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Singular Solutions in Problems of Optimal Control

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PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

Singular Solutions in Problems of Optimal Control

J. E. Gibson, Principal Investigator

C. D. Johnson

Control and Information Systems Laboratory

August, 1963

Lafayette, Indiana

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ABSTRACT

Johnson, Carroll D., Ph. D., Purdue University, August, 1963. Singular Solutions in Problems of Optimal Control. Major Professor: John E. Gibson.

The contribution of this thesis is the somewhat general analysis of singular solutions which arise in problems of optimal control and the development of certain analytical procedures for detecting and calculating singular solutions.

The basic optimal control problem considered in this study is the task of choosing a control u(t) which will a) transfer the state of a system, described by the n first order ordinary differential equations,

$$\dot{x}_i = f_i(x_1, \ldots, x_n, t, u) \qquad (i = 1, \ldots, n) \tag{1}$$

from some prescribed initial state to some prescribed final (terminal) state and b) simultaneously minimize (with respect to the control u) an index of performance J of the form

$$J[u] = \int_{t_o}^{T} f_o(x_1, \ldots, x_n, t, u)dt \tag{2}$$

It is assumed that the allowable values of the control u may be constrained to lie in some set U.

The conventional mathematical techniques presently being used in optimal control theory are discussed. It is shown that for a certain class of optimal control problems, which are characterized by the control u appearing linearly in the system state equations (1) and the integrand

of the index of performance (2), the optimal control $u^*(t)$ is found (formally) to be of the "bang-bang" type

$$u^*(t) = \begin{cases} A & \text{if } F(t) > 0 \\ B & \text{if } F(t) < 0 \end{cases} \tag{3}$$

In (3), A and B are, respectively, the upper and lower bounds on the admissible control u and F(t) is a certain function of time which is called the switching function. When the switching function becomes identically zero over a finite time interval the conventional mathematical techniques fail to yield any information about the desired optimal control. The solution in this case is said to be "singular" and the corresponding control is termed "singular control".

The nature of singular solutions is investigated in detail and the apparent failure of the conventional mathematical techniques is shown to be due to the fact that singular optimal controls lie in the interior (rather than on the boundary) of the admissible set U. The concept of a singular control surface in the system state space is introduced and is used to examine the geometry of singular solutions. Some mathematical properties of the singular control surface are derived and a backward tracing scheme is used to aid in establishing the role of singular subarcs in the solution of optimal control problems. It is shown that the singular control $u^*(t)$ can be obtained from the condition $F(t) \equiv 0$ and in some cases can be synthesized as a function of the system state variables.

The proposed techniques for solving optimal control problems with singular solutions are illustrated by means of four examples which are worked out in detail.

Chapter 1

INTRODUCTION

1.1. The Problem of Optimal Control

The control of processes by means of automatic sensing and regulating devices has been one of the most active areas in recent scientific research. The original applications of automatic control techniques were primarily concerned with replacing human effort by more reliable and less expensive machine effort. Recently, interest has centered around the possibility of designing automatic control systems which will perform their operations in an optimum manner with respect to some given figure of merit. As a result, a new approach to automatic control has emerged, the Theory of Optimal Control.

The basic problem of optimal control may be stated as follows: Given a plant (process) described by the set of differential equations

$$\dot{x}_i = f_i(x_1, \ldots, x_n, t, u) \qquad (i = 1, \ldots, n) \qquad (\cdot = \frac{d}{dt}) \tag{1.1}$$

determine a control $u^*(t)$ that will transfer the plant (1.1) from some prescribed initial condition

$$x_i(t_o) = x_{io} \qquad (t_o = \text{starting time}) \qquad (i = 1, \ldots, n) \tag{1.2}$$

to some prescribed terminal (final) condition

$$x_i(T) = x_{iT} \qquad (T = \text{terminal time}) \qquad (i = 1, \ldots, n) \tag{1.3}$$

and simultaneously minimize an index of performance (figure of merit) given by

$$J[u] = \int_{t_o}^{T} f_o(x_1,\ldots,x_n, t, u)dt \quad . \tag{1.4}$$

In (1.1) the x_i are state variables, the variables (position, velocity, temperature, etc.) required to specify the state or condition of the plant, and u is the control effort (force, voltage, etc.) by which the state of the plant may be changed. Some processes may permit simultaneous application of several different control efforts. In this thesis, primary concern is devoted to processes with a single control variable. The independent variable t is usually time. In general, some of the initial and terminal conditions (1.2), (1.3) (and possible t_o and/or T) may be unspecified a priori. The index of performance (1.4) represents the measure of goodness for the solution of the task of transferring the plant (1.1) from condition (1.2) to condition (1.3). Physically, (1.4) may represent such quantities as cost, time expended, energy expended, accumulated error, etc. In some processes, it is desired to maximize a certain index of performance. In this case (1.4) is selected as the negative of the quantity to be maximized.

The problem of optimal control as described above may be considerably complicated by certain physical requirements. First, the control effort u is usually bounded or constrained so that only certain finite values of u are allowed. The set from which allowable values of u may be selected is denoted by U, and the notation

$$u \in U \tag{1.5}$$

represents the condition that u is contained in the set U. Further complications arise when the set U varies with time and/or the x_i. Another factor which may complicate the problem of optimal control is bounded state variables; i.e., the specification that certain of the state variables x_i should not exceed given values X_i. The bounds X_i may, in general, depend on time and possibly other state variables. Finally, the f_i in (1.1) may be discontinuous with respect to one or more of the arguments x_i, t, and u. In this study only one of the factors listed above, the case of bounded control $u \in U$ in which U is constant, is considered.

1.2. Techniques for Solving the Problem of Optimal Control

The problem of optimal control outlined above differs from problems of maximization and minimization in the ordinary calculus in that the desired solution $u^*(t)$ is a function rather than a set of numbers. There are several mathematical techniques which can be used to determine optimal functions. The basic notions of the older and more widely known techniques are outlined below. The newer techniques, developed within the last decade are then presented in some detail.

1.2.1. The Classical Calculus of Variations

The problem of determining optimal functions was investigated by Lagrange and others in the latter part of the 18th century. Their results led to the formulation of a new branch of mathematics called the "Calculus of Variations". In its classical form, the calculus of variations can be applied only to those problems in which the control u(t) and the state variables $x_i(t)$ are unconstrained and in which the f_i $(i = 0, 1, \ldots, n)$ in (1.1) and (1.4) possess continuous partial derivatives (in all arguments) up to and including those of the third order.

Basically, in the calculus of variations (as well as in the other techniques to be discussed) one seeks to characterize the optimal function by means of certain necessary (but usually not sufficient) conditions. In the classical calculus of variations, the most important necessary conditions for the problem[1] (1.1)-(1.4) are: the Euler equations, the Weierstrass-Erdmann corner condition, and the necessary condition of Weierstrass.

The Euler equations which must be satisfied by the optimal solution are

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}_i}\right) = \frac{\partial G}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{1.6}$$

and

$$\frac{\partial G}{\partial u} = 0 \tag{1.7}$$

where

$$G \triangleq f_o + \sum \lambda_i (f_i - \dot{x}_i) \qquad (i = 1, \ldots, n) \tag{1.8}$$

The $\lambda_i = \lambda_i(t)$ in (1.8) are referred to as Lagrange multipliers and must be determined from (1.6).

The Weierstrass-Erdmann corner condition states that at corners of the optimal trajectory $x_i(t)$ $(i = 1, \ldots, n)$ the quantities $\partial G/\partial \dot{x}_i$, $\partial G/\partial u$, and $(G - \dot{x}_i \frac{\partial G}{\partial \dot{x}_i})$ have well defined one sided limits that are equal.

The necessary condition of Weierstrass requires that for all $\bar{\dot{x}}_i \neq \dot{x}_i$ $\bar{u} \neq u^*$

---

[1] In the classical calculus of variations this problem is referred to as the "problem of Bolza" [1].

$$G(t, x_1,\ldots,x_n, \bar{\dot{x}}_1,\ldots,\bar{\dot{x}}_n, \bar{u}, \lambda_i) - G(t, x_1,\ldots,x_n, \dot{x}_1,\ldots,\dot{x}_n, u^*, \lambda_i)$$

$$- \sum (\bar{\dot{x}}_i - \dot{x}_i) \frac{\partial G}{\partial \dot{x}_i} \geq 0 \tag{1.9}$$

at every point along the optimal trajectory $x_i(t)$. In (1.9), the bars denote quantities which are not associated with the optimal trajectory. A complete discussion of the necessary conditions given above (and also other necessary conditions) can be found in [1], [2].

1.2.2. Extensions of the Classical Calculus of Variations

Valentine [3] has proposed a method by which the classical calculus of variations can be used to solve the problem (1.1)-(1.4) with bounded control $u \in U$ where $U = U(t, x, u)$. In this case, the constraint on the control is written as

$$R(x_1,\ldots,x_n, t, u) \geq 0 \tag{1.10}$$

and a new (real) control variable z is defined as

$$\dot{z}^2 = R(x_1, \ldots, x_n, t, u) \qquad (z(t_o) \triangleq 0) \tag{1.11}$$

Equation (1.11) can now be treated as an additional state variable equation and appended to the set (1.1). With the new state variable (1.11) added to (1.1), the G in (1.8) becomes

$$G = f_o + \sum \lambda_i(f_i - \dot{x}_i) + \mu(R - \dot{z}^2) \qquad (\mu \leq 0) \tag{1.12}$$

resulting in the additional Euler equation

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \dot{z}}\right) = 0 \tag{1.13}$$

In (1.12), $\mu = \mu(t)$ is a non-positive undetermined multiplier. With the addition of (1.11), the Weierstrass-Erdmann corner condition requires that for the optimal solution the quantities $\partial G/\partial \dot{z}$ and $(G - \dot{x}_i \frac{\partial G}{\partial \dot{x}_i} - \dot{z} \frac{\partial G}{\partial \dot{z}})$ have well defined one sided limits that are equal. The necessary condition of Weierstrass (1.9) is not effected by the additional equation (1.11). A thorough treatment of the method of Valentine applied to the problem of Bolza can be found in [2]. Miele [4] has proposed another technique by which the problem (1.1)-(1.4) with bounded control can be solved by the classical calculus of variations. Miele's method, however, is restricted to control constraints of the form

$$B \leq u^*(t) \leq A \tag{1.14}$$

where A and B are constants.

1.2.3. Dynamic Programming

In 1957, Bellman [5] introduced a somewhat different approach to the problem of optimal control. Using his Principle of Optimality, Bellman derived a functional equation which can be used to solve, in a discrete manner, a large class of optimal control problems. More recently, it has been shown [6] that Bellman's recurrence equation is a finite difference version of the classical Hamilton-Jacobi partial differential equation. A treatment of Dynamic Programming theory for the problem (1.1)-(1.4) is given below. This presentation differs somewhat from that originally given by Bellman.

It has been mentioned previously that some of the initial and terminal conditions (1.2), (1.3) may be unspecified a priori. This condition may be generalized by replacing the conditions (1.2), (1.3) with the requirements that the equations

$$\eta(x_{1o}, \ldots, x_{no}, t_o) = 0 \tag{1.15}$$

and

$$\xi(x_{1T}, \ldots, x_{nT}, T) = 0 \tag{1.16}$$

be satisfied at the initial and terminal conditions respectively. Equations (1.15) and (1.16) represent hypersurfaces in the $(n + 1)$ dimensional (x, t) space. Thus, the desired optimal trajectory $x_i(t)$ $(i = 1, \ldots, n)$ in this space is required to have its initial and terminal points lying on the hypersurfaces (1.15), (1.16). For this reason, (1.15) and (1.16) are referred to as initial and terminal "manifolds". It is remarked that in some cases, the initial and/or terminal manifolds may be defined by a set of equations of the type (1.15) or (1.16). With the introduction of initial and terminal manifolds, the basic problem of optimal control to be considered in this investigation can be described as the problem of transferring the state of the plant (1.1) from the initial manifold (1.15) to the terminal manifold (1.16) and simultaneously minimizing (1.4).

It will be assumed in the following that an optimal solution does exist for the problem being considered. Furthermore, it will be assumed that if the given initial and/or terminal conditions are varied over some region R in the (x, t) space, an optimal solution exists for each set of initial and terminal conditions in R.

Consider the problem of minimizing (1.4) for a fixed terminal manifold (1.16) and variable initial conditions

$$\begin{aligned} x_1(t_o) &= x_{1o} \\ &\vdots \\ x_n(t_o) &= x_{no} \\ t_o &= t_o \end{aligned} \tag{1.17}$$

in the region R of the (x, t) space. Under the assumptions given above, it should be possible to assign to every initial point (1.17) in R a unique number representing the minimum value of the index of performance (1.4) (corresponding to the plant (1.1)). This value of min J[u] associated with any arbitrary point (x_o, t_o) in R is a property of the point (x_o, t_o) and will be denoted by V. Thus

$$V(x_{1o},\dots,x_{no},t_o) = \min_{u\in U} \int_{t_o}^{T} f_o(x_1,\dots,x_n,t,u)dt \qquad (1.18)$$

where the integral is evaluated between any arbitrary initial point (1.17) and the fixed terminal manifold (1.16). The locus of points (1.17) in R which have the same value for V will form an "isovee" contour as shown in Figure 1.1. Note that the two dimensional coordinate system of Figure 1.1 is used to represent the n + 1 dimensional (x, t) space. If the initial manifold (1.15) is now superimposed on Figure 1.1, a graphic representation of the optimization problem is obtained as shown in Figure 1.2. Note that from the definition of $V(x_{1o},\dots,x_{no},t_o)$, the addition of the initial manifold (1.15) to Figure 1.1 does not alter the shape or value of the field of isovee contours. That is, the initial manifold simply restricts the allowable set of starting points in Figure 1.1.

Referring to Figure 1.2, the original optimization problem can now be described geometrically as the problem of joining the η and ξ manifolds with a line (trajectory) (subject to the conditions (1.1) and (1.5)) such that the value of (1.4) computed along that trajectory is less than the value of (1.4) computed along any other trajectory joining η and ξ and compatible with (1.1) and (1.5). From the definition of ξ and V, the isovee

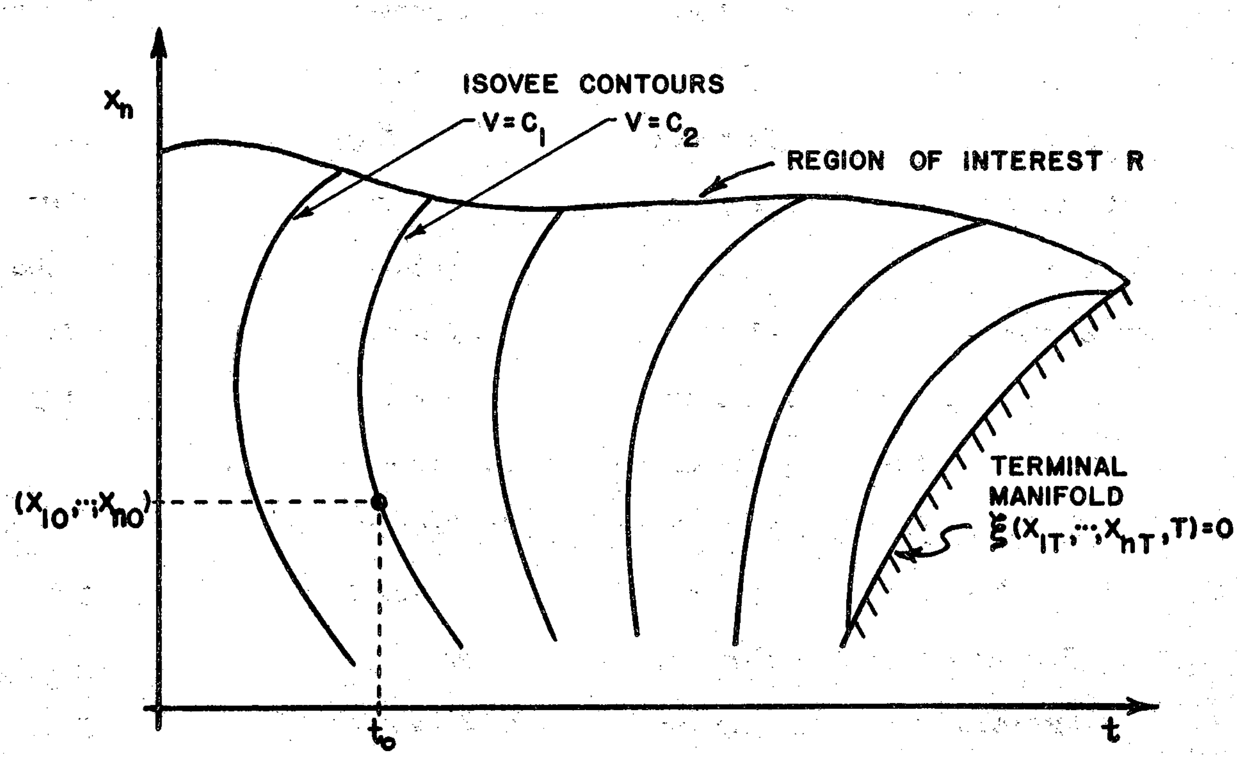

Figure 1.1. Isovee Contours and Terminal Manifold in the x-t Space

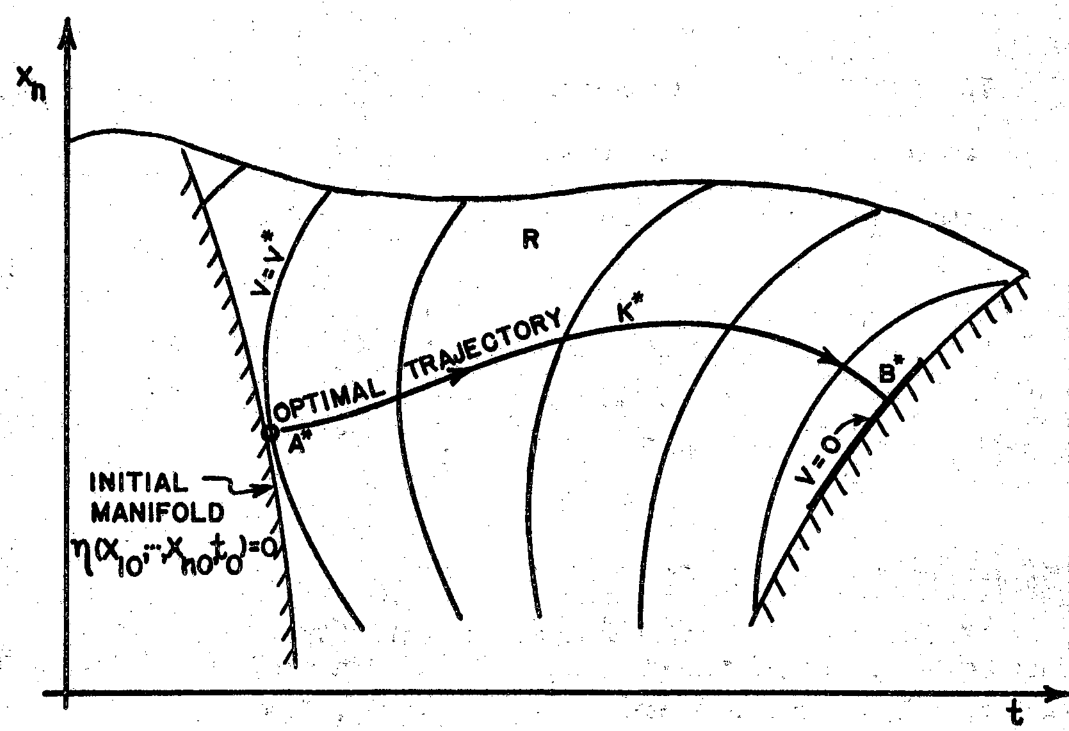

Figure 1.2. Initial Manifold and Optimal Trajectory in the x-t Space

contour V = 0 must[2] coincide with at least a portion of the terminal manifold ξ as shown in Figure 1.2. Clearly, the optimal starting point $(x_{1o},\ldots,x_{no}, t_o)$ for this problem is the point A^* shown in Figure 1.2. Further, the minimum value of the index of performance for this problem is the value V^* of the isovee contour which passes through point A^*. The actual form of the optimal trajectory which starts at A^* and joins ξ is not evident from Figure 1.2. However, it is possible, from Figure 1.2, to state a fundamental property of the optimal trajectory:

Fundamental Property of the Optimal Trajectory

The optimal trajectory K^* joining η and ξ has the property that at any intermediate point $E = (x_1^*, \ldots, x_n^*, t)$ along K^* the value of (1.4) computed from A^* to E (along K^*) must satisfy the relation

$$\int_{A^*}^{E} f_o(x_1,\ldots,x_n, t, u)dt \Big|_{K^*} = V^*(A^*) - V(E) \tag{1.19}$$

If there is only one optimal trajectory between η and ξ (i.e., if the solution to the problem is unique) then for any other path $K \not\equiv K^*$ starting at A^* and joining ξ (and compatible with (1.1) and (1.5)) the following inequality is satisfied at every point E' on K

$$\int_{A^*}^{E'} f_o(x_1,\ldots,x_n, t, u)\, dt \Big|_{K} > V^*(A^*) - V(E') \tag{1.20}$$

---

[2]This assumes, that V is not defined as the minimum value of some function ψ of the coordinates x_i, t. If $V \triangleq \min_{u\in U} \psi(x_i(T), T)$ then V can be zero only at points where $\psi = 0$. (However, V need not be zero at all points where $\psi = 0$).

The quantities $V^*(A^*)$, $V(E)$, and $V(E')$ in (1.19) and (1.20) are the values of the isovee contours passing through the points A^*, E, and E' respectively.

The proof of (1.19) and (1.20) follows immediately from the definition and assumed uniqueness of the optimal trajectory K^*. Equations (1.19) and (1.20) express the fundamental property of optimal trajectories for the optimization problem being considered. From Figure 1.2 and the rather obvious statement of fact represented by (1.19) and (1.20), the theory of Dynamic Programming and Pontryagin's Maximum Principle will be derived. It may be noted that relations (1.19) and (1.20) are quite similar to the relations between entropy and the integral $\int \frac{dQ}{T}$ for reversible and non-reversible thermodynamic processes. This similarity is not entirely superficial.

Since the trajectory K which the system follows between η and ξ is determined by the control variable $u(t)$, (1.19) and (1.20) may be restated as follows:

For optimal control $u(t) = u^*(t)$ $(K = K^*)$

$$V^*(x_{1o}, \ldots, x_{no}, t_o) - V(x_1^*, \ldots, x_n^*, t) - \int_{t_o}^{t} f_o(x_1^*, \ldots, x_n^*, t, u^*)dt \equiv 0 \qquad (1.21)$$

$$(t_o \le t \le T)$$

and for non-optimal control $u(t) \not\equiv u^*(t)$ $(K \not\equiv K^*)$

$$V^*(x_{1o},\dots,x_{no},\ t_o) - V(x_1,\dots,x_n,\ t) - \int_{t_o}^{t} f_o(x_1,\ \dots,\ x_n,\ t,\ u)dt < 0 \qquad (1.22)$$

$$(t_o < t \leq T)$$

where $(x_{1o},\ \dots,\ x_{no},\ t_o)$ is the optimal starting point A on η and $(x_1*,\ \dots,\ x_n*,\ t)$ is a point on the optimal path K*.

A typical plot of (1.21) and (1.22) versus time is shown in Figure 1.3. Note that at $t = t_o$ both curves in Figure 1.3 coincide,[3] and from the definition of K*, V*, and V neither curve can ever enter the region $\left[V^* - V - \int_{t_o}^{t} f_o\,dt\right] > 0$. Also, for $t_o < t \leq T$, the non-optimal (dotted) curve in Figure 1.3 must always lie entirely below the optimal (solid) curve. Note also, the slope of (1.22) in Figure 1.3 can never be positive. The optimal curve in Figure 1.3 is thus characterized by the fact that along K*

$$\frac{d}{dt}\left[V^*(x_{1o},\dots,x_{no},t_o) - V(x_1*,\dots,x_n*,t) - \int_{t_o}^{t} f_o(x_1*,\dots,x_n*,t,u*)dt\right] \equiv 0 \qquad (1.23)$$

$$(t_o \leq t \leq T)$$

and furthermore, at any point along the optimal trajectory K* we must have

---

[3]If the non-optimal trajectory K begins at any point other than A^*, then in Figure 1.3 the curve for (1.22) will be below (1.21) at <u>all</u> times. In any case, (1.19), (1.20), (1.21) and (1.22) are always satisfied.

