x_k)\,. \qquad (4.31)$$

To complete the proof, we note that if $||B^T PAx_k||_{R^{-1}} = 0$ or equivalently, $x_k \in \text{Ker}(B^T PA)$ then $u_k^* = 0$ and

$$\Delta V^*(x_k, u_k^*, e(k,x_k)) = -\,x_k^T Q x_k + e^T(k,x_k)B^T PBe(k,x_k)\,. \qquad (4.22)$$

Again, using the definition $R = B^T PB$ and the fact that $\lambda_{min}(M)||x_k||^2 \leq x_k^T M x_k \leq \lambda_{max}(M)||x_k||^2$ for a r.s.p.d. matrix M, [41] we obtain

$$\Delta V^*(x_k, e(k,x_k)) \leq -\,\lambda_{min}(Q)||x_k||^2 + \lambda_{max}(R)||e(k,x_k)||^2\,. \qquad (4.23)$$

Substituting (4.8) into equation (4.23) we get

$$\Delta V^*(x_k, e(k,x_k)) \leq -\,\lambda_{min}(Q)||x_k||^2 + \lambda_{max}(R)\xi^2(k,x_k)\,, \qquad (4.24)$$

whenever $||B^T PAx_k||_{R^{-1}} = 0$. Hence Theorem 4.1 is proved.

□

The following Proposition is concerned with some minimization properties of the controller (4.11).

Proposition 4.1: The controller given by (4.11a) minimizes (4.13) subject to the constraint

$$u_k^T B^T PBu_k = \gamma^2(k,x_k)\,, \qquad (4.25)$$

whenever $||B^T PAx_k||_{R^{-1}} \neq 0$.

Proof: We first form the Lagrangian

$$\ell(u_k,\nu;x_k) = \Delta V_N(x_k,u_k) + \nu(u_k^T B^T P B u_k - \gamma^2(k,x_k)) \ , \quad \nu \in \mathbb{R} \ . \tag{4.26}$$

The first-order necessary conditions for an extremum are [22]

$$\nabla_{u_k}\ell(u_k,\nu;x_k) = 0 \tag{4.27}$$

and

$$\nabla_{\nu}\ell(u_k,\nu;x_k) = 0 \ , \tag{4.28}$$

in other words,

$$\nabla_{u_k}\ell(u_k,\nu;x_k) = 2B^T P A x_k + 2B^T P B u_k + 2\nu B^T P B u_k = 0 \ ,$$

which implies that

$$u_k^* = -\frac{(B^T P B)^{-1} B^T P A x_k}{1+\nu} \ . \tag{4.29}$$

Likewise,

$$\nabla_{\nu}\ell(u_k,\nu;x_k) = u_k^T B^T P B u_k - \gamma^2(k,x_k) = 0 \ ,$$

which results in equation (4.25).

Thus, the following relation holds

$$u_k^{*T} B^T P B u_k^* = \gamma^2(k,x_k) = \frac{x_k^T A^T P B (B^T P B)^{-1} B^T P A x_k}{(1+\nu)^2} = \frac{\|B^T P A x_k\|^2_{(B^T P B)^{-1}}}{(1+\nu)^2} \ .$$

We therefore have

$$1 + \nu = \frac{\pm \|B^T P A x_k\|_{(B^T P B)^{-1}}}{\gamma(k,x_k)} \ .$$

If we use the negative of the square root of $(1+\nu)^2$ in (4.29), i.e.,

$$u_k^* = \frac{(B^TPB)^{-1}B^TPAx_k}{||B^TPAx_k||_{(B^TPB)^{-1}}}\gamma(k,x_k)\ ,\ \text{if}\ \ ||B^TPAx_k||_{(B^TPB)^{-1}} \neq 0 \qquad (4.30)$$

then we find that, although the constraint equation (4.25) is satisfied, $\Delta V_N(x_k,u_k)$ does not achieve a minimum. On the other hand, utilizing the positive of the square root of $(1 + \nu)^2$ in equation (4.29), yields

$$u_k^* = \frac{-\ (B^TPB)^{-1}B^TPAx_k}{||B^TPAx_k||_{(B^TPB)^{-1}}}\gamma(k,x_k)\ ,\ \text{if}\ \ ||B^TPAx_k||_{(B^TPB)^{-1}} \neq 0 \qquad (4.31)$$

and does indeed result in an extremum for $\Delta V_N(x_k,u_k)$ while (4.25) is satisfied at the same time. Hence, u_k given by equation (4.31) satisfies the first order necessary conditions for a minimum.

We now show that (4.31) also satisfies the second order sufficient conditions ([22], pp. 306), namely, that the matrix $L(u_k^*) = F(u_k^*) + \nu^TH(u_k^*)$ is positive definite on $M = \{y :\ \nabla h(u_k^*)y = 0\}$, where $F(u_k^*)$ and $H(u_k^*)$ are the Hessians of $\Delta V_N(u_k;x_k)$ and $u_k^TB^TPBu_k - \gamma^2(k,x_k)$, respectively, with respect to u_k and evaluated at u_k^*, and $\nabla h(u_k^*)$ is the gradient of $u_k^TB^TPBu_k - \gamma^2(k,x_k)$ evaluated at u_k^*.

Specifically,

$$\nabla_{u_k}(u_k^TB^TPBu_k - \gamma^2(k,x_k))\big|_{u_k^*} = -\frac{B^TPAx_k}{||B^TPAx_k||_{(B^TPB)^{-1}}}\gamma(k,x_k)\ . \qquad (4.32)$$

In other words,

$$M = \mathrm{Ker}(x^TA^TPB). \qquad (4.33)$$

Now,

$$L(u_k^*) = B^TPB + \nu B^TPB = (1 + \nu)B^TPB . \tag{4.34}$$

Clearly, $L(u_k^*)$ is positive definite everywhere if $1 + \nu > 0$, since B^TPB is positive definite on $\mathbb{R}^m$. Moreover, B^TPB is positive definite on M, because $M \subset \mathbb{R}^m$. But, $1 + \nu > 0$ implies that we must choose the positive of the square root of $(1 + \nu)^2$. Therefore, u_k^* given by equation (4.31) is a strict local minimum of $\Delta V_N(u_k;x_k)$ subject to $u_k^T B^T P B u_k = \gamma^2(k,x_k)$. Noticing further that $R = B^TPB$, then equation (4.31) becomes

$$u_k^* = -\frac{R^{-1}B^TPAx_k}{\|B^TPAx_k\|_{R^{-1}}}\gamma(k,x_k) , \text{ if } \|B^TPAx_k\|_{R^{-1}} \neq 0 ,$$

which is the same as equation (4.11a).

□

4.3. DETERMINATION OF STABILITY REGION

We again consider the class of discrete-time dynamical systems described by (4.6) with uncertainty $e(k,x_k)$ which is cone bounded by $\xi(k,x_k)$ defined by

$$\xi(k,x_k) \triangleq \xi_0 + \xi_1 \|x_k\| , \tag{4.35}$$

where ξ_0 and ξ_1 are given by

$$\xi_0 = \max_{v_k \in V}\{\|Hv_k\|\} , \tag{4.36}$$

$$\xi_1 = \max_{r_k \in \mathcal{R}}\{\|G(r_k)\|\} . \tag{4.37}$$

We first analyze the case when $x_k \notin Ker(B^TPA)$.

Substituting equation (4.35) into equation (4.31), we get

$$\Delta V^*(x_k,e(k,x_k)) \leq -\lambda_{min}(Q)\|x_k\|^2 + 4\,\lambda_{max}(R)[\xi_0^2 + 2\xi_0\xi_1\|x_k\| + \xi_1^2\|x_k\|^2] .$$

Rearranging the terms in the above equation yields

$$\Delta V^*(x_k,e(k,x_k)) \leq (4\,\lambda_{max}(R)\xi_1^2 - \lambda_{min}(Q))\|x_k\|^2 + 8\,\lambda_{max}(R)\xi_0\xi_1\|x_k\| + 4\,\lambda_{max}(R)\xi_0^2 . \tag{4.38}$$

Let

$$\beta \triangleq \frac{\lambda_{min}(Q)}{4\,\lambda_{max}(R)} , \tag{4.39}$$

then

$$\Delta V^*(x_k,e(k,x_k)) \leq 4\,\lambda_{max}(R)[(\xi_1^2 - \beta)\|x_k\|^2 + 2\xi_0\xi_1\|x_k\| + \xi_0^2] . \tag{4.40}$$

In order for the right hand side of equation (4.40) to be negative on some region of $\mathbb{R}^n$, it is necessary that $\xi_1 < \sqrt{\beta}$.

Proposition 4.2: If $\xi_1 < \sqrt{\beta}$, then $\Delta V^*(x_k,e(k,x_k))$ is negative definite on the region

$$\|x_k\| > \frac{\xi_0}{\beta^{1/2} - \xi_1} . \tag{4.41}$$

Proof: From equation (4.40) we have that $\Delta V^*(x_k,e(k,x_k))$ is negative definite on some region if $\xi_1 < \sqrt{\beta}$. To find the region, we proceed as follows (assuming that $\xi_1 < \sqrt{\beta}$).

If the right side of equation (4.40) is to be negative, then

$$(\xi_1^2 - \beta)\|x_k\|^2 + 2\xi_0\xi_1\|x_k\| + \xi_0^2 < 0$$

or equivalently,

$$-\beta\|x_k\|^2 + (\xi_0 + \xi_1\|x_k\|)^2 < 0 .$$

Thus

$$\beta\|x_k\|^2 - (\xi_0 + \xi_1\|x_k\|)^2 > 0 ,$$

which implies that

$$\beta\|x_k\|^2 > (\xi_0 + \xi_1\|x_k\|)^2 ,$$

or

$$\sqrt{\beta}\,\|x_k\| > \xi_0 + \xi_1\|x_k\| .$$

Therefore, $\|x_k\| > \dfrac{\xi_0}{\sqrt{\beta} - \xi_1}$.

□

If we define

$$\eta_0 \triangleq \frac{\xi_0}{\sqrt{\beta} - \xi_1} , \qquad (4.41)$$

then $\Delta V^*(x_k,e(k,x_k))$ is not negative definite for $x_k \in B(0,\eta_0)$, where $B(0,\eta_0) = \{x_k : \|x_k\| < \eta_0\}$ denotes the η_0-ball about $x = 0$.

We now consider the case when $x_k \in Ker(B^TPA)$. Proceeding in a similar manner as in the case when $x_k \notin Ker(B^TPA)$, we have

$$\Delta V^*(x_k,e(k,x_k)) \le -\lambda_{min}(Q)\|x_k\|^2 + \lambda_{max}(R)(\xi_0 + \xi_1\|x_k\|)^2 , \quad x_k \in Ker(B^TPA) .$$

Define

$$\beta' \triangleq \frac{\lambda_{min}(Q)}{\lambda_{max}(R)}\,, \tag{4.42}$$

then

$$\Delta V^*(x_k,e(k,x_k)) \leq \lambda_{max}(R)[-\beta'\|x_k\|^2 + (\xi_0 + \xi_1\|x_k\|)^2]\,. \tag{4.43}$$

Clearly, the region of $\mathbb{R}^n$ where $\Delta V^*(x_k,e(k,x_k))$ is negative is

$$\|x_k\| > \frac{\xi_0}{\sqrt{\beta'} - \xi_1}\,, \text{ if } \xi_1 < \sqrt{\beta'}\,. \tag{4.44}$$

Let η_0' be defined by

$$\eta_0' \triangleq \frac{\xi_0}{\sqrt{\beta'} - \xi_1}\,, \tag{4.45}$$

then noting that $\beta' = 4\beta$ enables us to conclude that $\eta_0 > \eta_0'$, which implies that whenever $x_k \in \text{Ker}(B^T PA)$, the region where $\Delta V^*(x_k,e(k,x_k))$ is negative is larger than that when $x_k \notin \text{Ker}(B^T PA)$. This illustrated in Figure 4.1, where $\xi_1 < \sqrt{\beta}$.

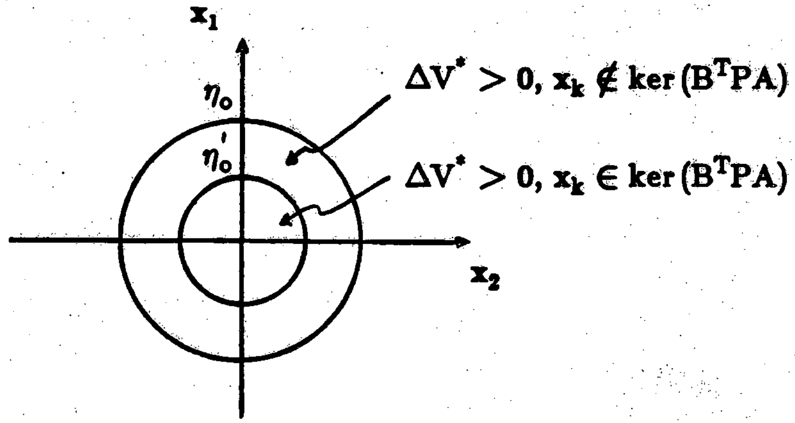

Figure 4.1. Illustration of Proposition 4.2.

Figure 4.2 further illustrates the behavior of $\Delta V^*(x_k,e(k,x_k))$ ($\xi_1 < \sqrt{\beta}$).