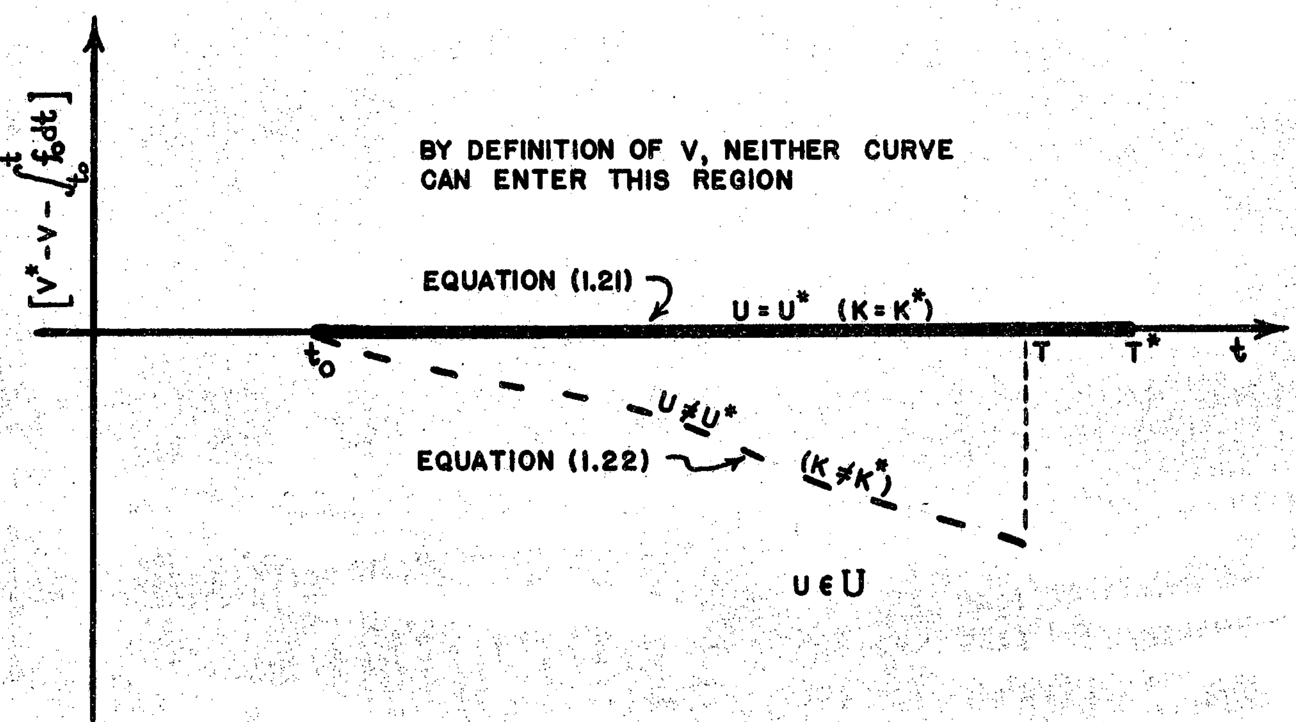

Figure 1.3. Plot of Equations (1.21) and (1.22) as Functions of Time

$$\max_{u \in U} \frac{d}{dt} \left[V^*(x_{1o},\ldots,x_{no},t_o) - V(x_1^*,\ldots,x_n^*,t) - \right.$$

$$\left. - \int_{t_o}^{t} f_o(x_1^*,\ldots,x_n^*,\ t,\ u^*)dt \right] \equiv 0 \qquad (1.24)$$

$$(t_o \leq t \leq T)$$

The quantity $V^*(x_{1o},\ldots,x_{no},t_o)$ corresponds to the value of the isovee contour which passes through the optimal starting point A on η. Thus, $V^*(x_{1o},\ldots,x_{no},t_o)$ is a constant with respect to time. Also, by the chain rule for differentiation

$$\frac{dV}{dt}(x_1,\ldots,x_n,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n \qquad (1.25)$$

Thus, (1.24) becomes (omitting the arguments)

$$\max_{u \in U} \left[- \frac{\partial V}{\partial t} - \frac{\partial V}{\partial x_1}\dot{x}_1 - \ldots - \frac{\partial V}{\partial x_n}\dot{x}_n - f_o \right] \equiv 0 \qquad (1.26)$$

$$(t_o \leq t \leq T)$$

However, since V is not a function of u explicitly, then $\partial V/\partial t$ is explicitly independent of u. Therefore, (1.26) can be rewritten

$$\frac{\partial V}{\partial t} = \max_{u \in U} \left[- \frac{\partial V}{\partial x_1}\dot{x}_1 - \ldots - \frac{\partial V}{\partial x_n}\dot{x}_n - f_o \right] \qquad (1.27)$$

$$(t_o \leq t \leq T)$$

where

$$V = V(x_1,\ldots,x_n,t)$$

$$\dot{x}_i = f_i(x_1,\ldots,x_n,\ t,\ u) \qquad (i = 1,\ \ldots,\ n)$$

$$f_o = f_o(x_1,\ldots,x_n,\ t,\ u)$$

Equation (1.27) is a form of the well known Hamilton-Jacobi partial differential equation [7], and is often referred to as Bellman's functional equation [8]. The dynamic programming method for solving the optimization problem formulated above is essentially a step-by-step (finite difference) technique for solving (1.27). Thus, if x_i, t, and $\partial V/\partial x_i$ for a certain point on K* are known (or assumed) then (1.27) can be used to determine u* (and thereby $\partial V/\partial t$) over a small time interval Δt. By this means, using (1.1), (1.27) may be numerically integrated. The main difficulty of this method is determining an initial set of values x_i, t, $\partial V/\partial x_i$ to start the numerical integration.

Equation (1.27) may also be solved (in principle) by the analytic techniques of partial differential equations. The solution $V = V(x_1, \ldots, x_n, t)$ of (1.27) by ordinary methods of partial differential equations may require rather involved "pieced solution" techniques if the right side of (1.27) is not differentiable at u = u* or if u = u* lies on the boundary of the admissible set U. In such a case, the pieced solutions for V should join if $V(x_1, \ldots, x_n, t)$ is continuous. It is remarked that for many practical problems, V is continuous in the (x, t) space. On the other hand, if the right side of (1.27) is continuously differentiable at u = u* and u* is not on the boundary of U then by setting

$$\frac{\partial}{\partial u}\left[-\frac{\partial V}{\partial x_1}\dot{x}_1 - \cdots - \frac{\partial V}{\partial x_n}\dot{x}_n - f_o\right] \equiv 0 \qquad (1.28)$$

in (1.27) and using ordinary theory of maxima and minima, the optimal control

$$u^* = u^*\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}, x_1, \ldots, x_n, t\right) \tag{1.29}$$

can be obtained. Then, (1.29) can be substituted into (1.27) to obtain the Hamilton-Jacobi equation in the more common form

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n + f_o = 0 \tag{1.30}$$

where

$$\dot{x}_i = f_i\left[x_1^*,\ldots,x_n^*,t,u^*\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}, x_1^*, \ldots, x_n^*, t\right)\right]$$

$$(i = 1, \ldots, n)$$

$$f_o = f_o\left[x_1^*, \ldots, x_n^*,t,u^*\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}, x_1^*, \ldots, x_n^*,t\right)\right]$$

It is seen that (1.30) will be a first order (usually nonlinear) partial differential equation explicitly independent of V and the control term u. After the solution $V = V(x_1^*, \ldots, x_n^*, t)$ of (1.30) is obtained, the optimal control in the form

$$u = u^*(x_1^*, \ldots, x_n^*, t) \tag{1.31}$$

can be obtained from (1.29). Equation (1.31) can then be expressed entirely as a function of time by substituting (1.31) into (1.1) and solving for the x_i as functions of time. This latter step yields the pre-programmed (or "open loop") control $u = u^*(t)$. In most cases, however, it is more desirable to leave the control in the "closed loop" form of (1.31). Note that t will not appear in (1.31) if all $\dot{x}_i$, f_o, and U are explicitly independent of time.

Although the functional equation (1.27) is the primary working "tool" of the Dynamic Programming method, several other important relations can be developed from (1.27) and Figure 1.2. These additional relations, developed below, form the basis of Pontriagin's Maximum Principle.

1.2.4. Characteristics of the Hamilton-Jacobi Equation

From (1.27), it is seen that for $u = u^*(t)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n + f_o \Big|_{u=u^*} \equiv 0 \tag{1.32}$$

Note that (1.32) holds even when the right side of (1.27) is not differentiable with respect to u at $u = u^*$ or when u^* is on the boundary of U. Equation (1.32) implies that the following relation is satisfied along the optimal trajectory K^*

$$\int_{t_o}^{t} \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_i}\dot{x}_1 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n + f_o\right] dt \equiv 0 \tag{1.33}$$

From (1.27), letting $u = u^*(t)$ $(t_o \leq t \leq T)$ the plant equations are obtained in the form

$$\frac{dx_i^*}{dt} = -\frac{\partial(\frac{\partial V}{\partial t})}{\partial(\frac{\partial V}{\partial x_i})} \qquad (i = 1, \ldots, n) \tag{1.34}$$

These equations should, of course, coincide with (1.1). From the fact that $V = V(x_1, \ldots, x_n, t)$ it is clear that

$$\frac{\partial V}{\partial x_i} = \frac{\partial V}{\partial x_i}(x_1, \ldots, x_n, t) \tag{1.35}$$

The total time derivative of (1.35) is then

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x_i}\right) = \frac{\partial^2 V}{\partial t \partial x_i} + \dot{x}_1 \frac{\partial^2 V}{\partial x_1 \partial x_i} + \dots + \dot{x}_n \frac{\partial^2 V}{\partial x_n \partial x_i} \tag{1.36}$$

However, from (1.27), when $u = u^*$

$$\frac{\partial^2 V}{\partial x_i \partial t} = -\frac{\partial V}{\partial x_1}\frac{\partial \dot{x}_1}{\partial x_i} - \dot{x}_1 \frac{\partial^2 V}{\partial x_i \partial x_1} - \dots - \frac{\partial V}{\partial x_n}\frac{\partial \dot{x}_n}{\partial x_i} - \dot{x}_n \frac{\partial^2 V}{\partial x_i \partial x_n} - \frac{\partial f_o}{\partial x_i} \tag{1.37}$$

where it has been assumed that the admissible set U is not a function of x_i (i.e., $U \neq U(x)$). It can be seen that when $U = U(x)$ equation (1.37) will contain additional terms. Now, if $V = V(x_1,\dots,x_n,t)$ has continuous mixed second partial derivatives with respect to all x_i and t then

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}$$

and (1.38)

$$\frac{\partial^2 V}{\partial x_i \partial t} = \frac{\partial^2 V}{\partial t \partial x_i}$$

If the conditions (1.38) hold, then (1.37) can be written

$$\frac{\partial^2 V}{\partial t \partial x_i} = -\frac{\partial V}{\partial x_1}\frac{\partial \dot{x}_1}{\partial x_i} - \dot{x}_1 \frac{\partial^2 V}{\partial x_1 \partial x_i} - \dots - \frac{\partial V}{\partial x_n}\frac{\partial \dot{x}_n}{\partial x_i} - \dot{x}_n \frac{\partial^2 V}{\partial x_n \partial x_i} - \frac{\partial f_o}{\partial x_i} \tag{1.39}$$

Substituting (1.39) in (1.36) gives

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x_i}\right)_{K^*} = -\frac{\partial V}{\partial x_1}\frac{\partial \dot{x}_1}{\partial x_i} - \dots - \frac{\partial V}{\partial x_n}\frac{\partial \dot{x}_n}{\partial x_i} - \frac{\partial f_o}{\partial x_i} \tag{1.40}$$

Equation (1.40) tells how $\partial V/\partial x_i$ varies with time along an optimal trajectory K* (assuming continuous mixed second partial derivatives of V with respect to all x_i and t along the optimal trajectory K*). Equations (1.34) and (1.40) are called the "equations of the characteristic strips" for the Hamilton-Jacobi equation (1.30).

Since $\partial V/\partial t = \partial V/\partial t(x_1,\ldots,x_n, t)$ then, assuming (1.38) holds,

$$\frac{d}{dt}\left(\frac{\partial V}{\partial t}\right) = \frac{\partial^2 V}{\partial t^2} + \dot{x}_1 \frac{\partial^2 V}{\partial t \partial x_1} + \ldots + \dot{x}_n \frac{\partial^2 V}{\partial t \partial x_n} \tag{1.41}$$

Along K* it is seen from (1.27) that

$$\frac{\partial^2 V}{\partial t^2} = -\frac{\partial V}{\partial x_1}\frac{\partial \dot{x}_1}{\partial t} - \dot{x}_1 \frac{\partial^2 V}{\partial t \partial x_1} - \ldots - \frac{\partial V}{\partial x_n}\frac{\partial \dot{x}_n}{\partial t} - \dot{x}_n \frac{\partial^2 V}{\partial t \partial x_n} - \frac{\partial f_o}{\partial t} \tag{1.42}$$

Substituting (1.42) in (1.41) yields

$$\frac{d}{dt}\left(\frac{\partial V}{\partial t}\right)_{K*} = -\frac{\partial V}{\partial x_1}\frac{\partial \dot{x}_1}{\partial t} - \ldots - \frac{\partial V}{\partial x_n}\frac{\partial \dot{x}_n}{\partial t} - \frac{\partial f_o}{\partial t} \tag{1.43}$$

Equation (1.43) tells how $\partial V/\partial t$ varies along the optimal trajectory K*. Note that although (1.40) and (1.43) are total derivatives, only partial derivatives appear on the right hand side. From (1.43) it can be seen that if all $\dot{x}_i$, f_o and U are explicitly independent of time then along K*

$$\frac{d}{dt}\left(\frac{\partial V}{\partial t}\right) = 0 \qquad (t_o \leq t \leq T) \tag{1.44}$$

and thus

$$\left.\frac{\partial V}{\partial t}\right|_{K*} = c \qquad \begin{matrix}(c = \text{constant}) \\ (t_o \leq t \leq T)\end{matrix} \tag{1.45}$$

Equations (1.44) and (1.45) apply for either T = free or T = fixed. From the definition of V, it can be seen that V will be an explicit function of time when any $\dot{x}_i$, f_o, or U is an explicit function of time, when f_o contains additive constants, and when the terminal time T is fixed. If T is fixed, but neither $\dot{x}_i$, f_o, or U are explicit functions of time then (1.45) applies so that V must be linear in t and of the form

$$V\Big|_{\substack{K^*\\ T=\text{fixed}}} = v(x_1, \ldots, x_n) + c\,t \qquad (1.46)$$

The relationship revealed by (1.46) has led to some interesting results in connection with optimization problems which have a fixed terminal time [9]. When V depends explicitly on time (as for example in (1.46)), the isovee contours when viewed in the x-space will appear to move with time. In the (x, t) space, however, they will remain fixed. If T is free, neither $\dot{x}_i$, f_o, or U are explicit functions of time, and f_o does not contain additive constants, then V will not depend on t explicitly, and the c in (1.45) becomes zero. Thus

$$\frac{\partial V}{\partial t}\Big|_{\substack{K^*\\ T=\text{free}}} \equiv 0 \quad (t_o \leq t \leq T) \qquad (1.47)$$

Equation (1.47) implies that the isovee surfaces are "parallel" to the t-axis of the x, t space. When (1.45) or (1.47) holds, then along K* (1.30) has the first integral

$$\frac{\partial V}{\partial x_1}\dot{x}_1 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n + f_o = C \qquad (1.48)$$

where $C \neq 0$ or $C = 0$ depending on whether (1.45) or (1.47) is applicable.

1.2.5. Terminal Conditions for the Optimal Trajectory K*

The given problem specifications require that the optimal trajectory K* should terminate somewhere on the terminal manifold (1.16). The particular point B^* on ξ at which K* will actually terminate is, of course, unknown a priori. It is possible, however, to derive some necessary conditions which the optimal terminal point B^* on ξ must satisfy.

It has been remarked earlier that except in the special case when $V \triangleq \min_{u \in U} \psi(x_i(T), T)$ at least a portion of the V = 0 contour must coincide with ξ. This portion may be only a point, line, etc. (or several isolated points, lines, etc.) or possibly the entire ξ manifold. From the definition of V, it is clear that B^* must lie on a portion of ξ for which V = 0. In the most general case, it is possible that 1) B^* may lie at the end (or edge) of ξ or, 2) the V = 0 contour may have a discontinuous gradient or ξ may have a corner at B^*. In either of these cases, the optimal terminal point B^* is characterized by the fact that any small (allowable) displacement (dx_i, dT) along (tangent to) the ξ manifold must yield $dV > 0$ or, since $V = V(x_1, \ldots, x_n, t)$

$$\left. \frac{\partial V}{\partial t} dT + \frac{\partial V}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n} dx_n \right|_{\text{at } B^* \text{ on } \xi} > 0 \tag{1.49}$$

If the V = 0 contour coincides with a finite portion of ξ and the V = 0 contour and ξ do not have corners, and if B^* does not lie on the edge of ξ, then any small displacements (dx_i, dT) tangent to the ξ manifold must yield dV = 0 so that (1.49) becomes

$$\left. \frac{\partial V}{\partial t} dT + \frac{\partial V}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n} dx_n \right|_{\text{at } B^* \text{ on } \xi} = 0 \tag{1.50}$$

It should be emphasized that the quantities dx_i, dT in (1.49) and (1.50) are not governed by (1.1) but rather are small arbitrary displacements tangent to ξ at B^*. In words, (1.50) states that at the optimal terminal point B^* on ξ the gradient of V ($\nabla V_{t,x}$) should be perpendicular to the ξ manifold. It is seen that any contour V = constant which happens to be tangent to ξ will satisfy (1.50) so that (1.50) is only a necessary condition. Actually, (1.50) is not even a necessary condition unless the above mentioned conditions leading to (1.50) are satisfied. For the special case in which $V \triangleq \min_{u \in U} \psi(x_i(T), T)$, a small change in the coordinates $x_i(T)$, T will not necessarily yield dV = 0. In this case, the optimal terminal point B^* is characterized by

$$dV = \frac{\partial \psi}{\partial T} dT + \frac{\partial \psi}{\partial x_1} dx_1 + \dots + \frac{\partial \psi}{\partial x_n} dx_n \Big|_{\text{at } B^* \text{ on } \xi} \geq 0 \qquad (1.51)$$

Suppose that (1.50) does apply, and let ξ be given in the form

$$\xi : \quad \begin{aligned} x_{1T} &= b_1 \\ x_{2T} &= b_2 \\ &\vdots \\ x_{kT} &= b_k \end{aligned} \qquad (k \leq n)$$

Then, $dx_i \Big|_{\xi} = 0$, $(i = 1, \dots, k)$ and (1.50) becomes

$$\frac{\partial V}{\partial t} dT + \frac{\partial V}{\partial x_{k+1}} dx_{k+1} + \dots + \frac{\partial V}{\partial x_n} dx_n \Big|_{\text{at } B^* \text{ on } \xi} = 0 \qquad (1.52)$$

for arbitrary values of $dx_{k+1}, \dots, dx_n$, dT. Since $dx_{k+1}, \dots, dx_n$ are arbitrary, (1.52) implies that

$$\frac{\partial V}{\partial x_j}\Big|_{\xi} = 0 \qquad (j = k+1, \dots, n) \qquad (1.53)$$

Also, if T = free, then $dT\Big|_{\xi} \neq 0$ so that from (1.52)

$$\frac{\partial V}{\partial t}\Big|_{\xi} = 0 \tag{1.54}$$

Note, that in contrast to (1.47), (1.54) applies even when some $\dot{x}_i$, f_o, or U depend on t explicitly.

When T = fixed, (1.52) does not yield any information about $\frac{\partial V}{\partial t}\Big|_{\xi}$. However, from the definition of V, when T = fixed then $V(x_1, \ldots, x_n, T) = 0$ for all allowable values of $x_1, \ldots, x_n$ at T. The reverse statement for x_{iT} fixed is not necessarily true since x_i need not be strictly increasing like time. Thus, if x_{iT} = fixed $(i = 1, \ldots, k)$ and T = free, then $V(x_{1T}, \ldots, x_{kT}, x_{k+1,T}, \ldots, x_{nT}, T)$ is not necessarily zero for any allowable $(x_{k+1,T}, \ldots, x_{nT}, T)$. In other words, if T = fixed, and the problem begins at some point on ξ then the problem must immediately end at that same point on ξ. But, if T = free and the problem begins at some point on ξ the optimal trajectory K* may leave and return at some later time to some other point on ξ.

When (1.51) applies, the terminal values of $\partial V/\partial t$ and $\partial V/\partial x_i$ are not necessarily zero even though T and x_i may be free. In this case

$$\begin{aligned} \frac{\partial V}{\partial t}\Big|_{\xi} &= \frac{\partial \psi}{\partial T} \\ \frac{\partial V}{\partial x_i}\Big|_{\xi} &= \frac{\partial \psi}{\partial x_i} \end{aligned} \tag{1.55}$$

1.2.6. Initial Conditions for the Optimal Trajectory K*

It has been remarked earlier that in general the starting point for the problem may not be completely fixed but instead may be required to lie on some initial manifold (1.15). The optimum starting point A^* on η is characterized by the fact that no other point on η lies on an isovee contour of lower value. As in the case of the terminal manifold there may, in general, be a corner in the η manifold at A^* or in the isovee contour V* which passes through A^*. Also, the point A^* may lie on the edge of the η manifold. In such a case, the optimal starting point A^* satisfies the necessary condition $dV > 0$ for any arbitrary (allowable) displacement (dx_i, dt_o) tangent to η. Or,

$$\frac{\partial V}{\partial t} dt_o + \frac{\partial V}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n} dx_n \Bigg|_{\text{at } A^* \text{ on } \eta} > 0 \qquad (1.56)$$

If the V* contour and η are "smooth" at the optimal point A^*, and A^* is not on the edge of η then any arbitrary displacement (dx_i, dt_o) tangent to η must yield $dV = 0$. Thus (1.56) becomes

$$\frac{\partial V}{\partial t} dt_o + \frac{\partial V}{\partial x_1} dx_1 + \ldots + \frac{\partial V}{\partial x_n} dx_n \Bigg|_{\text{at } A^* \text{ on } \eta} = 0 \qquad (1.57)$$

When (1.57) does apply, then for every x_{io} which is completely free at $t = t_o$ the condition $dx_{io} \neq 0$ holds so that

$$\frac{\partial V}{\partial x_i} \Bigg|_{\eta} = 0 \qquad (1.58)$$

If the initial time t_o is free, then (1.57) implies

$$\left.\frac{\partial V}{\partial t}\right|_{\eta} = 0 \tag{1.59}$$

When t_o is fixed, (1.57) yields no information about $\left.\frac{\partial V}{\partial t}\right|_{\eta}$. However, for the special case where all $\dot{x}_i$, f_o, and U are explicitly independent of time then, from (1.45)

$$\left.\frac{\partial V}{\partial t}\right|_{\eta} = \left.\frac{\partial V}{\partial t}\right|_{\xi} \tag{1.60}$$

Since the criteria for the optimal initial point A^* on η and the optimal terminal point B^* on ξ are similar, equations (1.56) and (1.57) are equivalent to (1.49) and (1.50). For this reason, the remarks below (1.50) also apply to equations (1.56) and (1.57). Equations (1.50) and (1.57) are sometimes referred to as the transversality conditions [1, pg. 162] at η and ξ. The conditions under which (1.50) and (1.57) do not apply are apparently not often encountered and therefore (1.50) and (1.57) are usually assumed (until proven otherwise) to be necessary conditions for the optimal trajectory K*.

1.2.7. Pontryagin's Maximum Principle

In 1958, the Russian mathematician, L. S. Pontryagin and his coworkers V. G. Boltyanskii and R. V. Gamkrelidze introduced a new technique for solving the general optimal control problem [10]. This technique, known as the Maximum Principle, can be derived by several methods [11], [12]. The presentation given below is based on the "isovee" concept introduced previously.

The optimization technique based on Pontryagin's Maximum Principle (PMP) is essentially a reformulation of the optimization theory given above. In the PMP technique, a new "independent" coordinate p_i $(i = 1, \ldots, n)$ is introduced[4] into the problem by defining

$$p_i \triangleq -\frac{\partial V}{\partial x_i} \tag{1.61}$$

Motivation for introducing this auxiliary coordinate follows from a similar technique used by Sir W. R. Hamilton (in 1834) for solving problems in what is now called Classical Mechanics [7]. For his dynamical systems, Hamilton introduced, by means of a contact (Legendre) transformation, an auxiliary coordinate p defined by

$$p_i \triangleq \frac{\partial L}{\partial \dot{q}_i} \tag{1.62}$$

where

L = Lagrangian function (the analog of f_o in (1.4))

q_i = generalized coordinates of the system.

If (1.61) is substituted into (1.27), the Hamilton-Jacobi equation becomes

$$\frac{\partial V}{\partial t} = \max_{u \in U} \left[p_1 \dot{x}_1 + \ldots + p_n \dot{x}_n - f_o \right] \tag{1.63}$$

In PMP, the bracket on the right side of (1.63) is called the Hamiltonian H so that

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[4] In essence, this changes the "basis" of the coordinate system from (x,t) to (p,x,t). There are several ways one could define p_i, (1.61) being the most convenient for optimization problems.

$$H(p,x,t,u) = p_1\dot{x}_1 + \dots + p_n\dot{x}_n - f_o \qquad (1.64)$$

With this definition, (1.63) becomes

$$H^*(p,x,t,u^*(p,x,t)) = \frac{\partial V}{\partial t} \qquad (1.65)$$

or

$$H^*(p,x,t,u^*(p,x,t)) = \max_{u \in U} \left[p_1\dot{x}_1 + \dots + p_n\dot{x}_n - f_o \right] \qquad (1.66)$$

where

$H^* = H$ with u replaced by the optimal control $u^* = u^*(p,x,t)$.

Equation (1.66) is the basic equation used in PMP and it is seen to be equivalent to the Hamilton-Jacobi equation (1.27). With the two changes in notation [(1.61) and (1.64)], the relations derived in the previous analysis can be converted into the corresponding relations used in PMP. For instance, if (1.66) is continuously differentiable at $u = u^*$ and u^* is not on the boundary of U, then as in (1.28) the optimal control is determined from

$$\left. \frac{\partial H}{\partial u} \right|_{u=u^*} \equiv 0 \qquad (1.67)$$

which, by (1.29), will yield

$$u^* = u^*(p, x, t) \qquad (1.68)$$

From (1.33) is obtained the relation

$$\int_{t_o}^{t} \left[H^* - p_1\dot{x}_1 - \dots - p_n\dot{x}_n + f_o \right] dt \doteq 0 \qquad (1.69)$$

From (1.34) and (1.40), the so-called <u>canonical equations</u> [11] of PMP are obtained as

$$\left.\frac{dx_i}{dt}\right|_{K^*} = \frac{\partial H^*(p,x,t,u^*)}{\partial p_i} \qquad (1.70)$$

$$\left.\frac{dp_i}{dt}\right|_{K^*} = \frac{-\partial H^*(p,x,t,u^*)}{\partial x_i} \qquad U \neq U(x) \qquad (1.71)$$

The p_i defined in (1.61) are referred to as "adjoint" or "costate" variables and are equivalent to the negative of the Lagrange multiplier function λ_i used in the Classical Calculus of Variations (see (1.8)). In this connection (1.71) corresponds to the Euler equations (1.6). The solutions $x_i = x_i(t)$ and $p_i = p_i(t)$ of (1.70) and (1.71) are actually the characteristics (characteristic strips) of the Hamilton-Jacobi equation [see (1.34) and (1.40)]. This result shows the interrelation between Dynamic Programming and the Maximum Principle. Dynamic Programming may be viewed as a numerical method for solving the Hamilton-Jacobi equation and the Maximum Principle as a method for determining the characteristic strips of the Hamilton-Jacobi equation. From (1.43) the relation between the total and partial time derivatives of H are

$$\frac{dH^*(p,x,t,u^*)}{dt} = \frac{\partial H^*(p,x,t,u^*)}{\partial t} \qquad (1.72)$$

where u* is as given in (1.68). It should be noted in (1.70), (1.71), and (1.72) that the p_i, x_i, and t are treated as <u>independent</u> coordinates in the (p,x,t) space.

If all x_i, f_o, and U are explicitly independent of time then, from (1.45), along K*

$$H^*(p,x,t,u^*(p,x,t)) = c \qquad \begin{matrix} (c = \text{constant}) \\ t_o \leq t \leq T \end{matrix} \tag{1.73}$$

which holds for either T = fixed or T = free. If, in addition, T = free and f_o does not contain additive constants then (1.73) becomes (following the reasoning of (1.47))

$$H^*(p,x,t,u^*(p,x,t)) \equiv 0 \qquad (t_o \leq t \leq T) \tag{1.74}$$

When (1.73) or (1.74) holds, then (following (1.48)), (1.66) has the first integral

$$p_1\dot{x}_1 + \ldots + p_n\dot{x}_n - f_o \Big|_{K^*} = C \qquad (C = \text{constant}) \tag{1.75}$$

Combining (1.49) (1.50), and (1.56) (1.57), the PMP versions of the transversality conditions are obtained as

$$H^*\, dT - p_1\, dx_1 - \ldots - p_n\, dx_n \Big|_{\text{at } B^* \text{ on } \xi} \geq 0 \tag{1.76}$$

and

$$H^*\, dt_o - p_1\, dx_1 - \ldots - p_n\, dx_n \Big|_{\text{at } A^* \text{ on } \eta} \geq 0 \tag{1.77}$$

where (dx_i, dT) and (dx_i, dt_o) are arbitrary (allowable) displacements tangent to the ξ and η manifolds respectively. The conditions under which equality holds in (1.76) and (1.77) are the same as discussed for (1.50) and (1.57). As mentioned previously, it is common practice to assume (until proven otherwise) that equality holds in (1.76) and (1.77). From (1.53) and (1.58) it is seen that for each x_i which is completely free at $t = T$ (or $t = t_o$) the corresponding p_i are given as [assuming (1.51) is not applicable]

$$\left. p_i \right|_\xi = 0 \qquad (t = T)$$
$$\left. p_i \right|_\eta = 0 \qquad (t = t_o) \tag{1.78}$$

Also, if T (or t_o) are completely free, then from (1.54) and (1.59)

$$\left. H^* \right|_\xi = 0 \qquad (T = \text{free})$$
$$\left. H^* \right|_\eta = 0 \qquad (t_o = \text{free}) \tag{1.79}$$

When $V \triangleq \min_{u \in U} \psi(x_i(T),T)$ then (1.78) and (1.79) do not apply at ξ because (1.50) is replaced by (1.51). In this case, (1.55) is used to obtain

$$\left. p_i \right|_\xi = -\frac{\partial \psi}{\partial x_i} \tag{1.80}$$

$$\left. H^* \right|_\xi = \frac{\partial \psi}{\partial T} \tag{1.81}$$

The basic optimal control problem formulated previously is solved by the PMP technique as follows:

a) Using (1.1) and (1.4), form the Hamiltonian H as given in (1.64).

b) Consider H in (1.64) as a function of u, and determine the $u^* = u^*(p,x,t)$ which maximizes H. If (1.67) is applicable, this maximization process is straightforward. Otherwise, it may be necessary to "inspect" (1.64) for various values of (x_i, p_i, t).

c) Substitute $u^* = u^*(p,x,t)$ into (1.64) to obtain H^*.

d) Obtain the canonical equations (1.70) and (1.71).

e) Integrate the canonical equations from t_o to T to determine $x_i = x_i(t)$ and $p_i = p_i(t)$. This latter step is difficult since (from (1.73) through (1.81) only <u>some</u> of the initial and terminal conditions for (1.70) and (1.71) are known a priori. Thus, trial and error techniques are usually required to determine the unknown p_{io}, x_{io}, t_o and p_{iT}, x_{iT}, T.

f) Substitute $x_i = x_i(t)$ and $p_i = p_i(t)$ into $u^* = u^*(p,x,t)$ to obtain $u^* = u^*(t)$ the "open loop" control.

g) Alternately, attempt to obtain the "closed loop" control $u^* = u^*(x)$ by eliminating the parameter t between the equations $p_i = p_i(t)$ and $x_i = x_i(t)$ obtained in (e).

1.3. <u>Research Objectives</u>

The analytical techniques outlined in the previous article constitute the primary methods of analysis presently used in optimal control theory. Each of these methods has been successfully applied to a large variety of optimal control problems. There is, however, a certain class of optimal control problems (this certain class is a sub-class of the general problem considered in the previous article) for which all the methods given above break down. This special class of problems is characterized by the fact that the control u enters the plant equations (1.1) and index of performance integrand (1.4) in a linear manner. That is, the plant equations (1.1) and index of performance (1.4) are of the form

$$\dot{x}_i = g_i(x_1,\ldots,x_n,t) + u\, h_i(x_1,\ldots,x_n,t) \quad (i = 1, \ldots, n) \tag{1.82}$$

$$J[u] = \int_{t_o}^{T} \left[g_o(x_1,\ldots,x_n,t) + u\, h_o(x_1,\ldots,x_n,t) \right] dt \tag{1.83}$$

Hereafter, an optimization problem characterized by (1.82) and (1.83) will be called a Linear Optimization Problem (LOP). An existence theorem for this class of problems has been given by Lee and Markus [13].

When the methods given above are used to determine the optimal control for a LOP, the formal solution for u(t) appears as (see Chapter 2)

$$u^*(t) = \begin{cases} A & \text{if} \quad F(t) > 0 \\ B & \text{if} \quad F(t) < 0 \end{cases} \tag{1.84}$$

where A and B are respectively the upper and lower bounds on the control u, and F(t) is a certain function of time called the "switching function". However, it is characteristic of the solutions to LOP that the switching function F(t) sometimes becomes identically zero over some finite time interval. In such a case, (1.84) fails to yield any information concerning the optimal control. Those LOP in which F(t) becomes identically zero over some finite time interval have been referred to as "singular" [14], [15], "degenerate" [16], "not normal" [17], and "ambiguous" [18]. Although the existence of singular solutions in the calculus of variations has been recognized for some time [1], apparently little is known about the general nature of such solutions.

The primary objective of the research discussed in this report is to examine, from a general point of view, singular solutions in LOP, and to develop analytic procedures which may be useful in detecting and calculating singular solutions. The techniques developed are illustrated by several detailed examples.

Chapter 2

LINEAR OPTIMIZATION PROBLEMS AND SINGULAR SOLUTIONS

2.1. Formal Solution of LOP by Conventional Methods

In order to establish that the optimal control $u^*(t)$ for a LOP is (formally) of the form (1.84), it is instructive to consider the solution to the general LOP (1.82), (1.83) as obtained by the various methods given in Chapter 1. When the control is unbounded, the solution to a LOP will involve infinite values of the control. For this reason, the formulation of a LOP should always be accompanied by constraints on the control $u(t)$. Hereafter, it will be assumed that the control u is constrained by the relation

$$B \leq u(t) \leq A \qquad (A > B) \tag{2.1}$$

where A and B are real constants.

In order to apply the classical calculus of variations to the general LOP (1.82), (1.83) with the constraint (2.1) one may employ the device of Valentine described in Chapter 1. In this case, the constraint (2.1) can be put into the form (1.10) by writing

$$R = (A - u)(u - B) \geq 0 \tag{2.2}$$

The auxiliary control variable z is then defined from (1.11) to be

$$\dot{z} = \sqrt{(A - u)(u - B)} \tag{2.3}$$

Substituting (1.82), (1.83), and (2.3) into (1.12), the function G for a LOP becomes

$$G = g_o(x_1,\dots,x_n,t) + u\,h_o(x_1,\dots,x_n,t)$$
$$+ \sum_{i=1}^{n} \lambda_i \left[g_i(x_1,\dots,x_n,t) + u\,h_i(x_1,\dots,x_n,t) - \dot{x}_i \right]$$
$$+ \mu \left[(A - u)(u - B) - \dot{z}^2 \right] \tag{2.4}$$

The Euler equations for (2.4) are, from (1.6), (1.7) and (1.13),

$$\frac{d\lambda_i}{dt} = - \frac{\partial g_o}{\partial x_i} - u \frac{\partial h_o}{\partial x_i} - \sum_{j=1}^{n} \lambda_j \left[\frac{\partial g_j}{\partial x_i} + u \frac{\partial h_j}{\partial x_i} \right] \tag{2.5}$$

$$h_o + \sum_{j=1}^{n} \lambda_j\, h_j + \mu \left[A - 2u + B \right] = 0 \tag{2.6}$$

and

$$\frac{d}{dt}\left(\mu\, \dot{z} \right) = 0 \tag{2.7}$$

where $\mu(t) \leq 0$. It can be shown [2], that at terminal time

$$\mu\, \dot{z} \Big|_{t=T} = 0 \tag{2.8}$$

and therefore (2.7) and the continuity requirement for $\mu(t)\,\dot{z}(t)$ requires

$$\mu(t)\,\dot{z}(t) \equiv 0 \qquad (t_o \leq t \leq T) \tag{2.9}$$

If μ is not zero, then (2.3) and (2.9) imply

$$(A - u)(u - B) = 0 \tag{2.10}$$

Using the fact that $A > B$ and $\mu < 0$, it is seen that (2.6) and (2.10) can only be satisfied by

$$u^*(t) = \begin{cases} A & \text{if:} \quad F_1(t) > 0 \\ B & \text{if:} \quad F_1(t) < 0 \end{cases} \tag{2.11}$$

where

$$F_1(t) \triangleq - h_o - \sum_{i=1}^{n} \lambda_i \, h_i \tag{2.12}$$

When $F_1(t) \equiv 0$ in (2.11), (2.12) then (2.6) and (2.9) can only be satisfied by

$$\mu(t) \equiv 0 \tag{2.13}$$

In this case, (2.4) becomes

$$G = g_o(x_1,\ldots,x_n,t) + \sum_{i=1}^{n} \lambda_i \left[g_i(x_1,\ldots,x_n,t) - \dot{x}_i \right] \tag{2.14}$$

and the necessary condition of Weierstrass (1.9) degenerates to the trivial identity 0 = 0. Thus, when $F_1(t) \equiv 0$ the necessary condition of Weierstrass also fails to yield any information about the optimal control $u^*(t)$.

The Dynamic Programming method for solving the general LOP (1.82), (1.83) consists of solving (1.27) in a discrete manner. Substituting (1.82) and (1.83) into (1.27) the functional equation of dynamic programming is

$$\frac{\partial V}{\partial t} = \max_{u \in U} \left[- g_o(x_1,\ldots,x_n,t) - u \, h_o(x_1,\ldots,x_n,t) \right.$$

$$\left. - \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \left[g_i(x_1,\ldots,x_n,t) + u \, h_i(x_1,\ldots,x_n,t) \right] \right] \tag{2.15}$$

It is clear that for any given values of x_i, $\partial V/\partial x_i$, and t, (2.15) is maximized with respect to the control u by choosing

$$u^*(t) = \begin{cases} A & \text{if: } F_2(t) > 0 \\ B & \text{if: } F_2(t) < 0 \end{cases} \tag{2.16}$$

where

$$F_2(t) \triangleq - h_o - \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} h_i \tag{2.17}$$

However, when $F_2(t) \equiv 0$ then (2.15) becomes explicitly independent of u and (2.16) [like (2.11)] fails to yield any information concerning the optimal control.

The solution of the general LOP (1.82), (1.83) by Pontryagin's Maximum Principle involves maximizing the Hamiltonian function H(1.64). Thus, substituting (1.82) and (1.83) into (1.64) yields

$$H = \sum_{i=1}^{n} p_i \left[g_i(x_1,\ldots,x_n,t) + u\, h_i(x_1,\ldots,x_n,t \right]$$

$$- g_o(x_1,\ldots,x_n,t) - u\, h_o(x_1,\ldots,x_n,t) \tag{2.18}$$

From (2.18), it is clear that the Hamiltonian is maximized by selecting

$$u^*(t) = \begin{cases} A & \text{if: } F_3(t) > 0 \\ B & \text{if: } F_3(t) < 0 \end{cases} \tag{2.19}$$

where

$$F_3(t) \triangleq -h_o + \sum_{i=1}^{n} p_i h_i \qquad (2.20)$$

When $F_3(t) \equiv 0$, the Hamiltonian (2.18) becomes explicitly independent of the control u and (2.19) fails to yield any information concerning the optimal control. The relationship between λ_i, $\partial V/\partial x_i$, and p_i is evident from comparison of (2.12), (2.17), and (2.20). Controls of the form (2.11), (2.16), and (2.19) are commonly referred to as "bang-bang" controls. The term $F_i(t)$ in (2.11), (2.16) and (2.19) is called the "switching function".

Equations (2.11), (2.16), and (2.19) verify that for the singular condition $F(t) \equiv 0$, formal solutions of LOP by conventional methods fail to yield any information about the desired optimal control. However, it will be shown below that the control u(t) which maintains the singular condition $F(t) \equiv 0$ may satisfy certain necessary conditions for an optimal control. For this purpose, the Maximum Principle is used to reexamine in detail the general LOP formulated above.

2.2. The Maximum Principle and Linear Optimization Problems

The general LOP formulated above permits the variable t (time) to appear explicitly in the system equations (1.82) and the integrand of the index of performance (1.83). However, if an auxiliary state variable x_{n+1} is defined as

$$x_{n+1} \triangleq t \qquad (2.21)$$

or

$$\dot{x}_{n+1} = 1 \tag{2.22}$$

then (2.21) can be substituted into (1.82), (1.83), [and (2.22) can be appended to the set (1.82)], so that (1.82) and (1.83) become explicitly independent of time. Further, if the terminal time T is explicitly fixed, then (2.21) and (2.22) can be used to convert the problem to one with a free terminal time and with the additional required boundary condition

$$x_{n+1}(T) = T \tag{2.23}$$

To simplify the index notation, it will be assumed in the following that whenever the auxiliary state variable (2.21) is used it will be included in the n original system equations.

When the system equations (1.82) and index of performance (1.83) are explicitly independent of time and terminal time is free, then from (1.74) the optimal value of the Hamiltonian is (noting the assumption (2.1))

$$H^*(p,x,u^*(p,x)) \equiv 0 \quad (t_o \leq t \leq T) \tag{2.24}$$

Since $H^* \triangleq \max_{u \in U} H(p,x,u)$, equation (2.24) indicates that at every instant of time along an optimal trajectory no portion of the curve representing the instantaneous plot of H vs. u $(u \in U)$ can lie in the upper half plane $H > 0$. Further, if an optimal control exists, then there must be at least one point u^* $(u^* \in U)$ at which the curve H vs. u touches the $H = 0$ axis. Since (2.24) is a necessary condition for an optimal control the Maximum Principle may be stated as follows:

<u>Statement 1</u>

If it has been established that $H^* \equiv 0$ then any control u^* which satisfies the relation

$$H(u^*) = \max_{u \in U} H(u) \equiv 0 \qquad (t_o \leq t \leq T),$$

and the boundary conditions of the problem, is a candidate for the optimal control.

Controls which satisfy the necessary condition of Statement 1 will be called "extremal controls". The trajectory produced by a system subjected to extremal control will be called an "extremal path". In general, if an optimal control is known to exist and it is possible to prove that there is only one extremal control then that (unique) control is optimal. If (1.82) and (1.83) are linear in the dependent variables and separable in the control variable [i.e., $h_i = h_i(t)$, $(i = 0, 1, \ldots, n)$ in (1.82), (1.83)] then satisfying the Maximum Principle is both a necessary and sufficient condition for the optimal control [18]. In this case, the Maximum Principle may be stated:

Statement 2

If (1.82) and (1.83) are linear in the dependent variables and separable in the control variable and if it has been established that $H^* \equiv 0$ then any control which satisfies the relation

$$H(u^*) = \max_{u \in U} H(u) \equiv 0 \qquad (t_o \leq t \leq T),$$

and the boundary conditions of the problem, is an optimal control.

If (2.21) and (2.22) are usedto make (1.82) and (1.83) explicitly independent of time then from (1.64) the Hamiltonian for the general LOP is

$$H(p,x,u) = I(p,x) + u\, F(p,x) \tag{2.25}$$

where[5]

$$I(p,x) \triangleq - g_o + p_1 g_1 + \dots + p_n g_n$$

$$F(p,x) \triangleq - h_o + p_1 h_1 + \dots + p_n h_n$$

$$g_i = g_i(x_1,\dots,x_n)$$

$$h_i = h_i(x_1,\dots,x_n) \qquad (i = 0, 1, \dots, n)$$

and, from (1.71),

$$\frac{dp_i}{dt} = - \frac{\partial H^*(p,x,u^*)}{\partial x_i} \qquad (i = 1, \dots, n) \tag{2.26}$$

If (2.23) is used to make the terminal time free then the optimal value of (2.25) is

$$\left[I(p,x) + u^* F(p,x) \right] = \max_{u \in U} \left[I(p,x) + u F(p,x) \right] \equiv 0 \qquad (t_o \leq t \leq T) \tag{2.27}$$

In (2.27) it is understood that the arguments of I and F are p(t) and x(t). According to Statement 1, any control u* which satisfies (2.27) and the boundary conditions of the problem is an extremal control. In general, there are three controls which could satisfy (2.27):

1) The control

$$u_1^*(t) \equiv 0 \tag{2.28}$$

if it is compatible with (2.19) and if it will satisfy $F(t) \not\equiv 0$, $I(t) \equiv 0$.

---

[5] Note that F_3 in (2.19), (2.20) is the same as F in (2.25).

2) Any control $u_2^*(t)$ which will satisfy

$$I(t) = - u_2^*(t)\ F(t) \qquad [I(t),\ F(t) \not\equiv 0] \qquad (2.29)$$

Note that for the particular bounded control $|u^*(t)| \le M$, (2.19) and (2.29) require $I(t) = - M\,|F(t)|$ or

$$I(t) < 0 \qquad (M > 0) \qquad (2.30)$$

3) Any control $u_3^*(t)$ which will satisfy

$$I(t) \equiv F(t) \equiv 0 \qquad (B \le u_3^*(t) \le A) \qquad (2.31)$$

The particular conditions for which u_1^*, u_2^*, and u_3^* will satisfy Statement 1 are illustrated in Figure 2.1. The control $u_1^*(t) \equiv 0$ corresponds to zero control effort. The control

$$u_2^*(t) = - \frac{I(t)}{F(t)} \qquad (2.32)$$

must correspond to switching between maximum and minimum control effort as given by (2.19). During this type of control $I(t)$ and $F(t)$ must vanish simultaneously at each switch. Note that the control $u_1^*(t)$ is actually a special case of the u_2^* control where either A or B (or both) become zero, as in problems with control energy (fuel) constraints. The control u_3^* which satisfies $I(t) \equiv F(t) \equiv 0$, also satisfies the relation (see (1.67))

$$\frac{\partial H}{\partial u}(x,\ p,\ u) \equiv 0 \qquad (2.33)$$

and usually corresponds to continuously variable control effort in the interior of the admissible set U. The condition $I(t) \equiv F(t) \equiv 0$ corres-

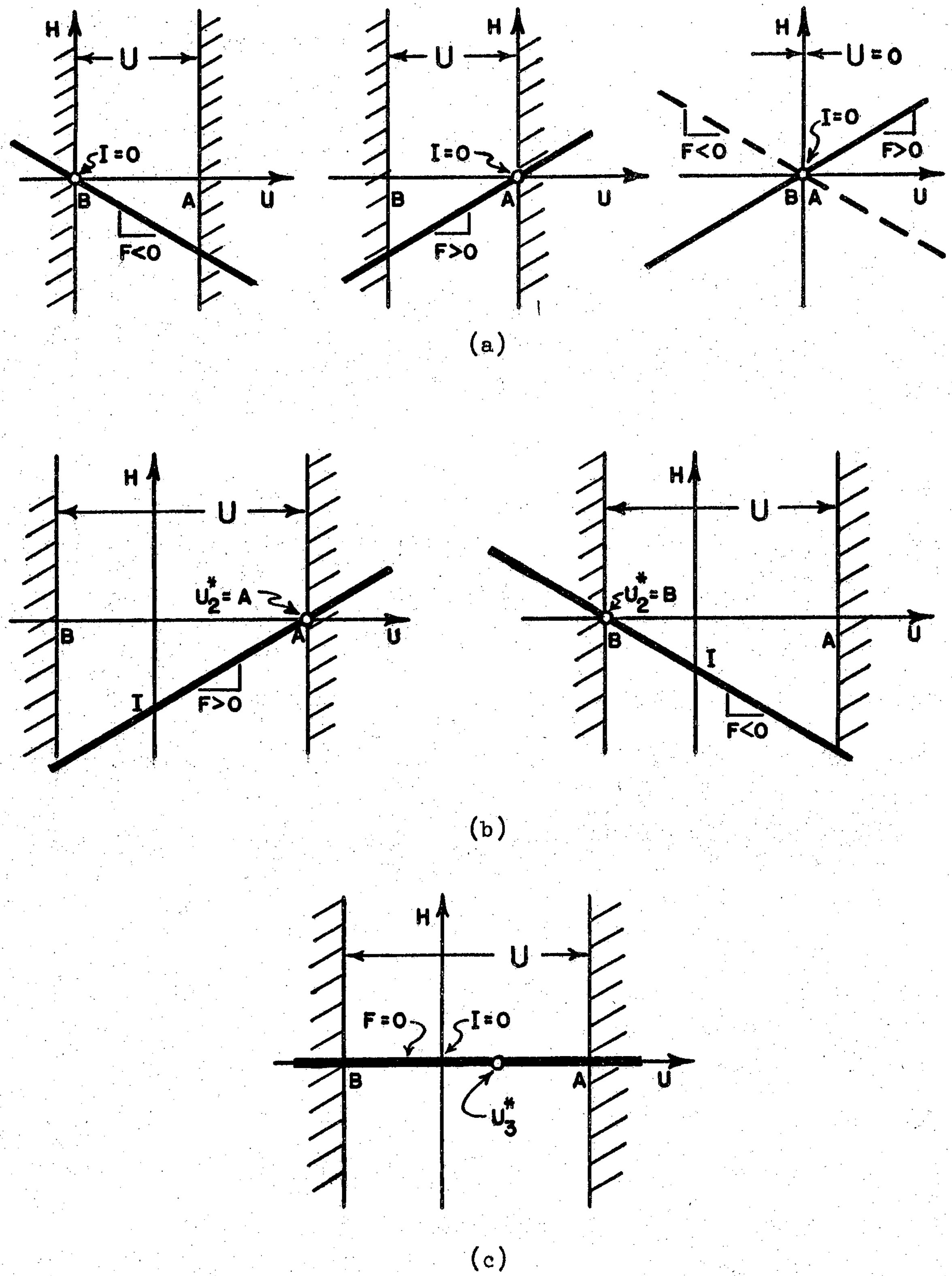

Figure 2.1. Conditions for which the Controls u_1*, u_2*, and u_3* will Satisfy Statement 1; (a) u_1* = 0, (b) u_2* = - I/F, (c) u_3*

ponds to the singular condition described previously. It is seen, therefore, that singular control (u_3*) satisfies the necessary condition (2.27) and may constitute a sub-arc of an extremal control.

In addition to satisfying (2.27), an optimal control must satisfy the specified initial and terminal values of the system state variables. However, it is generally impossible to satisfy these required boundary conditions by exclusive use of either u_1*, u_2*, or u_3*. In this case, the optimal control for a LOP, if it exists, must consist of some combination (sequence of sub-arcs) of the u_1*, u_2*, and u_3* controls which satisfies the required state variable boundary conditions. The selection of this optimal sequence is complicated by the fact that, in general, there may be several different combinations of these sub-arcs which satisfy the physical boundary conditions of the problem.

The results given above indicate that the optimal control for a LOP will, in general, consist of a combination of bang-bang type control (2.19) and singular control (2.31). Therefore, in order to compute the optimal control for a LOP one must determine the following:

1) Is singular control admissible as a candidate for the optimal control? That is, is it possible for the singular condition $F(t) \equiv 0$ to occur?
2) How may the bang-bang and singular control sub-arcs be combined in order to satisfy necessary conditions for the optimal control?
3) What is the functional form of the singular control?
4) Which of the possible combinations of extremal controls is actually the optimal control? This latter question includes

the question of whether a singular extremal control sub-arc does in fact appear in the optimal control.

Some techniques which may be helpful in answering these questions are discussed in Chapter 3.

Chapter 3

THE NATURE OF SINGULAR SOLUTIONS AND COMPUTATIONAL TECHNIQUES

3.1. Necessary Conditions for Pieced Extremal Paths

It has been shown in the previous chapter that, in general, the optimal control for a LOP will consist of both bang-bang (u_1^*, u_2^*) and singular (u_3^*) sub-arcs pieced together so as to satisfy the given initial and terminal conditions of the state variables. However, in addition to satisfying the physical constraints of the problem, an optimal control must satisfy certain other requirements. These additional requirements serve to reduce the number of pieced extremal paths which may be candidates for the optimal control.

The fundamental requirement of an optimal control for a LOP is that the Maximum Principle (2.27) be satisfied at all times $t_o \leq t \leq T$. Thus, the continuity property[6] of $H^*(t)$ as indicated in (2.27) prevents any changes in the control u as long as $F(t) \neq 0$. For instance, if it can be established that $F(t)$ does not change sign in a particular region Q of the n-dimensional x-space then it can be concluded that the optimal control in the region Q must be either $u^*(t) \equiv A$ or $u^*(t) \equiv B$.

Another requirement which must be satisfied by the optimal control is the specified initial and terminal values $p_i(t_o)$ and $p_i(T)$ of certain of the adjoint variables. As shown in Chapter 1, the required values of

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[6] If some of the state variables x_i are bounded, or if g_i or h_i $(i = 0, 1, \ldots, n)$ in (1.82), (1.83) do not possess continuous partial derivatives with respect to x_i and t then H^* might not be continuous [11], [19].

$p_i(t_o)$ and $p_i(T)$ are determined from the transversality conditions (1.76) and (1.77) and the given boundary conditions of the state variables. The requirement that the transversality conditions be satisfied is especially useful in singular control problems since this necessary condition often provides the only information concerning if, when, and how singular control sub-arcs should be joined with bang-bang sub-arcs. Application of this technique is demonstrated in Example 4.2 of Chapter 4.

In some problems, either the physical constraints or mathematical requirements will not allow certain of the controls u_1^*, u_2^* and u_3^*. For instance, it may be found that in order to have the singular condition $I(t) \equiv F(t) \equiv 0$ all the adjoint variables $p_i(t)$ $(i = 1, \ldots, n)$ must be identically zero. From (2.27), this condition implies that the integrand of the index of performance (1.83) is identically zero

$$g_o(x_1(t), \ldots, x_n(t)) + u(t) \; h_o(x_1(t), \ldots, x_n(t)) \equiv 0 \qquad (3.1)$$

If, for a particular problem, the condition (3.1) is known to be physically impossible then it may be concluded that a singular control sub-arc is not allowable.

3.2. The Singular Control Surface

The study of LOP in which singular solutions appear is simplified if it is possible to determine a surface S in the x-space which represents the condition

$$\begin{aligned} I(p(t), x(t)) &\equiv 0 \\ F(p(t), x(t)) &\equiv 0 \end{aligned} \qquad (3.2)$$

That is, if the condition (3.2) can be reduced to a relation

$$S(x_1(t), \ldots, x_n(t)) \equiv 0 \tag{3.3}$$

which defines a surface (hypersurface) in the n-dimensional x-space. The ability to express S as a function of state variables alone depends on the number of independent relations which can be obtained by taking successive time derivatives of (3.2) using the system equations (1.82) and the adjoint equations (2.26). General expressions for S in the case of first and second order systems (1.82) and for a particular class of third order systems, are given in Appendix I. In Example 4.4 of Chapter 4, a particular class of LOP is examined and general expressions for S are obtained for the n^{th} order system. For some higher order systems ($n \geq 3$) it may be impossible to eliminate all the p_i from the expression for S. The surface S does not exist if (3.2) leads to vacuous or impossible conditions. In particular, if (3.2) implies $p_i(t) \equiv 0$ $(i = 1, \ldots, n)$ then S exists only if (3.1) can be satisfied.

The surface S will be called the "singular control surface" since the state variable trajectory corresponding to singular control $u_3{*}$ must lie on this surface. For this reason, only those regions of S corresponding to $B \leq u_3{*} \leq A$ are considered. The surface S is also the "singular control switching boundary" since any point in the state space which is not on S must be associated with bang-bang control. A typical singular control surface is shown in Figure 3.1. For higher order systems, it is convenient to construct projections of the hypersurface S on various state variable planes. In some cases, a suitable coordinate transformation of the x-space may allow the hypersurface S to project into a surface of lower dimension.