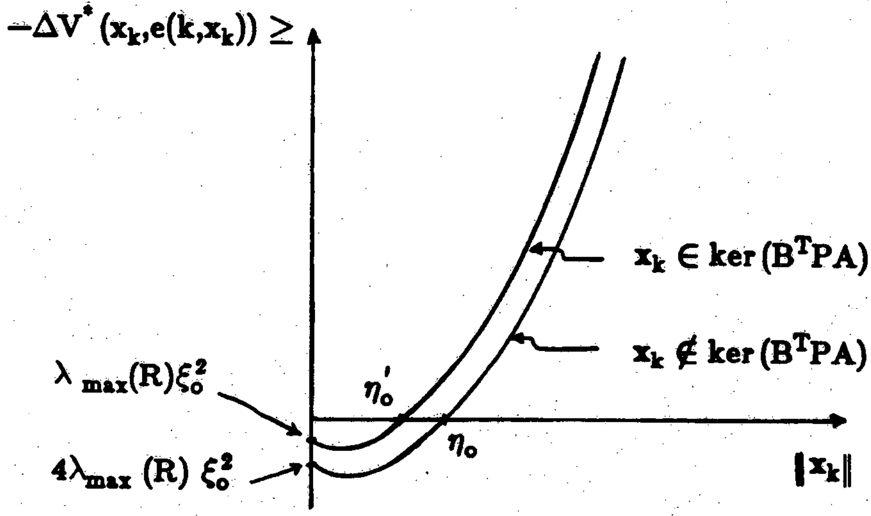

Figure 4.2. Estimates of - ΔV^*

Theorem 4.2. Consider the linear discrete-time uncertain dynamical system

$$x_{k+1} = Ax_k + B(u_k + e(k,x_k)) \ , \ x(k_0) = x_0 \ , \tag{4.46}$$

with control

$$u_k = \begin{cases} -\dfrac{R^{-1}B^TPAx_k}{\|B^TPAx_k\|_{R^{-1}}}\gamma(k,x_k) \ , & \text{if } x_k \notin \text{Ker}(B^TPA) \\ 0 \ , & \text{if } x_k \in \text{Ker}(B^TPA) \end{cases} \tag{4.47}$$

where $\gamma(k,x_k) = \lambda_{max}^{1/2}(R)\xi(k,x_k)$, satisfying Assumptions (1)-(3), with A a convergent matrix and $\xi_1 < \sqrt{\beta}$. If $x(\cdot) : [k_0,k_1] \rightarrow \mathbb{R}^n$, $x(k_0) = x_0$ is a solution of equation (4.46), then

$$\|x_0\| \le s \Longrightarrow \|x_k\| \le d(s) \ , \ \forall \ k \in [k_0,k_1] \ ,$$

where

$$d(s) = \begin{cases} \sqrt{\dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ s \ , & \text{if } s > \eta_0 \\ \sqrt{\dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ \eta_0 \ , & \text{if } s \le \eta_0 \end{cases} \tag{4.48}$$

whenever $x_k \notin \text{Ker}(B^TPA)$, and

$$d'(s) = \begin{cases} \sqrt{\dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ s \ , & \text{if } s > \eta_0' \\ \sqrt{\dfrac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ \eta_0' \ , & \text{if } s \le \eta_0' \end{cases} \tag{4.49}$$

whenever $x_k \in \text{Ker}(B^TPA)$.

Proof: Since the "free" nominal system $x_{k+1} = Ax_k$ is asymptotically stable, then given a r.s.p.d. matrix Q, there exists a r.s.p.d. matrix P which uniquely solves the discrete Lyapunov equation

$$A^TPA - P = -Q \tag{4.50}$$

with $V(x_k) = x_k^TPx_k$ a Lyapunov function for $x_{k+1} = Ax_k$.

Using the above Lyapunov function candidate in equation (4.46) along with the cone bounded uncertainty assumption, we obtained equations (4.31) and (4.34).

Once again, utilizing the well-known fact that $\lambda_{min}(P)||x_k||^2 \leq x_k^TPx_k \leq \lambda_{max}(P)||x_k||^2$, define

$$\alpha_1(||x_k||) \triangleq \lambda_{min}(P)||x_k||^2 \tag{4.51}$$

$$\alpha_2(||x_k||) \triangleq \lambda_{max}(P)||x_k||^2 \, . \tag{4.52}$$

We now consider the case where $x_k \notin Ker(B^TPA)$. Suppose $||x_0|| \leq s$ and $s > \eta_0$.

Let

$$d(s) \triangleq (\alpha_1^{-1} o \alpha_2)\,(s)\,, \tag{4.53}$$

then from equations (4.51) and (4.52) we have

$$d(s) = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\; s\,. \tag{4.54}$$

Clearly, $d(s) \geq s$.

Now,

$$\alpha_1(d(s)) = \alpha_2(s) \geq x_0^T P x_0 \ .$$

But for the time index $k \in [k_0,k_1]$ and initial condition $x_0 \in \mathbb{R}^n \setminus \overline{B(0,\eta_0)}$, $\Delta V^*(x_k,e(k,x_k))$ is negative definite, therefore

$$\alpha_1(d(s)) \geq x_0^T P x_0 \geq x_k^T P x_k \geq \alpha_1(\|x_k\|) \ , \tag{4.55}$$

thus,

$$\|x_k\| \leq d(s) \qquad \forall \ k \in [k_0,k_1] \ ,$$

with d(s) given by equation (4.54), where $\overline{B(0,\eta_0)}$ refers to the closed η_0-ball about $x = 0$.

Similarly, for $x_k \in \mathrm{Ker}(B^T PA)$ we replace η_0 by η_0' and proceed in the same fashion as above.

Note that $\|x_k\|$ remains bounded from above by d(s) and from below by η_0 or η_0'.

Suppose now that $\|x_0\| \leq s$ but $s \leq \eta_0$. Assuming $x_k \notin \mathrm{Ker}(B^T PA)$, let

$$\alpha_1(d(s)) \triangleq \alpha_2(\eta_0) \ , \tag{4.56}$$

then from equations (4.51) and (4.52) we obtain

$$d(s) = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \ \eta_0 \ . \tag{57}$$

Again, it is easy to see that $d(s) \geq \eta_0$.

From equation (4.56) and the fact that the representative point cannot leave the ball $\overline{B(0,\eta_0)}$ whenever $x_0 \in \overline{B(0,\eta_0)}$, we conclude the following

$$\alpha_1(d(s)) = \alpha_2(\eta_0) \geq x_k^T P x_k \geq \alpha_1(\|x_k\|) \ ,$$

or

$$\lambda_{min}(P)d^2(s) \geq \lambda_{min}(P)\|x_k\|^2 .$$

Therefore,

$$\|x_k\| \leq d(s) \ \forall \ k\in[k_0,k_1] ,$$

with d(s) given by equation (4.57).

For the case when $x_k \in Ker(B^TPA)$, we replace η_0 by η_0' and follow the same reasoning as above.

□

Theorem 4.3: Consider the system given by (4.46) with state feedback control (4.47) satisfying Assumptions (1)-(3), with A a convergent matrix and $\xi_1 < \sqrt{\beta}$. If $x(\cdot)$: $[k_0,\infty) \rightarrow \mathbb{R}^n$, $x(k_0) = x_0$, is a solution of (4.46) with $\|x_0\| \leq s$, then for given $\bar{d} > (\alpha_1^{-1} o \alpha_2)(\eta_0)$, $\|x_k\| \leq \bar{d} \ \forall \ k \geq k_0 + K(\bar{d},s)$ where

$$K(\bar{d},s) = \begin{cases} 0 , & \text{if } s \leq \bar{\eta}_0 \\ \left\lceil \dfrac{\alpha_2(s) - \alpha_1(\bar{\eta}_0)}{\alpha_2(\bar{\eta}_0)} \right\rceil , & \text{if } s > \bar{\eta}_0 \end{cases} \tag{4.58}$$

where $\lceil \cdot \rceil$: $\mathbb{R} \rightarrow \mathbb{N}$ is the ceiling function, i.e., if g(s) = 3.2, then $\lceil 3{\cdot}2 \rceil = 4$, and $\mathbb{N}$ is the set of natural numbers. If $x_k \notin Ker(B^TPA)$, then

$$\alpha_3(\|x_k\|) = 4 \ \lambda_{max}(R)(\beta - \xi_1^2)\|x_k\|^2 - 4 \ \lambda_{max}(R) \ (\xi_0^2 + 2\xi_0\xi_1\|x_k\|) , \tag{4.59}$$

and

$$\bar{\eta}_0 = (\alpha_2^{-1} o \alpha_1) \ (\bar{d}) . \tag{4.60}$$

Where $\alpha_3(||x_k||)$ is the negative of the upper bound of $\Delta V^*(x_k,e(k,x_k))$.

Proof: Consider $\alpha_1(\overline{d}) > d_2(\eta_0)$. By (60), $\alpha_2(\overline{\eta}_0) = \alpha_1(\overline{d})$, thus $\alpha_2(\overline{\eta}_0) > \alpha_2(\eta_0)$. Since $\alpha_2(\cdot)$ is continuous and strictly increasing, then $\overline{\eta}_0 > \eta_0$. This is illustrated in figure 4.3.

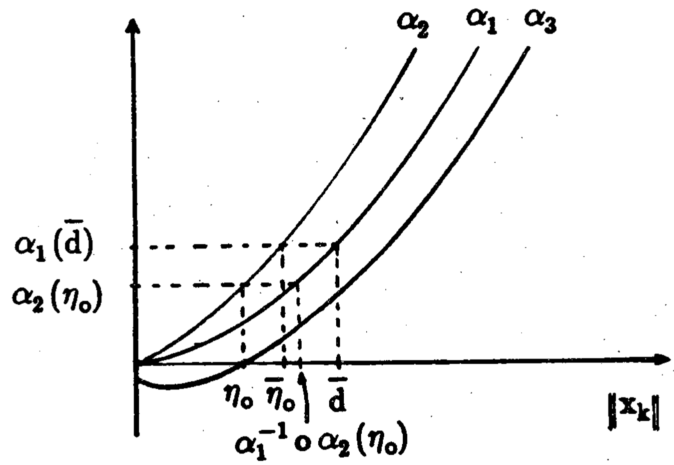

Figure 4.3. Functions used in the proof of Theorem 4.3.

Now, if $s \le \overline{\eta}_0$, then $||x_0|| \le \overline{\eta}_0$; therefore, from the results of the previous Theorem, we conclude that

$$||x_k|| \le \overline{d}\ ,\ \forall\ k \in [k_0,\infty) \Longrightarrow K(\overline{d},s) = 0\ .$$

We next look at the case when $s > \overline{\eta}_0$. Suppose that

$$||x_k|| > \overline{\eta}_0\ ,\ \ \forall\ k \in [k_0, k_0 + K(\overline{d},s)] \tag{4.61}$$

If $K(\overline{d},s) = \left\lceil \dfrac{\alpha_2(s) - \alpha_1(\overline{\eta}_0)}{\alpha_3(\overline{\eta}_0)} \right\rceil$, then because of equations (4.51), (4.52) we have

$$\alpha_1(\|x_{k_m}\|) \leq x_{k_m}^T P x_{k_m} = V(x_{k_m})$$

where

$$k_m = k_0 + K(\bar{d},s)\ . \tag{4.62}$$

But,

$$V(x_{k_m}) = V(x_0) + \sum_{i=k_0}^{k_m-1} \Delta V(x_i)\ ,$$

thus

$$\alpha_1(\|x_{k_m}\|) \leq V(x_0) + \sum_{i=k_0}^{k_m-1} \Delta V(x_i)$$

$$\leq \alpha_2(\|x_0\|) - \sum_{i=k_0}^{k_m-1} \alpha_3(\|x_i\|)\ ,$$

since $\alpha_3(\|x_i\|) > 0$ and $\Delta V(x_i) \leq -\alpha_3(\|x_i\|)$ for $\|x_i\| > \overline{\eta}_0$. Also, $\alpha_3(\|x_i\|) > \alpha_3(\overline{\eta}_0) > 0$, and $\|x_0\| \leq s$ therefore $\alpha_1(\|x_{k_m}\|) \leq \alpha_2(s) - \sum_{i=k_0}^{k_m-1} \alpha_3(\overline{\eta}_0)$. Hence

$$\alpha_1(\|x_{k_m}\|) \leq \alpha_2(s) - \alpha_3(\overline{\eta}_0)\,(k_m - k_0) = \alpha_2(s) - K(\bar{d},s)\alpha_3(\overline{\eta}_0)$$

$$\leq \alpha_2(s) - \alpha_3(\overline{\eta}_0) \left\lceil \frac{\alpha_2(s) - \alpha_1(\overline{\eta}_0)}{\alpha_3(\overline{\eta}_0)} \right\rceil .$$

If we observe that $\dfrac{\alpha_2(s) - \alpha_1(\overline{\eta}_0)}{\alpha_3(\overline{\eta}_0)} > 0$, $\alpha_3(\overline{\eta}_0) > 0$ for $s > \overline{\eta}_0$ and $\lceil f \rceil \geq f$, for $f > 0$, then

$$\alpha_1(||x_{k_m}||) \leq \alpha_2(s) - \alpha_3(\overline{\eta}_0)\left(\frac{\alpha_2(s) - \alpha_1(\overline{\eta}_0)}{\alpha_3(\overline{\eta}_0)}\right) = \alpha_1(\overline{\eta}_0) ,$$

which implies $||x_{k_m}|| \leq \overline{\eta}_0$, which contradicts supposition (4.61). Therefore, there is a $k_i \in [k_0, k_0 + K(\overline{d},s)]$ such that $||x_{k_i}|| \leq \overline{\eta}_0$. From equation (4.60) we infer that $\overline{d} \geq \overline{\eta}_0$. Hence, $||x_{k_i}|| \leq \overline{d}$. As a consequence of the previous theorem, we have

$$||x_k|| \leq \overline{d} \;\forall\; k \geq k_i ,$$

and consequently,

$$||x_k|| \leq \overline{d} \;\forall\; k \geq k_0 + K(\overline{d},s)$$

Notice that if $x_k \in Ker(B^T PA)$, then we replace η_0 by η_0', $\overline{\eta}_0$ by $\overline{\eta}_0'$, $\overline{d}$ by $\overline{d}'$ K by K' and proceed exactly in the same fashion.