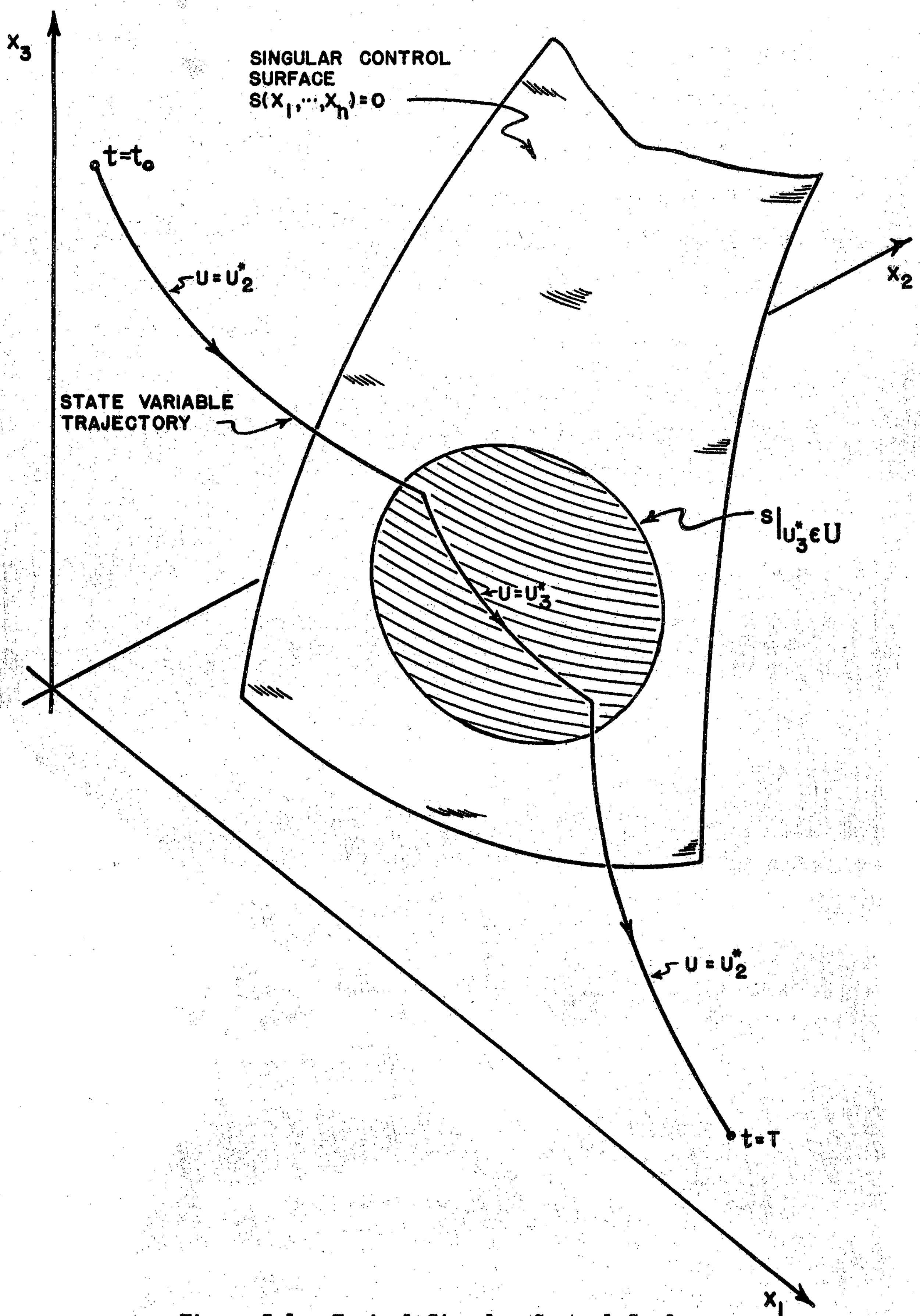

Figure 3.1. Typical Singular Control Surface

In [20] this technique is used to transform a particular class of S hypersurfaces into lines.

It is not unusual for S to be a multi-sheet surface in the state space. In this case there may be several distinct singular control trajectories each corresponding to a different singular control function u_3^*. The given boundary conditions for the problem help determine if and when each sheet of S may be used in the optimal control sequence. It should be noted, however, that the existence and location of the surface S is not dependent upon the particular boundary conditions of the problem.

With the S surface constructed in the state space, it is relatively easy to fill in the region around S with state variable trajectories corresponding to bang-bang control (2.19). The resulting field or network of extremal paths provides a clear picture of all possible optimal control sequences. From such a representation (see Fig. 4.2), it may be seen that the singular control trajectory is a compaction or locus of many extremal paths.

3.3. Characteristics of the Singular Control Surface

For the general LOP (1.82) the index of performance (1.83) may be written in terms of the Hamiltonian (2.25) as

$$J[u] = \int_{t_o}^{T} - H(p, x, u)dt + p_1 \, dx_1 + \dots + p_n \, dx_n \qquad (3.4)$$

Equation (3.4) is a line integral in the n + 1 dimensional x-t space. In general, (3.4) will depend on the path of integration if the term H is an explicit function of u. However, if H in (3.4) is explicitly independent of u, then the necessary and sufficient conditions that (3.4) be independent of path in a simply connected region R of the x-t space are [1, p. 91].

$$\left.\begin{aligned} -\frac{\partial H}{\partial x_i} &= \frac{\partial p_i}{\partial t} \\ \frac{\partial p_i}{\partial x_k} &= \frac{\partial p_k}{\partial x_i} \end{aligned}\right\} \quad (t, x \in R) \tag{3.5}$$

In the Maximum Principle formulation, the conditions (3.5) are automatically satisfied when the $p_i \triangleq p_i(t)$ are computed from the adjoint equations (1.71). Furthermore, on the singular control surface S, the Hamiltonian H* is explicitly independent of u. Thus, the singular control surface S is characterized by the fact that the index of performance (1.83) is formally independent of path for all paths contained in the surface S. In particular, when the integrand of the index of performance is positive definite, the S surface(s) will include all surfaces N on which that integrand is identically zero.[7]

The "exact differential" nature of J[u] on S can be demonstrated by expressing the index of performance (1.83) directly as a line integral in the x-space. For simplicity, it will be assumed that (2.21) has been used to make (1.82) and (1.83) explicitly independent of time. In the case of a second order LOP, (1.82) and (1.83) become

$$\begin{aligned} \dot{x}_1 &= g_1(x_1, x_2) + u\, h_1(x_1, x_2) \\ \dot{x}_2 &= g_2(x_1, x_2) + u\, h_2(x_1, x_2) \end{aligned} \tag{3.6}$$

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[7]This follows from the fact that in such a case the N surfaces must contain optimal paths for at least some initial and terminal conditions.

$$J[u] = \int_{t_o}^{T} \left[g_o(x_1, x_2) + u\, h_o(x_1, x_2)\right] dt \tag{3.7}$$

Substituting (3.6) into (3.7) the index of performance (3.7) can be written as (omitting the arguments)

$$J[x_1, x_2] = \int_{x_{1o}, x_{2o}}^{x_{1T}, x_{2T}} \left(\frac{g_o h_2 - g_2 h_o}{g_1 h_2 - g_2 h_1}\right) dx_1 + \left(\frac{g_1 h_o - g_o h_1}{g_1 h_2 - g_2 h_1}\right) dx_2 \tag{3.8}$$

The integral (3.8) is a line integral in the $x_1 - x_2$ plane and will be independent of path in the region where [1, pg. 91]

$$\frac{\partial}{\partial x_2}\left(\frac{g_o h_2 - g_2 h_o}{g_1 h_2 - g_2 h_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{g_1 h_o - g_o h_1}{g_1 h_2 - g_2 h_1}\right) \tag{3.9}$$

It is remarked that (3.9) is precisely the necessary Euler equation for minimizing the integral (3.8) without side constraints [1]. Performing the indicated differentiations in (3.9), the region in which (3.8) is independent of path is found to be given by the expression

$$\begin{aligned}
&\left(g_o h_2 - h_o g_2\right)\left(-g_1 \frac{\partial h_1}{\partial x_1} - g_2 \frac{\partial h_1}{\partial x_2} + h_1 \frac{\partial g_1}{\partial x_1} + h_2 \frac{\partial g_1}{\partial x_2}\right) + \\
&\left(g_1 h_o - h_1 g_o\right)\left(-g_1 \frac{\partial h_2}{\partial x_1} - g_2 \frac{\partial h_2}{\partial x_2} + h_1 \frac{\partial g_2}{\partial x_1} + h_2 \frac{\partial g_2}{\partial x_2}\right) + \\
&\left(g_2 h_1 - h_2 g_1\right)\left(-g_1 \frac{\partial h_o}{\partial x_1} - g_2 \frac{\partial h_o}{\partial x_2} + h_1 \frac{\partial g_o}{\partial x_1} + h_2 \frac{\partial g_o}{\partial x_2}\right) = 0
\end{aligned} \tag{3.10}$$

Comparison of (3.10) with (A.9) [in Appendix I] shows that (3.10) does coincide with the general expression for the singular control surface S of the

LOP (3.6), (3.7). In the case of higher order LOP, it may be more difficult to find the equivalent x-space line integral corresponding to a given index of performance (1.83). However, as shown in Appendix II, all multi-dimensional line integrals of the form (3.8) with differential side constraints of the form (1.82) can be transformed into a unique LOP integral of the form (1.83).

The denominator terms in the integrand of (3.8) must be non-zero in order that the previous analysis be valid. This denominator term can be written as the functional determinant

$$\begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix} \neq 0 \qquad (3.11)$$

Comparing (3.6) and (3.11) it can be seen that (3.11) is the determinant of the elements of the right hand sides of the system equations with the control term u omitted. The physical significance of requiring (3.11) to be non-zero can be seen by writing (3.6) in the form

$$\frac{dx_1}{dx_2} = \frac{g_1 + u\,h_1}{g_2 + u\,h_2} \qquad (3.12)$$

If the determinant (3.11) is zero, then (3.12) reduces to

$$\frac{dx_1}{dx_2} = \frac{h_1(x_1, x_2)}{h_2(x_1, x_2)} \qquad (3.13)$$

The trajectory represented by (3.13) is seen to be completely independent of the control u. Thus, when the functional determinant (3.11) becomes zero, the system (3.6) becomes uncontrollable [21] with respect to u.

Some further insight into the nature of singular solutions can be obtained from the theory of curves of quickest descent. The isovee concept, developed in Chapter 1, indicated that the relation

$$- f_o \leq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial V}{\partial x_n} \dot{x}_n \quad (t_o \leq t \leq T) \tag{3.14}$$

is satisfied along any path between η and ξ. Further, the equality in (3.14) is only satisfied when the (assumed unique) optimal control $u^*(t)$ is used. Thus (3.14) may be written

$$\max_{u \in U}/\min \left[\frac{- f_o}{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial V}{\partial x_n} \dot{x}_n} \right] \equiv +1 \quad (t_o \leq t \leq T) \tag{3.15}$$

In (3.15), the max operation applies when for $u \neq u^*$, the denominator is positive and the min operation applies when, for $u \neq u^*$, the denominator is negative. At each point $x_i(t)$ $(i = 1, \ldots, n)$ along an optimal trajectory the relation (3.15) determines a (assumed unique) control u^*. This u^*, when substituted into the system equations (1.1), determines a unique direction $\dot{x}_i(t)$ $(i = 1, \ldots, n)$ in the x - t space. The direction $\dot{x}_i(t)$ determined by u^* has been called, by Caratheodory, the "direction of quickest descent" [22]. For the general LOP (1.82), (1.83), equation (3.15) becomes

$$\max_{u \in U}/\min \left[\frac{- g_o - u\, h_o}{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} g_1 + \ldots + \frac{\partial V}{\partial x_n} g_n + u\left(\frac{\partial V}{\partial x_1} h_1 + \ldots + \frac{\partial V}{\partial x_n} h_n\right)} \right] \equiv +1 \tag{3.16}$$

If (2.21) is used to make $H^*(t) \equiv 0$ (and therefore $\partial V/\partial t \equiv 0$) then from (2.25) it is seen that during singular control, (3.16) becomes (noting that $p_i \triangleq -\partial V/\partial x_i$)

$$\max_{u \in U}/\min \left[\frac{\pm g_o - u\, h_o}{- g_o - u\, h_o} \right] \equiv +1 \qquad (3.17)$$

It is clear from (3.17) that, during singular control, the direction of quickest descent is not uniquely defined.

In terms of the Hamilton-Jacobi theory, the direction of quickest descent is determined by the u* which maximizes equation (1.27). If an auxiliary coordinate x_o is defined such that

$$x_o(t) \triangleq \int_{t_o}^{t} \left[g_o(x_1,\ldots,x_n) + u\, h_o(x_1,\ldots,x_n) \right] dt \qquad (3.18)$$

then

$$\dot{x}_o = g_o(x_1,\ldots,x_n) + u\, h_o(x_1,\ldots,x_n) \qquad (3.19)$$

It should be noted from the definition (1.18) of $V(x_1,\ldots,x_n,t)$ that

$$\frac{\partial V}{\partial x_o} = +1 \qquad (3.20)$$

Using (3.19) and (3.20), the Hamilton-Jacobi equation (1.27) for the LOP (1.82), (1.83) becomes

$$\max_{u \in U} \left[- \frac{\partial V}{\partial x_o}\dot{x}_o - \frac{\partial V}{\partial x_1}\dot{x}_1 - \ldots - \frac{\partial V}{\partial x_n}\dot{x}_n \right] \equiv 0 \quad (t_o \leq t \leq T) \qquad (3.21)$$

where it is assumed that (2.21) has been used to make $\partial V/\partial t \equiv 0$. The terms $\partial V/\partial x_i$ and $\dot{x}_i$ $(i = 0, 1, \ldots, n)$ in (3.21) can be considered as vector components in the n + 1 dimensional x_i space. In this case, (3.21) can be written as the inner product of the two vectors $\underset{\sim}{\nabla} V_x$ and $\dot{\underset{\sim}{x}}$

$$\max_{u \in U} < -\underset{\sim}{\nabla} V_x, \dot{\underset{\sim}{x}} > \stackrel{-}{=} 0 \tag{3.22}$$

Since $\underset{\sim}{\nabla} V_x$ is not a function of u, (3.22) becomes

$$|\underset{\sim}{\nabla} V_x| \max_{u \in U} \left[|\dot{\underset{\sim}{x}}| \cos \Theta \right] \stackrel{-}{=} 0 \tag{3.23}$$

where Θ is the angle between the vectors $-\underset{\sim}{\nabla} V_x$ and $\dot{\underset{\sim}{x}}$ in the two dimensional subspace spanned by $\underset{\sim}{\nabla} V_x$ and $\dot{\underset{\sim}{x}}$. From (3.20) it is clear that $|\underset{\sim}{\nabla} V_x|$ cannot be zero. Also, the condition $|\dot{\underset{\sim}{x}}| = 0$ implies the unlikely condition that all $x_i(t)$ $(i = 0, 1, \ldots, n)$ are constant. Thus, the solution of (3.23) which is of practical interest is

$$\max_{u \in U} (\cos \Theta) \stackrel{-}{=} 0 \tag{3.24}$$

The condition (3.24) implies that the optimal control u* should be selected so that the vector $\dot{\underset{\sim}{x}}$ is perpendicular to the vector $\underset{\sim}{\nabla} V_x$. Furthermore, (assuming uniqueness) any control $u \neq u^*$ should result in $\cos \Theta < 0$. The direction of quickest descent is thus seen to be that direction $\dot{\underset{\sim}{x}}(t)$ which is tangent to an isovee contour in the n + 1 dimensional x_i $(i = 0, 1, \ldots, n)$ space.

In the more general case in which u enters the system equations and index of performance in an arbitrary nonlinear manner, the vector $\dot{\underset{\sim}{x}}$ (for any fixed $\underset{\sim}{x}$) will vary over a cone shaped bundle as u ranges over the set U. This cone, which will be called the "state velocity cone" is illustrated in Figure 3.2 for n = 2. The term velocity refers, of course, to the time rate of change of the vector $\underset{\sim}{x}(t)$, and has no relation with the physical velocity

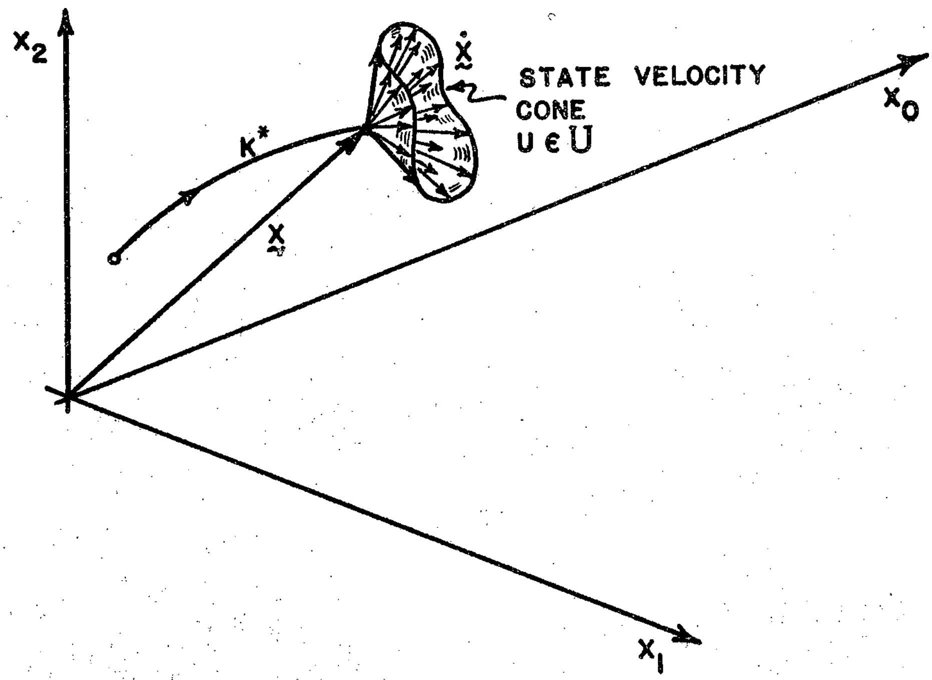

Figure 3.2. The State Velocity Cone in the System State Space

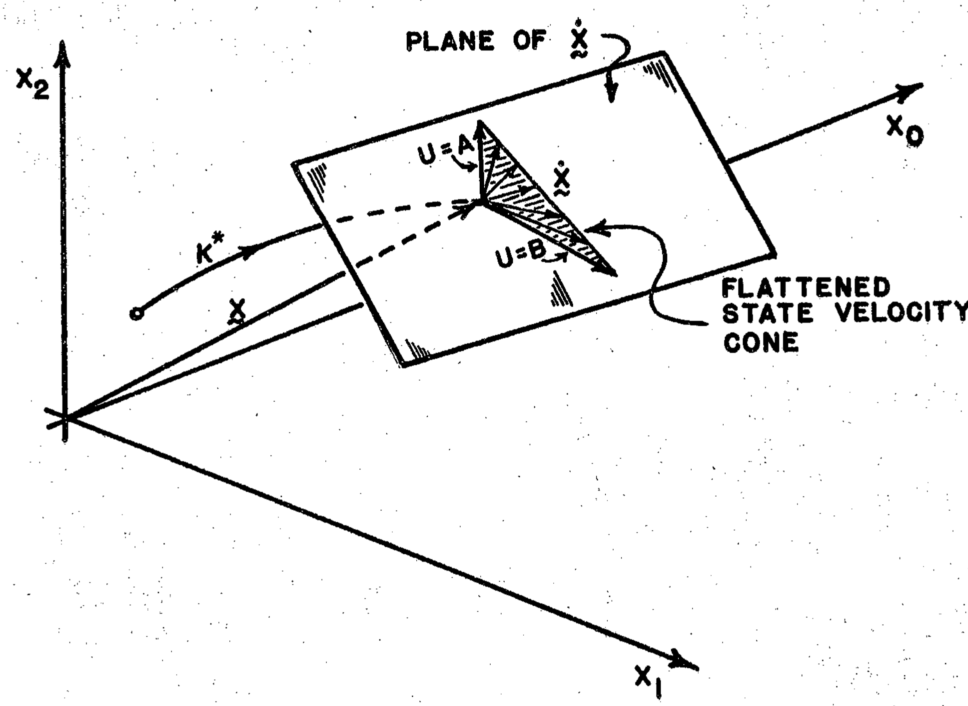

Figure 3.3. Flattened State Velocity Cone for a LOP

of the system. In the case of a LOP, the components of the vector $\dot{\tilde{x}}$ are, from (1.82)

$$\dot{\tilde{x}} = \tilde{g}_0 + \tilde{g}_1 + \dots + \tilde{g}_n + u(\tilde{h}_0 + \tilde{h}_1 + \dots + \tilde{h}_n) \tag{3.25}$$

where

$$\begin{aligned} \tilde{g}_i &= \tilde{g}_i(x_1, \dots, x_n) \\ \tilde{h}_i &= \tilde{h}_i(x_1, \dots, x_n) \end{aligned} \qquad (i = 0, 1, \dots, n)$$

In (3.25) it is understood that the quantities $\tilde{g}_i$ and $\tilde{h}_i$ $(i = 0,1,\dots,n)$ are vector components directed along the x_i axis. It is clear from (3.25) that as the scalar u varies over the interval A, B the vector $\dot{\tilde{x}}$ will (for any fixed $\tilde{x}$) always lie in a fixed two dimensional plane. Further, this plane is determined solely by the vector $\tilde{x}$. Thus, in the case of a LOP, the state velocity cone flattens out to a triangular element lying in a plane. The flattened state velocity cone for a LOP is illustrated (for the case $n = 2$) in Figure 3.3.

When $F(t) \not\equiv 0$, the u* for a LOP is determined by that unique $\dot{\tilde{x}}$ in the state velocity cone which is perpendicular to $\nabla V_{\tilde{x}}$. This situation is illustrated in Figure 3.4. When the singular condition, $F(t) \equiv 0$, occurs in a LOP, then the two dimensional plane which contains the flattened state velocity cone is perpendicular to the vector $\nabla V_{\dot{x}}$. In this case, any value of u will yield a vector $\dot{\tilde{x}}$ which is perpendicular to the vector ∇V_x. This situation is illustrated in Figure 3.5. The singular control function u_3* is defined as that control which will maintain the condition shown in Figure 3.5.

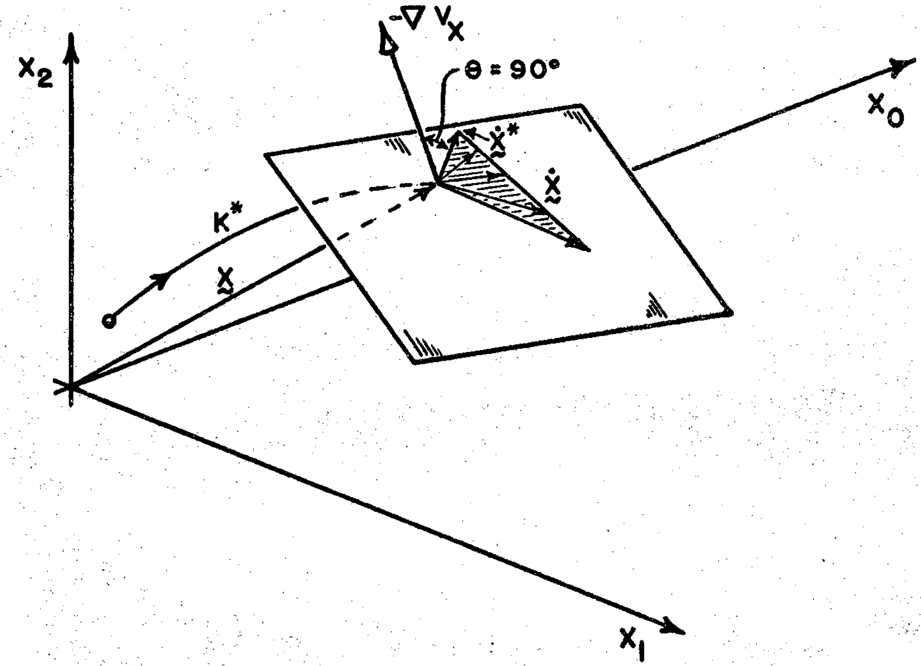

Figure 3.4. Flattened State Velocity Cone for $F(t) \gtrless 0$

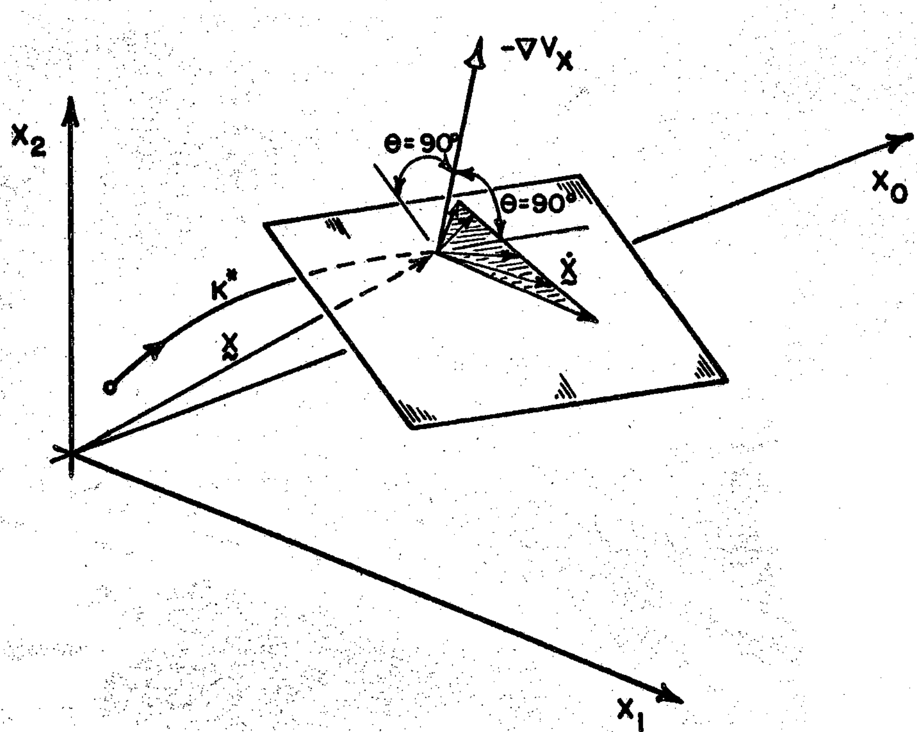

Figure 3.5. Flattened State Velocity Cone for $F(t) \equiv 0$

From the definition of S, it is clear that during singular control the two dimensional plane which contains the flattened state velocity cone must intersect or be tangent to the singular control surface.

It has been shown that the index of performance (1.83) is independent of path for all paths lying entirely on the surface S. This property alone, however, does not necessarily imply that a sub-arc of singular control will constitute part of the optimal control. There is apparently no general criteria by which one may determine a priori whether a singular control sub-arc (when it is allowable) will or will not constitute part of the optimal control. However, for certain classes of LOP (such as considered in Example 4.4 of Chapter 4) it may be possible to establish, by special methods, somewhat general criteria for the optimality of singular solutions.

One factor which complicates the problem of determining optimality of singular sub-arcs is the fact that optimality may depend critically on the particular boundary conditions specified in the problem. That is, the extremal paths on S may only be locally optimum. In this case, if the initial and terminal conditions lie on or near S, then motion along S may very well be optimum. But, if the initial and terminal conditions are not in the neighborhood of S then motion along S may cease to be optimum.

The optimality of trajectories on S may also depend upon the allowable signs of the $\dot{x}_i$. Consider for instance a second order LOP (3.6) which when written in the form of (3.8) yields the line integral

$$J[x_1, x_2] = -\int_{\substack{x_{1o}=0 \\ x_{2o}=0}}^{\substack{x_{1T}=1 \\ x_{2T}=1}} x_1^2 x_2 \, dx_1 + x_1 x_2 \, dx_2 \qquad (3.26)$$

The singular control surface (line) for (3.26) is determined from (3.10) to be

$$S: \quad x_2 = x_1^2 \tag{3.27}$$

It is assumed that the limits A and B on u will allow motion along (3.27). Along S, the value of (3.26) is (formally) independent of path and given by

$$J = {}^3/_5\,(x_{1o} - x_{1T}) \tag{3.28}$$

The singular control surface (3.27) is shown in the $x_1 - x_2$ state plane of Figure 3.6. It is easy to verify that, between the two points $x_{1o} = x_{2o} = 0$ and $x_{1T} = x_{2T} = +1$, the value of J computed along S is smaller than the value of J computed along the two neighboring paths $x_2 = x_1$ and $x_2 = x_1^3$ as shown in Figure 3.6. However, it is clear that a path α as shown in Figure 3.6 can be chosen such that the value of (3.26) computed along α can be made as small as desired. Thus, the integral (3.26) does not really possess a finite minimum when the signs of the increments dx_1 and dx_2 are unrestricted. If, however, the signs of dx_1 and dx_2 along the optimal path are restricted to be positive, then the allowable paths between $x_{1o} = x_{2o} = 0$ and $x_{1T} = x_{2T} = +1$ must lie in the square $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$ and in this case, the singular control trajectory S is indeed the optimal trajectory. In this example one can see why ordinary variations taken about the path S (as classically used in deriving the Euler equations and the Weierstrass condition) will fail to detect paths such as α which yield lower values of J than does S.

The optimality of the singular extremal paths on S is further complicated by the restriction that $B \leq u_3^* \leq A$. That is, certain regions of

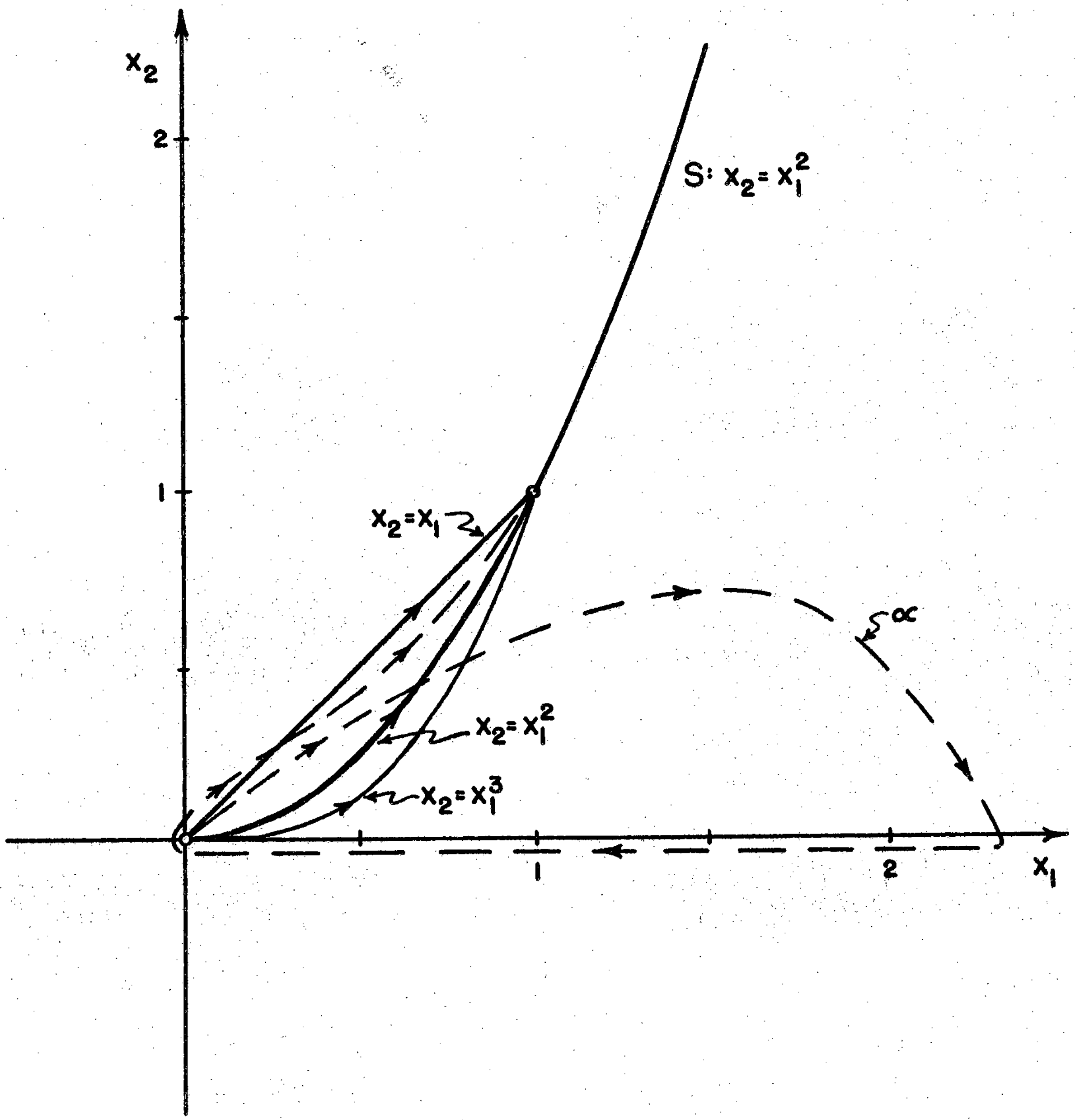

Figure 3.6. Singular Control Surface for Equation (3.26)

the S surface may contain optimal extremal paths which cannot be utilized because of the particular constraints on the magnitude of u*. In the more common cases, these singular extremal paths cannot be utilized because they require a $u_3^*(t)$ larger (or smaller) than the allowable values of u*. However, in some cases the optimal singular extremal paths correspond to $B \leq u_3^*(t) \leq A$ and still they cannot be utilized because of the constraints on u*. Example 4.2 in Chapter 4 illustrates this latter situation.

A necessary condition for a singular control sub-arc to be optimal is that the singular control condition $F(t) \equiv 0$ should be attainable by optimal control. That is, the particular conditions on $p_i(t)$, and $x_i(t)$ required to make $F(t) \equiv 0$ must be obtainable by starting on the initial manifold (1.15), using u* as determined by (2.19), and satisfying the canonical equations (1.70), (1.71) at all times. Since some of the initial values $p_i(t_o)$ are unknown a priori it would appear to be rather difficult to establish if the condition $F(t) \equiv 0$ is reachable by optimal control. This difficulty has been overcome by using a "backward tracing" technique.

The backward tracing procedure depends upon knowledge of the correct values of $x_i(t)$ and $p_i(t)$ for at least one point E on the unknown optimal trajectory. Then, by solving the canonical equations (1.70), (1.71) in reverse time (with E considered as the initial condition), the Maximum Principle can be used to determine u*(- t) and thereby determine the optimal trajectory K* between E and the initial manifold (1.15). A similar technique can be used to extend K* from E to the terminal manifold ξ.

In the case of LOP with singular solutions, the values of $p_i(t)$ corresponding to the surface S can be determined from (3.2). Thus, by starting at various points on S the backward tracing method may be used to "flood" the x-space with trajectories connecting S and η.[8] These trajectories will be called "flood paths". If this flooding technique reveals that η cannot be intersected by flood paths from S than it can be concluded that a singular sub-arc is not optimal. The flood paths that do connect S and η are potential candidates for an optimal trajectory which includes a singular sub-arc. Flood paths from S to the terminal manifold (1.16) may be traced out in a similar manner. In either case, the required transversality conditions on $p_i(t_o)$ and $p_i(T)$ help to reduce the number of flood paths which can be candidates for the optimal path. Some illustrations of flood paths are given in Figures 4.2 and 4.7 of Chapter 4.

This flooding technique, of course, does not settle the question of whether the candidate singular sub-arcs so determined are in fact optimal. It does, however, reduce the number of candidate solutions to a small number which can be individually compared by analytical or computer techniques. The "best" of the extremal control sequences which use singular control can then be compared with the "best" of the pure bang-bang extremal control sequences. In the cases in which η and ξ are points in the x-space, the flooding technique may reveal the existence of only one flood path connecting η, S and ξ. Thus, if singular control is optimal this particular flood path must be the optimal trajectory. The flooding technique may also be used to determine possible extremal paths when two or more singular control surfaces exist. In this case, flood paths connecting two singular control surfaces

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8 This may be accomplished by analytical methods in some cases, but computer solutions are usually required for most practical problems.

must be compatible with the known values of $p_i(t)$ on each S surface.

The locus of points in the x-space at which the bang-bang switching function F [see (2.19)] changes sign form what is called the bang-bang switching boundaries (hypersurfaces). When singular control is optimal, the flooding technique described above will automatically determine (numerically) these bang-bang switching boundaries. This latter application is demonstrated in Example 4.4 of Chapter 4.

The flooding technique described above will indicate how bang-bang extremal control (2.19) can be used to reach the singular control surface. However, motion along a singular sub-arc on S is usually unstable with respect to the bang-bang control law (2.19). That is, the control law (2.19) will not "chatter" along a singular path on S.[9] In order to follow a singular sub-arc on S, the bang-bang control (2.19) must be replaced by the singular control function $u_3^*(t)$ which, in general, is a continuously variable control $B \leq u_3^*(t) \leq A$.

3.4. Synthesis of the Singular Control Function

The functional form of the singular control u_3^* can be determined from the condition (3.2). That is, equations (3.2), together with the canonical equations, will in general yield either algebraic or differential equations involving u_3^*, x_i, p_i. The solution of these equations will yield

$$u_3^* = u_3^*(x_1,\ldots,x_n, p_1, \ldots, p_n) \tag{3.29}$$

If the expression $S(x_1,\ldots,x_n) = 0$ for the singular control surface can be

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[9] For this reason, ordinary computer searching methods may fail to detect the presence of a singular sub-arc.

obtained, then by taking one time derivative of S and substituting the system equations (1.82), the singular control function (3.29) can be obtained as a function only of state variables

$$u_3^* = u_3^*(x_1,\dots,x_n) \tag{3.30}$$

The control (3.29) can be substituted into the canonical equations (1.82) and (2.26) to obtain $x_i(t)$ and $p_i(t)$. By this means, u_3^* may be expressed entirely as a function of time. In some cases, however, it may be desirable to leave u_3^* in the form (3.29) or (3.30).

The above technique for obtaining u_3^* in the form (3.29) is demonstrated in the examples of Chapter 4.

3.5. Allowable Switching Direction Regions

It has been shown in 3.1 that in the case of a LOP the continuity requirement of H* prevents the control u* from changing value except at the instants where $F(t) = 0$. At the isolated points where $F(t) = 0$ the control u* may change from $u^* = A$ to $u^* = B$ or from $u^* = B$ to $u^* = A$. If $F(t) \equiv 0$, then u* may change from A or B to u_3^* or from u_3^* to A or B. The latter case, in which the control changes from bang-bang to singular and vice versa, has been discussed in the previous article. Some information concerning the nature of the isolated points where $F(t) = 0$ can be obtained by examining the sign of $dF(t)/dt$ at the points where $F(t) = 0$.

If the LOP under consideration has been suitably augmented so that $H^*(t) \equiv 0$ then at the isolated points where $F(t) = 0$ the condition $I(t) = 0$ must also be satisfied. Consider the expression

$$\sigma \triangleq \text{sgn}\left(\frac{dF}{dt}\right)\Bigg|_{\substack{F(t)=0 \\ I(t)=0}} = \text{sgn}\ \phi(x_1,\ldots x_n,\ p_1,\ldots,p_n) \quad (3.31)$$

obtained from (1.82), (2.25) and (2.26). If $\sigma = +1$ in a certain region P of the x-p space then it is clear that the control u*(t) can only switch from u*(t) = B to u*(t) = A in P. Likewise, if $\sigma = -1$ in a certain region N of the x-p space then the control u*(t) can only switch from u*(t) = A to u*(t) = B in N. If dF(t)/dt = 0 in (3.31) then σ will be defined as

$$\sigma(0) \triangleq 0 \quad (3.32)$$

In the regions Z of the x-p space where $\sigma = 0$ the control may switch from either (a) A to B, (b) B to A, (c) A or B to u_3*, (d) from u_3* to A or B or, there may be no switch at all. The conditions $\sigma = +1$, $\sigma = -1$ and $\sigma = 0$ are illustrated in Figure 3.7. The conditions $\sigma = +1$, $\sigma = -1$ and $\sigma = 0$ can be used to divide the x-p space into P, N, and Z regions. Since the allowable direction in which u*(t) can switch is completely specified in the P and N regions, this information can be used to test and eliminate many of the possible sequences of bang-bang extremal paths. For instance, it is clear that only one consecutive switch is allowed in each of the P and N regions. These allowable switching direction regions represent another necessary (but not sufficient) condition for the optimal control. For instance, the condition $\sigma = +1$ in a region P of the x-p space is not sufficient to conclude that the optimal control must switch in that region. In some cases, the function ϕ in (3.31) can be expressed solely as a function of the state variables x_i. This condition is especially useful since the P, N, and Z regions can then be constructed in

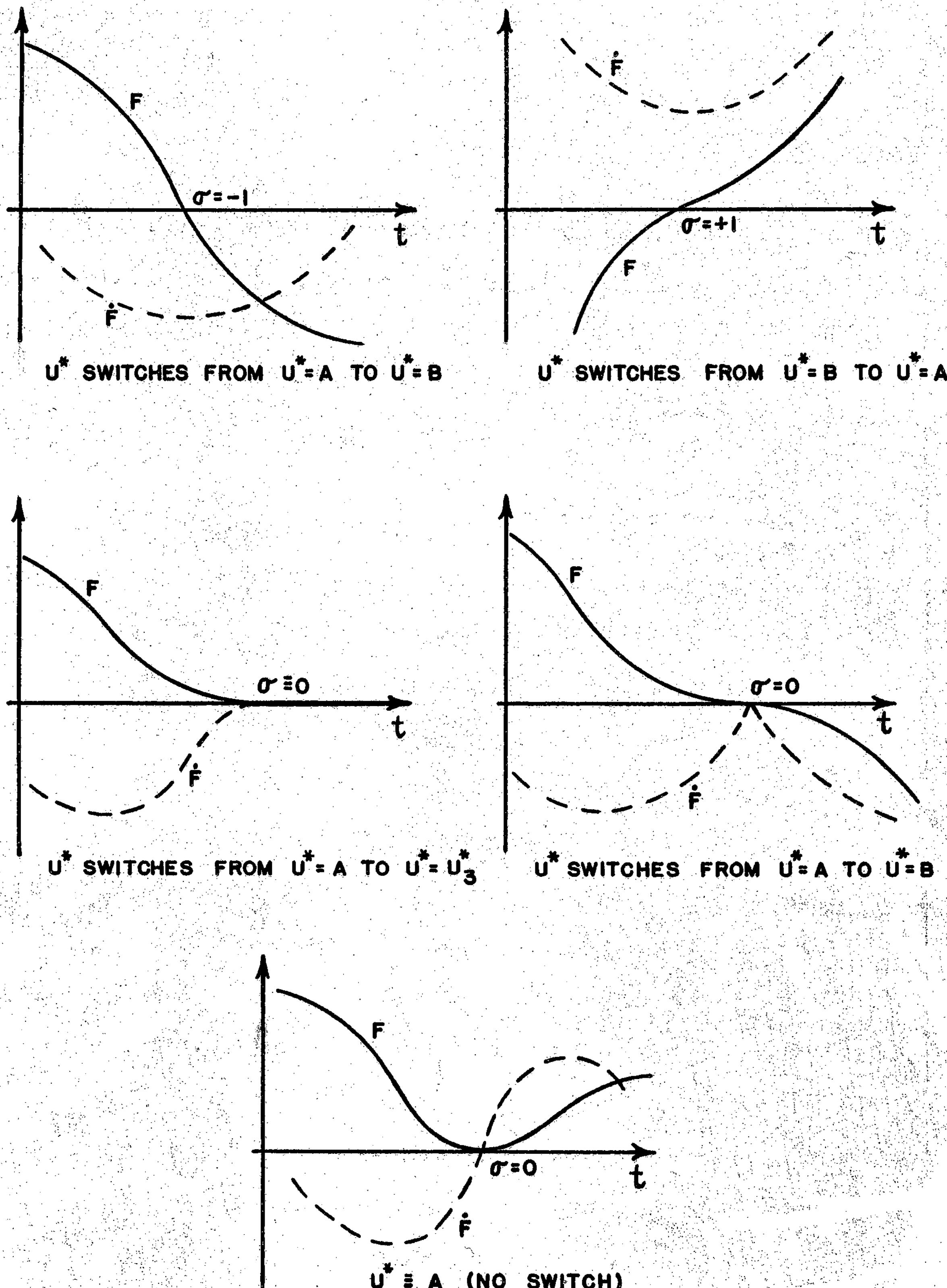

Figure 3.7. Conditions for Allowable Switching Directions

the x-space. Application of this concept of allowable switching direction regions is illustrated in the examples of Chapter 4 and also in [23]. It is remarked that a somewhat similar technique has been used by Miele [24].

3.6. Miele's Method

Miele [24] has examined a particular class of LOP (1.82), (1.83) in which (1.83) may be written as a line integral in a plane. That is, where (1.83) may be written in the form (3.8). In this case, the singular control surface S becomes a line in the plane and sufficient conditions for optimality of a singular sub-arc may be established (under certain conditions) by using Green's theorem. Application of Milele's method is somewhat limited, however, because of certain restrictions which must be imposed on the line integral to be minimized. The most important restriction is that the initial point A^* and the terminal point B^* of the line integral (3.8) must be absolutely fixed a priori. Another restriction is that the allowable paths of integration must be contained in a closed, finite region Ω of the plane. The boundary of Ω must be known a priori and the (fixed) initial and terminal points A^* and B^* must lie on this boundary. If the above restrictions are satisfied, and if S divides Ω into two regions $\sigma = +1$ and $\sigma = -1$ [σ defined by (3.31)] and if A^* and B^* have a certain orientation with respect to S then Miele has shown (by somewhat formal arguments) that the singular sub-arc is optimal. Conversely, if σ has the same sign on both sides of S then, under certain conditions, the singular sub-arc can be shown to be non-optimal. In spite of the limitations, Miele's method has proven quite useful in certain optimization problems in flight mechanics. The importance of the restriction that A^* and B^* should be fixed points is illustrated in Example 4.2 of Chapter 4. In that example, the terminal point B^* depends on the path of integration and purely formal application of Miele's method leads to an incorrect answer.

3.7. LOP with Multivariable Control

The class of LOP considered so far have been characterized by having only one (scalar) control variable. A more general class of LOP would be one in which the system equations (1.82) and index of performance (1.83) are of the form

$$\dot{x}_i = g_i(x_1,\ldots,x_n,t) + \sum_{j=1}^{r} u_j\, h_{ij}(x_1,\ldots,x_n,t) \quad (i = 1, \ldots, n) \tag{3.33}$$

$$J[u] = \int_{t_o}^{T} \left[g_o(x_1,\ldots,x_n,t) + \sum_{j=1}^{r} u_j\, h_{oj}(x_1,\ldots,x_n,t) \right] dt \tag{3.34}$$

The study of LOP with multivariable control u_j $(j = 1, \ldots, r)$ is rather complex and is not an objective of this thesis. The formulation given below is only intended to demonstrate the fact that singular solutions may also occur in LOP with multivariable control.

The Hamiltonian (1.64) for the LOP (3.33), (3.34) is (omitting the arguments)

$$\begin{aligned} H = p_1 g_1 + \cdots + p_n g_n - g_o &+ u_1 \left[p_1 h_{11} + p_2 h_{21} + \cdots + p_n h_{n1} - h_{o1} \right] + \\ &+ u_2 \left[p_1 h_{12} + p_2 h_{22} + \cdots + p_n h_{n2} - h_{o2} \right] + \\ &\;\vdots \\ &+ u_r \left[p_1 h_{1r} + p_2 h_{2r} + \cdots + p_n h_{nr} - h_{or} \right]. \end{aligned} \tag{3.35}$$

The Maximum Principle requires that the Hamiltonian (3.35) should be maximum with respect to all $u_j \in U_j$ $(j = 1, \ldots, r)$. Thus, the optimal controls u_j for the LOP (3.33) (3.34) are given by

$$u_j^*(t) = \begin{cases} A_j & \text{if } F_j(t) > 0 \\ B_j & \text{if } F_j(t) < 0 \end{cases} \qquad (j = 1, \ldots, r) \tag{3.36}$$

where A_j and B_j are, respectively, the upper and lower bounds on the control u_j and

$$F_j \triangleq p_1 h_{1j} + p_2 h_{2j} + \ldots + p_n h_{nj} - h_{oj} \tag{3.37}$$

Equations (3.36) and (3.37) indicate that in the case of LOP with multi-variable control, u_j $(j = 1, \ldots, r)$, the singular condition $F_j(t) \equiv 0$ may occur in one or more of the controls u_j.

Chapter 4

SOME EXAMPLES OF SINGULAR SOLUTIONS IN LINEAR OPTIMIZATION PROBLEMS

The following examples demonstrate the techniques developed in the previous chapters for analyzing linear optimization problems with singular solutions. The first two examples are taken from the field of flight mechanics, an area in which LOP are frequently encountered. The remaining examples are representative of the LOP that arise in modern problems of automatic control.

Example 4.1. Goddard's Problem

One of the classic problems in rocketry is the problem of determining a thrust program which will maximize the height achieved by a vertical sounding rocket. Goddard [25] was one of the first to suggest that an optimum thrust program should exist for this problem but he was unable to obtain a rigorous mathematical solution. The most complete solution to Goddard's problem has been given by Tsien and Evans [26], using the classical Calculus of Variations.

The solution to Goddard's problem involves a singular sub-arc along which the thrust is varied continuously. This particular solution was one of the first examples to demonstrate the practical importance of singular sub-arcs in variational problems.

The vertical sounding rocket to be considered in this example is assumed to be described by the nonlinear, non-autonomous dynamical equation

$$m(t)\,\frac{dv}{dt} + c\,\frac{dm}{dt} + C_D\,\frac{A\rho_0\,e^{-\alpha x}\,v^2}{2} + m(t)\,g = 0 \tag{4.1}$$

where

x = vertical height of rocket $[x(t = 0) = 0]$

v = absolute velocity of rocket $[v(t = 0) = 0]$

C_D = drag coefficient (assumed positive constant)

A = cross-sectional area of rocket

$\rho_0 e^{-\alpha x}$ = air density (ρ_0 and α are assumed positive constants)

$m(t)$ = instantaneous mass of rocket $[m(t = 0) = m_0]$

g = acceleration of gravity (assumed positive constant)

c = velocity of exhaust gas with respect to rocket (assumed positive constant).

To simplify equation (4.1), the following constants are defined

$$c \frac{dm}{dt} \triangleq - u(t)$$
$$C_D \frac{A\rho_0}{2} \triangleq k \qquad (4.2)$$

The state variables x_i are defined as

$$x_1 = x$$
$$x_2 = v \qquad (4.3)$$
$$x_3 = m(t)$$

Using (4.2) and (4.3), equation (4.1) can be written in the form of (1.82) as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = - g - \frac{k e^{-\alpha x_1} (x_2)^2}{x_3} + \frac{u(t)}{u_3} \qquad (x_2(0) = 0) \qquad (4.4)$$
$$\dot{x}_3 = - \frac{u(t)}{c} \qquad (c > 0)$$

The problem to be considered may be stated as follows: Determine the thrust control $u(t)$ which will maximize the vertical height $(x_1)_{max}$ attained by the rocket (4.4) with the constraints

$$0 \leq u(t) \leq u_m \qquad (u_m = \text{constant}) \tag{4.5}$$

$$\int_0^T u(t)\,dt \leq b \qquad (b > 0) \tag{4.6}$$

where T is the time corresponding to $x_1 = (x_1)_{max}$. The time T is assumed free. The constraints (4.5) and (4.6) represent, respectively, limitations on the magnitude of thrust and amount of fuel. The fuel constraint is incorporated into the problem by writing (4.6) as

$$-\int_{x_3(0)}^{x_3(T)} \frac{dx_3}{dt}\,dt \leq \frac{b}{c} \tag{4.7}$$

Equation (4.7) implies an inequality constraint on the state variable x_3

$$x_3(T) \geq \left[m_o - \frac{b}{c}\right] \tag{4.8}$$

where m_o = initial mass of rocket $(m_o > \frac{b}{c})$. The index of performance to be minimized is

$$J[u] = \int_0^T - x_2\,dt \tag{4.9}$$

Using (4.4) and (4.9) the Hamiltonian (1.64) is written

$$H = p_1\,x_2 - p_2\left[g + \frac{k\,e^{-\alpha x_1}}{x_3}\,(x_2)^2\right] + x_2 + u\left[\frac{p_2}{x_3} - \frac{p_3}{c}\right] \tag{4.10}$$

The adjoint equations are obtained from (1.71) as

$$\dot{p}_1 = - p_2 \frac{\alpha k e^{-\alpha x_1}}{x_3} (x_2)^2$$

$$\dot{p}_2 = - p_1 + 2p_2 \frac{k e^{-\alpha x_1}}{x_3} x_2 - 1 \qquad (4.11)$$

$$\dot{p}_3 = p_2 \left[\frac{- k e^{-\alpha x_1} (x_2)^2 + u^*}{(x_3)^2} \right]$$

Since T and $x_2(T)$ are free, the transversality condition (1.76) requires

$$p_2(T) = 0$$
$$H^*(T) = 0 \qquad (4.12)$$

The coordinate $- x_1(T)$ is the quantity to be minimized. Thus from (1.80) the terminal value of p_1 must be

$$p_1(T) = + 1 \qquad (4.13)$$

Since H is explicitly independent of t, the last of (4.12) implies [see (1.74)]

$$H^*(t) \equiv 0 \qquad (4.14)$$

From (2.19) and (4.10), the optimal control $u^*(t)$ is

$$u^*(t) = \begin{cases} u_m & \text{if:} \quad F(t) > 0 \\ 0 & \text{if:} \quad F(t) < 0 \end{cases} \qquad (4.15)$$

where

$$F(t) \triangleq \frac{p_2}{x_3} - \frac{p_3}{c} \tag{4.16}$$

Following the procedure of (3.2), the test for a singular solution is carried out by setting

$$\begin{aligned} I(t) &= \dot{I}(t) = \ddot{I}(t) = \ldots \equiv 0 \\ F(t) &= \dot{F}(t) = \ddot{F}(t) = \ldots \equiv 0 \end{aligned} \tag{4.17}$$

where

$$I \triangleq x_2(p_1 + 1) - p_2 \left[g + \frac{k\, e^{-\alpha x_1}}{x_3} (x_2)^2 \right] \tag{4.18}$$

Thus

$$I \equiv 0 \Rightarrow x_2(p_1 + 1) \equiv p_2 \left[g + \frac{k\, e^{-\alpha x_1}}{x_3} (x_2)^2 \right]$$

$$\begin{aligned} \dot{I} \equiv 0 &\Rightarrow \frac{u}{x_3}(p_1 + 1) \equiv \frac{u\, p_2\, x_2\, k\, e^{-\alpha x_1}}{(x_3)^2} \left[2 + \frac{x_2}{c} \right] \\ F \equiv 0 &\Rightarrow \frac{p_2}{x_3} \equiv \frac{p_3}{c} \end{aligned} \tag{4.19}$$

$$\dot{F} \equiv 0 \Rightarrow \frac{1}{x_3}(p_1 + 1) \equiv \frac{p_2\, x_2\, k\, e^{-\alpha x_1}}{(x_3)^2} \left[2 + \frac{x_2}{c} \right]$$

The simultaneous solution of (4.19) yields the expression for the singular control surface S

$$S: \quad x_3 = \frac{k\, e^{-\alpha x_1}}{g\, c} (x_2)^2 (x_2 + c) \tag{4.20}$$

By taking one time derivative of (4.20) and substituting (4.4), the singular control u_3^* is obtained as

$$u_3^* = k\, e^{-\alpha x_1} (x_2)^2 \left[\frac{3(x_2)^2 + 8c\, x_2 + 4c^2 + \frac{\alpha}{g}[(x_2)^4 + 2c(x_2)^3 + c^2(x_2)^2]}{(x_2)^2 + 4c\, x_2 + 2c^2} \right] \tag{4.21}$$

The allowable switching direction regions are determined by using (3.31) and (4.19) to obtain

$$\sigma = \operatorname{sgn} \left[g(p_1 + 1) \frac{-x_3 + \frac{ke^{-\alpha x_1}}{gc}(x_2)^2 (x_2 + c)}{x_3[g\, x_3 + k\, e^{-\alpha x_1}(x_2)^2]} \right] \tag{4.22}$$

Since $V \triangleq \min_{u \in U} (-x_1(T))$ and $p_1 \triangleq -\partial V/\partial x_1$ it is clear that whenever an increase in $x_1(t)$ will cause an increase in $x_1(T)$, the $p_1(t)$ will be positive. In this problem, an increase in $x_1(t)$ always reduces the aerodynamic drag by reducing the air density $\rho_0 e^{-\alpha x_1}$. Since this is the only manner in which dx_1 affects the problem it can be concluded that along the optimal trajectory K* the value of $p_1(t)$ is always positive.[10] The denominator of (4.22) is always positive from the definition of x_3, g, and k. Thus

---

[10] This technique of obtaining signum information from the physical interpretation of the adjoint variables can be quite useful as demonstrated in the present example. See also, [27] and [28].