□

Theorem 4.4: Consider system given by (4.46) with state feedback control (4.47) satisfying Assumptions (1)-(3), with A a convergent matrix, $\xi_1 < \sqrt{\beta}$ and $\xi_0 = 0$. If $x(\cdot) : [k_0,\infty) \rightarrow \mathbb{R}^n$, $x(k_0) = x_0$ is a solution of (4.46), then the origin of (46) is uniformly asymptotically stable in the large.

Proof: Suppose $x_k \notin Ker(B^T PA)$, then using the Lyapunov function candidate

$$V(x_k) = x_k^T P x_k , \tag{4.63}$$

where P is the unique solution of equation (4.9) for a given r.s.p.d. matrix Q, we found that

$$\Delta V^*(x_k,e(k,x_k)) \leq -\lambda_{min}(Q)\|x_k\|^2 + 4\,\lambda_{max}(R)\xi^2(k,x_k)\,,$$
$$\leq -\lambda_{min}(Q)\|x_k\|^2 + 4\,\lambda_{max}(R)\xi_1^2\|x_k\|^2\,,$$
$$\leq -4\,\lambda_{max}(R)\,(\beta - \xi_1^2)\|x_k\|^2\,, \tag{4.64}$$

where $R = B^TPB$.

Let

$$\alpha_4(\|x_k\|) \triangleq 4\,\lambda_{max}(R)(\beta - \xi_1^2)\|x_k\|^2\,, \tag{4.65}$$

then for $\xi_1 < \sqrt{\beta}$, α_4 is a strictly increasing function, and $\Delta V^*(x_k,e(k,x_k)) \leq -\alpha_4(\|x_k\|)$.

If $x_k \in Ker(B^TPA)$, then from equation (4.34)

$$\Delta V^*(x_k,e(k,x_k)) \leq -\lambda_{min}(Q)\|x_k\|^2 + \lambda_{max}(R)\xi^2(k,x_k)$$
$$\leq -\lambda_{min}(Q)\|x_k\|^2 + \lambda_{max}(R)\xi_1^2\|x_k\|^2$$
$$\leq -\lambda_{max}(R)(\beta' - \xi_1^2)\|x_k\|^2 \tag{4.66}$$

Let

$$\alpha_4'(\|x_k\|) \triangleq \lambda_{max}(R)\,(\beta' - \xi_1^2)\|x_k\|^2\,. \tag{4.67}$$

Again, if $\xi_1 < \sqrt{\beta'}$, then as in the case when $x_k \notin Ker(B^TPA)$, we conclude that the origin of the system given by (4.46) is uniformly asymptotically stable in the large.

□

4.4. EXAMPLE

We will now illustrate the level of robustness that we can achieve with the controller derived in this Chapter. Consider the discrete-time dynamical system

$$x_{k+1} = [A + \Delta A(r_k)]x(k) + Bu_k \ ,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\Delta A(r_k) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} r_k \ ,$$

with $|r_k| \leq 0.1$.

We note that the uncertainty matrix $\Delta A(r_k)$ is matched, i.e.,

$$\Delta A(r_k) = BG(r_k) \ ,$$

where

$$G(r_k) = r_k[1 \ \ 1] \ .$$

Since A is an asymptotically stable matrix with poles located at 0.93 and -0.43, we can always find a r.s.p.d. matrix P which uniquely solves equation (4.9) for a given r.s.p.d. matrix Q. Let $Q = I_2$, then

$$P = \begin{bmatrix} 2.247 & 2.597 \\ 2.597 & 7.792 \end{bmatrix}.$$

The uncertainty $e(k,x_k)$ is given by

$$e(k,x_k) = r_k[1 \;\; 1]x_k \; .$$

Clearly

$$\|e(k,x_k)\| \leq |r_k| \, \|[1 \;\; 1]\| \, \|x_k\| = |r_k| \sqrt{2}\|x_k\| \leq 0.1\sqrt{2}\|x_k\| = 0.1414\|x_k\| \, ,$$

which implies that $\xi_0 = 0$ and $\xi_1 = 0.1414$. Now

$$R = B^T P B = 7.792 \, ,$$

$$\beta^{1/2} = \frac{1}{2}\sqrt{\frac{\lambda_{min}(Q)}{\lambda_{max}(R)}} = \frac{1}{2}\sqrt{\frac{1}{7.792}} = 0.179 \, .$$

The condition for ultimate boundedness is satisfied since $\beta^{1/2} > \xi_1 \Longrightarrow (\beta')^{1/2} > \xi_1$. Moreover, $\xi_0 = 0$ implies that the system is uniformly asymptotically stable.

For simulation purposes we let $r_k = 0.1$. Under this condition, $A + \Delta A(r_k) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.6 \end{bmatrix}$ is unstable with poles located at 1.07 and -0.45.

The initial conditions are $x_1(0) = 2$ and $x_2(0) = 1$ and the controller is given by

$$u_k = \begin{cases} -0.1414 \, \text{sgn}\,[3.117x_1(k) + 6.494x_2(k)]\,\|x_k\| \, , & \text{for } \; x_k \notin \text{Ker}[3.117 \;\; 6.494] \\ 0 \; , \;\; \text{for } \; x_k \in \text{Ker}[3.117 \; , \; 6.494] & \end{cases}$$

Figures 4.4 and 4.5 show the time histories of the state variables $x_1(k)$ and $x_2(k)$ of the unforced (free) and controlled uncertain systems. Figure 4.6 displays the time history of the control action applied to the uncertain system.

It is clear from Figure 4.4 and 4.5 that the free uncertain system is unstable and that the above controller yields an asymptotically stable system when the uncertainty is constant. However, we point out that the nominal system could have been asymptotically stabilized using linear state feedback and that the above controller would have then served to robustly maintain the desired level of stability.

4.5. CONCLUSIONS

We considered a class of uncertain discrete-time dynamic systems given by equation (4.1) for which assumptions (1)-(3) were valid. It was noted that the only information required about these uncertainties way their possible size. Synthesis of the controller to stabilize system (4.6) was based on the premise that the overall uncertainty $e(k,x_k)$ belonged to a class of cone bounded functions (4.8) over $\mathbb{R}^n$. It was deduced that $\xi_1 < \sqrt{\beta}$, was a sufficient condition for uniform boundedness and uniform ultimate boundedness of the solution x_k. Finally, we showed that uniform asymptotic stability could be achieved if $\xi_0 = 0$ and $\xi_1 < \sqrt{\beta}$, i.e., if the uncertainty due to the external disturbance Hv_k were zero. The proposed controller (4.11) suffers from the drawback that it is discontinuous in nature, which means that chattering problems would occur if the solution x_k enters and exits the subspace $\mathrm{Ker}(B^TPA)$. Moreover, controller (4.11) also depends on the choice of the matrix Q, which means that one would have to devise an algorithm to choose a Q such that $\lambda_{min}(Q)$ is indeed the largest over all possible choices of Q.

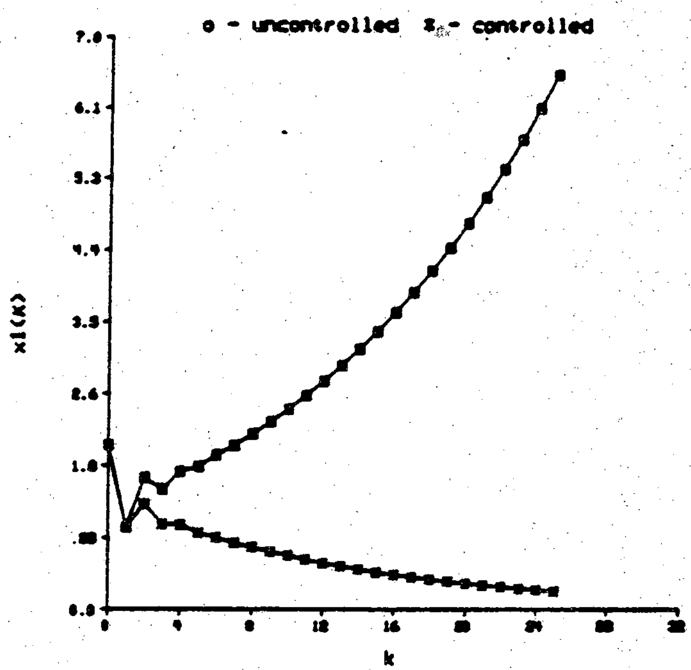

Figure 4.4. Time history of x_1, $x_1(0) = 2$.

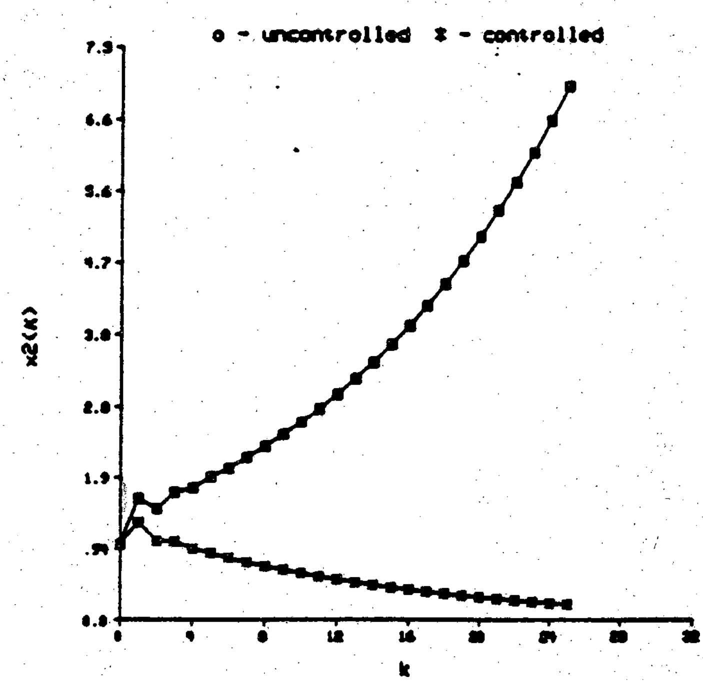

Figure 4.5. Time history of x_2, $x_2(0) = 1$.

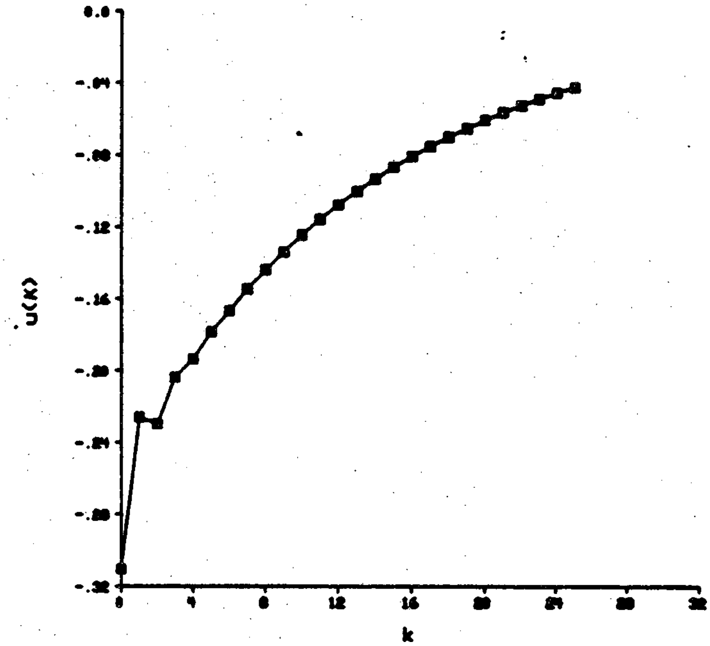

Figure 4.6. Time history of control effort.

Another possible approach to the control problem of discrete uncertain system is via discrete variable structure systems (DVSS) techniques [26] which are also based on the second method of Lyapunov. Preliminary investigations indicate that there is a link between the DVSS approach and our method.

CHAPTER V

ROBUST OUTPUT FEEDBACK STABILIZATION OF DISCRETE-TIME UNCERTAIN DYNAMICAL SYSTEMS

5.1. INTRODUCTION

Recently, there has been a lot of activity in the area of state-feedback stabilization of discrete-time control systems ([10], [20], [30]).

If not all state variables are available; as is usually the case in practice, because either some of them are not accessible or the cost makes it impractical for the designer to utilize measuring devices for every state variable, then a prediction estimator, or a current estimator [10] is used to reconstruct the state vector to implement a feedback control law. Such estimators, however, are dynamic in nature and usually of high order, thus their use is not practical when the designer deals with a high dimensional system.

In this Chapter we shall use the available outputs to stabilize a class of uncertain discrete-time dynamic systems. The approach we shall use to solve this stabilization problem will require no prior statistical information about such uncertainties, except the bounding compact sets where they belong to.