(4.22) reduces to

$$\sigma = \text{sgn}\left[-x_3 + \frac{k\,e^{-\alpha x_1}}{gc}(x_2)^2\,(x_2 + c)\right] \tag{4.23}$$

Comparing (4.20) and (4.23) it is seen that the singular control surface S divides the $x_1 - x_2 - x_3$ state space into two regions

$$\sigma = +1 \quad \text{where} \quad x_3 < \frac{k\,e^{-\alpha x_1}}{gc}(x_2)^2\,(x_2 + c)$$

and

$$\sigma = -1 \quad \text{where} \quad x_3 > \frac{k\,e^{-\alpha x_1}}{gc}(x_2)^2\,(x_2 + c) \tag{4.24}$$

Since the initial condition $x_1(0) = x_2(0) = 0$, $x_3(0) = m_o$ is in the $\sigma = -1$ region it is clear that the initial value of u* must be $u^*(0) = u_m$. In fact, if $u^*(0) = 0$ then, because $\sigma \equiv -1$, the Maximum Principle will not allow switching to $u^* = u_m$ at any time and the rocket will never get off the ground.

From physical considerations it is unlikely that the maximum height can be achieved without using all available fuel. Thus, since $u^*(0) = u_m$ and only one switch is allowed in the $\sigma = -1$ region the optimal control in the $\sigma = -1$ region must be $u^*(t) \equiv u_m$ as long as $x_3(t) > m_o - \frac{b}{c}$. However, if the system trajectory enters the $\sigma = +1$ region with $u^* = u_m$ then no further changes in the control are allowed. The allowable combinations of extremal sub-arcs are therefore limited to a small number which can be readily compared by computer solution. By this means it is found that the optimal solution which satisfies the required boundary conditions (4.12) and (4.13) is $u^*(t) = u_m$ until S is reached and then singular control u_3^* along S until $x_3(t) = m_o - \frac{b}{c}$. The final sub-arc is a coasting sub-arc $u^* = 0$ along the plane $x_3(t) \equiv m_o - \frac{b}{c}$. The maximum height is obtained when $x_2(T) = 0$. The optimal trajectory in the $x_1 - x_2 - x_3$ state space is shown in Figure 4.1. Further discussion of the performance optimization of vertical sounding rockets is given in [29].

Example 4.2.

In the previous example, the optimal control included a singular sub-arc which caused the system trajectory to follow along S for as long as physically possible. Many problems with singular solutions are characterized by the fact that the optimal solution utilizes singular control as much as possible. In the present example, which is taken from [15], the optimal solution requires the system trajectory to leave the S surface at a certain point even though continued motion along S does not violate any physical constraints.

The system to be considered is a considerably simplified model of the vertical sounding rocket of Example 4.1. The rocket is assumed to be described by the dynamical equation

$$\frac{dv}{dt} + k(v)^2 + g = u \tag{4.25}$$

where k and g are positive constants, u is the control variable, and v is velocity. Berkovitz [30] has used this equation to represent a constant mass, constant weight sounding rocket moving in a constant density atmosphere. Following (4.3), the state variable equations for (4.25) are written

$$\begin{aligned} \dot{x}_1 &= x_2 \qquad (x_1(0) = 0) \\ \dot{x}_2 &= -k(x_2)^2 - g + u \qquad (x_2(0) = 0) \end{aligned} \tag{4.26}$$

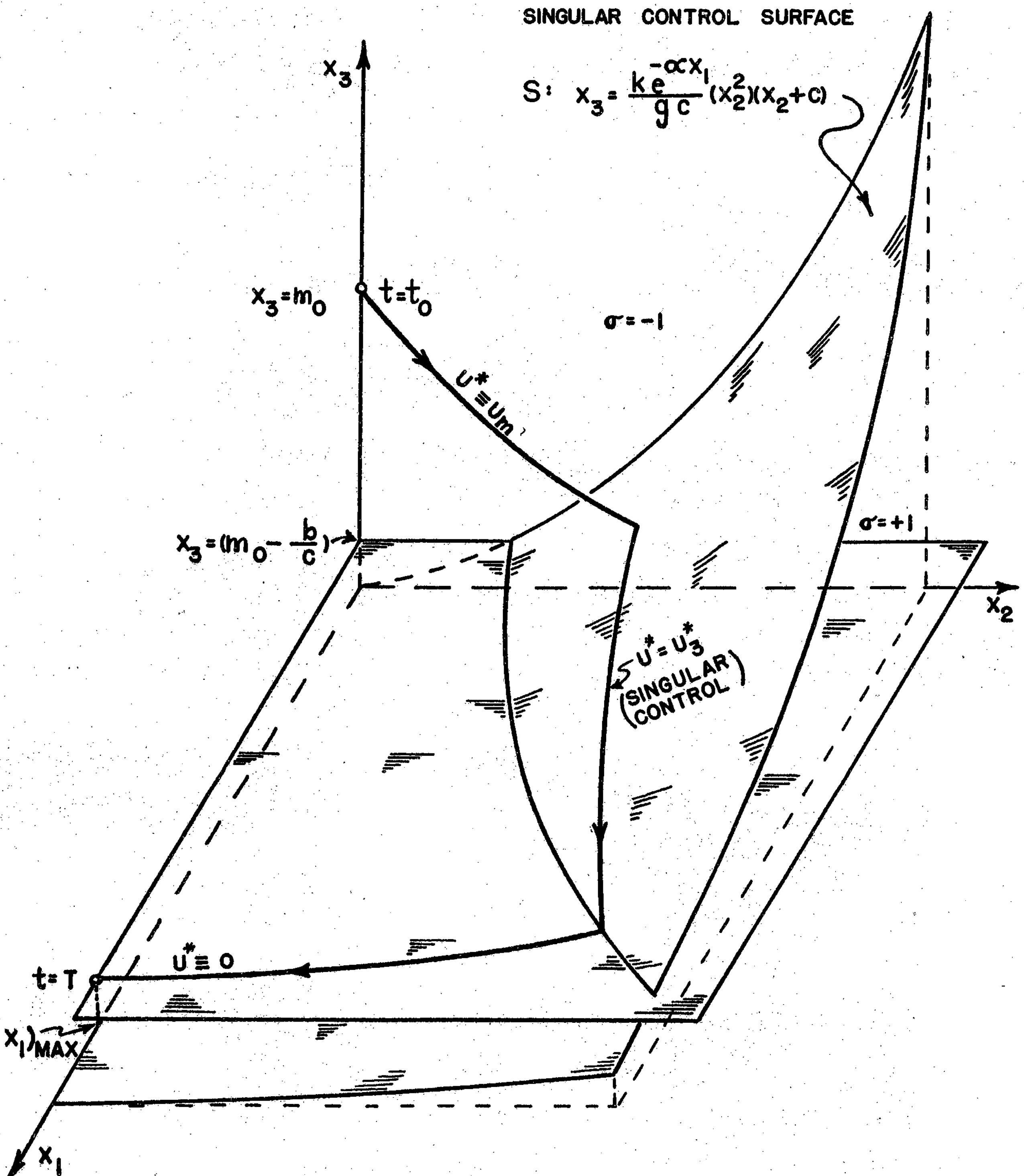

Figure 4.1. Singular Control Surface for Example 4.1

The control u is bounded and given as

$$u_{min} \leq u(t) \leq u_{max} \qquad \begin{matrix}(u_{max} > u_{min} > 0) \\ (u_{max}, u_{min} = \text{constant})\end{matrix} \tag{4.27}$$

A fuel constraint requires that

$$\int_0^T u \, dt \leq b \qquad (b > 0) \tag{4.28}$$

where T is the terminal time for the problem. In this problem, T will be defined as "burnout time", the instant at which (4.28) first becomes an equality. Thus, T is explicitly free but is implicitly defined by

$$\int_0^T u \, dt = b \tag{4.29}$$

The constraint (4.28) is incorporated into the problem of defining

$$x_3(t) \triangleq \int_0^t u \, dt \tag{4.30}$$

$$\dot{x}_3(t) = u(t)$$

and specifying

$$\begin{aligned} x_3(0) &= 0 \\ x_3(T) &= b \end{aligned} \tag{4.31}$$

The index of performance to be minimized is

$$J[u] = -\int_0^T x_2 \, dt$$

or

$$J[u] = -x_1(T) \tag{4.32}$$

That is, it is desired to maximize the height x_1 at burnout.

The Hamiltonian is written

$$H = (p_1 + 1)x_2 + p_2(-k(x_2)^2 - g) + u(p_2 + p_3) \qquad (4.33)$$

The adjoint equations are, from (1.71)

$$\begin{aligned} \mathring{p}_1 &= 0 & (p_1 = \text{constant}) \\ \mathring{p}_2 &= -(p_1 + 1) + 2p_2\, k\, x_2 & \qquad (4.34) \\ \mathring{p}_3 &= 0 & (p_3 = \text{constant}) \end{aligned}$$

From the transversality conditions the terminal values of the adjoint variables are

$$\begin{aligned} p_1(T) &= +1 \qquad (\therefore\ p_1(t) \equiv 1) \\ p_2(T) &= 0 \end{aligned} \qquad (4.35)$$

and, from (1.74),

$$H^*(t) \equiv 0 \qquad (4.36)$$

The Hamiltonian (4.33) can thus be written

$$H = I(p, x) + u\, F(p, x) \qquad (4.37)$$

where

$$\begin{aligned} I(p, x) &\triangleq 2x_2 - p_2(k(x_2)^2 + g) \\ F(p, x) &\triangleq p_2 + p_3 \end{aligned}$$

The test for a singular solution is carried out by applying the conditions (3.2) to obtain

$$
\begin{aligned}
I &\equiv 0 \quad \Rightarrow p_2(k(x_2)^2 + g) \equiv 2x_2 \\
\dot{I} &\equiv 0 \quad \Rightarrow u(1 - p_2\, k\, x_2) \equiv 0 \\
F &\equiv 0 \quad \Rightarrow p_2 = - p_3 \\
\dot{F} &\equiv 0 \quad \Rightarrow p_2 \equiv \frac{1}{k\, x_2}
\end{aligned}
\tag{4.38}
$$

The singular control surface is obtained from simultaneous solution of (4.38)

$$
S: \qquad x_2 = \sqrt{g/k} \qquad \text{(i.e., } x_2 = \text{constant)} \tag{4.39}
$$

The singular control is obtained from (4.26) and (4.39)

$$
u_3{}^* = + 2g \qquad \text{(i.e., } u_3{}^* = \text{constant)} \tag{4.40}
$$

Along S the adjoint variables are, from (4.38)

$$
\begin{aligned}
p_1 &= + 1 \\
p_2 &= \frac{1}{\sqrt{g\, k}} \\
p_3 &= \frac{-1}{\sqrt{g\, k}}
\end{aligned}
\tag{4.41}
$$

The allowable switching direction regions (3.31) are determined from (4.38)

$$
\sigma = \operatorname{sgn}\left[\frac{-g + k(x_2)^2}{g + k(x_2)^2}\right] \tag{4.42}
$$

From (4.42) it is clear that only one consecutive switch of the control is allowed in the regions above and below the singular control surface S in the $x_1 - x_2 - x_3$ state space. Thus, the allowable combinations of extremal paths are limited to a small number which can be compared by computer solution. By this means it is determined that, if $u_{min} < 2g < u_{max}$, the required terminal conditions (4.35) cannot be satisfied by exclusive bang-bang extremal control

$$u^*(t) = \begin{cases} u_{max} & \text{if} \quad F(t) > 0 \\ u_{min} & \text{if} \quad F(t) < 0 \end{cases} \tag{4.43}$$

This result serves to indicate that a singular extremal sub-arc will enter into the optimal control sequence. However, since (4.35) is not compatible with the singular conditions (4.41), the problem solution cannot end ($t = T$) with the system trajectory on S. The flooding technique can now be used to fill the $x_1 - x_2 - x_3$ state space with flood paths flowing from S. For this purpose, the system and adjoint equations (4.26) and (4.34) are solved in forward and reverse time ($\tau = \pm t$) starting at various points on S and using (4.41) to compute the initial conditions of the adjoint variables. A portion of the field of flood paths (on both sides of S) is shown in the x_1-x_2 state plane of Figure 4.2. By this means it is determined that the optimal control sequence is: 1) $u^* = u_{max}$ until S is reached, 2) $u^* = 2g$ (singular control) along S until the "line" $x_3 = m$ is reached, 3) $u^* = u_{min}$ until $x_3 = b$ (at which time the problem ends). The value of m depends upon the values of k, g, b, and u_{min}. The optimal trajectory and the flattened state velocity cone are shown in the $x_1 - x_2 - x_3$ state space in Figure 4.3.

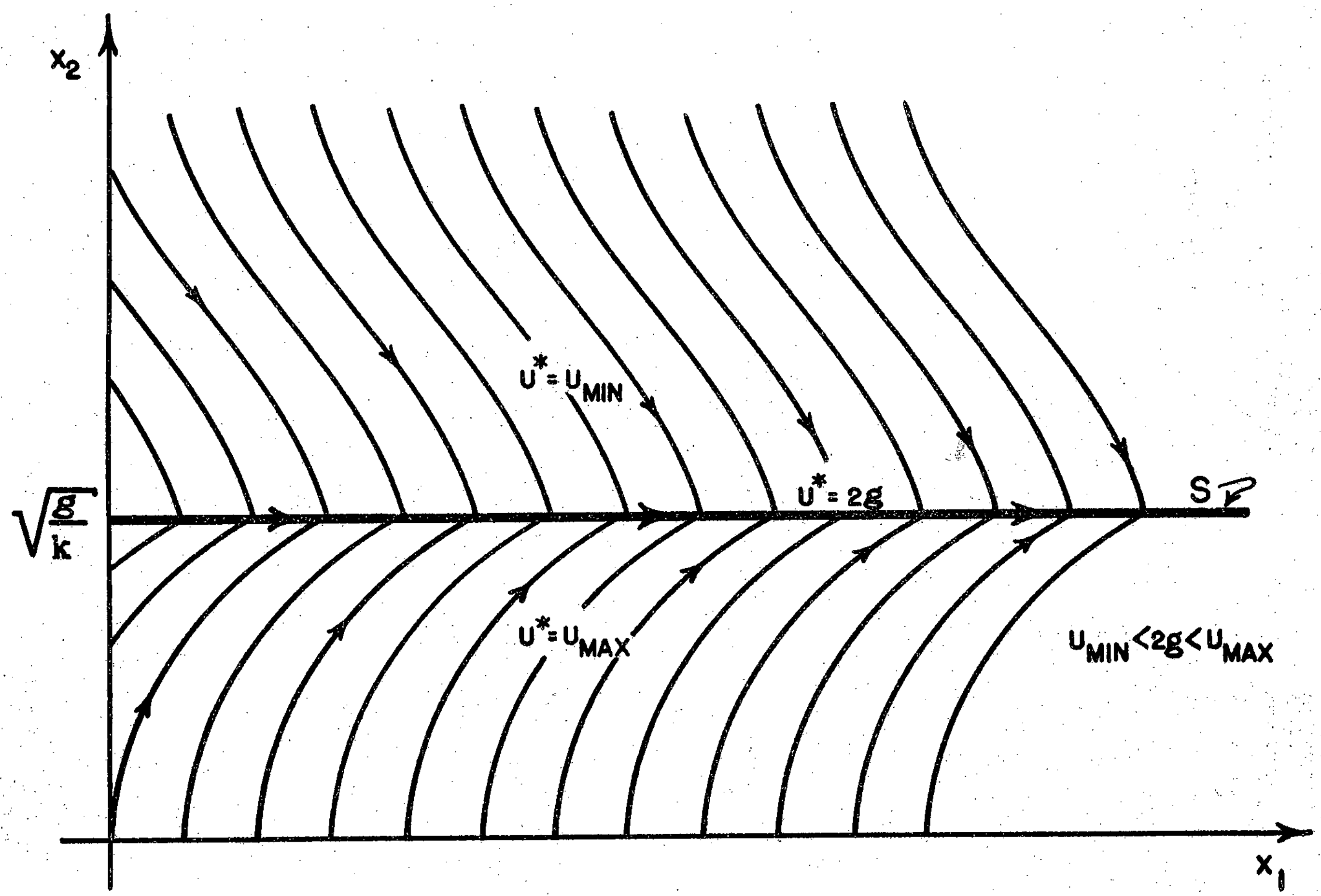

Figure 4.2. Portion of Field of Flood Paths for Example 4.2
(Flood Paths Leaving S Not Shown)

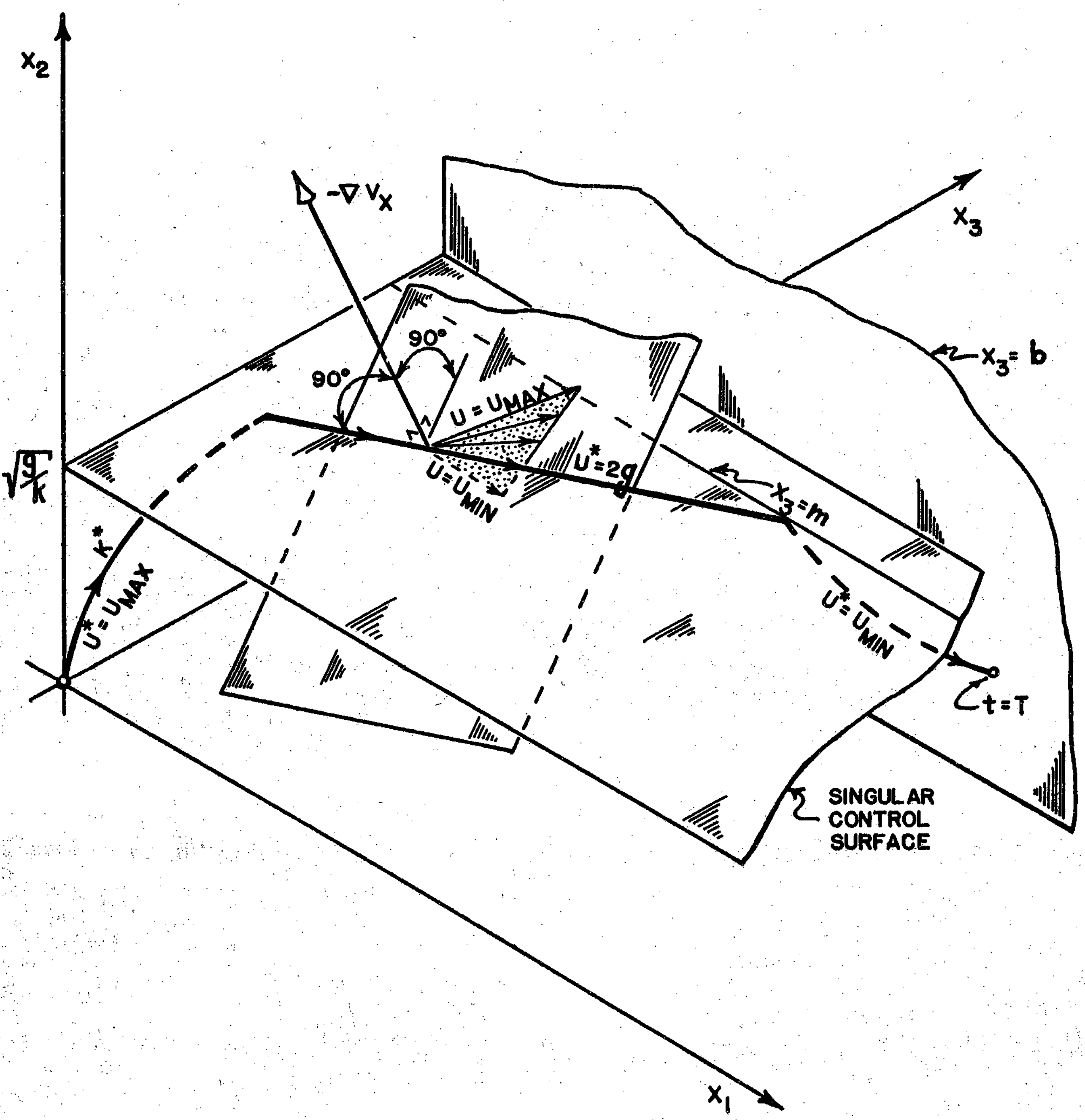

Figure 4.3. Singular Control Surface for Example 4.2

If $u_{max} < 2g$ then, from (4.42), the optimal control must be exclusively bang-bang with no more than one switch. The switching time is again determined by the required terminal conditions (4.41).

It should be noted that for the optimal controls described above, the Hamiltonian H* will experience a jump at t = T. This does not violate the continuity requirement for H* since in problems of this type, [i.e., T defined implicitly by (4.29)], the continuity of H* is only required over the interval $0 \leq t < T$.

If, in this example, it is desired to maximize the burnout velocity rather than the burnout height then a similar analysis will show that no singular control surface exists and the optimal control is $u^* \equiv u_{max}$. The problem of maximizing burnout velocity has been considered by Berkovitz [30] and Kalman [31].

Example 4.3.

A problem in automatic control which has received considerable attention is the problem of "time optimal control". That is, the problem of transferring a system between given initial and terminal states in minimum time. La Salle [17] has shown that for linear systems, time optimal control can always be achieved by a bang-bang control sequence. The present example, which is taken from [32], demonstrates that for nonlinear systems the time optimal control might not include any bang-bang sub-arcs at all but instead may consist entirely of singular control.

Consider a nonlinear system described by the state variable equations

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^2 x_2 u \\ \dot{x}_2 &= - x_2 + u \end{aligned} \qquad (4.43)$$

where the control u is bounded and given as

$$|u(t)| \leq 1 \tag{4.44}$$

It should be noted that $x_2 \neq \dot{x}_1$ in (4.43).