5.2. PROBLEM STATEMENT

Consider a class of discrete-time dynamical systems modeled by the following difference equation

$$\left.\begin{aligned} x_{k+1} &= Ax_k + B(u_k + e(k,x_k)) \ , \ x_{k_0} = x_0 \\ y_k &= Cx_k \end{aligned}\right\} \tag{5.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$; $p \geq m$, A, B and C are constant matrices of appropriate dimensions. Moreover, matrices B and C are assumed to have full rank. The m-valued vector function $e(k,x_k)$ represents the lumped uncertainties of the plant [20].

Let the nominal system, namely, the system without uncertainity be described by

$$x_{k+1} = Ax_k + Bu_k \ , \ x_{k_0} = x_0 \tag{5.2}$$

We now consider the following assumptions:

A.1. The nominal system is stable. If **A** is not stable then we assume that (5.2) is output feedback stabilizable, i.e., there exists a constant matrix $G \in \mathbb{R}^{m \times p}$ such that the spectrum of $A_0 = A - BGC$, $\sigma(A_0)$, is contained in the unit circle, in other words, $\rho(A_0) < 1$, where $\rho(A_0)$ is the spectral radius of A_0.

A.2. There exists a r.s.p.d. matrix $Q \in \mathbb{R}^{n \times n}$, and a matrix $F \in \mathbb{R}^{m \times p}$ such that

$$B^T P A_0 = FC \ ,$$

where P is the unique r.s.p.d. matrix which solves the discrete Lyapunov equation

$$A_0^T P A_0 - P = - Q .$$

A.3. The uncertainity $e(\cdot) : \mathbb{N}\times\mathbb{R}^n \rightarrow \mathbb{R}^m$ is not known but $e(k,x_k)$ belongs to a known compact set $E(k,x_k)$, $\forall (k,x_k) \in \mathbb{N}\times\mathbb{R}^n$. To be exact, the uncertainty $e(\cdot)$ is a cone bounded function over $\mathbb{R}^n$, i.e., $\|e(k,x_k)\| \leq \xi_0 + \xi_1 \|x_k\|$, $\forall$ $k \in \mathbb{N}$ and $x_k \in \mathbb{R}^n$, where $\mathbb{N}$ denotes the set of natural numbers.

Let the Lyapunov function candidate be given by

$$V(x_k) \triangleq x_k^T P x_k , \tag{5.3}$$

where for a given $Q = Q^T > 0$, P solves the discrete Lyapunov equation

$$A_0^T P A_0 - P = -Q . \tag{5.4}$$

The existence of the Lyapunov function given by equation (5.3) is guaranteed by assumption A.1.

We now state the problem: Given system (5.1) subject to the assumption that the matrices B and C have full rank and the assumptions A1-A3 hold, and given the Lyapunov function (5.3), we want to find a function $p(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that if we choose

$$u_k = u_k^* = p(x_k), \tag{5.5}$$

we obtain a minimum bound for $\max_{e \in E} \Delta V(x_k, u_k^*, e(k,x_k))$,

where

$$\Delta V(x_k, u_k, e(k,x_k)) \triangleq V(x_{k+1}) - V(x_k) . \tag{5.6}$$

5.3. DERIVATION OF OUTPUT FEEDBACK CONTROLLER

To find the controller u_k which minimizes $\max_{e \in E} \Delta V(x_k, u_k, e(k,x_k))$ we proceed in the following manner.

Theorem 5.1: Given a discrete-time dynamical system modeled by equation (5.1) and the Lyapunov function (5.3), then if the constant matrices B and C have full rank and if assumptions A1 and A3 hold, the controller

$$u_k = u_k^* = -GCx_k - (B^TPB)^{-1}B^TPA_0x_k \tag{5.7}$$

yields to a minimum bound for $\max_{e \in E} \Delta V$, which is given by

$$\max_{e \in E} \Delta V(x_k, u_k^*, e(k,x_k)) \leq -x_k^TQx_k - x_k^TA_0^TPB(B^TPB)^{-1}B^TPA_0x_k$$

$$+ \lambda_{max}(B^TPB)\xi^2(k,x_k)\ , \tag{5.8}$$

where $\lambda_{max}(B^TPB)$ is the maximum eigenvalue of the symmetric, positive definite matrix B^TPB and

$$\xi(k,x_k) \triangleq \xi_0 + \xi_1 \|x_k\|\ . \tag{5.9}$$

Proof: The proof is basically the same as the one in Manela [20]. The only difference is that the first term in equation (5.7) is used to ensure that the spectral radius of A_0 is strictly less than 1 and that A_0 is used in the second term instead of A for obvious reasons.

Remark: The controller given by (5.7) does not guarantee the negative definiteness of the first forward difference of the Lyapunov function (5.6) for all $x_k \neq 0$. However, when certain conditions (which we shall discuss later) are met by the uncertainty $e(k,x_k)$, $\max_{e \in E} \Delta V$ can be negative for all $x_k \neq 0$.

Theorem 5.2: Given a discrete-time dynamical system modeled by equation (5.1) and the Lyapunov function defined by equation (5.3). If assumption A2 along with the assumptions of Theorem 5.1 hold, and if

$$u_k = u_k^* = -GCx_k - (B^TPB)^{-1}FCx_k \tag{5.10}$$

then

$$\max_{e \in E} \Delta V(x_k, u_k^*, e(k,x_k)) \leq -x_k^TQx_k - x_k^TC^TF^T(B^TPB)^{-1}FCx_k + \lambda_{max}(B^TPB)\xi^2(k,x_k)\,. \tag{5.11}$$

Proof: Without loss of generality, assume that $\rho(A) < 1$, in which case $G = 0$, $A_0 = A$ and $u_k = u_k^* = -(B^TPB)^{-1}FCx_k$.

Explicitly, the first forward difference of the Lyapunov function (equation (5.3)) becomes

$$\begin{aligned}\Delta V(x_k, u_k, e(k,x_k)) &= V(x_{k+1}) - V(x_k) \\ &= -x_k^TQx_k + 2u_k^TB^TPA_0x_k + 2e^T(k,x_k)B^TPA_0x_k \\ &\quad + 2u_k^TB^TPBe(k,x_k) + u_k^TB^TPBu_k \\ &\quad + e^T(k,x_k)B^TPBe(k,x_k)\,. \end{aligned} \tag{5.12}$$

Let

$$R \triangleq B^T P B . \tag{5.13}$$

Substituting $u_k = u_k^* = -R^{-1}FCx_k$ into equation (5.12), we get

$$\Delta V(x_k,u_k^*,e(k,x_k)) = -x_k^T Q x_k - 2x_k^T C^T F^T R^{-1} B^T P A_0 x_k + 2e^T(k,x_k) B^T P A_0 x_k$$
$$- 2x_k^T C^T F^T e(k,x_k) + x_k^T C^T F^T R^{-1} F C x_k + e^T(k,x_k) R e(k,x_k) .$$

Using Assumption A2, i.e., $B^T P A_0 = FC$, we get

$$\Delta V(x_k,u_k^*,e(k,x_k)) = -x_k^T Q x_k - x_k^T C^T F^T R^{-1} F C x_k + e^T(k,x_k) R e(k,x_k) .$$

Maximizing ΔV over all values of e, $e \in E$, yields

$$\max_{e \in E} \Delta V(x_k,u_k^*,e(k,x_k)) = -x_k^T Q x_k - x_k^T C^T F^T R^{-1} F C x_k$$
$$+ \max_{e \in E} \{e^T(k,x_k) R e(k,x_k)\}$$
$$\leq -x_k^T Q x_k - x_k^T C^T F^T R^{-1} F C x_k + \lambda_{max}(R) \xi^2(k,x_k) ,$$

where R is given by equation (5.13).

□

Manela [20] has already shown that if $e(\cdot)$ is a cone bounded function, i.e.,

$$\max_{e \in E} \|e(k,x_k)\| \leq \xi(k,x_k) = \xi_0 + \xi_1 \|x_k\| ,$$

and if the matrix A in the nominal system is asymptotically stable, that one can achieve uniform boundedness and uniform ultimate boundedness using full state feedback if $\xi_0 \neq 0$ and $\frac{\lambda_{min}(Q)}{\lambda_{max}(R)} > \xi_1^2$, and that asymptotic stability can be attained if $\xi_0 = 0$ and $\frac{\lambda_{min}(Q)}{\lambda_{max}(R)} > \xi_1^2$. Therefore, it is clear that if assumptions (A1) and (A2) hold, then we can obtain the same results using output feedback, i.e.,

$$u_k = -\,Gy_k - (B^TPB)^{-1}Fy_k \,. \tag{5.14}$$

5.4. CONTROLLER DESIGN

So far nothing has been said about the conditions under which the matrices Q and F exist such that assumption A2 holds. We shall address this issue later in the report.

For the time being, however, we shall present one possible algorithm [27] that the designer can use to obtain the matrices F and Q such that

$$B^TPA_0 = FC \,, \tag{5.15}$$

where P is the unique, r.s.p.d. matrix which solves the discrete Lyapunov equation

$$A_0^TPA_0 - P = -\,Q \,. \tag{5.16}$$

ALGORITHM

Step 1. Pick a constant matrix G such that the spectral radius of $A_0 = A - BGC$ is strictly less than one.

Note that in Step 1 we assume that the system modeled by equation (5.1) is output feedback stabilizable.

Step 2. Solve the matrix equation

$$B^TPA = FC \,,$$

such that the matrix P can be expressed in terms of the components of F and P is symmetric.

Step 3. Express the matrix Q in terms of P, i.e., $Q(P) = P - A_0^T P A_0$.

Step 4. Choose the components of Q such that its leading principal minors are greater than zero.

Execution of Step 4 results in the determination of the numerical values of the components of the matrix F and therefore of the matrix P.

We showed in Theorem 5.2 that uniform boundedness and uniform ultimate boundedness (see Appendix) can be achieved if the condition

$$\frac{\lambda_{min}(Q)}{\lambda_{max}(R)} > \xi_1^2 \tag{5.17}$$

holds, where R is given by (5.13). This suggests that Step 4 could be modified in such a way that $\lambda_{min}(Q)$ is as large as possible to accommodate for larger uncertainties.

5.5. AN EXAMPLE

Consider the following second order linear discrete-time uncertain dynamical system.

$$\left.\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ 0.4 & 0.5 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u_k + e(k,x_k)) \\ y_k &= [1 \ \ 0] \, x_k \end{aligned}\right\} \tag{5.18}$$

where

$$e(k,x_k) = r_k[1 \ \ 1] \, x_k \, . \tag{5.19}$$

Here, the uncertainty satisfies the matching condition [21].

Now,

$$||e(k,x_k)|| \le \sqrt{2}\ |r_k|\ ||x_k|| = \xi_1\ ||x_k|| , \tag{5.20}$$

thus, $\xi_0 = 0$ and $\xi_1 = \sqrt{2}\ |r_k|$.

We now compute matrices F and P.

Step 1. Since A is already a convergent matrix with eigenvalues located at 0.93 and -0.43, we can choose G equal to zero. Therefore, $A_0 = A$.

Step 2. Equating B^TPA to FC and solving P in terms of F we get

$$B^TPA = [0.4\ p_3\ \ p_2 + 0.5\ p_3] = [f\ \ 0] = FC ,$$

thus,

$$P = \begin{bmatrix} p_1 & -1.25f \\ -1.25f & 2.5f \end{bmatrix}.$$

Step 3. Form the matrix Q(P).

$$Q(P) = \begin{bmatrix} p_1 - 0.4f & -1.25f \\ -1.25f & 3.125f - p_1 \end{bmatrix}.$$

Step 4. Choosing the components of Q(P) such that the leading principal minors are positive yields the following conditions.

(i) $p_1 > 0.4f$,

and

(ii) $(p_1 - 1.22f)(2.3f - p_1) > 0$, or $p_1 \in (1.22f, 2.3f)$. Clearly, condition (ii) implies condition (i), hence, we have to choose p_1 such that $p_1 \in (1.22f, 2.3f)$. Letting $f = 1$, we have that $p_1 \in (1.22, 2.3)$, $p_2 = -1.25$ and $p_3 = 2.5$.

From equation (5.13) we find that $R = p_3 = 2.5$ and therefore $\lambda_{max}(R) = 2.5$. To get $\lambda_{min}(Q)$ to be as large as possible, one can show that $p_1 = 1.7625$ yields such maximum. Hence, the matrices P and Q are finally given by

$$P = \begin{bmatrix} 1.7625 & -1.25 \\ -1.25 & 2.5 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 1.3625 & -1.25 \\ -1.25 & 1.3625 \end{bmatrix}.$$

For simulation purposes, we let $r_k = 0.1$, which implies that the state equation (5.18) can be rewritten as

$$x_{k+1} = A_1 x_k + B u_k ,$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A_1 are 1.07 and -0.45, therefore, A_1 is unstable.

Now,

$$\sqrt{\frac{\lambda_{min}(Q)}{\lambda_{max}(R)}} = \sqrt{\frac{0.1125}{2.5}} = 0.212 ,$$

thus

$$\xi_1 = 0.1414 < 0.212 = \sqrt{\frac{\lambda_{min}(Q)}{\lambda_{max}(R)}} ,$$

which implies that the controller

$$u_k = -R^{-1}Fy_k = -R^{-1}FCx_k = -0.4x_1(k)\ , \tag{5.21}$$

will yield a closed-loop asymptotically stable system, (see Figures 5.1 and 5.2 for initial condition $x_0 = [2\ \ 1]^T$). Figure 5.3 shows the time history of the control effort (5.21) necessary to drive the system (5.18) to the origin.

5.6. COMMENTS ON ASSUMPTION A2

Steinberg and Corless [25] showed that the output stabilization of a class of continuous-time uncertain dynamical systems problem can be solved if there exist real matrices $F_c \in \mathbb{R}^{m\times p}$ and $Q_c \in \mathbb{R}^{n\times n}$, $Q_c = Q_c^T > 0$ such that

$$B_c^T P_c = F_c C_c\ , \tag{5.22}$$

$$P_c A_c + A_c^T P_c = -Q_c\ , \tag{5.23}$$

where the subindex c stands for continuous-time and A_c is asymptotically stable.

They showed that the sufficient condition for the existence of such matrices is that the transfer function matrix

$$G_F(s) = F_c C_c (sI - A_c)^{-1} B_c \tag{5.24}$$

be strictly positive real [29].

In the light of the results obtained by Steinberg and Corless for the continuous-time case, one would be tempted to extend their results to the discrete-time case. However, as Hitz and Anderson [30] show, the conditions under which the transfer function matrix $G_D(z)$ of a discrete-time dynamical system is positive real, do not lead to the conclusion of the existence of the real matrices F and Q that satisfy assumption A2.

Consequently, other avenues have to be searched to determine the conditions under which the real matrices F and Q that satisfy assumption A2 exist.

5.7. CONCLUSIONS

We showed that the problem of robustly stabilizing the class of discrete-time uncertain dynamical systems described by equation (5.1), where the uncertainty was of the cone bounded type, could be solved by using output feedback provided that the algebraic constraint described in Assumption 2 were satisfied and that $\dfrac{\lambda_{min}(Q)}{\lambda_{max}(R)} > \xi_1^2$. However, as was pointed out in the last Section, the question of a system theoretic interpretation of the existence of the real matrices F and Q that satisfy assumption A2 has not yet been resolved and remains an open problem.

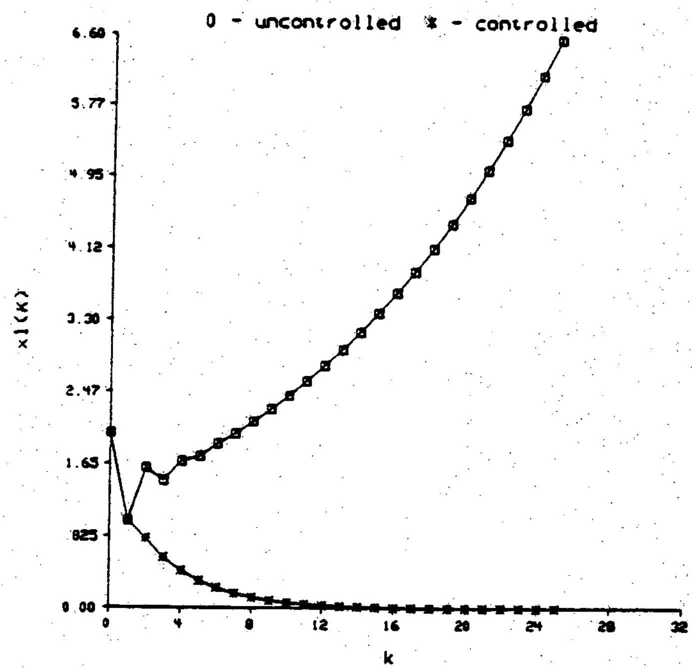

Fig. 5.1. Time history of x_1, $x_1(0) = 2$.

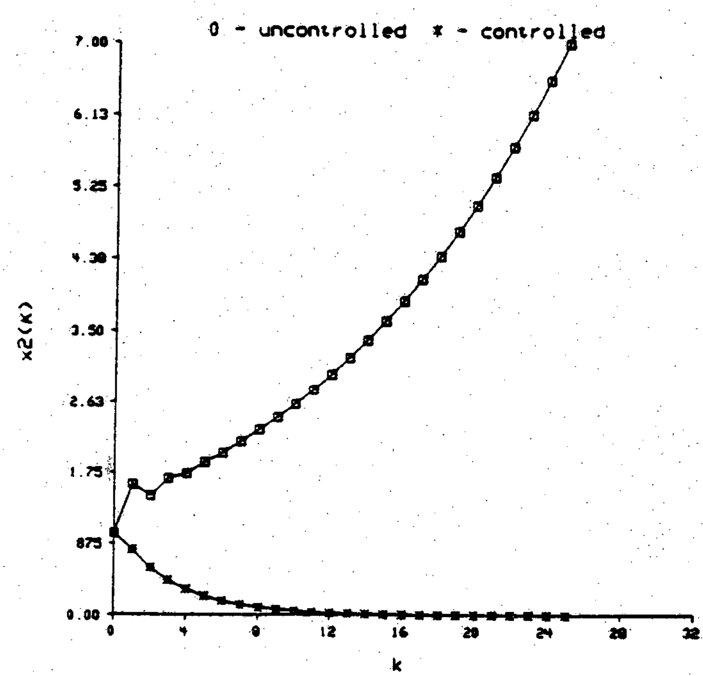

Fig. 5.2. Time history of x_2, $x_2(0) = 1$.

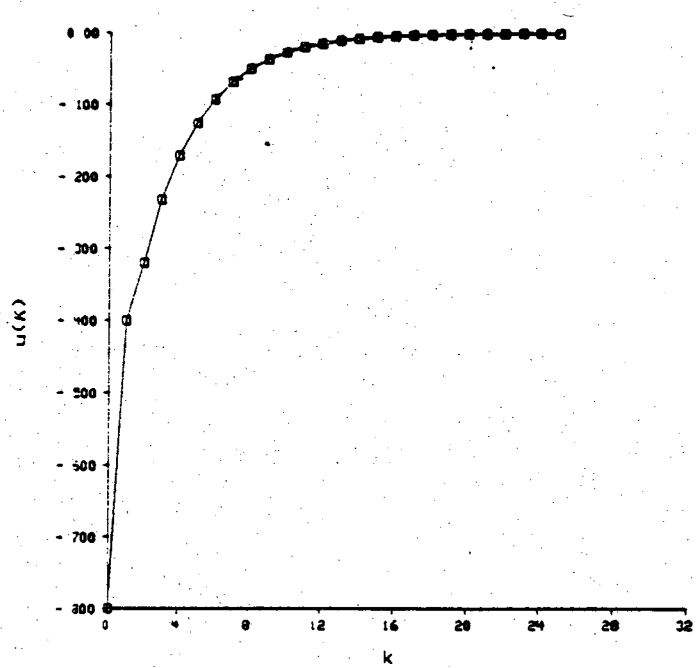

Fig. 5.3. Time history of the control effort u(k).

CHAPTER VI

ROBUST STABILITY OF DISCRETE-TIME DYNAMICAL SYSTEMS PROJECTED ONTO A DESIRED HYPERPLANE

6.1. INTRODUCTION

Up to now we were concerned with the problem of steering the state trajectory of linear time-invariant discrete dynamical systems onto desired hyperplanes where they possess certain stability properties and reduced dimensionality. We also analyzed the problem of robust stabilization of a class of discrete-time uncertain dynamical systems whose "nominal" system is linear, stable and the uncertainties do not depend on the input.

In this Chapter we make an attempt at putting together the theories proposed in Chapters 3 and 4.

Before we go on any further, we should relize that the feedback control laws derived in Chapter 3 can only be applied to the "nominal" system since they were not designed to handle parameter uncertainties or external disturbances. To resolve the uncertainties problem, we shall utilize the controller derived in Chapter 4.

6.2. COMPOSITE CONTROLLER

Let a linear time invariant discrete dynamical system be governed by the following equation

$$x_{k+1} = Ax_k + B[u_k + e(k,x_k)] \; , \tag{6.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ are the state and control vectors, respectively, $e(k,x_k) \in \mathbb{R}^m$ represents the uncertainties and A and B are constant matrices of appropriate dimensions.

As in Chapter 4, we shall assume that $e(k,x_k)$ is a cone bounded uncertainty, i.e.,

$$\|e(k,x_k)\| \leq \xi(k,x_k) = \xi_0 + \xi_1 \|x_k\| \; . \tag{6.2}$$

Define the "nominal" system by

$$x_{k+1} = Ax_k + Bu_k \; . \tag{6.3}$$

We would like to drive the state trajectory of system (6.1) onto the hyperplane Ker(S) as fast as possible and in such a way that once it reaches it, it slides on it towards the origin. However, we now have to resolve the additional problem of the presence of the uncertainty $e(k,x_k)$. If we were to try to solve this problem by merely applying any of the controllers proposed in Chapter 3 to system (6.1) we would soon find out that Ker(S) would not be reached because of the uncertainties.

A possible solution to the above problem is to apply a controller which is a hybrid combination of those developed in Chapters 3 and 4.

To use the controller proposed in Chapter 4, it was assumed that the free "nominal" system was asymptotically stable, therefore, we shall first stabilize the "nominal" system by applying the feedback control strategies derived in Chapter 3. Because of practical reasons, however, we will exclude the time depending controller in order to avoid the problem of having to compute the solution to the Lyapunov equation at every time step.

6.2.1. Composite Controller I

Let

$$u_k^h = (SB)^{-1}[\Lambda S - SA]x_k \;, \tag{6.4}$$

be the linear feedback controller that drives the state trajectory of the "nominal" system onto the hyperplane

$$\sigma_k = Sx_k \;, \tag{6.5}$$

where $S \in \mathbb{R}^{m \times n}$ is a constant matrix whose components are picked such that the inverse of the matrix product SB exists and the "nominal" system, when constrained to the hyperplane (6.5), possesses certain predetermined stability characteristics. Moreover, the matrix $\Lambda \in \mathbb{R}^{m \times m}$ is a convergent matrix whose components are chosen according to how fast we want the state trajectory of (6.3) to reach the hyperplane (6.5).

Let

$$u_k^r = \begin{cases} -\dfrac{R^{-1}B^TP\,A_0x_k}{\|B^TPA_0x_k\|_{R^{-1}}}\,\gamma(k,x_k)\;, & \text{if } x_k \notin \mathrm{Ker}(B^TPA_0) \\ 0\;, & \text{if } x_k \in \mathrm{Ker}(B^TPA_0) \end{cases} \tag{6.6}$$

be the feedback controller that stabilizes the system (6.1) assuming that the "nominal" system has been asymptotically stabilized by applying u_k^h to (6.3), where $P \in \mathbb{R}^{n \times n}$ is the unique r.s.p.d. solution to the discrete Lyapunov equation

$$A_0^T P A_0 - P = -Q\ , \tag{6.7}$$

for a given $Q = Q^T > 0$, $R = B^T P B$, B has rank m, $\gamma(k,x_k) = \lambda_{max}^{1/2}(R)\xi(k,x_k)$, $\|B^T P A_0 x_k\|_{R^{-1}} = (x_k^T A_0^T P B R^{-1} B^T P A_0 x_k)^{1/2}$, and $A_0 = A + B(SB)^{-1}[\Lambda S - SA]$.

Theorem 6.1: Consider the system (6.1) and the state feedback control

$$u_k = u_k^h + u_k^r\ . \tag{6.8}$$

If $\xi_1 < \sqrt{\beta}$, and $\xi_0 = 0$ where $\beta \triangleq \dfrac{\lambda_{min}(Q)}{4\lambda_{max}(R)}$, then if the controller (6.8) is applied to the system (6.1), then the resulting closed-loop system is asymptotically stable. Furthermore, the origin may be reached via a sliding mode.

Proof: See the proofs of Theorem 3.5 and Theorem 4.4.

Corollary 6.1: If $\xi_0 > 0$ and $\xi_1 < \sqrt{\beta}$, then the application of the controller (6.8) to the system (6.1) results in a closed-loop system which is at least uniformly ultimately bounded.

Proof: See the proofs of Theorems 3.5, 4.2 and 4.3.

Example 6.1: Let us consider the discrete-time dynamical system modeled by the equation

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10+r_k & 9 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_k \, . \tag{6.9}$$

Rewriting the above system equations we get

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 9 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} [u_k + e(k,x_k)] \, , \tag{6.10}$$

where $e(k,x_k) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r_k & 0 \end{bmatrix} x_k$, which implies that $\|e(k,x_k)\| \leq |r_k| \, \|x_k\|$.

The free nominal system has its eigenvalues located at -1, 1, 5 and 10. We want the equivalent second order nominal system to have its eigenvalues at 0.1 and 0.2. The following choice of S will yield such eigenvalues

$$S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1.32 & 0 & -1.3 & 1 \end{bmatrix} .$$

Let

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in (0,1) \, ,$$

then

$$u_k = u_k^h = \begin{bmatrix} \lambda_1 + 5 & \lambda_1 - 7 & \lambda_1 - 1 & -2 \\ -1.32\lambda_2 & 1.32 & -1.3\lambda_2 - 10 & \lambda_2 - 7.7 \end{bmatrix} x_k \, . \tag{6.11}$$

Application of the above controller to (6.10), yields

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_1 & \lambda_1 - 1 & \lambda_1 & -1 \\ 0 & 0 & 0 & 1 \\ -1.32\lambda_2 & 1.32 & -1.3\lambda_2 & \lambda_2 + 1.3 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} e(k,x_k) \ . \quad (6.12)$$

The eigenvalues of the compensated free nominal system are located at $\lambda_1 = 0.1$, $\lambda_2 = 0.2$. Hence the nominal system is asymptotically stable since $\lambda_1, \lambda_2 \in (0,1)$.

Letting $\lambda_1 = 0.5$, $\lambda_2 = 0.4$, $r_k = \pm\ 0.11$ $(\beta^{1/2} > \xi_1 = |r_k|)$ and $x_0 = [5 \ \ -1 \ \ 2 \ \ 1]^T$, we can see in Figures 6.1 through 6.4 that the application of the controller (6.6) to the system (6.9), after the controller (6.4) has been applied, does indeed yield a closed-loop system that is asymptotically stable.