The problem is to transfer the system (4.43) from the initial condition

$$x_{1o} = +1 \quad , \quad x_{2o} = 0 \tag{4.45}$$

to the terminal condition

$$x_{1T} = 2 \qquad x_{2T} = 0 \tag{4.46}$$

in minimum elapsed time. Thus

$$J[u] = \int_{t_o}^{T} 1 \, dt \tag{4.47}$$

The Hamiltonian is written

$$H = p_1(x_1^2 - x_1^2 x_2 u) + p_2(- x_2 + u) - 1 \tag{4.48}$$

and the adjoint equations are

$$\begin{aligned} \dot{p}_1 &= p_1(- 2x_1 + 2x_1 x_2 u^*) \\ \dot{p}_2 &= p_1 x_1^2 u^* + p_2 \end{aligned} \tag{4.49}$$

Since the problem is to minimize the elapsed time $T - t_o$, it is seen from (1.73) and (1.79) that

$$H^*(t) \equiv 0 \tag{4.50}$$

The singular control surface is determined from the conditions (3.2) where

$$I(x, p) = p_1 x_1^2 - p_2 x_2 - 1$$
$$F(x, p) = p_2 - p_1 x_1^2 x_2 \qquad (4.51)$$

Thus

$$S: \quad x_1^2 x_2 = 0 \qquad (4.52)$$

From (4.52), the singular control surface for this problem has two branches

$$S_1: \quad x_1 \equiv 0$$
$$S_2: \quad x_2 \equiv 0 \qquad (4.53)$$

The singular control for each branch of S is determined from (4.43) and (4.53)

$$S_1: \quad u_3^* = \frac{1}{x_2}$$
$$S_2: \quad u_3^* = 0 \qquad (4.54)$$

Because of the constraint (4.44) on u the S_1 surface (line) must be truncated as shown in Figure 4.4. The singular sub-arc S_1 has no significance in this problem because of the particular initial and terminal points (4.45) and (4.46) which have been specified. It is clear from Fig. 4.4 however, that the singular sub-arc S_2 connects the specified initial and terminal points and thus qualifies as an extremal control. The allowable switching direction regions are determined from (3.31)

$$\sigma = \operatorname{sgn}\left[\frac{x_2}{1 - (x_2)^2}\right] \qquad (4.55)$$

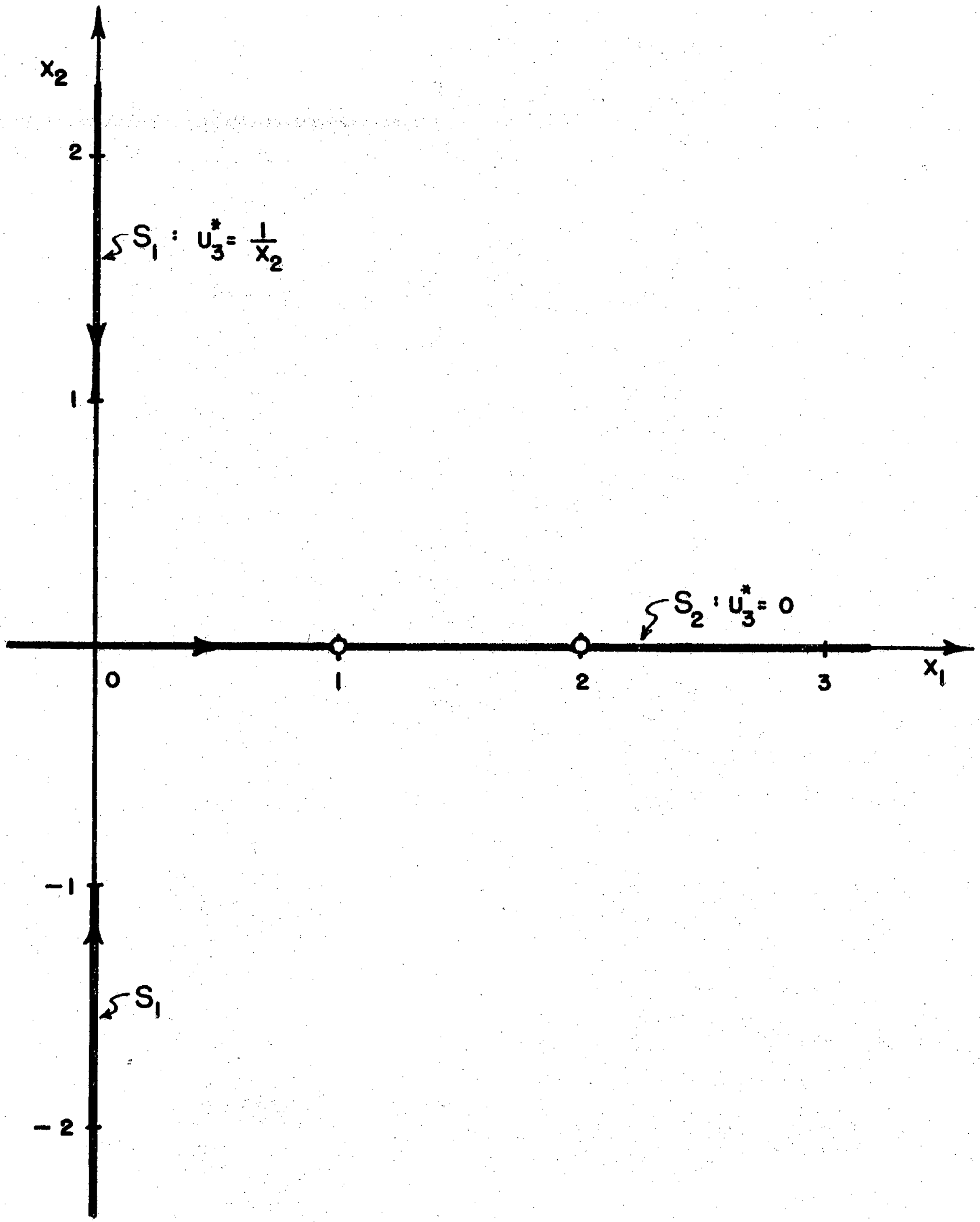

Figure 4.4. Singular Control Surface for Example 4.3

It is clear from (4.55) that only one consecutive switch of the control is allowed in each of the regions $0 < x_2 < 1$, and $-1 < x_2 < 0$. The possible bang-bang extremal controls can be determined by tracing, from the given initial and terminal points, the system trajectories corresponding to $u(t) \equiv +1$ and $u(t) \equiv -1$. By this means, using both forward and reverse time, it is determined that the only possible bang-bang extremal controls are those shown in Figure 4.5. It is seen from Figure 4.5 that the bang-bang extremal sub-arcs are not compatible with the allowable switching direction regions. Therefore the time optimal control for this problem must be exclusively singular control along S_2

$$u^*(t) \equiv 0 \qquad (4.56)$$

Example 4.4.

Singular solutions usually arise in those LOP in which the system equations (1.82) and/or the integrand of the index of performance (1.83) are nonlinear in the x_i. Because of this nonlinear character, it is usually difficult to establish (by analytical means) general conclusions about the optimality of singular solutions which appear in a given class of LOP.

In this example, a particular class of n^{th} order LOP are considered in which

a) the system equations (1.82) are linear in the state variables x_i and separable [e.g., h_i = constant] in the control variable and

b) the integrand of the index of performance (1.83) is a positive semi-definite quadratic form in the x_i with $h_o = 0$.

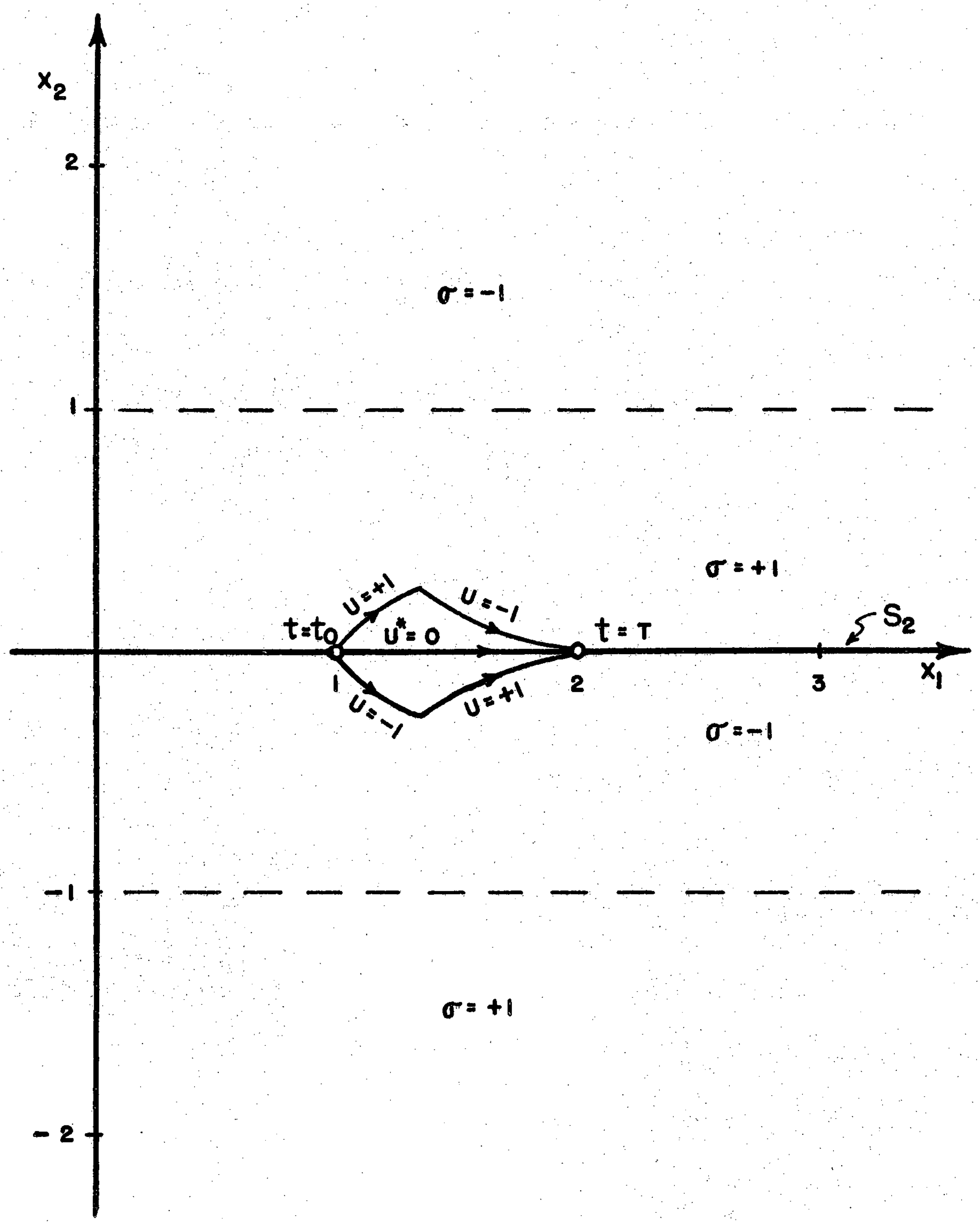

Figure 4.5. Bang-Bang and Singular Extremal Paths for Example 4.3

The initial conditions on all the individual x_i are considered given and the specified terminal condition is $x_1 = x_2 = \dots = x_n = 0$. For this particular class of n^{th} order LOP it is possible to draw somewhat general conclusions concerning the optimality of the singular solutions which appear. The material in this example is a condensed version of [20].

The system equations and index of performance for this particular class of LOP are of the form

$$\dot{x}_i = a_{i1}\, x_1 + \dots + a_{in}\, x_n + u\, b_i \qquad \begin{array}{l} (i = 1, \dots, n) \\ (a_{ij},\ b_i = \text{constant}) \\ (j = 1, \dots, n) \end{array} \tag{4.57}$$

$$J[u] = \int_0^T \left(\frac{1}{2} \sum q_{ij}\, x_i\, x_j \right) dt \qquad \begin{array}{l} (i,\ j = 1, \dots, n) \\ (q_{ij} = \text{constant}) \end{array} \tag{4.58}$$

It is assumed that the quadratic form in (4.58) is positive semi-definite. The control is constrained by the relation

$$|u(t)| \leq 1 \qquad 0 \leq t \leq T \tag{4.59}$$

The initial conditions $x_i(0)$ $(i = 1, \dots, n)$ are assumed to be individually specified and the desired terminal state is assumed to be the origin $x_1(T) = x_2(T) = \dots = x_n(T) = 0$. The terminal time T is assumed free.

It is assumed that the system (4.57) is controllable [20] so that the vectors

$$\underset{\sim}{b}, \ \underset{\sim}{ab}, \ \dots, \ \underbrace{a^{n-1}\, b} \tag{4.60}$$

are linearly independent. In (4.60) the quantities $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are defined as the matrix of coefficients of (4.57). Thus,

$$\underset{\sim}{a} \triangleq \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{4.61}$$

$$\underset{\sim}{b} \triangleq \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{4.62}$$

When the system (4.57) is controllable it can be shown [20], [33], that there is no loss of generality in assuming (4.57) is of the phase variable form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= a_1 x_1 + a_2 x_2 + \dots + a_n x_n + u \end{aligned} \tag{4.63}$$

$$(a_i = \text{constant})$$

and that (4.58) is of the diagonal form

$$J[u] = \int_0^T \frac{1}{2} (q_1 x_1^2 + q_2 x_2^2 + \dots + q_n x_n^2) dt \tag{4.64}$$

$$(q_i = \text{constant} \geq 0)$$

The Hamiltonian for the system (4.63), (4.64) is

$$H = p_1 x_2 + p_2 x_3 + \dots + p_{n-1} x_n + p_n(a_1 x_1 + \dots + a_n x_n)$$
$$- \frac{1}{2} (q_1 x_1^2 + \dots + q_n x_n^2) + u p_n \qquad (4.65)$$

and the adjoint equations are

$$\begin{aligned} \dot{p}_1 &= - a_1 p_n + q_1 x_1 \\ \dot{p}_2 &= - p_1 - a_2 p_n + q_2 x_2 \\ &\vdots \\ \dot{p}_n &= - p_{n-1} - a_n p_n + q_n x_n \end{aligned} \qquad (4.66)$$

From (1.74), it is seen that for this problem

$$H^*(t) \equiv 0 \qquad (0 \leq t \leq T) \qquad (4.67)$$

The optimal control u* is, from (4.65),

$$u^*(t) = \operatorname{sgn} p_n(t) \qquad (4.68)$$

unless the singular condition $p_n(t) \equiv 0$ occurs. The test for a singular solution is carried out by applying the conditions (3.2) where

$$I = p_1 x_2 + \dots + p_{n-1} x_n + p_n(a_1 x_1 + \dots + a_n x_n)$$
$$- \frac{1}{2} (q_1 x_1^2 + \dots + q_n x_n^2) \qquad (4.69)$$

$$F = p_n \qquad (4.70)$$

The condition $F(t) = \dot{F}(t) = \ddot{F}(t) = \dots \equiv 0$ together with the adjoint equations (4.66) yields the linear differential equation

$$q_n \frac{d^{2(n-1)}x_1}{dt^{2(n-1)}} - q_{n-1} \frac{d^{2(n-2)}x_1}{dt^{2(n-2)}} + q_{n-2} \frac{d^{2(n-3)}x_1}{dt^{2(n-3)}} - \dots + (-1)^{n-2} q_2 \frac{d^2x_1}{dt^2} +$$

$$+ (-1)^{n-1} q_1 x_1 = 0 \tag{4.71}$$

The characteristic equation of (4.71) is

$$q_n \lambda^{2(n-1)} - q_{n-1} \lambda^{2(n-2)} + q_{n-2} \lambda^{2(n-3)} - \dots + (-1)^{n-2} q_2 \lambda^2$$

$$+ (-1)^{n-1} q_1 = 0 \tag{4.72}$$

It is clear that the $2(n-1)$ eigenvalues of (4.72) occur in pairs $(\lambda_m, -\lambda_m)$. Thus, although (4.72) is of order $2(n-1)$, it can be treated as an equation of $(n - 1)$ order in the λ^2. It will be assumed hereafter that $q_1 > 0$, $q_n > 0$ and that the $(n-1)$ eigenvalues λ_m are distinct. The assumption $q_1 > 0$ assures that (4.72) does not possess zero or pure imaginary roots of the form $\lambda = i\,\omega$ $(i = \sqrt{-1}$, $\omega =$ real). The assumption $q_n > 0$ assures that (4.72) is of degree $2(n-1)$ and thus possesses $2(n-1)$ roots.

Under the assumptions given above, the solution to (4.71) can be written

$$x_1 = \Theta_1 e^{\lambda_1 t} + \Theta_2 e^{\lambda_2 t} + \dots + \Theta_{n-1} e^{\lambda_{n-1} t} + \Theta_n e^{-\lambda_1 t} +$$

$$+ \Theta_{n+1} e^{-\lambda_2 t} + \dots + \Theta_{2(n-1)} e^{-\lambda_{n-1} t} \tag{4.73}$$

where the Θ_j are constants of integration. The expression for the singular control surface $S(x_1,\dots,x_n) = 0$ is simply a first integral of (4.63) corresponding to the condition (4.73). It is assumed that the terms in

(4.73) are grouped so that the first (n-1) terms on the right correspond to the λ_m having negative real parts. Then, (4.73) can be considered as the sum of two parts; a stable part

$$x_1 = \theta_1 e^{\lambda_1 t} + \theta_2 e^{\lambda_2 t} + \dots + \theta_{n-1} e^{\lambda_{n-1} t} \tag{4.74}$$

and an unstable part

$$x_1 = \theta_n e^{-\lambda_1 t} + \theta_{n+1} e^{-\lambda_2 t} + \dots + \theta_{2(n-1)}{}^{-\lambda_{n-1} t} \tag{4.75}$$

Each of the solutions (4.74) and (4.75) will furnish a first integral[11] $S(x_1, \dots, x_n) = 0$ for (4.63). For the particular boundary conditions considered in this example, only the first integral corresponding to (4.74) is of interest [20].

The first integral of (4.63) corresponding to (4.74) is given by the linear expression

$$S: \quad c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 0 \tag{4.76}$$

which defines an (n-1) dimensional hyperplane passing through the origin of the x-space. The c_i $(i = 1, \dots, n)$ in (4.76) are determined from the set of (n-1) equations

$$\begin{aligned} c_1 + c_2\lambda_1 + c_3\lambda_1^2 + \dots + c_n\lambda_1^{n-1} &= 0 \\ c_1 + c_2\lambda_2 + c_3\lambda_2^2 + \dots + c_n\lambda_2^{n-1} &= 0 \\ \vdots \qquad\qquad\qquad &\ \vdots \\ c_1 + c_2\lambda_{n-1} + c_3\lambda_{n-1}^2 + \dots + c_n\lambda_{n-1}^{n-1} &= 0 \end{aligned} \tag{4.77}$$

[11]However, these are not the only possibilities, in general. For specified terminal states other than the origin, some of the other possible first integrals of (4.73) may be of interest.

It may be noted from (4.77) that one of the c_i may be chosen arbitrarily. The singular control function $u_3{}^*$ corresponding to the singular control surface (4.76) is obtained by taking one time derivative of (4.76) and substituting (4.63). Thus the singular control in state variable feedback form is

$$u_3{}^* = - a_1 x_1 - \left(\frac{a_2 c_n + c_1}{c_n}\right) x_2 - \left(\frac{a_3 c_n + c_2}{c_n}\right) x_3 - \cdots - \left(\frac{a_n c_n + c_{n-1}}{c_n}\right) x_n \tag{4.78}$$

The assumption $q_1 > 0$, $q_n > 0$ assures that $c_1 \neq 0$, $c_n \neq 0$. It is interesting to note that the singular control (4.78) effectively cancels out all the existing feedback terms on the right of (4.63) and leaves the system (4.63) in the new linear form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \frac{1}{c_n}(- c_1 x_2 - c_2 x_3 - \cdots - c_{n-1} x_n) \end{aligned} \tag{4.79}$$

The values of the adjoint variables p_i corresponding to points on the singular control surface S can be determined by setting $p_n = \dot{p}_n = \ddot{p}_n = \cdots \equiv 0$ in (4.66) and substituting (4.79). By this means, the p_i can be determined as unique linear combinations of the state variables $x_1, \ldots, x_n$

$$p_i = \sum_{j=1}^{n} b_{ij} x_j \qquad \begin{matrix} (i = 1, \ldots, n) \\ (b_{ij} = \text{constant}) \end{matrix} \tag{4.80}$$

From the relation (4.76), x_n on the singular hypersurface can be expressed as a linear function of the x_i $(i = 1, \ldots, n-1)$

$$x_n = \frac{1}{c_n}(-c_1 x_1 - c_2 x_2 - \ldots - c_{n-1} x_{n-1}) \tag{4.81}$$

By means of (4.81), the singular control (4.78) and the adjoint variables (4.80) may be expressed as linear functions of $x_1, \ldots, x_{n-1}$. It is remarked, however, that the substitution of (4.81) into (4.78) may yield a singular control which is unstable. That is, a control which causes the system trajectory to diverge from, rather than follow, S. The singular control defined by (4.78) will be real-valued if all complex λ occur in conjugate pairs. This is assured if the q_i $(i = 1, \ldots, n)$ in (4.64) are all real-valued. In particular, if $q_1 = q_2 = \ldots = q_n = + 1$ then the λ_m in (4.74) are the stable roots of the equation

$$\lambda^{2(n-1)} - \lambda^{2(n-2)} + \lambda^{2(n-3)} - \ldots + (-1)^{n-2}\lambda^2 + (-1)^{n-1} = 0 \tag{4.82}$$

Setting

$$r \triangleq \lambda^2 \tag{4.83}$$

in (4.82), and multiplying (4.82) by the factor $(r + 1)$ it is seen that

$$r = \left[(-1)^n\right]^{1/n} \tag{4.84}$$

Thus, the $(n - 1)$ roots r_j are simply the n roots of $(-1)^n$ less the one artificially introduced root $r = -1$. Graphically, the n roots of (4.84) lie evenly spaced on the unit circle of the Argand diagram [34] as shown in Figure 4.6-a. The desired $2(n - 1)$ roots $\lambda_j, -\lambda_j$ are therefore given by

$$\lambda_j = +\sqrt{r_j} \qquad \begin{matrix}(j = 1, \ldots, n-1) \\ (r_j \neq -1)\end{matrix} \qquad (4.85)$$

where the r_j are given by (4.84). It is clear from (4.84) and (4.85) that the $2(n-1)$ roots λ_j, $-\lambda_j$ also lie on the unit circle of the Argand diagram and all complex λ_j appear in conjugate pairs as shown in Figure 4.6-b.

In [20], it is shown[12] that the value of the index of performance (4.64) for any arbitrary path $x_1(t), \ldots, x_n(t)$ between any two points α and β on S is given by

$$J[u] = \int_{\alpha}^{\beta} \frac{1}{2}(q_1x_1^2 + q_2x_2^2 + \ldots + q_nx_n^2)dt$$

$$= \int_{\alpha}^{\beta} \left[\frac{1}{2}(c_1x_1 + c_2x_2 + \ldots + c_nx_n)^2 - \frac{dV}{dt}\right]dt \qquad (4.86)$$

where $\sum_i^n c_ix_i = 0$ is the expression for S given by (4.76) and V is the value of min $J[u]$ for paths on S. It may be recalled from art. 3.3 that V is an exact differential for all paths lying entirely on S. Since the desired terminal point $x_{1T} = x_{2T} = \ldots = x_{nT} = 0$ is a point on S it is clear from (4.86) that, when the restriction (4.59) is absent, the singular sub-arcs on S are strictly optimal with respect to all other paths between any given point $\alpha \in S$ and the origin $x_1 = x_2 = \ldots = x_n = 0$. Then, the value $V(\alpha) = V(x_1,\ldots,x_n)$ corresponding to any point α on S is

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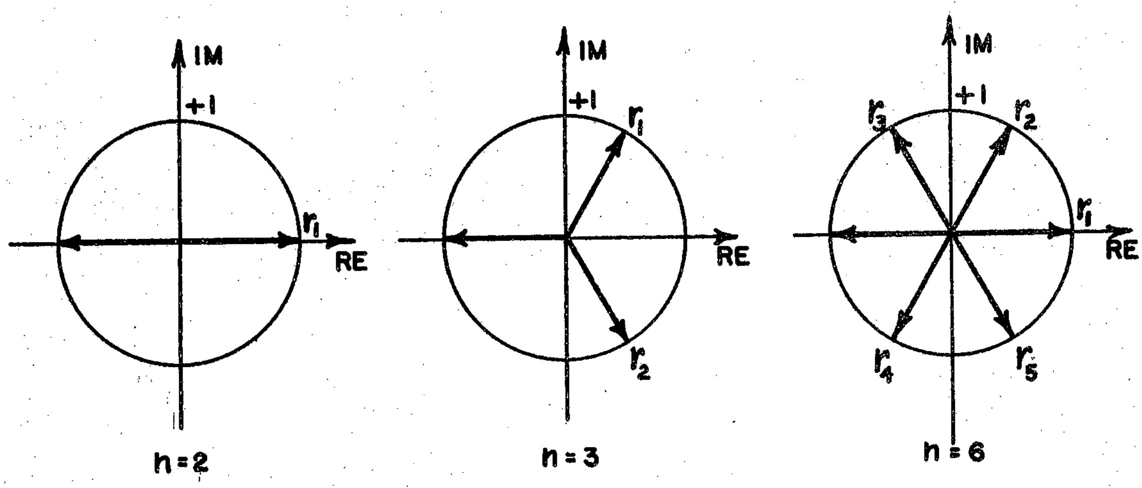
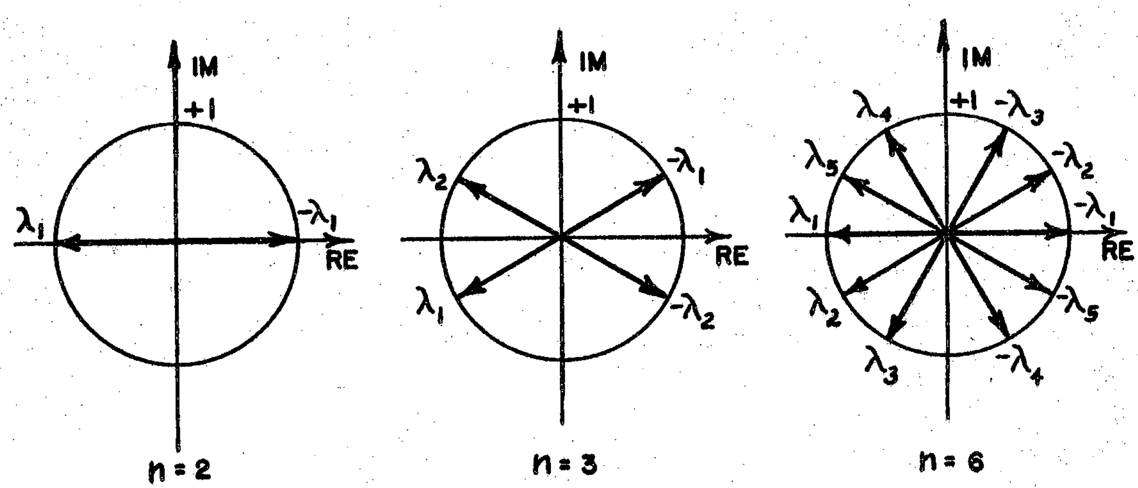
[12] The proof of (4.86), given in [20], is due to W. M. Wonham



(a)
Roots of (4.84)



(b)
Roots of (4.85)

Figure 4.6. Roots of Equations (4.84) and (4.85) on the Argand Diagram

the optimal (minimum) value of $J[u]$.

When the control constraint (4.59) is introduced, the points α corresponding to $|u_3^*(t)| \leq 1$ will lie in a strip $\bar{S}$ of S. The expressions for the boundaries of S are determined by setting $u = \pm 1$ in (4.78). In order that the previously derived expressions $V(\alpha)$ (for points in $\bar{S}$) remain equal to min $J[u]$ it is necessary that the singular trajectory (which starts at a point α in $\bar{S}$) remain in $\bar{S}$ at all times. In [20] it is shown that the region R in which the above condition is satisfied is an $(n - 1)$-dimensional convex subset of the strip $\bar{S}$. If at $t = t_1$ a bang-bang extremal arc should intersect R, then the optimal control for $t_1 \leq t \leq T$ is singular control u_3^* as given by (4.78). Since (4.79) is linear, it is seen that $T = \infty$ when the optimal control sequence includes a singular sub-arc.

From (4.80), the p_i $(i = 1, \ldots, n)$ are known for all points of the optimal paths on R. Thus, the flooding technique of art. 3.3 may be used to trace out (in backward time) the optimal bang-bang sub-arcs which lead to R. By this means, the optimal bang-bang switching boundaries (hypersurfaces) in the x-space may be determined.

The form of (4.76) suggests that a suitable linear transformation of the x-space may permit the S hyperplane to project into a surface of lower dimension. In [20] this technique is employed with the result that (under certain conditions) both the singular control hyperplane S and the bang-bang switching hypersurfaces, corresponding to an n^{th} order system, can be studied as lines and curves in a two-dimensional coordinate system.