We note that for this particular example $\gamma(k,x_k)$ is given by

$$\gamma(k,x_k) = 0.11\ \lambda_{max}^{1/2}(R)\ ||x_k||$$

Furthermore, for $Q = I_4$ the ratio $\lambda_{min}(Q)/\lambda_{max}(R)$ is maximum and the matrices P and R are found to be

$$P = \begin{bmatrix} 5.065 & 1.421 & -4.885 & -1.461 \\ 1.421 & 4.065 & -4.284 & -4.885 \\ -4.885 & -4.284 & 1.036 & 6.741 \\ -1.461 & -4.885 & 6.741 & 9.366 \end{bmatrix},$$

$$R = \begin{bmatrix} 4.065 & -4.885 \\ -4.885 & 9.366 \end{bmatrix}.$$

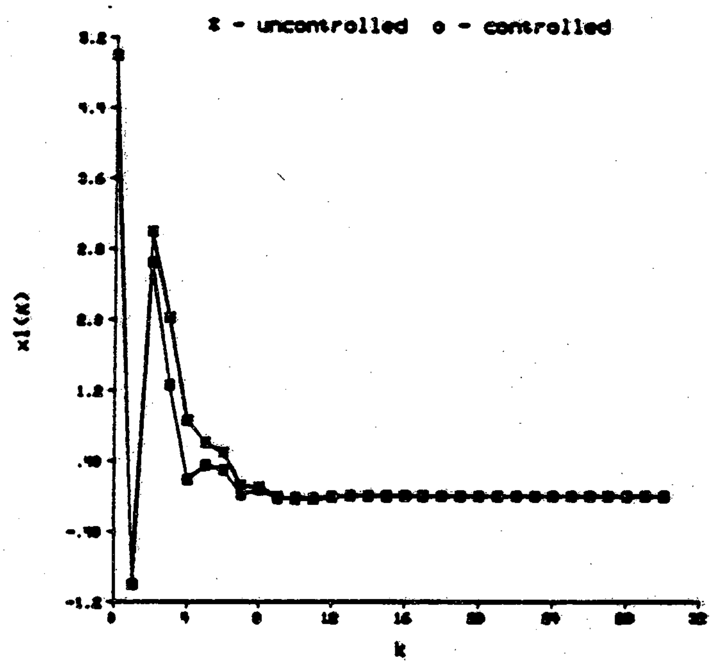

Fig. 6.1. Time evolution of x_1.

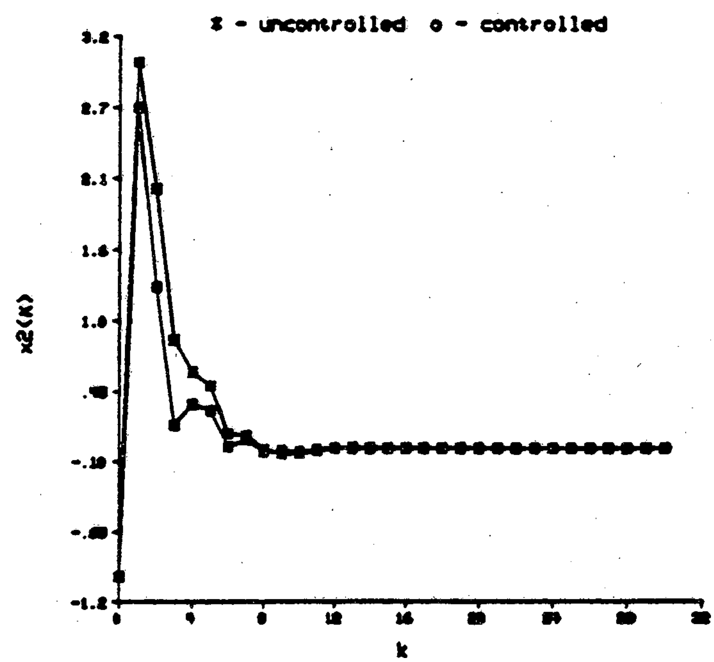

Fig. 6.2. Time evolution of x_2.

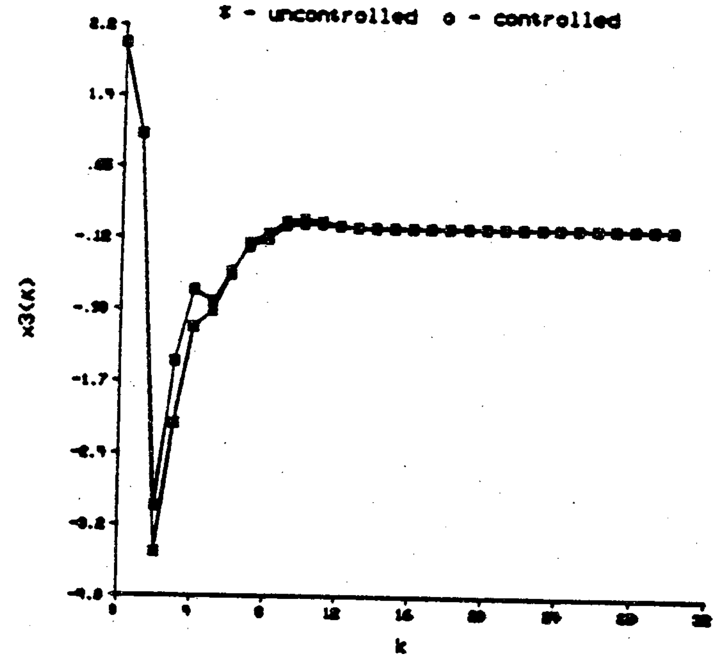

Fig. 6.3. Time evolution of x_3.

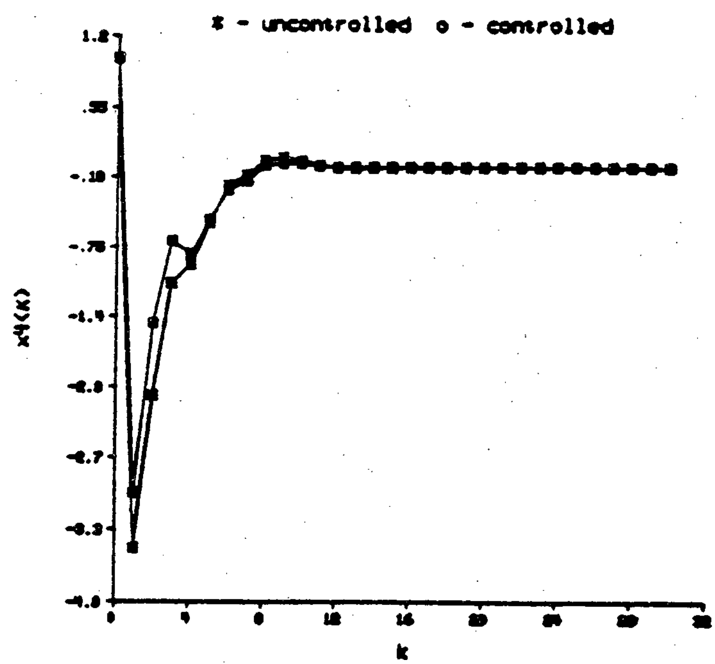

Fig. 6.4. Time evolution of x_4.

Example 6.2: Let us now consider the discrete-time dynamical system given by

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 9 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} [u_k + e(k,x_k)] \ , \qquad (6.13)$$

where

$$e(k,x_k) = \begin{bmatrix} 0 \\ 0.5 \sin(0.1\pi k) \end{bmatrix}, \qquad (6.14)$$

and $e||(k,x_k)|| \leq 0.5$.

If we first apply the controller (6.11) to the system (6.13) we find by looking at Figures 6.5 through 6.8 that the external disturbance goes through the system without being attenuated. However, after applying controller (6.11) along with controller (6.6) to system (6.13) we see that the disturbance is attenuated.

In this example,

$$\gamma(k,x_k) = 0.5 \ \lambda^{1/2}_{max}(R) \ .$$

Also, matrices P and R are the same as those used in the previous example.

Observation: Whenever an external disturbance is applied to the system (6.1), the controller proposed here decreases the effects of such a disturbance. However the controller is still unable to drive the state trajectory onto the desired hyperplane.

6.2.2. Composite Controller II

We now let

$$u_k^h = -(SB)^{-1}SAx_k \, . \tag{6.15}$$

Theorem 6.2: If we apply the controller

$$u_k = u_k^h + u_k^r \, ,$$

where u_k^h is now given by equation (6.15) and u_k^r by equation (6.6), to the system (6.1), then the closed-loop system is asymptotically stable whenever $\xi_0 = 0$ and $\xi_1 < \sqrt{\beta}$.

Proof: See the proofs of Theorems 3.7 and 4.4.

Corollary 6.2: If $\xi_0 \neq 0$, then the application of the above controller to the system (6.1) yields a closed-loop system that is at least uniformly ultimately bounded.

Proof: See the proofs of Theorems 3.7, 4.2 and 4.3

Example 6.3: We again consider the system as in Example 6.1, except that $r_k = \pm\, 0.18$ since the application of controller (6.15) to the "nominal" system in (6.9) produces a maximum parameter β such that $\sqrt{\beta} > 0.18$ when our choice of the hyperplane $\sigma(x_k) = 0$ is

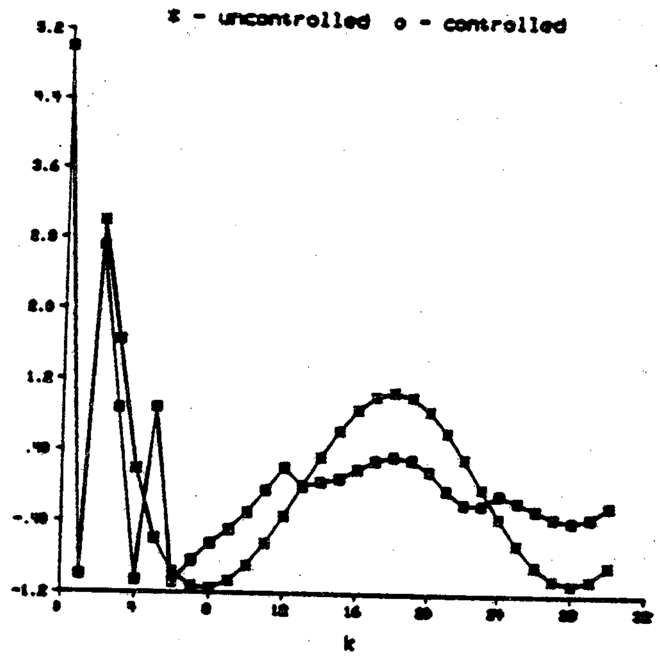

Fig. 6.5. Time evolution of x_1.

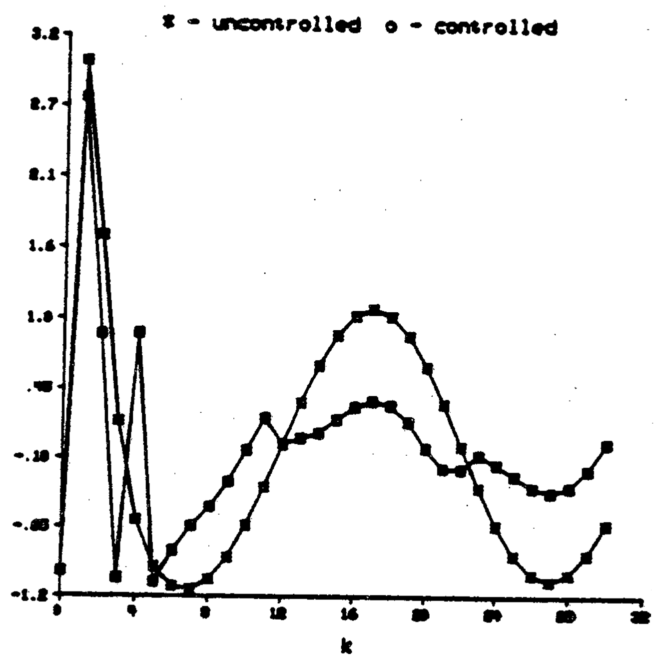

Fig. 6.6. Time evolution of x_2.

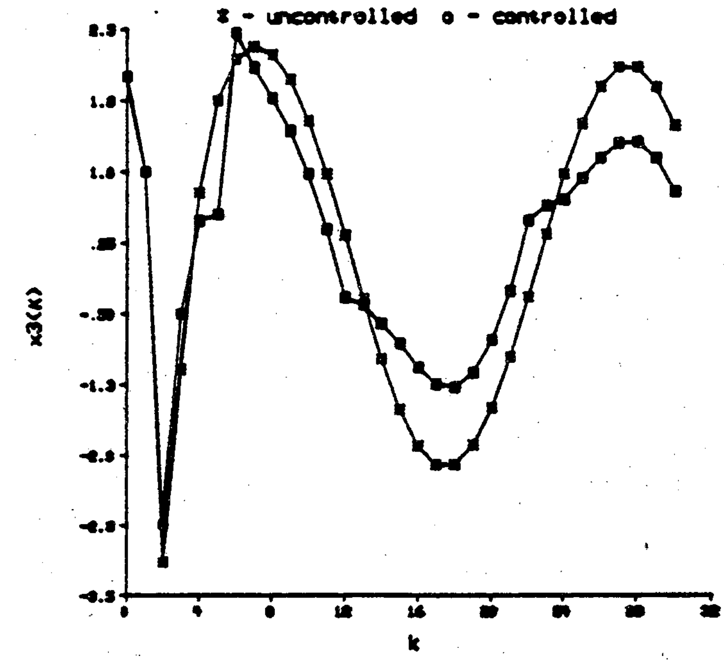

Fig. 6.7. Time evolution of x_3.

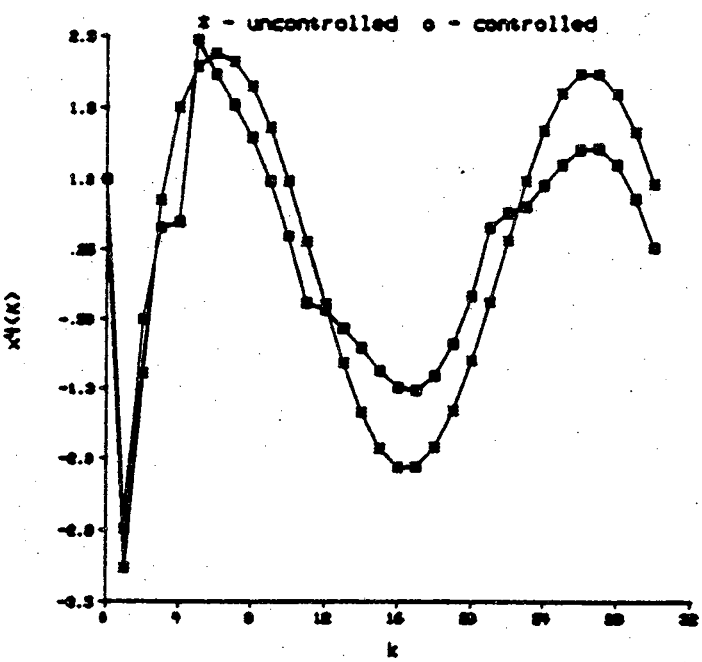

Fig. 6.8. Time evolution of x_4.

$$\sigma(x_k) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1.32 & 0 & -1.3 & 1 \end{bmatrix} x_k = 0.$$

Figures 6.9 through 6.12 show that the origin is reached faster when controller $u_k = u_k^h + u_k^r$ is applied to the system in question.

6.3. CONCLUSIONS

The controllers we proposed in this Chapter enable the class of linear time-invariant discrete dynamical system modeled by (6.1) to be robustly stabilized. However, the size of the uncertainty is limited by the constraint $\sqrt{\beta} > \xi_1$. Furthermore, the hyperplane $\sigma(x_k) = 0$ can not be reached by the system when an external disturbance is applied even though its effect is greatly reduced.

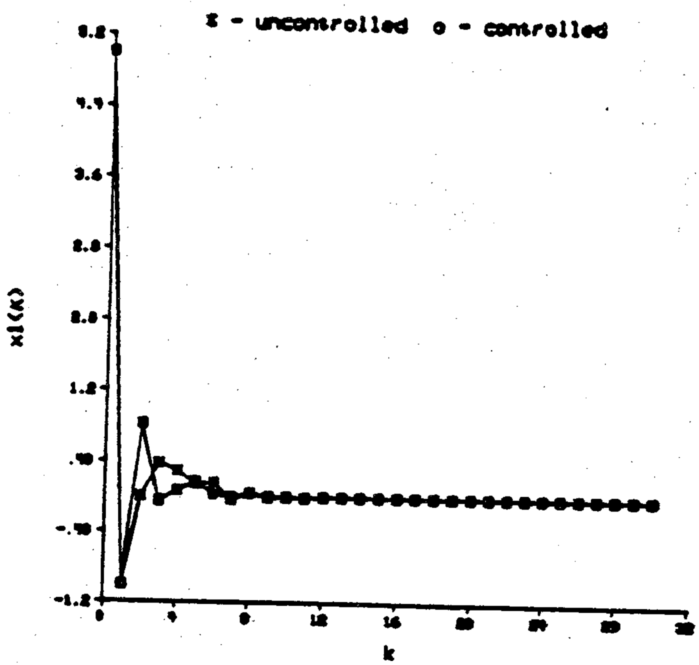

Fig. 6.9. Time evolution of x_1.

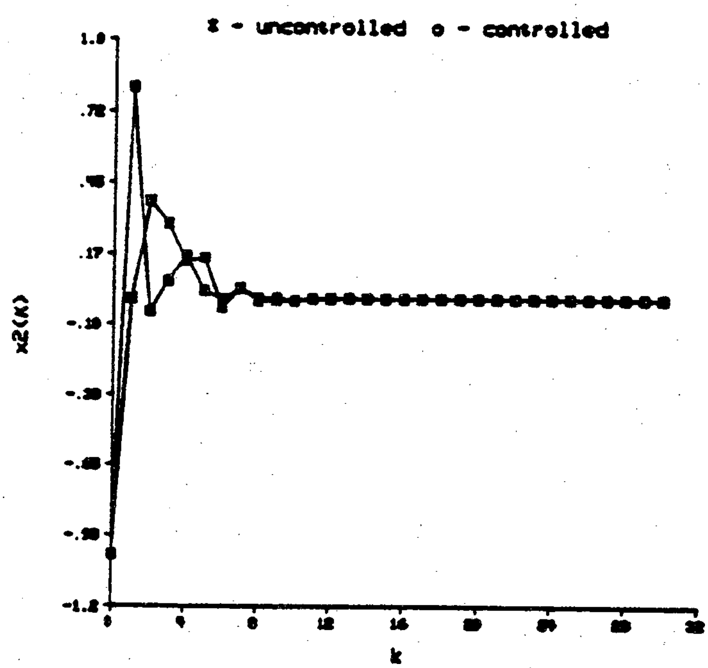

Fig. 6.10. Time evolution of x_2.

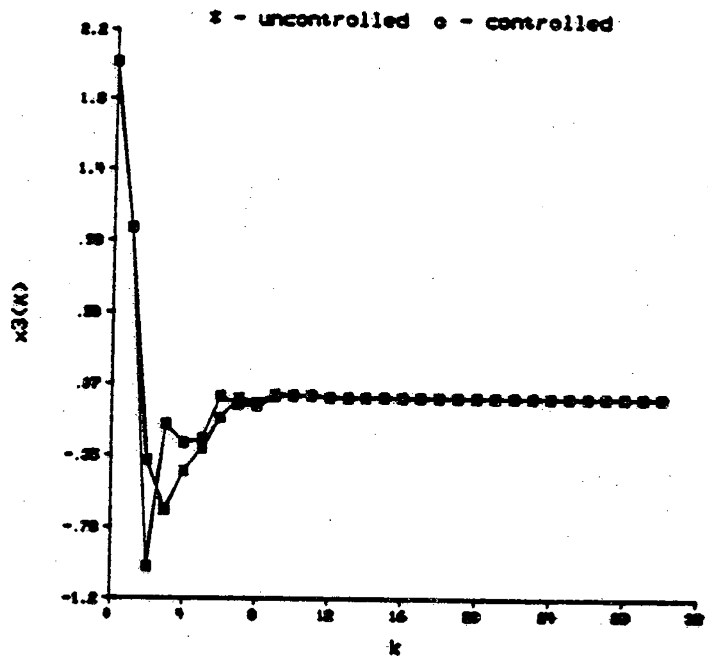

Fig. 6.11. Time evolution of x_3.

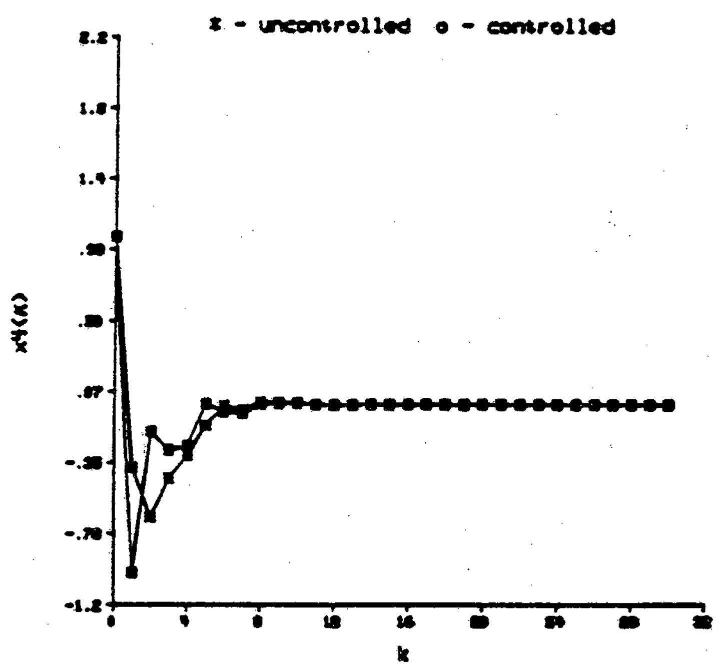

Fig. 6.12. Time evolution of x_4.

CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1. SUMMARY

Motivated by the fact that the goal of this research was to design stabilizing controllers for a class of discrete-time uncertain dynamical systems via the second method of Lyapunov, we presented a review of Lyapunov stability theory of discrete-time dynamical systems in Chapter 2. In this chapter, we selected and presented the definitions and theorems which we considered to be the most useful to our purposes. Next, we introduced the notions of uniform boundedness and uniform ultimate boundedness since they were at the heart of the developments in Chapters 4 and 5.

Our quest to try to extend the idea of a sliding mode of continuous-time variables structure systems led us to develop, in Chapter 3, several control strategies which stabilized linear time invariant discrete dynamical systems by projecting their state trajectories onto hyperplanes where they were guaranteed to possess reduced dimensions along with prescribed degrees of stability. To be specific, we proposed three controllers that steer the state trajectory of these systems onto hyperplanes and keep them there until the origin is reached.

In Chapters 4 and 5 we concentrated our efforts on the development of full state feedback and output feedback controllers, respectively, to stabilize a class of linear time invariant discrete uncertain dynamical systems where the "nominal" system was asymptotically stable and the uncertainties did not depend on the control input and belonged to known compact bounding sets. We found in these chapters that if the uncertainties were of the cone bounded type, i.e., the uncertainty vector $e(k,x_k)$ was bounded by $\xi(k,x_k)$, where

$$\xi(k,x_k) \triangleq \xi_0 + \xi_1 \|x_k\| ,$$

and $\xi_1 < \sqrt{\beta}$, where

$$\beta \triangleq \frac{\lambda_{min}(Q)}{4\,\lambda_{max}(R)} ,$$

then uniform boundedness and uniform ultimate boundedness could be guaranteed. Additionally, we found that if $\xi_0 = 0$ and $\xi_1 < \sqrt{\beta}$, then we could achieve asymptotic stability. We also found that the size of the uncertainty was limited by the constraint that ξ_1 must be strictly less than $\sqrt{\beta}$.

Finally, in Chapter 6 we attempted to unify the theories developed in Chapters 3 and 4 in order to robustly stabilize the class of systems discussed in Chapter 4.

7.2. CONCLUSIONS AND OPEN PROBLEMS

7.2.1. Conclusions

We have devised in this work a new solution to the problem of stabilizing discrete-time dynamical systems by projecting their state trajectories onto prespecified hyperplanes where such systems possess desired levels of stability as well as reduced dimensions.

We have also proposed a method to stabilize a class of discrete-time dynamical systems with uncertainties that can be characterized by cone bounded functions. The main feature of this approach is that it does not require knowledge of the statistics of the uncertainties, it only assumes that such uncertainties lie in known closed and bounded sets.

We also put the two theories together and succeeded in driving the state trajectories of discrete-time dynamical systems with uncertainties in the system matrix onto prespecified hyperplanes. However, we were not successful in steering such trajectories to the hyperplanes when external disturbances were applied, even though their effects were substantially reduced.

7.2.2. Open Problems

During the course of investigation we encountered many interesting problems. Many of them remain to be solved. Among more interesting open problems, in our opinion, are

(i) Justification of assumption A2 in Chapter 5 from the system theoretic point of view, specifically the problem of the existence of real matrices

F and $Q = Q^T > 0$ such that

$$B^T PA = FC ,$$

where $P = P^T > 0$ solves the discrete Lyapunov equation

$$A^T PA - P = - Q ,$$

where A is assumed to be a convergent matrix, remains open.

(ii) We need to design a controller such that the trajectories of the systems we have studied can be driven onto prespecified hyperplanes when the systems are subjected to external disturbances. The results in [40] should be of help in this endeavor.

(iii) Investigation of the Lie algebraic approach to the control and synthesis of nonlinear discrete-time systems seems to be another fertile area of study. Methods developed in [37], [38], [39], and [42] constitute a nice starting point in this direction. Preliminary results are quite encouraging. Our approach can be summarized as follows. For a given nonlinear discrete-time system we first find a transformation bringing the system into a canonical form. Then we design a controller for the system in the new coordinates. From the above considerations it follows that the problem of the existence of a "nice" transformation is central in the design process. To be more specific let us consider a dynamical system modeled by the following equations

$$x(k+1) = a(x(k)) + b(x(k))u(k) \tag{7.1}$$

where a and b are C^∞ vector fields on $\mathbb{R}^n$ with $a(0) = 0$.