As a concrete example[13] of the problem discussed above, consider the

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[13] Taken from [20].

system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \qquad |u| \leq 1 \tag{4.87}$$

with

$$J[u] = \int_0^T \frac{1}{2}(x_1^2 + x_2^2)dt \tag{4.88}$$

$$T = \text{free}$$
$$x_{1o}, x_{2o} = \text{given}$$
$$x_{1T} = x_{2T} = 0$$

From (4.72) the λ are determined from $\lambda^2 - 1 = 0$ so that

$$\lambda = \pm 1 \tag{4.89}$$

The c_i are determined from (4.77) (setting $\lambda_1 = -1$)

$$c_1 = c_2 \tag{4.90}$$

Choosing $c_1 = +1$, the singular control surface S is determined from (4.76)

$$S: \quad x_1 + x_2 = 0 \tag{4.91}$$

and the singular control function u_3^* is obtained from (4.78)

$$u_3^* = -x_2 \tag{4.92}$$

From the constraint $|u| \leq 1$, the S surface (line) must be truncated at $|x_2| = 1$. The region R and the neighboring field of optimal bang-bang sub-arcs (flood paths) are shown in the $x_1 - x_2$ state plant in Figure 4.7. The optimal bang-bang switching boundaries, determined by the flooding technique, are shown in Figure 4.8.

It should be noted that, following (4.86), the index of performance (4.88) can be written in the form

$$J[u] = \int_0^T \frac{1}{2}(x_1 + x_2)^2\, dt + \frac{1}{2} x_1^2(0) \qquad (4.93)$$

It is clear from (4.93) that the minimum of J[u] occurs when

$$x_1(t) = - x_2(t)$$

which defines the singular control trajectory. Thus, on R (4.93) becomes

$$\min J[u] \Big|_R = \frac{1}{2} x_1^2(0) \qquad (4.94)$$

The nature of the S, $\overline{S}$ and R regions is perhaps more clearly seen by considering a third order example. Consider, for instance, the system (triple integrator)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \qquad |u(t)| \leq 1 \end{aligned} \qquad (4.95)$$

with

$$J[u] = \int_0^T \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)dt \qquad (4.96)$$

$$T = \text{free}$$

$$x_{1o}, x_{2o}, x_{3o} = \text{given (arbitrary)}$$

$$x_{1T} = x_{2T} = x_{3T} = 0$$

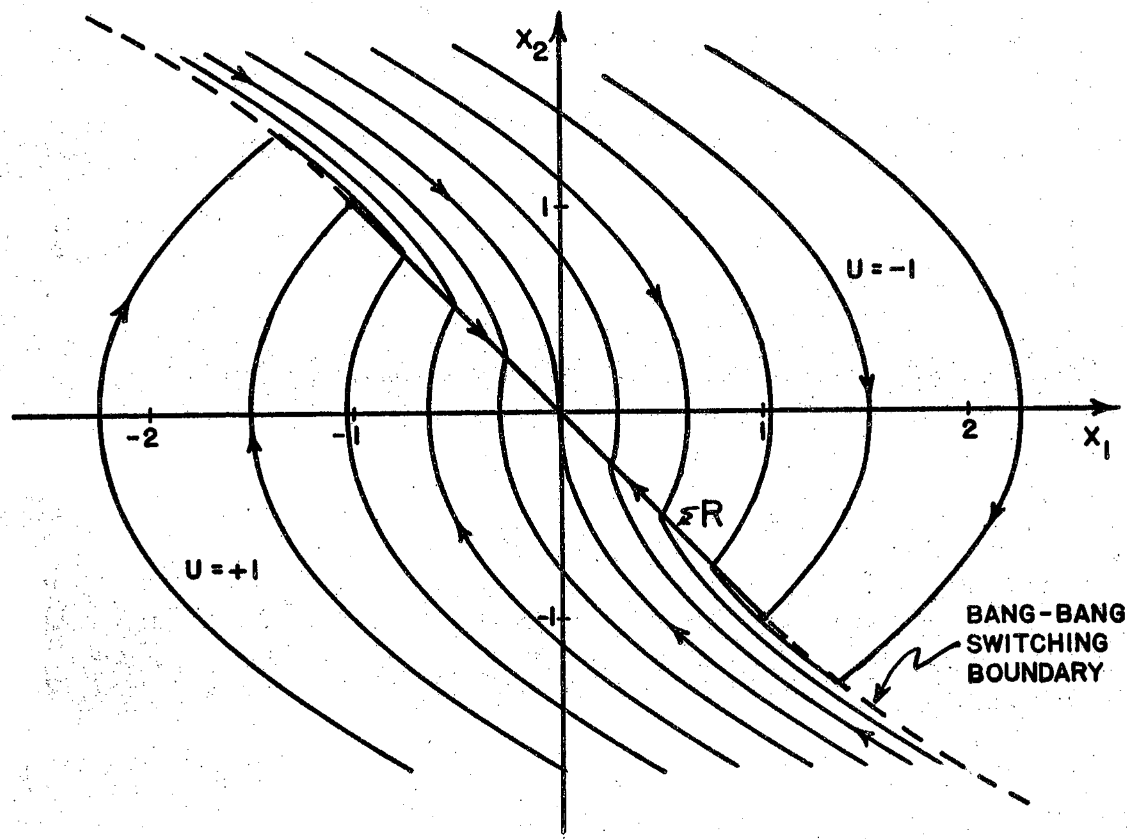

Figure 4.7. Field of Optimal Trajectories Near Origin for Second Order System of Example 4.4

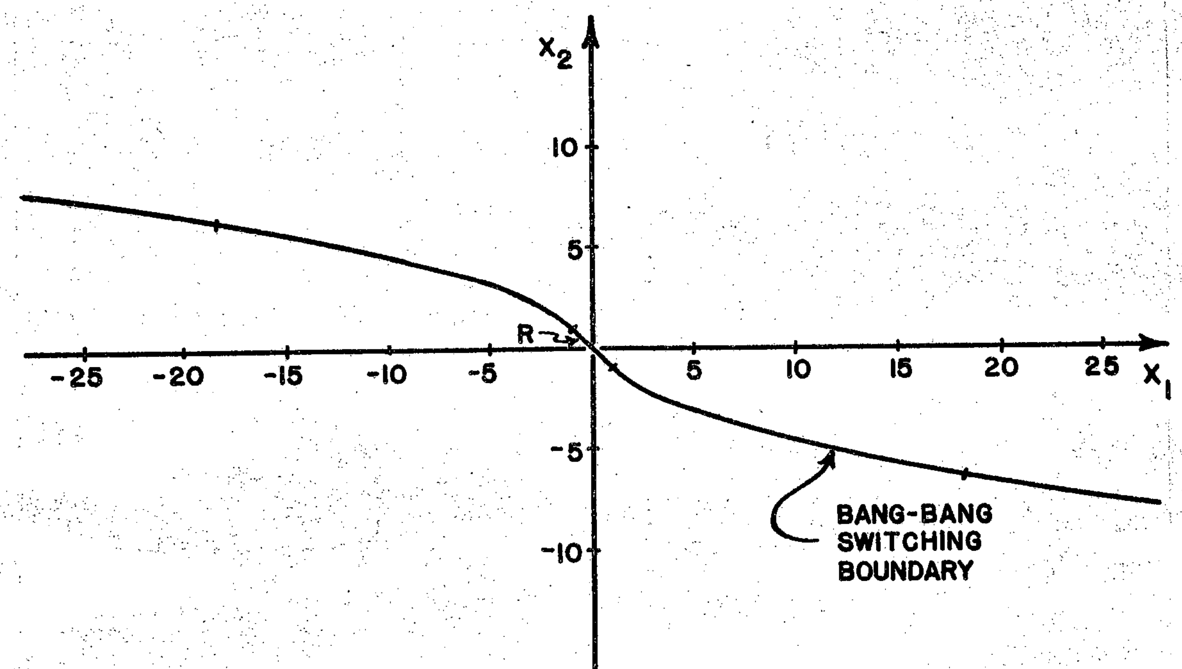

Figure 4.8. Optimal Bang-Bang Switching Boundaries for Second Order System of Example 4.4

Using (4.95), the index of performance (4.96) can be written in the form

$$J[u] = \int_0^T \frac{1}{2} (x_1 + \sqrt{3}\, x_2 + x_3)^2 \, dt + \frac{\sqrt{3}}{2} x_1^2(0) + x_1(0)\, x_2(0) + \frac{\sqrt{3}}{2} x_2^2(0) \tag{4.97}$$

It is clear from (4.97) that the 2-dimensional hyperplane S is given by the expression

$$S: \quad x_1 + \sqrt{3}\, x_2 + x_3 = 0 \tag{4.98}$$

and that the singular paths on S yield J[u] an absolute minimum.

The singular control function u_3^* is

$$u_3^* = - x_2 - \sqrt{3}\, x_3 \tag{4.99}$$

Setting $u_3^* = \pm 1$ in (4.99) the boundaries of $\bar{S}$ are obtained as the lines

$$\begin{aligned} x_2 &= -1 - \sqrt{3}\, x_3 \qquad (u = +1) \\ x_2 &= +1 - \sqrt{3}\, x_3 \qquad (u = -1) \end{aligned} \tag{4.100}$$

on the plane S.

On S, the optimal (singular) trajectories are given by

$$x_1 = e^{-0.5\sqrt{3}\, t} \left[\theta_1 e^{-j0.5t} + \theta_2 e^{+j0.5t} \right] \tag{4.101}$$

so that during singular control, the original third-order system is equivalent to the second order system

$$\frac{d^2 x_1}{dt^2} + \sqrt{3} \frac{dx_1}{dt} + x_1 = 0 \tag{4.102}$$

The boundaries of the subset R of $\overline{S}$ are formed by the two trajectories (4.101) which are tangent to the two lines (4.100).

On R, the p_i are given by

$$
\begin{aligned}
p_1 &= 2x_2 + \sqrt{3}\, x_3 \\
p_2 &= x_3 \\
p_3 &= 0
\end{aligned}
\qquad (4.103)
$$

The singular control surface S, the strip $\overline{S}$, and the region R of optimal singular trajectories are shown in the $x_1 - x_2 - x_3$ state space of Figure 4.9.

Points on the bang-bang switching surface which connects with R can be obtained by using (4.103) as initial conditions and integrating the system and adjoint equations in reverse time (starting at various points on R). The locus of points where $p_3(t) = 0$ determines the bang-bang switching surface.

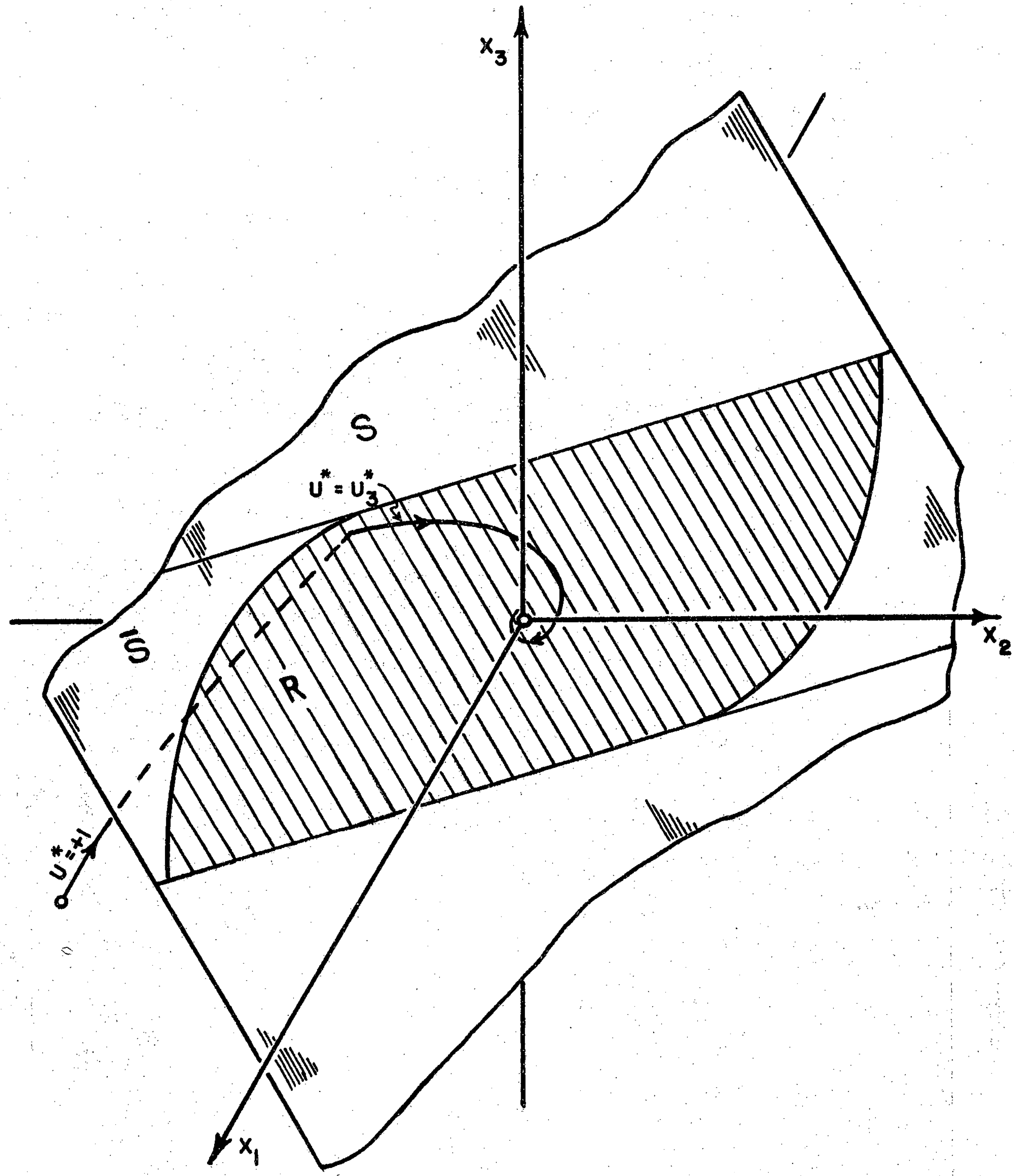

Figure 4.9. Regions S, $\bar{S}$, and R for the Third Order System of Example 4.4

Chapter 5

SUMMARY AND CONCLUSIONS

5.1. Summary

The study of optimal control problems has received considerable attention in recent times. Several mathematical techniques have been used to examine problems of optimal control, the three most important being; the classical Calculus of Variations (with extensions), Dynamic Programming, and the Maximum Principle. Each of these techniques has been discussed, and their close relationship pointed out.

It has been shown that for a certain class of optimal control problems the mathematical techniques given above sometimes fail (formally) to yield any information about the desired optimal control. This particular class of optimal control problems is characterized by the control appearing linearly in the state equations and index of performance. A problem of this type has been called a "linear optimization problem" (LOP).

Usually, the optimal control for a LOP is of the bang-bang type ($u^* = u_{max}$ or $u^* = u_{min}$) and the above mathematical techniques yield this solution quite readily. The apparent failure of these techniques, in the case of certain LOP, has been shown to be caused by optimal controls which are not of the bang-bang type at all times. The mathematical form of the solution to LOP cannot explicitly define the optimal control when it is not bang-bang. Optimal controls (for LOP) which are not of the bang-bang type are called "singular controls", and the corresponding solutions are called "singular solutions".

It has been shown that singular solutions are characterized by the bang-bang switching function becoming identically zero. This characteristic allows the construction of a surface S in the x-space (or x, p-space) which is the locus of all singular paths. The control $u^*(x, p)$ which maintains the singular condition can be obtained from the expression for S. A scheme involving the backward time solution of the canonical equations has been used to help establish the role of singular sub-arcs in the solution of LOP. It has been shown that the physical realizability of singular control conditions is not sufficient to establish the optimality of singular control.

Several examples with varying degree of complexity have been worked in detail to illustrate the proposed techniques for solving LOP with singular solutions.

5.2. Suggestions for Further Work

The methods given in this report are primarily intended for analytical studies of optimal controls for LOP. There is, at present, a large amount of effort being directed toward the study of optimal control processes by means of numerical searching methods employing digital computers [35], [36]. The detection of singular controls by such methods has received very little attention, and further work in this area would seem to be of practical importance [37].

In general, the optimality of singular solutions is difficult to establish except by actual numerical comparison. The development of analytical techniques to determine the optimality of singular solutions would be an important contribution.

This study has been primarily concerned with LOP having one control variable (with constant constraints). The study of LOP having multivariable control (with variable constraints on the controls) would seem appropriate in view of the present trends in optimal control theory.

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Appendix I

GENERAL EXPRESSIONS FOR THE SINGULAR CONTROL SURFACE S

The General First Order LOP

In the case of a general first order LOP, the system equations (1.82) and index of performance (1.83) are

$$\dot{x}_1 = g_1(x_1) + u\, h_1(x_1) \tag{A.1}$$

$$J[u] = \int_0^T \left[g_o(x_1) + u\, h_o(x_1) \right] dt \tag{A.2}$$

The singular control conditions (3.2) are then given by

$$I \equiv 0 \Rightarrow p_1 g_1 - g_o \equiv 0$$

$$F \equiv 0 \Rightarrow p_1 h_1 - h_o \equiv 0$$

$$\dot{I} \equiv 0 \Rightarrow u \left[p_1 \left(h_1 \frac{\partial g_1}{\partial x_1} - g_1 \frac{\partial h_1}{\partial x_1} \right) + g_1 \frac{\partial h_o}{\partial x_1} - h_1 \frac{\partial g_o}{\partial x_1} \right] \equiv 0$$

$$\dot{F} \equiv 0 \Rightarrow \left[p_1 \left(h_1 \frac{\partial g_1}{\partial x_1} - g_1 \frac{\partial h_1}{\partial x_1} \right) + g_1 \frac{\partial h_o}{\partial x_1} - h_1 \frac{\partial g_o}{\partial x_1} \right] \equiv 0 \tag{A.3}$$

It may be noted that when $u \neq 0$, the expressions for $\dot{I}$ and $\dot{F}$ are not independent.[14] The simultaneous solution of (A.3) yields the following ex-

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[14] This is true for any order LOP of the form (1.82), (1.83).

pression for S.

$$S: \quad h_1\left(-g_1\frac{\partial h_o}{\partial x_1}+h_1\frac{\partial g_o}{\partial x_1}\right)+h_o\left(g_1\frac{\partial h_1}{\partial x_1}-h_1\frac{\partial g_1}{\partial x_1}\right)=0 \tag{A.4}$$

The general expression for the singular control function u_3^* can be obtained by taking the time derivative of (A.4) and substituting (A.1). The result is

$$u_3^* = -\frac{g_1}{h_1} \tag{A.5}$$

If a singular solution does not exist for a particular first order LOP then (A.4) will degenerate to vacuous or impossible conditions. For instance, if $g_o = 1$, and $h_o = 0$ (i.e., for time optimal control) then (A.4) reduces to $0 = 0$ and no singular solutions exist. It is clear from (A.4) and (A.5) that when singular solutions do appear in first order LOP they correspond to the trajectory $x_1(t) \equiv$ constant.

The General Second Order LOP

In the case of a general second order LOP, the system equations (1.82) and index of performance (1.83) are

$$\dot{x}_1 = g_1(x_1, x_2) + u\, h_1(x_1, x_2) \tag{A.6}$$

$$\dot{x}_2 = g_2(x_1, x_2) + u\, h_2(x_1, x_2) \tag{A.7}$$

$$J[u] = \int_o^T \left[g_o(x_1, x_2) + u\, h_o(x_1, x_2)\right] dt \tag{A.8}$$

The expressions for I, $\dot{I}$, F, and $\dot{F}$ are obtained in the same manner as for the first order LOP and their simultaneous solution yields the following expression for S.

$$S: \left(g_o h_2 - h_o g_2\right)\left(- g_1 \frac{\partial h_1}{\partial x_1} - g_2 \frac{\partial h_1}{\partial x_2} + h_1 \frac{\partial g_1}{\partial x_1} + h_2 \frac{\partial g_1}{\partial x_2}\right) +$$

$$\left(g_1 h_o - h_1 g_o\right)\left(- g_1 \frac{\partial h_2}{\partial x_1} - g_2 \frac{\partial h_2}{\partial x_2} + h_1 \frac{\partial g_2}{\partial x_1} + h_2 \frac{\partial g_2}{\partial x_2}\right) +$$

$$\left(g_2 h_1 - h_2 g_1\right)\left(- g_1 \frac{\partial h_o}{\partial x_1} - g_2 \frac{\partial h_o}{\partial x_2} + h_1 \frac{\partial g_o}{\partial x_1} + h_2 \frac{\partial g_o}{\partial x_2}\right) = 0 \qquad \text{(A.9)}$$

The general expression for the singular control function $u_3{}^*$ can be obtained by taking the total time derivative of (A.9) and substituting (A.6) and (A.7). The resulting expression is rather long and is not given here.

A Particular Third Order LOP

Consider the particular class of third order LOP having the phase variable form

$$\dot{x}_1 = x_2 \qquad \text{(A.10)}$$

$$\dot{x}_2 = x_3 \qquad \text{(A.11)}$$

$$\dot{x}_3 = g(x_1, x_2, x_3) + u\, h(x_1, x_2, x_3) \qquad \text{(A.12)}$$

with the index of performance

$$J[u] = \int_o^T g_o(x_1, x_2, x_3)dt \qquad \text{(A.13)}$$

Following the same procedure as above, the results are:

a) If

$$\frac{\partial^2 g_o}{\partial x_3{}^2} \neq 0$$

then

$$u_3^* = \frac{-\frac{g_o}{x_2} + \frac{x_3}{x_2}\frac{\partial g_o}{\partial x_3} + \frac{\partial g_o}{\partial x_2} - x_2 \frac{\partial^2 g_o}{\partial x_1 \partial x_3} - x_3 \frac{\partial^2 g_o}{\partial x_2 \partial x_3} - g \frac{\partial^2 g_o}{\partial x_3^2}}{h \frac{\partial^2 g_o}{\partial x_3^2}} \tag{A.14}$$

b) If

$$\frac{\partial^2 g_o}{\partial x_3^2} = 0$$

then

$$S: \quad g_o - x_3 \frac{\partial g_o}{\partial x_3} - x_2 \frac{\partial g_o}{\partial x_2} + x_2^2 \frac{\partial^2 g_o}{\partial x_1 \partial x_3} + x_2 x_3 \frac{\partial^2 g_o}{\partial x_2 \partial x_3} = 0 \tag{A.15}$$

Appendix II

TRANSFORMATION OF MULTIDIMENSIONAL LINE INTEGRALS INTO LOP

Consider the problem of minimizing (or maximizing) the ordinary multidimensional line integral in the $(x_1,\ldots,x_n)$ space

$$J = \int_a^b f_1(x_1,\ldots,x_n)dx_1 + f_2(x_1,\ldots,x_n)dx_2 + \ldots + f_n(x_1,\ldots,x_n)dx_n \tag{A.16}$$

According to the classical Calculus of Variations, a necessary condition for the integral (A.16) to be a minimum (or maximum) is that Euler's equation (1.6) should be satisfied. Thus, if (say) x_1 is chosen as the independent variable[15] in (A.16) then the Euler equations for (A.16) are

$$\frac{d}{dx_1}\left(\frac{\partial G}{\partial \dot{x}_i}\right) = \frac{\partial G}{\partial x_i} \qquad (i = 2, 3, \ldots, n) \quad \left(\cdot \triangleq \frac{d}{dx_1}\right) \tag{A.17}$$

where

$$G = f_1 + f_2 \frac{dx_2}{dx_1} + \ldots + f_n \frac{dx_n}{dx_1} \tag{A.18}$$

Expanding (A.17), the Euler equations become

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15 The choice of the x_i which will be the independent variable is somewhat arbitrary. The only requirement is that the x_i chosen should be strictly increasing along the optimal path.

$$\left(\frac{\partial f_i}{\partial x_1} - \frac{\partial f_1}{\partial x_i}\right) + \left(\frac{\partial f_i}{\partial x_2} - \frac{\partial f_2}{\partial x_i}\right) \frac{dx_2}{dx_1} + \dots + \left(\frac{\partial f_i}{\partial x_n} - \frac{\partial f_n}{\partial x_i}\right) \frac{dx_n}{dx_1} = 0 \qquad \text{(A.19)}$$

$$(i = 2, 3, \dots, n)$$

The (n - 1) equations (A.19) can be satisfied if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad \begin{matrix} (i = 2,3,\dots,n) \\ (j = 1,2,\dots,n) \end{matrix} \qquad \text{(A.20)}$$

It is clear that the solutions (A.20) do not involve any arbitrary constants and therefore the rigid trajectory defined by (A.20) cannot be made to pass through arbitrary points a and b in the x-space. In the Calculus of Variations, the solutions (A.20) are known as degenerate solutions to the Euler equations. It is well known that line integrals of the form (A.16) will become independent of path when the integrand becomes an exact differential. The degenerate Euler equations (A.20) are recognized as the necessary and sufficient conditions that (A.16) be independent of path in a simply connected region of the x-space [1, pg. 91]. Thus, if the equations (A.20) define a surface S in the x-space, then the integral (A.16) will be independent of path for all paths lying entirely on S.

Consider now, the problem of minimizing the integral (A.16) under the additional constraints that the differentials dx_i $(i = 1, \dots, n)$ must satisfy the following parametric relations along the optimal trajectory

$$\frac{dx_i}{d\sigma} = g_i(x_1,\dots,x_n) + u\, h_i(x_1,\dots,x_n) \quad (i = 1,\dots,n) \qquad \text{(A.21)}$$

where $u = u(\sigma)$ can be chosen arbitrarily subject to $u \in U$ and σ is a parameter which is monotonic increasing along the optimal trajectory. Substituting the parametric constraint equations (A.21) into (A.16), the integral to be minimized can be written as

$$J[u] = \int_{\sigma=\sigma_a}^{\sigma=\sigma_b} \left[g_o(x_1,\ldots,x_n) + u\, h_o(x_1,\ldots,x_n) \right] d\sigma \tag{A.22}$$

where

$$g_o \triangleq f_1\, g_1 + \ldots + f_n\, g_n$$

$$h_o \triangleq f_1\, h_1 + \ldots + f_n\, h_n$$

The problem of minimizing (A.22) subject to the constraints (A.21) is recognized as the general LOP (1.82), (1.83).

The rigid trajectories defined by the degenerate Euler equations (A.20) represent candidates for unconstrained optimal trajectories which minimize the integral (A.16). Thus, when the singular condition occurs during the optimization of the (constrained) LOP (A.21), (A.22) it implies that the particular trajectory being followed simultaneously satisfies necessary conditions for both constrained and unconstrained optimal trajectories. The surface of exact differential defined by (A.20) is the singular control surface S corresponding to the singular condition (3.2). Since the Euler equations (A.20) are necessary conditions for the optimal trajectory it is clear that if the integral (A.16) has a finite minimum then the unconstrained optimal trajectory must be given by (A.20) (i.e., it must be singular). It is entirely possible, however, that (A.20) will define a trajectory(s) even when the integral (A.16) has no finite minimum for unconstrained trajectories. When the specified boundary conditions a, and b do not lie on the surface (A.20) then, since Euler's equations cannot be satisfied, an unconstrained optimal trajectory does not exist. However, when the constraints (A.21) are imposed, the integral (A.16) may very well have a finite minimum and in this case the singular trajectories defined by (A.20) may constitute part of the constrained optimal trajectory.

Every line integral of the type (A.16) with the constraints (A.21) can be easily transformed to the form (A.22). From this, it would appear that the Euler equations (A.20) should furnish a convenient method for obtaining the expression for S in LOP. However, when the integral (A.22) is given first, the reverse transformation from (A.22) to (A.16) is unique only in the case of first and second order LOP.