The problem is to find sufficient conditions on a and b so that there exists a C^∞ transformation

$$x^*(k) = T(x(k)) \; : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that the system (7.1) can be transformed into the controller canonical form

$$x^*(k+1) = \begin{bmatrix} x_2^*(k) \\ \vdots \\ x_n^*(k) \\ f(x^*(k)) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k) \, . \qquad (7.2)$$

In further considerations the following notation and definitions are used. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^∞ vector fields on $\mathbb{R}^n$. For f and g the Lie bracket is

$$[f,g] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f \, ,$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial x}$ are the Jacobian matrices of f and g, respectively. Using an alternative notation, one can represent the Lie bracket as follows

$$[f,g] = (ad^1 f,g) \, .$$

We define

$$(ad^k f,g) = [f,(ad^{k-1} f,g)] \, ,$$

where

$$(ad^0 f,g) = g \, .$$

Next, consider a C^∞ function $h: \mathbb{R}^n \longrightarrow \mathbb{R}$. Let $dh = \nabla^T h$ be the derivative of h with respect to x, where ∇h is the gradient of h with respect to x. Then the Lie derivative of h with respect to f is defined by

$$L_f h \triangleq L_f(h) = \langle dh,f \rangle = \nabla^T h \cdot f \,,$$

and

$$L_f^0 h = h \,,$$
$$L_f^k h = L_f(L_f^{k-1} h) \,.$$

The Lie derivative of dh with respect to the vector field f is defined by

$$L_f(dh) \triangleq \left(\frac{\partial (dh)^T}{\partial x} f \right)^T + (dh) \frac{\partial f}{\partial x} \,.$$

One may easily verify that these Lie derivatives obey the following so-called Leibnitz formula

$$L_{[f,g]} h = \langle dh,[f,g] \rangle = L_g L_f h - L_f L_g h \,.$$

Furthermore, the following relation is valid

$$dL_f h = L_f(dh) \,.$$

Duly armed with the Lie derivativeds we may proceed further. Taking the differential of (7.1) yields

$$dx^* = \frac{\partial T}{\partial x} dx \,. \tag{7.3}$$

If we now use the following approximations

$$dx^* = \Delta x^* = x^*(k+1) - x^*(k) \,,$$
$$dx = \Delta x = x(k+1) - x(k) \,,$$

then (7.3) can be represented as

$$x^*(k+1) - x^*(k) = \frac{\partial T}{\partial x} (x(k+1) - x(k)) \,. \tag{7.4}$$

Substituting $x(k+1) = a(x(k)) + b(x(k))u(k)$ into (7.4) gives

$$x^*(k+1) = \frac{\partial T}{\partial x}\,[a(x(k)) + b(x(k))u(k) - x(k)] + x^*(k) \tag{7.5}$$

Comparing (7.5) and (7.2) yields

$$\frac{\partial T}{\partial x}\Big[a(x(k)) - x(k)\Big] + x^*(k) + \begin{bmatrix} x_2^*(k) \\ x_3^*(k) \\ \vdots \\ x_n^*(k) \\ f(x^*(k)) \end{bmatrix} = \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_n \\ f(x^*(k)) \end{bmatrix}, \tag{7.6}$$

and

$$\frac{\partial T}{\partial x}\,b(x(k)) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{7.7}$$

Hence from (7.6) we get

$$\frac{\partial T_i}{\partial x}\,[a(x(k)) - x(k)] + x_i^*(k) = T_{i+1}\,,$$

$$i = 1,2,\ldots,n-1\,. \tag{7.8}$$

Let

$$\bar{a}(x) \triangleq a(x) - x\,, \tag{7.9}$$

then (7.8) can be represented as

$$\frac{\partial T_i}{\partial x}\,\bar{a} = T_{i+1} - T_i\,,\ \ i = 1,2,\ldots,n-1\,, \tag{7.10}$$

or equivalently

$$<dT_i,\bar{a}> = T_{i+1} - T_i \;, \quad i = 1,2,\ldots,n-1 \;. \qquad (7.11)$$

Equation (7.11) can be rewritten as follows

$$\begin{aligned} &T_1 = T_1 \;, \\ &T_2 = <dT_1,\bar{a}> + T_1 = L_{\bar{a}}T_1 + T_1 \;, \\ &T_3 + <dT_2,\bar{a}> + T_2 = L_{\bar{a}}L_{\bar{a}}T_1 + L_{\bar{a}}T_1 + T_2 \;, \\ &\quad\vdots \end{aligned}$$

Therefore, the transformation matrix T can be represented as

$$T = \begin{bmatrix} T_1 \\ L_{\bar{a}}T_1 + T_1 \\ L_{\bar{a}}L_{\bar{a}}T_1 + 2L_{\bar{a}}T_1 + T_1 \\ L_{\bar{a}}^3T_1 + 2L_{\bar{a}}^2T_1 + 3L_{\bar{a}}T_1 + T_1 \\ \vdots \end{bmatrix},$$

where T_1 is called the starting function. Thus, finding the transformation T is reduced to finding T_1. In order to find T_1 we first analyze equation (7.7) which can alternatively be represented as

$$\left.\begin{aligned} &<dT_i,b> = 0 \;, \quad i = 1,2,\ldots,n-1 \\ &<dT_n,b> = 1 \end{aligned}\right\} \qquad (7.12)$$

Thus, in particular $<dT_1,b> = 0$. We now look at the following equation

$$<dT_2,b> = 0 \;. \qquad (7.13)$$

From (7.11) we have

$$T_2 = <dT_1,\bar{a}> + T_1 \;. \qquad (7.14)$$

Substituting (7.14) into (7.13) gives

$$<dT_2,b> = <d(<dT_1,\bar{a}> + T_1),b>$$

$$= <d<dT_1,\bar{a}>,b> + <dT_1,b>$$

$$= <d<dT_1,\bar{a}>,b> = L_b L_{\bar{a}} T_1 \ . \tag{7.15}$$

On the other hand

$$<dT_1,[\bar{a},b]> = L_b L_{\bar{a}} T_1 - L_{\bar{a}} L_b T_1 = L_b L_{\bar{a}} T_1 \ . \tag{7.16}$$

From (7.15) and (7.16) we conclude that

$$<dT_2,b> = <dT_1,[\bar{a},b]> = \frac{\partial T_1}{\partial x} (ad^1\bar{a},b) = 0 \ . \tag{7.17}$$

Similarly we can show that

$$<dT_3,bg> = <dT_2,[\bar{a},b]>$$

$$= <dT_1,(ad^2\bar{a},b)> = \frac{\partial T_1}{\partial x} (ad^2\bar{a},b) \ . \tag{7.18}$$

Proceeding as above we arrive at a set of equations which can be represented in the following form

$$\frac{\partial T_1}{\partial x} \ b,(ad^1\bar{a},b), (ad^2\bar{a},b),\ldots,(ad^{n-1}\bar{a},b)] \triangleq \frac{\partial T_1}{\partial x} \ C_1$$

$$= [0,0,0,\ldots,1] \ . \tag{7.19}$$

If C_1^{-1} exists then

$$\frac{\partial T_1}{\partial x} = [0,0,0,\ldots,1] C_1^{-1} \ . \tag{7.20}$$

which implies that $\frac{\partial T_1}{\partial x}$ is the last row of C_1^{-1}. Let q(x) be a vector such that

$$q(x) = \frac{\partial T_1}{\partial x} . \tag{7.21}$$

A vector $q(x)$ for which there exists a real-valued function $T_1(x)$ such that equation (7.21) holds is called a conservative vector field or a gradient field. The function T_1 is referred to as the field potential of $q(x)$.

In summary, a sufficient condition for the existence of the transformation $x^* = T(x)$ bringing the system $x(k+1) - a(x(k)) + b(x(k))u(k)$ into the controller canonical form (7.2) is

(i) invertibility of the matrix C_1,

and

(ii) solvability of equation (7.21). Conditions for satisfaction of requirements (i) and (ii) can be deduced from the complete integrability theorem of Frobenius concerning integral manifolds.

Example

Consider a dynamical system modeled by the following difference equation

$$x(k+1) = \begin{bmatrix} x_2 \\ K_1 \sin x_1 + K_2 x_3 \\ K_3 x_2 + K_4 x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} u(k) , \tag{7.22}$$

where K_i ($i = 1,\dots,5$) are constants.

Our goal is to transform (7.22) into the controller canonical form. First we form

$$\bar{a} = \begin{bmatrix} x_2 - x_1 \\ K_1 \sin x_1 + K_2 x_3 - x_2 \\ K_3 x_2 - K_4 x_3 - x_3 \end{bmatrix}. \tag{7.23}$$

Next, we compute the matrix C_1. Note that

$$(ad^1\bar{a},b) = \frac{\partial \bar{a}}{\partial x}\, b - \frac{\partial b}{\partial x}\, \bar{a} = \frac{\partial \bar{a}}{\partial x}\, b$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ K_1 \cos x_1 & -1 & K_2 \\ 0 & K_3 & K_4 - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} = \begin{bmatrix} 0 \\ K_2 \\ K_5 \\ (K_4 - 1)K_5 \end{bmatrix}. \tag{7.24}$$

Next

$$(ad^2\bar{a},b) = [\bar{a},[\bar{a},b]] = \frac{\partial \bar{a}}{\partial x}\, [\bar{a},b]$$

$$= \begin{bmatrix} K_2K_5 \\ -K_2K_5 + K_2(K_4-1)K_5 \\ K_2K_3K_5 + (K_4-1)^2K_5 \end{bmatrix}. \tag{7.25}$$

Hence

$$C_1 = \begin{bmatrix} 0 & 0 & K_2K_5 \\ 0 & K_2K_5 & -K_2K_5 + K_2(K_4-1)K_5 \\ K_5 & (K_4-1)K_5 & K_2K_3K_5 + (K_4-1)^2K_5 \end{bmatrix}. \tag{7.26}$$

The last row of C_1^{-1} is

$$q = \frac{\partial T_1}{\partial x} = \begin{bmatrix} \frac{1}{K_2K_5} & 0 & 0 \end{bmatrix}. \tag{7.27}$$

Therefore

$$\left.\begin{aligned}
x_1^* &= T_1 = \frac{1}{K_2K_5}\, x_1\,, \\
x_2^* &= T_2 = \langle dT_1,\bar{a}\rangle + T_1 = \frac{1}{K_2K_5}\, x_2 \\
x_3^* &= T_3 = \langle dT_2,\bar{a}\rangle + T_2 = \frac{1}{K_2K_5}\, (K_1 \sin x_1 + K_2 x_3)
\end{aligned}\right\}. \tag{7.28}$$

From (7.28) we can also compute the inverse of T(x)

$$\left.\begin{aligned}
x_1 &= K_2K_5x_1^* \\
x_2 &= K_2K_5x_2^* \\
x_3 &= \frac{K_2K_5x_3^* - K_1\sin(K_2K_5x_1^*)}{K_2}
\end{aligned}\right\}. \tag{7.29}$$

Observe that

$$\frac{\partial T}{\partial x}\,\bar{a} = \frac{1}{K_2K_5}\begin{bmatrix} x_2 - x_1 \\ K_1\sin x_1 - x_2 + K_2x_3 \\ K_1(x_2 - x_1)\cos x_1 + K_2(K_4-1)x_3 + K_2K_3x_2 \end{bmatrix}. \tag{7.30}$$

Hence

$$x^*(k+1) = \left(\frac{\partial T}{\partial x}\,\bar{a}\right)\Bigg|_{x=\tilde{T}(x^*)} + x^*$$

$$= \begin{bmatrix} x_2^* \\ x_3^* \\ K_1(x_2^* - x_1^*)\cos(K_2K_5x_1^*) - \dfrac{(K_4-1)}{K_2K_5}\, K_1\sin(K_2K_5x_1^*) + K_4x_3^* + K_2K_3x_2^* \end{bmatrix}$$

$$= \begin{bmatrix} x_2^*(k) \\ x_3^*(k) \\ f(x^*(k)) \end{bmatrix}, \tag{7.31}$$

and

$$\frac{\partial T}{\partial x}\, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{7.32}$$

In a similar fashion we can proceed to transform the system equations into the observer canonical form. This form then can be utilized in the output feedback control design.

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APPENDICES

APPENDIX A

A.1. DEFINITION OF A CONVERGENT MATRIX

Consider an m×m constant matrix Λ.

Definition: Matrix Λ is convergent if $\lim_{k\to\infty} \Lambda^k = 0$.

Theorem A.1: Let $\Lambda \in \mathbb{R}^{m\times m}$. Then $\lim_{k\to\infty} \Lambda^k = 0$ if and only if $\rho(\Lambda) < 1$, where $\rho(\Lambda) = \max\{|\lambda|: \lambda \text{ is an eigenvalue of } \Lambda\}$ is the spectral radius of Λ.

Proof: See [35] p. 298.

APPENDIX B

B.1. COMPUTATION OF Λ^k

Suppose that $\Lambda \in \mathbb{R}^{m \times m}$ is diagonalizable, i.e., $\Lambda = N D N^{-1}$, where D is diagonal.

Define.

$$N \triangleq [c_1 \mid c_2 \mid \ldots \mid c_m],$$

$$N^{-1} \triangleq \begin{bmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \\ \hline r_m \end{bmatrix},$$

where $c_1, c_2, \ldots, c_m$ are the columns of N and $r_1, r_2, \ldots, r_m$ are the rows of N^{-1}, and

$$B_i \triangleq c_i r_i.$$

The representation $\Lambda = N D N^{-1}$ can be written as (see [36], pp. 367-368)

$$\Lambda = \lambda_1 B_1 + \lambda_2 B_2 + \ldots + \lambda_m B_m.$$

Moreover,

$$\Lambda^k = \lambda_1^k B_1 + \lambda_2^k B_2 + \ldots + \lambda_m^k B_m,$$

where λ_i, $i=1,2,\ldots,m$ are the eigenvalues of Λ.